①

Lagrange Interpolating polynomial

x	x_0	x_1	x_2
y	$f(x_0)$	$f(x_1)$	$f(x_2)$

$$P_2(x) = a_0(x-x_1)(x-x_2) + a_1(x-x_0)(x-x_2) + a_3(x-x_0)(x-x_1)$$

$$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

Ex Using Lagrange interpolation find $y(9)$

x	5	7	11	13
$y(x)$	150	392	1452	2366

$$P_3(x) = \frac{(x-7)(x-11)(x-13)}{(5-7)(5-11)(5-13)}(150) +$$

Ex Evaluate $\log_{10} 300$ using Lagrange's formula from the given data $\log_{10} 300 = 2.4771$

$\log_{10} 304 = 2.4829$ $\log_{10} 305 = 2.4843$

$\log_{10} 307 = 2.4871$

x	300	304	305	307
$y = \log_{10} x$	2.4771	2.4829	2.4843	2.4871

The calculus of finite differences

Forward Differences

$$\Delta f(a) = f(a+h) - f(a)$$

$$\Delta f(x) = f(x+h) - f(x)$$

Backward Differences

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(a+h) = f(a+h) - f(a)$$

$$\nabla f(a+2h) = f(a+2h) - f(a+h)$$

$$\vdots$$

$$\nabla f(a+nh) = f(a+nh) - f(a+(n-1)h)$$

Operator of E and Δ

$$E f(x) = f(x+h) \qquad E^{-1} f(x) = f(x-h)$$

$$E^{n} f(x) = f(x+nh) \qquad E^{-n} f(x) = f(x-nh)$$

Relation between the operators

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = E f(x) - f(x)$$

$$\Delta f = (E - I) f$$

$$\boxed{\Delta = E - I} \Rightarrow E = I + \Delta$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x) = f(x) - E^{-1} f(x) \Rightarrow \boxed{\nabla = I - E^{-1}}$$

$$\Rightarrow E^{-1} = I - \nabla$$

Ex prove that $\cancel{x}\left(\frac{\Delta^2}{E}\right)e^x$ (3)

(2)

① Find the value of E^2x^2 when the value of x vary by a constant increament of 2.

Soln:
$$E(x) = (x+2)^2$$
$$E^2x = (x+4)^2 = x^2 + 8x + 16$$

② Find $E^n e^x = e^{x+nh}$

③ Find the value of $\Delta f(x) = \Delta \tan^{-1}x$

Ans
$$\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$$
$$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$$
$$= \tan^{-1}\left\{\frac{h}{1+x^2+hx}\right\}$$

(4) prove that $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$

Soln
$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$
$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$
$$= \log\left[\frac{E f(x)}{f(x)}\right]$$
$$= \log\left[\frac{(I+\Delta) f(x)}{f(x)}\right]$$
$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Ex Prove that $\left(\frac{\Delta^2}{E}\right)e^x$

$E(e^x) = e^{x+h}$

$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$\Delta^2 e^x = e^{x+2h} - e^{x+h} - e^{x+h} + e^x$

$= e^{x+2h} - 2e^{x+h} + e^x$

$= e^x(e^h - 1)^2$.

$\left(\frac{\Delta^2}{E}\right)e^x = \Delta^2 E^{-1} e^x = \Delta^2 e^{x-h}$

$= e^x\{e^{-2h} - 2e^{-h} + 1\}$

$= e^x\{e^{-h} - 1\}^2$

Ex Evaluate $\Delta^2 \cos 2x$?

$\Delta(\Delta \cos(2x)) = \Delta(\cos(2x+2h) - \cos 2x)$

$= \Delta\{\cos(2x+2h)\} - \Delta \cos(2x)$

$= \cos(2x+4h) - \cos(2x+2h) -$

$\cos(2x+2h) + \cos 2x$

$= \cos(2x+4h) - 2\cos(2x+2h) + \cos 2x$

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right)$

Ex $\Delta^n\{\frac{1}{x}\} = \Delta^{n-1}\{\Delta(\frac{1}{x})\} = \Delta^{n-1}\{\frac{1}{x+h} - \frac{1}{x}\}$

