2023年1月5日 17:27

分部积分求定积分

$$\int_a^b u v' dx = uv\Big|_a^b - \int_a^b u' v dx$$

例 $\int_0^1 x e^x dx$　　反对幂三指 (x e^x)

别把e^x当成对数了！

设 $u=x$　$v'=e^x$

$u'=1$　$v=e^x$

$= xe^x\Big|_0^1 - \int_0^1 e^x dx = e-(e-1)=1$

例 $\int_1^e x\ln x\, dx$　　$\ln x$ 对　x幂

设 $u=\ln x$　$v=x$

$u'=\frac{1}{x}$　$v'=\frac{1}{2}x^2$

$= \frac{1}{2}x^2\ln x\Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{1}{2}x^2 dx$

$= \frac{1}{2}x^2\ln x\Big|_1^e - \boxed{\int_1^e \frac{x}{2} dx}$

$= \frac{1}{2}e^2 - \frac{1}{4}x^2\Big|_1^e = \frac{1}{4}e^2+\frac{1}{4}$

别直接代　积分完再代

例 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x\cos 2x\, dx$

设 $u=x$　$v'=\cos 2x$

$u'=1$　$v=\frac{1}{2}\sin 2x$

$= \frac{1}{2}x\sin 2x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}\sin 2x\, dx$　　接积分的第一换元法

第二换元法（含$\sqrt{\ }$，消$\sqrt{\ }$）

① 直接换元法 消$\sqrt{\ }$

例 $\int_0^3 \frac{1}{1+\sqrt{1+x}} dx$

设 $\sqrt{1+x}=t$　　$x=t^2-1$

设 $\sqrt{1+x}=t$ $\quad x=t^2-1$

$$\int_1^2 \frac{1}{1+t}\,d(t^2-1) = \int_1^2 \frac{1}{1+t}\cdot 2t\,dt$$

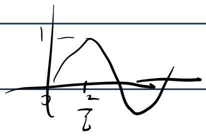
$$=2\int_1^2 \frac{t}{1+t}\,dt = 2\int_1^2 \frac{t+1-1}{t+1}\,dt$$

$$=2\int_1^2 \left(1-\frac{1}{t+1}\right)dt = 2\left(1-\ln(t+1)\Big|_1^2\right)$$

$$=2-2(\ln 3-\ln 2)=2-2\ln\frac{3}{2}$$

三角换元

$$\int_0^1 \sqrt{1-x^2}\,dx$$ 设 $x=\sin t$



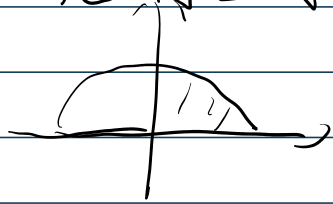
$$=\int_0^{\frac{\pi}{2}} \cos t\, d\sin t$$

$$=\int_0^{\frac{\pi}{2}} \cos t\cdot\cos t\,dt = \int_0^{\frac{\pi}{2}} \frac{\cos 2t+1}{2}\,dt = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \cos 2t\,dt+\frac{\pi}{2}\right]$$

$$=\frac{\pi}{4}+\frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}$$

法二：几何意义

$y=\sqrt{1-x^2}$



广义积分

极限算不出来叫发散

$$\int_{-\infty}^{b} f(x)\,dx = F(x)\Big|_{-\infty}^{b} = F(b) - \lim_{x\to-\infty} F(x)$$

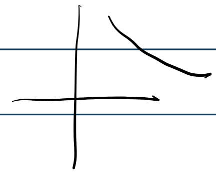
$$\int_{a}^{+\infty} f(x)\,dx = F(x)\Big|_{a}^{+\infty} = \lim_{x\to+\infty} F(x) - F(a)$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = F(x)\Big|_{-\infty}^{+\infty} = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$$

极限存在，广义积分称为收敛

不存在　　　　为发散（算不出来）

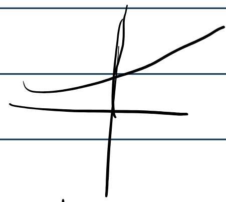
例 $\int_{1}^{+\infty} \frac{1}{x^3}\,dx$



$$= -\frac{1}{2}x^{-2}\Big|_{1}^{+\infty} = \lim_{x\to+\infty} -\frac{1}{2}x^{-2} + \frac{1}{2}$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

