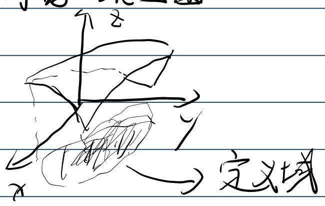
多元函数与微分

$z=f(x,y)$ 是空间内的一张曲面



偏导数

$z=f(x,y)$ 关于 x 的偏导数

记为 z_x' 或 $f_x(x,y)$ 或 $\frac{\partial z}{\partial x}$

关于 y 的偏导数

z_y' 或 $f_y'(x,y)$ 或 $\frac{\partial z}{\partial y}$

计算 $\frac{\partial z}{\partial x}$ 时，把 y 看成常数，x 同理.

例 $z=2xy^2-3x^2y+4x-7y^3$

$$\frac{\partial z}{\partial x}=2y^2-6xy+4$$

$$\frac{\partial z}{\partial y}=4xy-3x^2-21y^2$$

例 $z=\sqrt{x^2-y}$

$$\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x^2-y}}\cdot 2x=\frac{x}{\sqrt{x^2-y}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{x^2-y}}\cdot(-1)=-\frac{1}{2\sqrt{x^2-y}}$$

例 $z=x^y$

$$\frac{\partial z}{\partial x}=yx^{y-1}$$

$$\frac{\partial z}{\partial y}=x^y\ln x$$

3. 全微分

(二元函数的微分)

$z=f(x,y)$ 的微分 dz 或 $df(x,y)$

$$dz=\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}dy$$

例 $z=\sin(x^2+y)$ 求 dz

$dz=2x\cos(x^2+y)dx+\cos(x^2+y)dy$

例 $z=\arctan\frac{y}{x}$ 求 $dz|_{\substack{x=1\\y=1}}$ 也可以写成 $dz|_{(1,1)}$

$dz=\frac{1}{1+(\frac{y}{x})^2}\cdot(-\frac{y}{x^2})|_{(1,1)}dx+\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}|_{(1,1)}dy$

x偏导　　　　y偏导

$=-\frac{1}{2}dx+\frac{1}{2}dy$ （(1,1)代入得到）

例 $z=\ln(x^2-y^3)$ 求 dz

$dz=\frac{2x}{x^2-y^3}dx+\frac{-3y^2}{x^2-y^3}dy$

二阶偏导数

$z=f(x,y)$ 一阶偏导 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$

$\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$，$\frac{\partial^2 z}{\partial x^2}$，$z''_{xx}$ 或 $f''_{xx}(x,y)$ 先对x偏导 再对x偏导

$\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$，$\frac{\partial^2 z}{\partial x\partial y}$，$z''_{xy}$ 或 $f''_{xy}(x,y)$

先对x求偏导
再对y求偏导

例 $z=3x^2y-4xy^3+5x^3-6y^2$

$\frac{\partial z}{\partial x}=6xy-4y^3+15x^2$

$\frac{\partial z}{\partial y}=3x^2-12xy^2-12y$

$\frac{\partial^2 z}{\partial x^2}=6y+30x$　　$\frac{\partial^2 z}{\partial x\partial y}=6x-12y^2$

$\frac{\partial^2 z}{\partial y\partial x}=6x-12y^2$　　$\frac{\partial^2 z}{\partial y^2}=-24xy-12$

5. 多元复合求导

$z=f(u,v)$　　$u=u(x,y)$
　　　　　　$v=v(x,y)$

$$z\begin{cases}u\begin{cases}x\\y\end{cases}\\v\begin{cases}x\\y\end{cases}\end{cases}$$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

$\frac{\partial z}{\ }=\frac{\partial z}{\ }\cdot\frac{\partial v}{\ }+\frac{\partial z}{\ }\cdot\frac{\partial u}{\ }$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}$$

例 $z=u^2+\ln v$ $\begin{cases} u=x^2+y^3 \\ v=xy \end{cases}$

$$z\begin{cases} u\begin{cases} x \\ y \end{cases} \\ v\begin{cases} x \\ y \end{cases} \end{cases}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x}=2u\cdot 2x+\frac{1}{v}\cdot y=2(x^2+y^3)\cdot 2x+\frac{1}{xy}\cdot y$$
$$=4(x^3+xy^3)+\frac{1}{x}$$

$$\frac{\partial z}{\partial y}=2u\cdot 3y^2+\frac{1}{v}\cdot x$$
$$=2(x^2+y^3)\cdot 3y^2+\frac{1}{xy}\cdot x$$
$$=6(x^2y^2+y^5)+\frac{1}{y}$$

