Deduction

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Following the same notation as in HPIPM document, the cost function of optimal control function is:

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_k & S_k & r_k \\ S_k^{\mathrm{T}} & Q_k & q_k \\ r_k^{\mathrm{T}} & q_k^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix} + \frac{1}{2} \Delta x_N^{\mathrm{T}} Q_N \Delta x_N + \Delta x_N^{\mathrm{T}} q_N \tag{1}$$

which is equivalent to

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_k & S_k \\ S_k^{\mathrm{T}} & Q_k \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix} + \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} + \frac{1}{2} \Delta x_N^{\mathrm{T}} Q_N \Delta x_N + \Delta x_N^{\mathrm{T}} q_N \tag{2}$$

The dynamic constraints are

$$x_{k+1} = f(x_k, u_k) \tag{3a}$$

$$\Delta x_{k+1} = A_k \Delta x_k + B_k \Delta u_k + b_k \tag{3b}$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \tag{3c}$$

The equality constraints are

$$g(x_k, u_k) = 0 \quad g \in \mathbb{R}^p \tag{4a}$$

$$C_k \Delta x_k + D_k \Delta u_k + e_k = 0 \tag{4b}$$

$$C_k = \frac{\partial g}{\partial x_k}(x_k, u_k) \quad C_k \in \mathbb{R}^{p \times n} \quad D_k = \frac{\partial g}{\partial u_k}(x_k, u_k) \quad D_k \in \mathbb{R}^{p \times m} \quad e_k = g(x_k, u_k) \tag{4c}$$

Now do the QR decomposition on the equality constraints

$$D_k \Delta u_k = -C_k \Delta x_k - e_k \tag{5a}$$

$$D_k^{\mathrm{T}} = \begin{bmatrix} Q_k^1 & Q_k^2 \end{bmatrix} \begin{bmatrix} R_k^1 \\ 0 \end{bmatrix} \quad Q_k^1 \in \mathbb{R}^{m \times p} \quad Q_k^2 \in \mathbb{R}^{m \times (m-p)} \quad R_k^1 \in \mathbb{R}^{p \times p} \tag{5b}$$

$$\text{The upper indices of } Q_k^1\text{, } Q_k^2 \text{ and } R_k^1 \text{ matrices do not mean power!} \tag{5c}$$

$$\begin{aligned} \Delta u_k &= Q_k^2 \tilde{\Delta u_k} + Q_k^1 (R_k^1)^{-\mathrm{T}} (-C_k \Delta x_k - e_k) && (5\mathrm{d}) \\ &= Q_k^2 \tilde{\Delta u_k} - P_{ke}(C_k \Delta x_k + e_k) && (5\mathrm{e}) \\ &= Q_k^2 \tilde{\Delta u_k} - P_{kx} \Delta x_k - P_{ke} e_k && (5\mathrm{f}) \\ \tilde{\Delta u_k} &\in \mathbb{R}^{(m-p) \times 1} && (5\mathrm{g}) \end{aligned}$$

with the following for simpler notations:

$$P_{kx} = Q_k^1 (R_k^1)^{-\mathrm{T}} C_k \qquad P_{ke} = Q_k^1 (R_k^1)^{-\mathrm{T}} \tag{6}$$

The new variable $\tilde{\Delta u_k}$ is not constrained anymore. Now plug the new Δu_k into the cost terms. Begin with the first order terms

$$\begin{bmatrix}\Delta u_k \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} = \begin{bmatrix} Q_k^2 \tilde{\Delta u_k} - P_{kx}\Delta x_k - P_{ke} e_k) \\ \Delta x_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_k \\ q_k \end{bmatrix} \tag{7a}$$

$$= \left(\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} - \Delta x_k^{\mathrm{T}} P_{kx}^{\mathrm{T}} - e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}}\right) r_k + \Delta x_k^{\mathrm{T}} q_k \tag{7b}$$

$$= \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (Q_k^2)^{\mathrm{T}} r_k \\ q_k - P_{kx}^{\mathrm{T}} r_k \end{bmatrix} - e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}} r_k \tag{7c}$$

$$= \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{r_k^1} \\ \tilde{q_k^1} \end{bmatrix} + C_k^1 \tag{7d}$$

$$\tilde{r_k^1} = (Q_k^2)^{\mathrm{T}} r_k \tag{7e}$$

$$\tilde{q_k^1} = q_k - P_{kx}^{\mathrm{T}} r_k \tag{7f}$$

$$C_k^1 = -e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}} r_k \tag{7g}$$

Then the second order term $\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k$

$$\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k \tag{8a}$$

$$= \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k Q_k^2 \tilde{\Delta u_k} + \frac{1}{2}\Delta x_k^{\mathrm{T}} P_{kx}^{\mathrm{T}} R_k P_{kx} \Delta x_k + \frac{1}{2} e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}} R_k P_{ke} e_k \tag{8b}$$

$$- \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k P_{kx} \Delta x_k - \frac{1}{2}\Delta x_k^{\mathrm{T}} P_{kx}^{\mathrm{T}} R_k (Q_k^2) \tilde{\Delta u_k} \tag{8c}$$

$$- \tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} R_k P_{ke} e_k - \Delta x_k^{\mathrm{T}} P_{kx}^{\mathrm{T}} R_k P_{ke} e_k \tag{8d}$$