$= \Delta^{n-1}\{\frac{-h}{(x+h)x}\}$

$= \frac{(-1)^n h^n}{x(x+h)\cdots(x+nh)}$

Qn Define operator E, ∇ and Δ and show that (4)

$$\nabla = \Delta E^{-1}$$

$$\nabla = I - E^{-1} = (E-I)E^{-1} = \Delta E^{-1}$$

Qn

$$\Delta^2 (ab^{cx}) = a\,\Delta^2 (b^{cx})$$

$$= a\,\Delta(\Delta b^{cx})$$

$$= a\,\Delta(b^{c(x+h)} - b^{cx})$$

$$= a\,\Delta\{b^{cx+ch}\} - a\,\Delta\{b^{cx}\}$$

$$= a\{b^{cx+2ch}\} - 2b^{cx+ch} + b^{cx}$$

$$= ab^{cx+2ch} - 2ab^{cx+ch} + ab^{cx}$$

$$= ab^{cx}\{b^{2ch} - 2b^{ch} + 1\}$$

$$= ab^{cx}\{(b^{ch}-1)^2\}$$

Qn Find $\Delta \cot 2x$?

<u>Centeral difference operator</u>

Centeral difference operator is denoted by δ.

Let $y = f(x)$

x	x_0	x_1	..	x_n
y	y_0	y_1	..	y_n

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1$$

. . . . .

. . . . .

$$\delta_{n-\frac{1}{2}} = y_n - y_{n-1}$$

$$\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}$$

$$\delta f(x) = f(x + \tfrac{h}{2}) - f(x - \tfrac{h}{2})$$

Mean operator / Averaging operator

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Centeral difference operator Tabular form

value of x	value of y	1st diff	2nd diff
x_0	y_0		
		$y_1 - y_0 = \delta y_{1/2}$	
$x_1 = x_0 + h$	y_1		
		$y_2 - y_1 = \delta y_{3/2}$	
$x_2 = x_0 + 2h$	y_2		
$\vdots$		$y_3 - y_2 = \delta y_{5/2}$	
	y_3		
$x_n = x_0 + nh$			

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

(1) $$\delta = \frac{E-1}{E^{1/2}} = \Delta E^{-\frac{1}{2}} = E^{-\frac{1}{2}}\Delta$$

(2) $$\delta = \nabla E^{\frac{1}{2}} = E^{\frac{1}{2}}\nabla$$

(3) $$\delta^n f(x) = \Delta^n f\left(x-\frac{nh}{2}\right) = \nabla^n f\left(x+\frac{nh}{2}\right)$$

Qn Show that $\mu^2 = 1 + \frac{1}{4}\delta^2$

$\mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})$

$\mu^2 = \frac{1}{4}\{E + E^{-1} + I\}$

$= \frac{1}{4}\{(E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2 + 4\}$

$= \frac{1}{4}\{\delta^2 + 4\}$

$$\boxed{\mu^2 = 1 + \frac{\delta^2}{4}}$$

Qn $E^{\frac{1}{2}} = \mu + \frac{1}{2}\delta$

consider R.H.S $\Rightarrow (\mu + \frac{1}{2}\delta)f(x) = \frac{1}{2}\{f(x+\frac{h}{2}) + f(x-\frac{h}{2})\}$
$+ \frac{1}{2}\{\delta\}$

$= \frac{1}{2}\{2f(x+\frac{h}{2})\}$

$= f(x+\frac{h}{2})$

$= E^{\frac{1}{2}}$

Qn Show that $\sqrt{1+\delta^2\mu^2} = 1 + \frac{1}{2}\delta^2$

Soln: We know $\mu^2 = 1 + \frac{\delta^2}{4}$

consider L.H.S $\sqrt{1+\delta^2\mu^2} = \sqrt{1+\delta^2(1+\frac{\delta^2}{4})}$

$= \sqrt{1+\delta^2+(\frac{\delta^2}{2})^2} = \sqrt{(1+\frac{\delta^2}{2})^2} = 1 + \frac{\delta^2}{2}$

Qn $\delta^2 y_0 = y_1 - 2y_0 + y_{-1}$

$(E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2 y_0 = (E + E^{-1} - 2)y_0 = Ey_0 + E^{-1}y_0 - 2y_0$

$= y_1 - 2y_0 + y_{-1}$

$E^n y_0 = y_{n+0}$

Lagrange Interpolation

Definition: Polynomial Interpolation

A polynomial $P_n(x)$ is said to be an interpolating polynomial if the following conditions are satisfied

- polynomial: $P_n(x)$ is of degree $\leq n$
- $P_n(x_i) = y_i$

Theorem: Lagrange form of Interpolating polynomial

Given data set

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

Lagrange form of interpolating polynomial is given by

$$P_n(x) = \sum_{k=0}^{n} y_k \, l_k(x) \, , \quad l_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{(x_k - x_i)}$$

Pf → Let $q(x) = \sum_{k=0}^{n} y_k \, l_k(x)$

Step.1 To prove that $q(x)$ is a polynomial of degree $\leq n$.