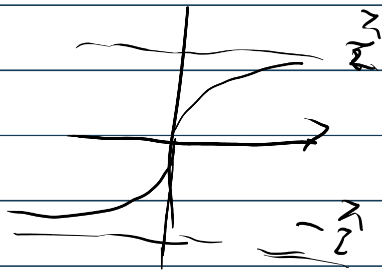
例 $\int_{-\infty}^{0} e^{5x+1}\,dx$



$$= \frac{1}{5}\left(e^{5x+1}\Big|_{-\infty}^{0}\right) = \frac{1}{5}\left(e - \lim_{x\to-\infty} e^{5x+1}\right) = \frac{1}{5}e$$

例 $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \arctan x\Big|_{-\infty}^{+\infty}$

$$= \lim_{x\to\infty} \arctan x - \lim_{x\to-\infty} \arctan x$$



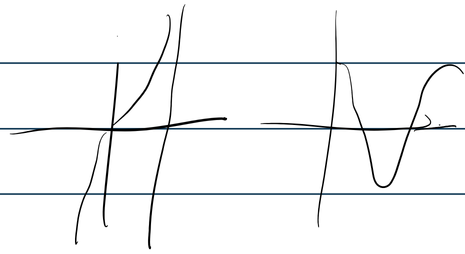
$$= \pi$$

奇偶积分　奇函数在左右向限一致的情况下 积分为0

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{奇函数} \\ 2\int_0^a f(x)\,dx & \text{偶函数} \end{cases}$$

例 $\int_{-1}^{1} \frac{\tan x}{x^2(1+\cos x)}\,dx$ 奇 ＝奇

偶　偶



奇函数，且对称 所以积分为0

例 $\int_{-1}^{1} (x+1)^2\,dx$

$= \int_{-1}^{1} x^2 + \underbrace{2x}_{\text{奇 为0}} + 1\,dx = \int_{-1}^{1} x^2+1\,dx$

$= \frac{1}{3}x^3 \Big|_{-1}^{1} + 2 = \frac{2}{3}+2$

例 $f(x) \begin{cases} x^2+1 & x \le 1 \\ 2x-1 & x>1 \end{cases}$ 求 $\int_0^2 f(x)\,dx$

$= \int_0^1 x^2+1\,dx + \int_1^2 2x-1\,dx$

$= \frac{1}{3}x^3\Big|_0^1 + 1 + x^2\Big|_1^2 - 1 = \frac{10}{3}$

~~$d f(x) = f(x)$~~ dx

② 求 $\int_?^3 f(x-1)\,dx$

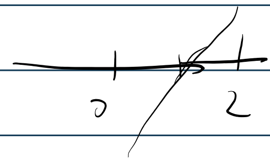
$\frac{d f(x)}{\ } = f(x)$

求 $\int_1^3 f(x-1)\,dx$ $\qquad\qquad \frac{df(x)}{dx} = f(x)$

令 $x-1=t$ $\quad x=t+1 \quad d(t+1)=dt$

$= \int_0^2 f(t)\,dt = \frac{10}{3}$

例 $\int_0^2 |x-1|\,dx$ $\qquad x$



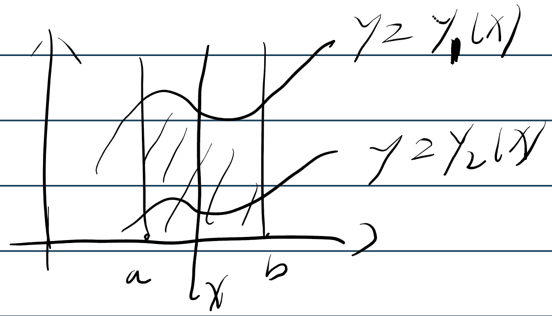
$$\begin{cases} -x+1 & 0 \le x \le 1 \\ x-1 & 1 < x \le 2 \end{cases}$$

$= \int_0^1 -x+1\,dx + \int_1^2 x-1\,dx$

$= -\frac{1}{2}x^2 \Big|_0^1 \cancel{+x} + \frac{1}{2}x^2 \Big|_1^2 \cancel{-x} = 1$