多元单一复合求全导

$$z\begin{cases} u - t \\ v - t \end{cases}$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

求偏导

因为u和v都是一个参数去定义的
因此可以求出这个t的全导数

例

$z=e^u\cdot\ln v$ $\quad u=\sin t$
$\quad v=t^2+2$

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$
$$=\ln v\cdot e^u\cdot\cos t+\frac{e^u}{v}2t$$
$$=\ln(t^2+2)\cdot e^{\sin t}\cdot\cos t+\frac{e^{\sin t}}{t^2+2}\cdot 2t$$
$$=e^{\sin t}\left[\ln(t^2+2)\cos t+\frac{2t}{t^2+2}\right]$$

隐函数求导

例 $x^2y-4xy^3=2x^3-5y^2$ 求 y'

老方法，两边求导提 y'

$$y'=\frac{2yx-4y^3-6x^2}{x^2-12xy^2+10y}$$

新方法 移一边令 $F(x,y)=0$

$$x^2y-4xy^3-2x^3+5y^2=0$$

z — x，y — x 由上面学的复合求导可推出

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\underbrace{\frac{dy}{dx}}_{y'}=0$$

$$y'=-\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

$$\boxed{y'=-\frac{F_x'(x,y)}{F_y'(x,y)}}$$

对 x 求偏导

对 y 求偏导

二元隐函数求导公式

$F(x,y,z)=0$　$z=z(x,y)$　z 可以由 x、y 表达

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}\qquad\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}$$

对 x，z 求偏导

例 $x^2+y^2+z^2=4xyz$ 隐含 $z(x,y)$

求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$

$$x^2+y^2+z^2-4xyz=0$$

$$\frac{\partial z}{\partial x}=-\frac{F'(x)}{F'(z)}=-\frac{2x-4yz}{2z-4xy}=\frac{-x+2yz}{z-2xy}$$

$$\frac{\partial z}{\partial y}=-\frac{F'(y)}{F'(z)}=-\frac{2y-4xz}{2z-4xy}=\frac{-y+2xz}{z-2xy}$$

二元函数的极值

求 $z=f(x,y)$ 的极值

① 计算 $\begin{cases} f'_x(x,y)=0 \\ f'_y(x,y)=0 \end{cases}$ $\Rightarrow$ 解得驻点 (x_0,y_0)

② 计算 $f''_{xx}(x_0,y_0)=A$

$f''_{xy}(x_0,y_0)=B$

$f''_{yy}(x_0,y_0)=C$

③ 判别式

(1) $B^2-AC>0$ (x_0,y_0) 不是极值点

(2) $B^2-AC<0$ 是极值点

$\begin{cases} A>0 & \text{极小值点} \\ A<0 & \text{极大值点} \end{cases}$

(3) $B^2-AC=0$ 不一定是极值点

例 $f(x,y)=x^3+y^3-3xy$ 求极值

解 ① $f'_x(x,y)=3x^2-3y=0$

$f'_y(x,y)=3y^2-3x=0$ $\Rightarrow$ 驻点 $(0,0)$ $(1,1)$

② $\underbrace{f''_{xx}=6x}_{A}$ $\underbrace{f''_{yy}=6y}_{C}$ $\underbrace{f''_{xy}=-3}_{B}$

$B^2-AC=9-6x\cdot 6y=9-36xy$

$(0,0)$ 点代入 $9-36\cdot 0\cdot 0>0$ 不是极值点

$(1,1)$ 代入 $9-36=-27<0$ 且 $A=6>0$ 是极小值

所以 $(1,1)$ 是极小值，极小值为 $\cancel{-27}$ -1

应用题

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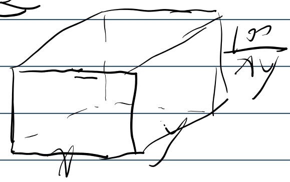

求表面积 z

$$z = \frac{2000}{y} + \frac{2000}{x} + xy$$

$(x>0,\ y>0)$

$$\begin{cases} z'_x = -\frac{2000}{x^2} + y = 0 \\ z'_y = -\frac{2000}{y^2} + x = 0 \end{cases} \Rightarrow x = 10\sqrt[3]{2} \quad y = 10\sqrt[3]{2}$$

$$A = z''_{xx} = \frac{4000}{x^3} \quad B = z''_{xy} = 1 \quad C = z''_{yy} = \frac{4000}{y^3}$$

$$B^2 - AC = 1 - \frac{4000 \cdot 4000}{x^3 y^3} = -3 \quad A = 2 > 0$$

小

极小值点为 $(10\sqrt[3]{2},\ 10\sqrt[3]{2})$ 且极小值为

代入回答