Consider $l_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i}\right)$ is a polynomial of degree n.

⇒ Since q is a combination of polynomial of degree n hence it is also a polynomial of degree $\leq n$.

Step: 2 $\quad l_K(x) = \prod_{\substack{i=0 \\ i \neq K}}^{n} \dfrac{x - x_i}{x_K - x_i}$

$$= \frac{(x-x_0)(x-x_1)\cdots(x-x_{K-1})(x-x_{K+1})\cdots(x-x_n)}{(x_K-x_0)(x_K-x_1)\cdots(x_K-x_{K-1})(x_K-x_{K+1})\cdots(x_K-x_n)}$$

$$l_K(x_K) = \prod_{i=0}^{n} \left(\frac{x_K - x_i}{x_K - x_i} \right) = 1$$

$$l_K(x_i) = \begin{cases} 1 & \text{if } i = K \\ 0 & \text{if } i \neq K \end{cases}$$

$\Rightarrow q(x) = y_0 l_0(x) + y_1 l_1(x) + \cdots + y_n l_n(x)$

$\Rightarrow q(x_1) = y_1 l_1(x_1) = y_1 = P_n(x_1)$

Similarly $\quad q(x_0) = y_0, \; q_1(x_1) = y_1, \; \ldots \; q(x_n) = y_n$

$\Rightarrow \quad q(x_j) = y_j \; , \; j = 0, 1, 2, \ldots n$

Thus $q(x)$ is an interpolating polynomial.

Ex Consider approximating $f(x) = e^x$

x	0.82	0.83
y	2.270500	$e^{0.83} = 2.293319$

To get the value of $e^{0.826}$ using $P_1(x)$

Sol^n $\rightarrow \quad P_1(x) = y_0 l_0(x) + y_1 l_1(x)$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \; , \; l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$P_1(x) = \cancel{0.82}\; 2.270500 \left(\frac{x - 0.83}{0.82 - 0.83} \right) + \cancel{(0.83)}\; 2.293319 \left(\frac{x - 0.82}{0.01} \right)$$

$$= 2.2819\, x + 0.399342$$

Interpolating polynomial

(3)

Di:

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To construct interpolating polynomial $P_{n+1}(x)$ for

x	x_0	x_1	x_3	$\cdots$	x_n	x_{n+1}
y	y_0	y_1	y_3	$\cdots$	y_n	y_{n+1}

Define a polynomial P_{n+1} by.

$$P_{n+1}(x) = P_n(x) + \text{something}$$

$$= P_n(x) + c\,(x-x_0)(x-x_1)\cdots(x-x_n)$$

$\Rightarrow$ $P_{n+1}(x) = P_n(x)$ for $x_0, x_1, \cdots x_n$

$\Rightarrow$ $P_{n+1}(x_i) = P_n(x_i)$ for $i = 0, 1, 2, \cdots n$

Also $P_{n+1}(x)$ should satisfy $P_{n+1}(x_{n+1}) = y_{n+1}$

$\Rightarrow$ $P_{n+1}(x_{n+1}) = P_n(x_{n+1}) + c\,(x_{n+1}-x_0)\cdots(x_{n+1}-x_n)$

$$\Rightarrow \quad c = \frac{-P_n(x_{n+1}) + P_{n+1}(x_{n+1})}{(x_{n+1}-x_0)(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)}$$

$$= \frac{P_{n+1}(x_{n+1}) - P_n(x_{n+1})}{(x_{n+1}-x_0)(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)}$$

Theorem: Newton form of interpolating polynomial

Given data set

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

The newton form of interpolating polynomial $P_n(x)$ can be written as

$$P_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$f[x_0, x_1,$

Divided Difference:

where $A_0, A_1, \ldots, A_n$ are constants obtained to satisfy interpolation conditions.