应用1：围面积 步骤

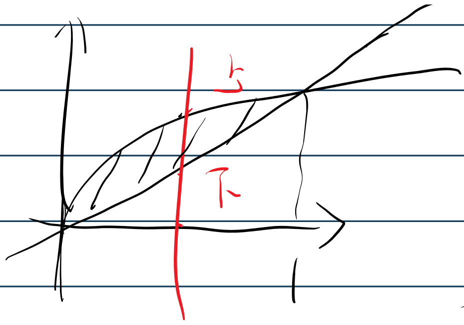
① x 型



面积

$S = \int_a^b [y_1(x) - y_2(x)]\,dx$

例求 $y=x$ 与 $y=\sqrt{x}$ 所围面积

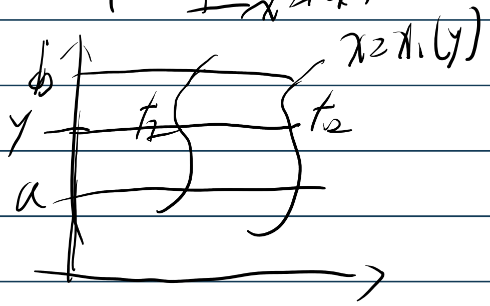


$$S=\int_0^1\left[\sqrt{x}-x\right]dx$$

$$=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1-\frac{1}{2}x^2\Big|_0^1=\frac{1}{6}$$

② Y 型

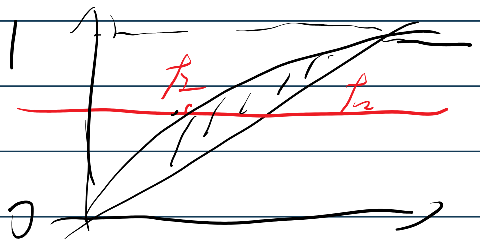
左 $x=x_2(y)$



$$S=\int_a^b\left[x_1(y)-x_2(y)\right]dy$$

例

$y=x$ 与 $y=\sqrt{x}$ $\Rightarrow x=y$ $x=y^2$

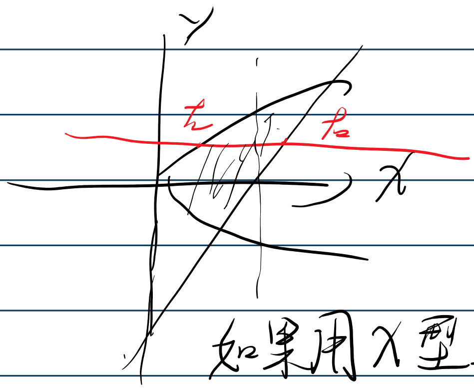


$$S=\int_0^1\left[y-y^2\right]dy$$

$$=\frac{1}{2}y^2\Big|_0^1-\frac{1}{3}y^3\Big|_0^1=\frac{1}{6}$$

例 求 $y^2=x$ 与 $y=x-2$ 所围的面积

$$\begin{cases}x=y^2\\x=y+2\end{cases}$$



$$S=\int_{-1}^{2}\left[y+2-y^2\right]dy$$

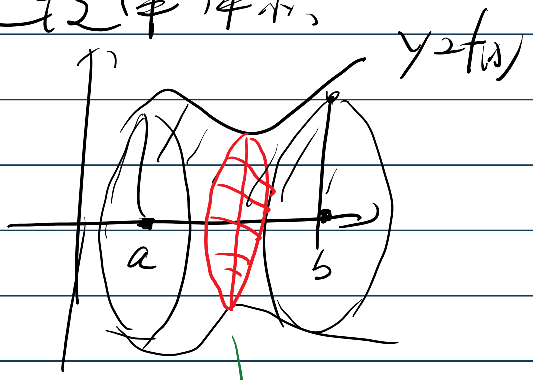
$$=\left(\frac{1}{2}y^2-\frac{1}{3}y^3\right)\Big|_{-1}^{2}+6=\frac{9}{2}$$