Then the second order term $\frac{1}{2}\Delta u_k^{\mathrm{T}} S_k \Delta x_k$ and $\frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} \Delta u_k$

$$\frac{1}{2}\Delta u_k^{\mathrm{T}} S_k \Delta x_k = \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} (Q_k^2)^{\mathrm{T}} S_k \Delta x_k - \frac{1}{2}\Delta x_k^{\mathrm{T}} P_{kx}^{\mathrm{T}} S_k \Delta x_k - \frac{1}{2} e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}} S_k \Delta x_k \tag{9a}$$

$$\frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} \Delta u_k = \frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} (Q_k^2) \tilde{\Delta u_k} - \frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} P_{kx} \Delta x_k - \frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} P_{ke} e_k \tag{9b}$$

Note there is no change to the second order term $\frac{1}{2}\Delta x_k^{\mathrm{T}} Q_k \Delta x_k$. Combining all above:

$$\frac{1}{2}\Delta u_k^{\mathrm{T}} R_k \Delta u_k + \frac{1}{2}\Delta x_k^{\mathrm{T}} Q_k \Delta x_k + \frac{1}{2}\Delta u_k^{\mathrm{T}} S_k \Delta x_k + \frac{1}{2}\Delta x_k^{\mathrm{T}} S_k^{\mathrm{T}} \Delta u_k \tag{10a}$$

$$= \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} \tilde{R_k} \tilde{\Delta u_k} + \frac{1}{2}\Delta x_k^{\mathrm{T}} \tilde{Q_k} \Delta x_k + \frac{1}{2}\tilde{\Delta u_k}^{\mathrm{T}} \tilde{S_k} \Delta x_k + \frac{1}{2}\Delta x_k^{\mathrm{T}} \tilde{S_k}^{\mathrm{T}} \tilde{\Delta u_k} + \begin{bmatrix}\tilde{\Delta u_k} \\ \Delta x_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{r_k^2} \\ \tilde{q_k^2} \end{bmatrix} + C_k^2 \tag{10b}$$

$$\tilde{R_k} = (Q_k^2)^{\mathrm{T}} R_k Q_k^2 \tag{10c}$$

$$\tilde{Q_k} = Q_k - P_{kx}^{\mathrm{T}} S_k - S_k^{\mathrm{T}} P_{kx} + P_{kx}^{\mathrm{T}} R_k P_{kx} \tag{10d}$$

$$\tilde{S_k} = (Q_k^2)^{\mathrm{T}} S_k - (Q_k^2)^{\mathrm{T}} R_k P_{kx} \tag{10e}$$

$$\tilde{r_k^2} = -(Q_k^2)^{\mathrm{T}} R_k P_{ke} e_k \tag{10f}$$

$$\tilde{q_k^2} = -S_k^{\mathrm{T}} P_{ke} e_k - P_{kx}^{\mathrm{T}} R_k P_{ke} e_k \tag{10g}$$

$$C_k^2 = \frac{1}{2} e_k^{\mathrm{T}} P_{ke}^{\mathrm{T}} R_k P_{ke} e_k \tag{10h}$$

The new cost is now

$$\sum_{k=0}^{N-1}\left(\frac{1}{2}\begin{bmatrix}\Delta\tilde{u}_k\\ \Delta x_k\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{R}_k & \tilde{Q}_k\\ \tilde{S}_k^{\mathrm{T}} & \tilde{Q}_k\end{bmatrix}\begin{bmatrix}\Delta\tilde{u}_k\\ \Delta x_k\end{bmatrix}+\begin{bmatrix}\Delta\tilde{u}_k\\ \Delta x_k\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{r}_k\\ \tilde{q}_k\end{bmatrix}+C_k^1+C_k^2\right)+\frac{1}{2}\Delta x_N^{\mathrm{T}}Q_N\Delta x_N+\Delta x_N^{\mathrm{T}}q_N \tag{11a}$$

$$\tilde{r}_k = \tilde{r}_k^1 + \tilde{r}_k^2 \tag{11b}$$

$$\tilde{q}_k = \tilde{q}_k^1 + \tilde{q}_k^2 \tag{11c}$$

Now we plug the new variable $\Delta u_k = Q_k^2\Delta\tilde{u}_k - P_{kx}\Delta x_k - P_{ke}e_k$ into the dynamic equality constraints

$$\Delta x_{k+1} = A_k\Delta x_k + B_k\Delta u_k + b_k \tag{12a}$$

$$= A_k\Delta x_k + B_kQ_k^2\Delta\tilde{u}_k - B_kP_{kx}\Delta x_k - B_kP_{ke}e_k + b_k \tag{12b}$$

$$= \left(A_k - B_kP_{kx}\right)\Delta x_k + B_kQ_k^2\Delta\tilde{u}_k + b_k - B_kP_{ke}e_k \tag{12c}$$

$$= \tilde{A}_k\Delta x_k + \tilde{B}_k\Delta\tilde{u}_k + \tilde{b}_k \tag{12d}$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \tag{12e}$$

$$\tilde{A}_k = (A_k - B_kP_{kx}) \quad \tilde{B}_k = B_kQ_k^2 \quad \tilde{b}_k = b_k - B_kP_{ke}e_k \tag{12f}$$