We observe that

$$A_0 = f(x_0)$$

$$A_1 = \frac{f(x_1) - P_0(x_1)}{x_1 - x_0}$$

$$A_2 = \frac{f(x_2) - P_1(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

In general

$$A_n = \frac{f(x_n) - P_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}$$

Note: A_2 depends only on f at x_0, x_1, x_2

A_n depends only on f at $x_0, x_1, \ldots x_n$

To compute Newton form of $P_n(x)$, it is enough to compute A_k, $k = 0, 1, 2, \ldots, n$

Divided Difference :

Given general data set

x	x_0	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

Newton form of interpolating polynomial

$$P_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \cdots + A_n \prod_{j=0}^{n-1} (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Introduce a new notation

$$P_n(x) = f[x_0] + \sum_{K=1}^{n} f[x_0, x_1, \ldots, x_K] \prod_{j=0}^{K-1} (x-x_j)$$

where $f[x_0, x_1, \ldots, x_n] = \dfrac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$

Definition : Divided Differences

- Let $x_0, x_1, \ldots, x_n$ be distinct nodes.
- Let $P_n(x)$ be the polynomial interpolating a function f at the nodes $x_0, x_1, \ldots, x_n$.

The coefficient of x^n in the polynomial $P_n(x)$ is denoted by $f[x_0, x_1, \ldots, x_n]$ is called nth divided difference of f.

Theorem: The divided difference is a symmetric function of its arguments. That is, if $z_0, z_1, \ldots z_n$ is a permutation of $x_0, x_1, \ldots, x_n$, then

$$f[x_0, x_1, \ldots, x_n] = f[z_0, z_1, \ldots, z_n]$$

Proof: $z_0, z_1, \dots, z_n$ is a permutation of $x_0, x_1, \dots, x_n$ (2) means that the nodes $x_0, x_1, \dots, x_n$ have only been relabelled as $z_0, z_1, \dots, z_n$.

- Hence the polynomial interpolating the function f at both of these sets of nodes is the same.
- By definition $f[x_0, x_1, \dots, x_n]$ is the coefficient of x^n in the polynomial interpolating the function at the nodes $x_0, x_1, \dots, x_n$, and $f[z_0, z_1, \dots, z_n]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $z_0, z_1, \dots, z_n$.
- Since both the interpolation polynomials are equal, so are the coefficients of x^n in them. Thus, we get

$$f[x_0, x_1, x_2, \dots, x_n] = f[z_0, z_1, \dots z_n]$$

Theorem: (Higher order divided differences)

Divided differences satisfy the equation

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

proof: Let us start the proof by setting up the following notations.

- Let $P_n(x)$ be the polynomial interpolating f at the nodes $x_0, x_1, x_2, \dots, x_n$

1

Ex Using Newton's divided difference formula find $f(8)$.

	x_0	x_1	x_2	x_3	x_4	x_5
x	4	5	7	10	11	13
$y=f(x)$	48	100	294	900	1210	2028
	y_0	y_1	y_2	y_3	y_4	y_5

$$y = P_5(x) = f(x_0) + f[x_0,x_1](x-x_0) +$$
$$f[x_0,x_1,x_2](x-x_0)(x-x_1) +$$
$$f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) +$$
$$f[x_0,x_1,x_2,x_3,x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3) +$$
$$f[x_0,x_1,x_2,x_3,x_4,x_5](x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

x	y	First order	2nd	3rd	4th
$x_0=4$	$y_0=48$				
		$\frac{100-48}{5-4}=52$			
			15		
$x_1=5$	$y_1=100$				
		97		1	
			21		0
$x_2=7$	$y_2=294$				
		202		1	
$x_3=10$	$y_3=900$		27		
					0
		310		1	
$x_4=11$	$y_4=1210$				
			33		
		409			
$x_5=13$	$y_5=2028$				