如果用X型来做

就要分两部分

应用2：求旋转体体积

绕x轴旋转



$y=f(x)=r$

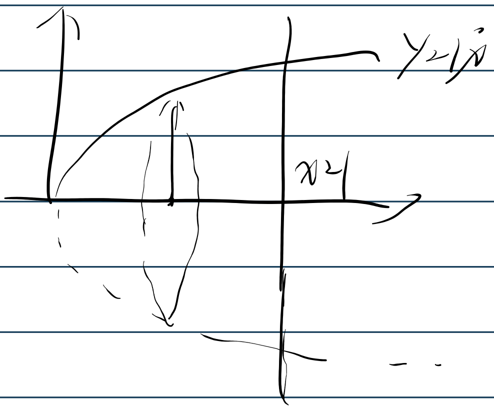
$$V=\int_a^b \pi[f(x)]^2dx$$

面积为 πr^2

体积就是所有面积求积分

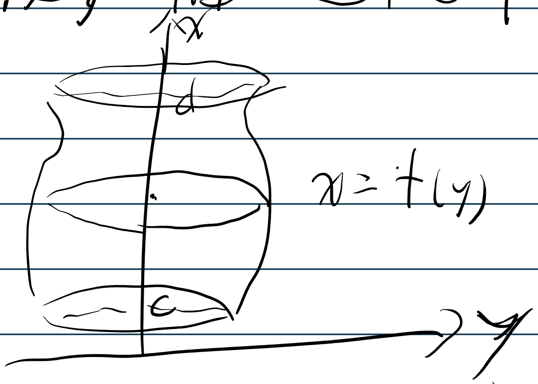
例 $y=\sqrt{x}$ 与 $x=1$ 及 $y=0$

$y=\sqrt{x}$ 旋转



$$V=\int_0^1 \pi(\sqrt{x})^2dx$$

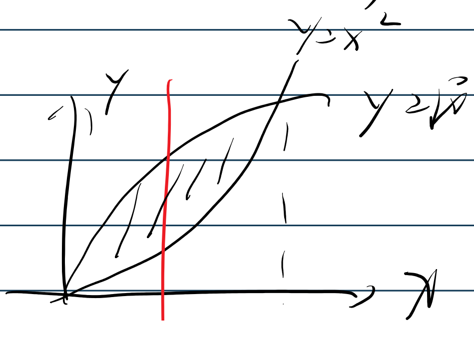
绕y轴旋转体积



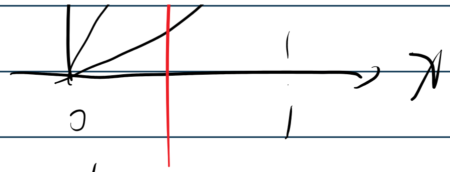
$x=f(y)$

$$V=\int_c^d \pi f(y)\,dx$$

例



绕x轴旋转

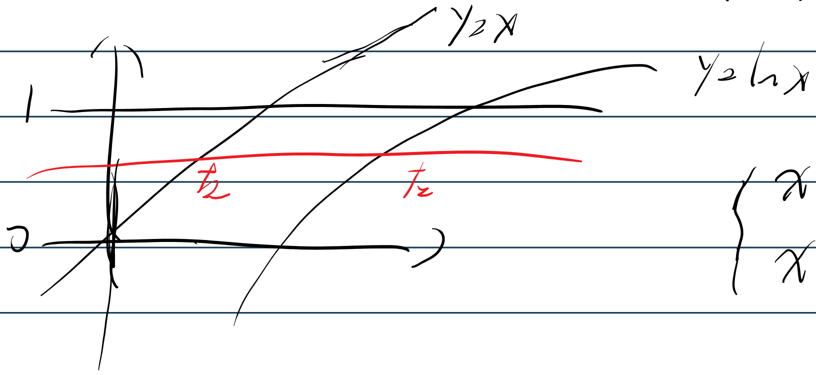


$$S=\int_0^1 \sqrt{x}-x^2\,dx$$

$$V=\int_0^1 \pi(x-x^4)\,dx$$

绕y轴旋转

例 $y=x$ 与 $y=\ln x$ ～ 所围成面积、体积～



$$\begin{cases} x=y \\ x=e^y \end{cases}$$

$$S=\int_0^1 y-e^y\,dy \quad =\left(\tfrac{1}{2}y^2-e^y\right)\Big|_0^1 \cdots$$

$$V=\int_0^1 \pi(e^{2y}-y^2)\,dy \quad =\pi\left(\tfrac{1}{2}e^{2y}-\tfrac{1}{3}y^3\right)\Big|_0^1$$