

Deduction

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Following the same notation as in HPIPM document, the cost function of optimal control function is:

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix}^T \begin{bmatrix} R_k & S_k & r_k \\ S_k^T & Q_k & q_k \\ r_k^T & q_k^T & 0 \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \\ 1 \end{bmatrix} + \frac{1}{2} \Delta x_N^T Q_N \Delta x_N + \Delta x_N^T q_N \quad (1)$$

which is equivalent to

$$\sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} R_k & S_k \\ S_k^T & Q_k \end{bmatrix} \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix} + \begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} r_k \\ q_k \end{bmatrix} + \frac{1}{2} \Delta x_N^T Q_N \Delta x_N + \Delta x_N^T q_N \quad (2)$$

The dynamic constraints are

$$x_{k+1} = f(x_k, u_k) \quad (3a)$$

$$\Delta x_{k+1} = A_k \Delta x_k + B_k \Delta u_k + b_k \quad (3b)$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \quad (3c)$$

The equality constraints are

$$g(x_k, u_k) = 0 \quad g \in \mathbb{R}^p \quad (4a)$$

$$C_k \Delta x_k + D_k \Delta u_k + e_k = 0 \quad (4b)$$

$$C_k = \frac{\partial g}{\partial x_k}(x_k, u_k) \quad C_k \in \mathbb{R}^{p \times n} \quad D_k = \frac{\partial g}{\partial u_k}(x_k, u_k) \quad D_k \in \mathbb{R}^{p \times m} \quad e_k = g(x_k, u_k) \quad (4c)$$

Now do the QR decomposition on the equality constraints

$$D_k \Delta u_k = -C_k \Delta x_k - e_k \quad (5a)$$

$$D_k^T = [Q_k^1 \quad Q_k^2] \begin{bmatrix} R_k^1 \\ 0 \end{bmatrix} \quad Q_k^1 \in \mathbb{R}^{m \times p} \quad Q_k^2 \in \mathbb{R}^{m \times (m-p)} \quad R_k^1 \in \mathbb{R}^{p \times p} \quad (5b)$$

$$\text{The upper indices of } Q_k^1, Q_k^2 \text{ and } R_k^1 \text{ matrices do not mean power!} \quad (5c)$$

$$\Delta u_k = Q_k^2 \tilde{\Delta u}_k + Q_k^1 (R_k^1)^{-T} (-C_k \Delta x_k - e_k) \quad (5d)$$

$$= Q_k^2 \tilde{\Delta u}_k - Q_k^1 (R_k^1)^{-T} (C_k \Delta x_k + e_k) \quad \tilde{\Delta u}_k \in \mathbb{R}^{(m-p) \times 1} \quad (5e)$$

The new variable $\tilde{\Delta u}_k$ is not constrained anymore. Now plug the new Δu_k into the cost terms. Begin with the first order terms

$$\begin{bmatrix} \Delta u_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} r_k \\ q_k \end{bmatrix} = \begin{bmatrix} Q_k^2 \tilde{\Delta u}_k - Q_k^1 (R_k^1)^{-T} (C_k \Delta x_k + e_k) \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} r_k \\ q_k \end{bmatrix} \quad (6a)$$

$$= \left(\tilde{\Delta u}_k^T (Q_k^2)^T - \Delta x_k^T (Q_k^1 (R_k^1)^{-T} C_k)^T - e_k^T (Q_k^1 (R_k^1)^{-T})^T \right) r_k + \Delta x_k^T q_k \quad (6b)$$

$$= \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} (Q_k^2)^T r_k \\ q_k - (Q_k^1 (R_k^1)^{-T} C_k)^T r_k \end{bmatrix} - e_k^T (Q_k^1 (R_k^1)^{-T})^T r_k \quad (6c)$$

$$= \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} \tilde{r}_k^1 \\ \tilde{q}_k^1 \end{bmatrix} + C_k^1 \quad (6d)$$

$$\tilde{r}_k^1 = (Q_k^2)^T r_k \quad (6e)$$

$$\tilde{q}_k^1 = q_k - (Q_k^1 (R_k^1)^{-T} C_k)^T r_k \quad (6f)$$

$$C_k^1 = -e_k^T (Q_k^1 (R_k^1)^{-T})^T r_k \quad (6g)$$

Then the second order term $\frac{1}{2} \Delta u_k^T R_k \Delta u_k$

$$\frac{1}{2} \Delta u_k^T R_k \Delta u_k \quad (7a)$$

$$= \frac{1}{2} \tilde{\Delta u}_k^T (Q_k^2)^T R_k Q_k^2 \tilde{\Delta u}_k + \frac{1}{2} \Delta x_k^T (Q_k^1 (R_k^1)^{-T} C_k)^T R_k (Q_k^1 (R_k^1)^{-T} C_k) \Delta x_k + \frac{1}{2} e_k^T (Q_k^1 (R_k^1)^{-T})^T R_k (Q_k^1 (R_k^1)^{-T}) e_k \quad (7b)$$

$$- \tilde{\Delta u}_k^T (Q_k^2)^T R_k (Q_k^1 (R_k^1)^{-T} C_k) \Delta x_k - \tilde{\Delta u}_k^T (Q_k^2)^T R_k (Q_k^1 (R_k^1)^{-T}) e_k - \Delta x_k^T (Q_k^1 (R_k^1)^{-T} C_k)^T R_k (Q_k^1 (R_k^1)^{-T}) e_k \quad (7c)$$