---ward difference interpolation ③

②

$$P_5(x) = 48 + 52(x-4) + 15(x-4)(x-5) +$$
$$1(x-4)(x-5)(x-7)$$

$$f(8) = 48 + 52(8-4) + 15(8-4)(8-5) +$$
$$(8-4)(8-5)(8-7)$$

Ex Find the ~~inte~~ Newton's divided difference interpolating polynomial for the given set of data

	x_0	x_1	x_2	x_3
x	-5	-1	0	2
$f(x)$	-2	6	1	3
	y_0	y_1	y_2	y_3

-5	-2			
		$\frac{8}{4} = 2$		
-1	6		$-\frac{7}{5}$	
		-5		$\frac{12}{5 \times 7} = \frac{12}{35}$
0	1		2	
		1		
2	3			

$$P_3(x) = -2 + (x+5)\, f[x_0, x_1] + (x+5)(x+1)\, f[x_0, x_1, x_2]$$
$$+ (x+5)(x+1)x\, f[x_0, x_1, x_2, x_3]$$

$$= -2 + 2(x+5) + \left(-\frac{7}{5}\right)(x+5)(x+1) +$$
$$\frac{12}{35}(x+5)(x+1)x$$

Newton Forward difference interpolation

(3)

Qn Form a difference table and interpolate the value of $f(x)$ when $x=4$, given

x	3	5	7	9
$f(x)$	180	150	120	90

Soln → $h=2$, $u = \frac{x-x_0}{h} = \frac{1}{2} = 0.5$

Difference table

x	y	Δy	$\Delta^2 y$
3	180		
		-30	
			0
5	150		
		-30	
			0
7	120		
		-30	
9	90		

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

Newton's forward is also called as Newton's Gregory's formula for forward interpolation with equal intervals.

For equal intervals

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}[y_1 - y_0] = \frac{\Delta y_0}{h}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h^2}[y_2 - y_1 - y_1 + y_0]$$

$$= \frac{1}{2h^2}[y_2 - 2y_1 + y_0]$$

$$= \frac{\Delta^2 y_0}{2h^2}$$

$$= \frac{1}{2}\frac{\Delta^2 y_0}{h^2}$$

We know Newton's divided differe formula

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) + \cdots +$$
$$f[x_0, x_1, x_2 \ldots x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

$$P_n(x) = f(x_0) + \frac{1}{h}(x - x_0)\Delta y_0 + \frac{(x - x_0)(x - x_1)}{h^2}\Delta^2 y_0$$
$$+ \cdots + \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{h^n}\Delta^n y_0$$

Thus the required polynomial is

$$f(x) = f(x_0) + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots +$$

$$\frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x - x_0}{h}$ and Δ represent the divided difference.

Ex Find the value of $\sin 52^\circ$ from the given table

x	45	50	55	60
$\sin x$	0.7071	0.7660	0.8192	0.8660

x	y	Δ	Δ^2	Δ^3
45	0.7071			
		0.0589		
50	0.7660		−0.0057	
		0.0532		−0.0007
55	0.8192		−0.0064	
		0.0468		
60	0.8660			

$$P_3(x) = 0.7071 + (x-45)(0.0589) -$$

$$(x-50)(x-45)(-0.0057) -$$

$$(x-50).(x-45)(x-55)(-0.0007)$$

Newton's Backward interpolation formula (6)

Newton's Backward is also called as Newton-Gregory's formula for backward interpolation with equal intervals.

x	y				
x_0	$f(x_0)$	40	204		
			100	20	
x_1	$f(x_1)$	50	224		2
			100	22	
x_2	$f(x_2)$	60	246		2
			100	24	
x_3		70	270		2
				26	
		80	100 296	28	2
x_4		90	100 324		

$$\nabla y_0 = y_0 - y_{-1} = f(x_0) - f(x_0 - h)$$

$$\nabla y_1 = y_1 - y_0 = f(x_0 + h) - f(x_0)$$

$$f(x) = P_n(x) = f(y_n) + (x - x_n) f[x_n, x_{n-1}] + \cdots + (x - x_n)(x - x_{n-1}) \cdots (x - x_1) f[x_n, x_{n-1}, \ldots, x_0]$$

$$= f(y_n) + (x - x_n)\frac{\nabla y_n}{h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!\,h^2} + \cdots + (x - x_n)(x - x_{n-1}) \cdots (x - x_1)\frac{\nabla^n y_n}{n!\,h^n}$$

$$= f(y_n) + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots + \frac{u(u+1)(u+2)\cdots(u+(n-1))}{n!}\nabla^n y_n$$

Newton's backward difference interpolation ⑪

Ex Find value of $f(84)$ for the data ⑦

x	$f(x)$	∇y	$\nabla^2 y$
40	204		
		20	
50	224		2
		22	
60	246		2
		24	
70	270		2
		26	
80	296		2
		28	
90	324		

$$P_5(x) = 324 + (x-90)\,28 + (x-90)(x-80)\,2$$

$$= 324 + (-0.6)\,28 + \frac{(-0.6)(-0.6+1)\,2}{2}$$

$$= 324 + (-0.6)\,28 + \frac{(-0.6)(0.4)\,2}{2}$$

$$= 324 - 28(0.6) - \frac{2(0.4)(0.6)}{2}$$

$$= 324 - 28(0.6) - (0.4)(0.6)$$

Newton's backward difference interpolation

Consider a polynomial

$$p(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots$$

put $x = x_n \Rightarrow p(x_n) = a_0 \Rightarrow a_0 = y_n$

put $x = x_{n-1}$

$$p(x_{n-1}) = y_n + a_1(x_{n-1} - x_n)$$

$$\Rightarrow y_{n-1} = y_n + a_1(x_{n-1} - x_n)$$

$$\Rightarrow y_{n-1} - y_n = a_1(x_{n-1} - x_n)$$

$$\Rightarrow a_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{1}{h}\nabla y_n$$

$$\Rightarrow p(x) = y_n + (x - x_n)\frac{1}{h}\nabla y_n$$

put. $x = x_{n-2}$:

$$\Rightarrow p(x_{n-2}) = y_n + (x - x_n)\frac{1}{h}(y_n - y_{n-1}) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow y_{n-2} - y_n = (x_{n-2} - x_n)\frac{1}{h}(y_n - y_{n-1}) + a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{y_{n-2} - y_n}{x_{n-2} - x_n} = \frac{1}{h}\{y_n - y_{n-1}\} + a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{y_n - y_{n-2}}{2h} = \frac{1}{h}\{y_n - y_{n-1}\} + a_2(x_{n-2} - x_{n-1})$$

$$\frac{y_n - y_{n-2}}{2h} = \frac{1}{h}\{y_n - y_{n-1}\} + a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{y_n}{2h} - \frac{y_n}{h} - \frac{y_{n-2}}{2h} + \frac{y_{n-1}}{h} = a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{1}{h}\left\{\frac{y_n}{2} - y_n - \frac{y_{n-2}}{2} + y_{n-1}\right\} = a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{1}{h}\left\{-\frac{y_n}{2} - \frac{y_{n-2}}{2} + y_{n-1}\right\} = a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{1}{h}\{-y_n - y_{n-2} + 2y_{n-1}\} = a_2(x_{n-2} - x_{n-1})$$

$$\Rightarrow \frac{1}{2h^2}\{y_{n-2} - 2y_{n-1} + y_n\} = a_2$$

$$\Rightarrow \frac{1}{2h^2}\nabla^2 y_n = a_2$$

$$\Rightarrow P_n(x) = y_n + (x - x_n)\nabla y_n + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2}\nabla^2 y_n$$

$$+ \cdots + \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{n!}\frac{\nabla^n y_n}{h^n}$$

$$= y_n + u\nabla y_n + u(x - x_n + x_n - x_{n-1})\nabla^2 y_n$$

$$= y_n + u\nabla y_n + u(uh + h)\frac{\nabla^2 y_n}{2h}$$

$$= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n$$

$$\nabla y_n = y_n - y_{n-1} \Rightarrow \nabla f(x_n) = f(x_n) - f(x_n - h)$$

$$= f(x_n) - f(x_{n-1})$$

Qn Find the value of $f(1.6)$ for the data given below

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

x	$f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

$$P(x) = y_n + u \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+3)}{3!} \nabla^2 y_n$$

$$= 6.5 + u(0.54) + \frac{u(u+1)}{2!}(-0.6) + \frac{u(u+1)(u+2)}{3!}(-0.41)$$

Qn

year x	1982	1992	2002	2012	2022
population $f(x)$	46	66	81	93	101

Estimate the population for the year 2016

Qn Find the value of $\tan 0.26$ from the given table

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

x	$\tan x$
0.10	0.1003
0.15	0.1511
0.20	0.2027
0.25	0.2553
0.30	0.3093

Qn From the following table estimate the the number of students who obtained marks in computer programming between 75 and 80.