Then the second order term $\Delta u_k^T S_k \Delta x_k$

$$\Delta u_k^T S_k \Delta x_k = \tilde{\Delta u}_k^T (Q_k^2)^T S_k \Delta x_k - \Delta x_k^T (Q_k^1 (R_k^1)^{-T} C_k)^T S_k \Delta x_k - e_k^T (Q_k^1 (R_k^1)^{-T})^T S_k \Delta x_k \quad (8a)$$

Note there is no change to the second order term $\frac{1}{2} \Delta x_k^T Q_k \Delta x_k$. Combining all above:

$$\frac{1}{2} \Delta u_k^T R_k \Delta u_k + \frac{1}{2} \Delta x_k^T Q_k \Delta x_k + \Delta u_k^T S_k \Delta x_k \quad (9a)$$

$$= \frac{1}{2} \tilde{\Delta u}_k^T \tilde{R}_k \tilde{\Delta u}_k + \frac{1}{2} \Delta x_k^T \tilde{Q}_k \Delta x_k + \tilde{\Delta u}_k^T \tilde{S}_k \Delta x_k + \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} \tilde{r}_k^2 \\ \tilde{q}_k^2 \end{bmatrix} + C_k^2 \quad (9b)$$

$$\tilde{R}_k = (Q_k^2)^T R_k Q_k^2 \quad (9c)$$

$$\tilde{Q}_k = Q_k - 2 (Q_k^1 (R_k^1)^{-T} C_k)^T S_k + (Q_k^1 (R_k^1)^{-T} C_k)^T R_k (Q_k^1 (R_k^1)^{-T} C_k) \quad (9d)$$

$$\tilde{S}_k = (Q_k^2)^T S_k - (Q_k^2)^T R_k (Q_k^1 (R_k^1)^{-T} C_k) \quad (9e)$$

$$\tilde{r}_k^2 = -(Q_k^2)^T R_k (Q_k^1 (R_k^1)^{-T}) e_k \quad (9f)$$

$$\tilde{q}_k^2 = -S_k (Q_k^1 (R_k^1)^{-T}) e_k - (Q_k^1 (R_k^1)^{-T} C_k)^T R_k (Q_k^1 (R_k^1)^{-T}) e_k \quad (9g)$$

$$C_k^2 = \frac{1}{2} e_k^T (Q_k^1 (R_k^1)^{-T})^T R_k (Q_k^1 (R_k^1)^{-T}) e_k \quad (9h)$$

The new cost is now

$$\sum_{k=0}^{N-1} \left(\frac{1}{2} \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} \tilde{R}_k & \tilde{Q}_k \\ \tilde{S}_k & \tilde{Q}_k \end{bmatrix} \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix} + \begin{bmatrix} \tilde{\Delta u}_k \\ \Delta x_k \end{bmatrix}^T \begin{bmatrix} \tilde{r}_k \\ \tilde{q}_k \end{bmatrix} + C_k^1 + C_k^2 \right) + \frac{1}{2} \Delta x_N^T Q_N \Delta x_N + \Delta x_N^T q_N \quad (10a)$$

$$\tilde{r}_k = \tilde{r}_k^1 + \tilde{r}_k^2 \quad (10b)$$

$$\tilde{q}_k = \tilde{q}_k^1 + \tilde{q}_k^2 \quad (10c)$$

Now we plug the new variable $\Delta u_k = Q_k^2 \tilde{\Delta} u_k - Q_k^1 (R_k^1)^{-T} (C_k \Delta x_k + e_k)$ into the dynamic equality constraints

$$\Delta x_{k+1} = A_k \Delta x_k + B_k \Delta u_k + b_k \quad (11a)$$

$$= A_k \Delta x_k + B_k Q_k^2 \tilde{\Delta} u_k - B_k Q_k^1 (R_k^1)^{-T} C_k \Delta x_k - B_k Q_k^1 (R_k^1)^{-T} e_k + b_k \quad (11b)$$

$$= (A_k - B_k Q_k^1 (R_k^1)^{-T} C_k) \Delta x_k + B_k Q_k^2 \tilde{\Delta} u_k + b_k - B_k Q_k^1 (R_k^1)^{-T} e_k \quad (11c)$$

$$= \tilde{A}_k \Delta x_k + \tilde{B}_k \tilde{\Delta} u_k + \tilde{b}_k \quad (11d)$$

$$A_k = \frac{\partial f}{\partial x_k}(x_k, u_k) \quad B_k = \frac{\partial f}{\partial u_k}(x_k, u_k) \quad b_k = f(x_k, u_k) - x_{k+1} \quad (11e)$$

$$\tilde{A}_k = (A_k - B_k Q_k^1 (R_k^1)^{-T} C_k) \quad \tilde{B}_k = B_k Q_k^2 \quad \tilde{b}_k = b_k - B_k Q_k^1 (R_k^1)^{-T} e_k \quad (11f)$$