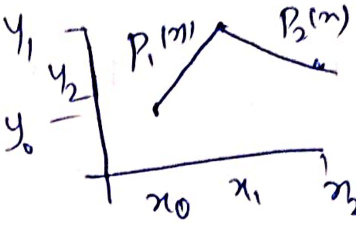
Marks	35-45	45-55	55-65	65-75	75-85
No. of Students	20	40	60	60	20

Soln :) Rewrite the given information as

Marks less than	45	55	65	75	85
No of students	20	60	120	180	200

Introduction to spline interpolation

$$S(x) = \begin{cases} P_1(x) \; ; \; x \in [x_0, x_1] \\ P_2(x) \; : \; x \in [x_1, x_2] \end{cases}$$



Ex Using linear spline interpolation find ;

a) the necessary interpolation functions

b) the outputs at $x = 2, 5$ and 10.

x	1	6	7	9	12	20
y	2	8	6	10	14	41

Ans

$$P_1(x) = 2 + \frac{6}{5}(x-1) =$$

$$P_1(2) = 2 + \frac{6}{5}(2-1) = 1 + \frac{6}{5} = \frac{11}{5} =$$

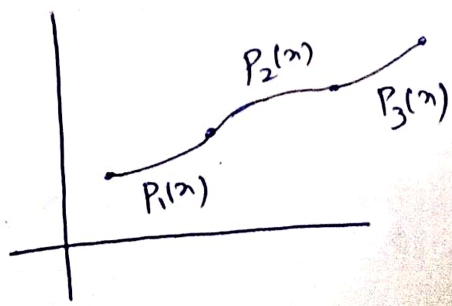
$$P_1(5) = 2 + \frac{6}{5}(5-4) = 1 + \frac{24}{5} = \frac{29}{5}$$

Quadratic spline interpolation Theory

Ex a) Find all the relevant interpolating functions between the following data points using quadratic splines.

b) Find outputs for $x = 2, 4, 7$

x	1	3	5	8
y	2	3	9	10



Sol^n ⇒ $P_1(x) = a_0 + a_1 x + a_2 x^2$ — ①

c

$P_2(x) = b_0 + b_1 x + b_2 x^2$ — ②

$P_3(x) = c_0 + c_1 x + c_2 x^2$ — ③

$P_1(1) = a_0 + a_1 + a_2 = 2$

$P_1(3) = a_0 + 3a_1 + 9a_2 = 3$

$P_2(3) = 3 + 3\cancel{b}_1 + 9b_2 = 3$

$P_2(5) = b_0 + 5b_1 + 25b_2 = 9$

$P_3(x_3) = P_3(5) = c_0 + 5c_1 + 25c_2 = 9$

$P_3(8) = c_0 + 8c_1 + 64c_2 = 10$

(ii) Smoothness @ interior data points (n-1) equations

$$\left.\frac{dP_1(x)}{dx}\right|_{x_1} = \left.\frac{dP_2(x)}{dx}\right|_{x_1} \Rightarrow a_1 + 2a_2 x = b_1 + 2b_2 x$$

$$\Rightarrow a_1 + 6a_2 = b_1 + 6b_2$$

$$\left.\frac{dP_2(x)}{dx}\right|_{x=x_2} = \left.\frac{dP_3(x)}{dx}\right|_{x=x_2} \Rightarrow b_1 + 10b_2 = c_1 + 10c_2$$

(iii) Assume $\frac{d^2P_1(x)}{dx} = 0 \Rightarrow 2a_2 = 0$

Cubic spline Interpolation:

A powerful technique used to approximate a smooth curve that passes through a set of data points.

Ex a) Find all the relevant interpolating functions between the following data points using cubic splines. (Using a natural cubic spline)

b) Using the interpolated cubic functions find the outputs at $x = 1.5, 4, 7$.

x	1	3	5	8
y	2	3	9	10

Cubic spline:

The spline is a piecewise continuous cubic polynomial.

Here $F''(x)$ is a linear function of x in all intervals.

Consider the interval $[x_{i-1}, x_i]$.

Using lagrange interpolation in this interval $F''(x)$ can be written as

$$F''(x) = \frac{x - x_i}{x_{i-1} - x_i} F''(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} F''(x_i)$$

$$= \frac{x_i - x}{x_i - x_{i-1}} F''(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} F''(x_i)$$

Denote $F''(x_{i-1}) = m_{i-1}$ $\quad F''(x_i) = m_i$

Integrating w.r.t x we get

$$F'(x) = -\frac{(x_i - x)^2}{2(x_i - x_{i-1})} m_{i-1} + \frac{(x - x_{i-1})^2}{2(x_i - x_{i-1})} m_i + a$$

Again integrating we get

$$F(x) = \frac{(x_i - x)^3}{6(x_i - x_{i-1})} m_{i-1} + \frac{(x - x_{i-1})^3}{6(x_i - x_{i-1})} m_i + ax + b$$

where a, b are arbitrary constants to be determined by using the condition

To ease the computations, we write

$$ax+b = c_j(x_j - x) + d(x - x_{j-1})$$

where $a = d - c$, $b = cx_j - dx_{j-1}$

Hence we can obtain

$$F(x) = \frac{(x_j - x)^3}{6h_j} m_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} m_j + c(x_j - x) + d(x - x_{j-1})$$

Using the condition $F(x_{j-1}) = f(x_{j-1}) = f_{j-1}$,

we get

$$f_{j-1} = \frac{(x - x_{j-1})^3}{6h_j} m_{j-1} + c(x_j - x_{j-1}) = \frac{h_j^3}{6h_j} m_{j-1} + ch_j$$

$$\Rightarrow c = \frac{1}{h_j}\left[f_{j-1} - \frac{h_j^2}{6} m_{j-1}\right]$$

Using the condition $F(x_j) = f_j$, we get

$$f_j = \frac{(x_j - x_{j-1})^3}{6h_j} m_j + d(x_j - x_{j-1}) = \frac{h_j^3}{6h_j} m_j + dh_j).$$

$$d = \frac{1}{h_j}\left[f_j - \frac{h_j^2}{6} m_j\right]$$

Substituting the expression for c and d

we obtain the spline in the interval

$[x_{j-1}, x_j]$ as

$$F_j(x) = \frac{1}{6h_j}\left[(x_j - x)^3 m_{j-1} + (x - x_{j-1})^3 m_j\right] + \frac{x_j - x}{h_j}\left[f_{j-1} - \frac{h_j^2}{6} m_{j-1}\right]$$
$$+ \frac{x - x_{j-1}}{h_j}\left[f_j - \frac{h_j^2}{6} m_j\right]$$

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Now we require that the derivative $F'(x)$ be continuous at $x = x_i$. Hence the left hand and right hand derivatives of $F'(x)$ at $x = x_i$ must be equal, that is

$$\lim_{\epsilon \to 0} F_i'(x_i - \epsilon) = \lim_{\epsilon \to 0} F_{i+1}'(x_i + \epsilon)$$

$$\Rightarrow \frac{h_i}{6} m_{i-1} + \frac{1}{3}(h_i + h_{i+1}) m_i + \frac{h_{i+1}}{6} m_{i+1} =$$

$$\frac{1}{h_{i+1}}(f_{i+1} - f_i) - \frac{1}{h_i}(f_i - f_{i-1})$$

$$i = 1, 2, \cdots (n-1)$$

Ex Obtain the cubic spline approximation for the following data

x	0	1	2
$f(x)$	-1	3	29

Soln → We have the equispaced data with $h = 1$.

$$m_{i-1} + 4m_i + m_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1}), \quad i = 1$$

For $i = 1$, we get

$$m_0 + 4m_1 + m_2 = 6(f_2 - 2f_1 + f_0)$$

Since $m_0, m_2 = 0$, $4m_1 = 6[29 - 6 - 1] = 132$

$$M_1 = 33$$

The spline is given by

$$F_i(x) = \frac{1}{6}\left[(x_i - x)^3 m_{i-1} + (x - x_{i-1})^3 m_i\right] +$$

$$(x_i - x)\left[f_{i-1} - \frac{1}{6} m_{i-1}\right] + (x - x_{i-1})\left[f_i - \frac{1}{6} m_i\right]$$

$$F(x) = \frac{1}{2}\left(11x^3 - 3x - 2\right)$$

on $[1,2]$ $\quad F(x) = \frac{11}{2}(2-x)^3 + \frac{83}{2}x - 34$

Ex Obtain the cubic spline approximation for the following data

x	0	1	2	3
$f(x)$	1	2	33	244