FREQUENCY DOMAIN INTERPRETATION OF MPE METHODS

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$\widehat{\theta}_N$: estimated from N data,

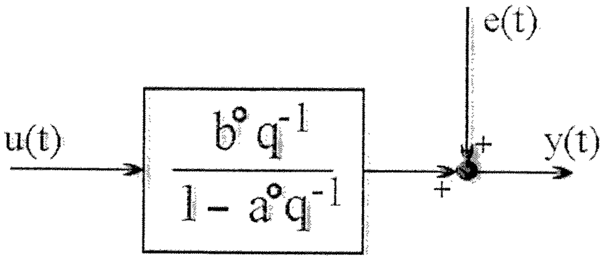
it is stochastic

$\widehat{\theta}_N \to \theta^\star$, $\theta^\star$ deterministic

$\mathcal{M}(\theta^\star)$: asymptotically estimated model

description of $\mathcal{M}(\theta^\star)$ in the frequency domain

An Introductory Example



$a^\circ = 0.5,\ b^\circ = 1$

$$\hat{y}(t|\theta) = \frac{bq^{-1}}{1 - aq^{-1}} u(t) \qquad \textbf{(O.E.)}$$

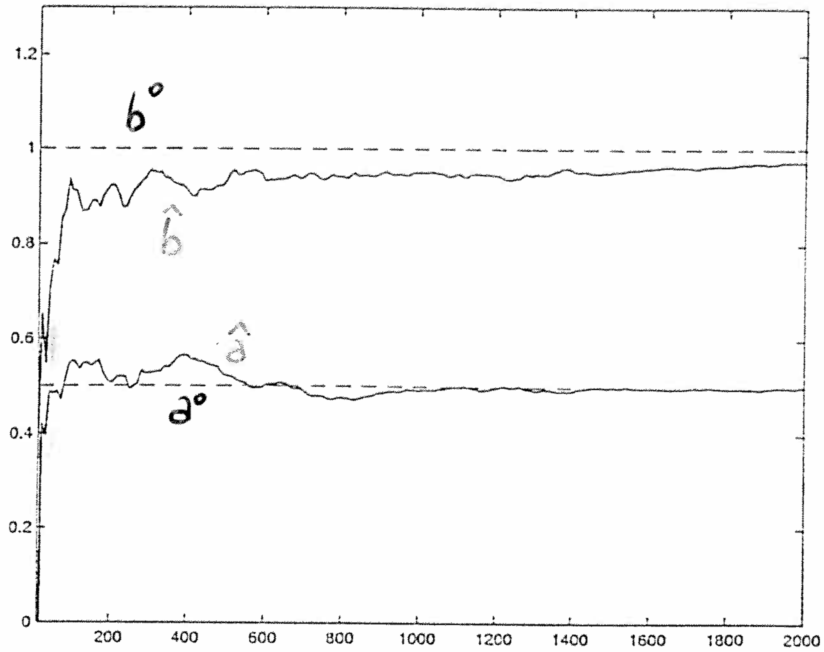
$$\hat{\theta}_N \text{ minimizes } \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t|\theta))^2$$

$\Rightarrow$ iterative minimization

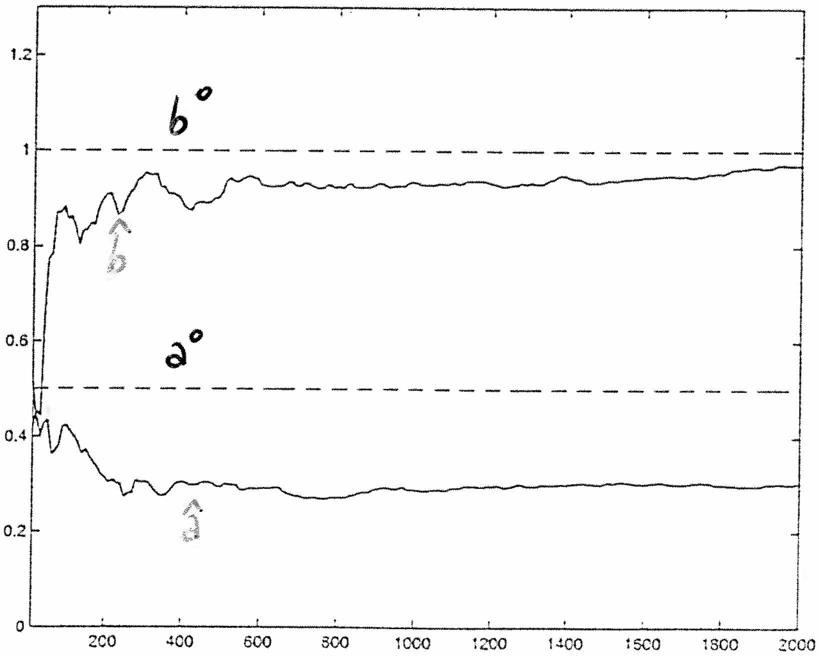
$$y(t) = a^\circ y(t-1) + b^\circ u(t-1) + \text{disturbance term}$$

$$\hat{y}(t|\theta) = ay(t-1) + bu(t-1) \qquad \textbf{(ARX)}$$

$\Rightarrow$ least squares



$$\hat{y}(t|\theta) = \frac{bq^{-1}}{1 - aq^{-1}} u(t)$$



$$\hat{y}(t|\theta) = ay(t-1) + bu(t-1)$$

MPE Methods

$$\mathcal{M}(\theta): \quad y(t) = G(\theta)u(t) + H(\theta)e(t)$$

$$\widehat{y}(t|\theta) = H^{-1}(\theta)G(\theta)u(t) + [1 - H^{-1}(\theta)]y(t)$$

$$\epsilon(t|\theta) := y(t) - \widehat{y}(t|\theta)$$

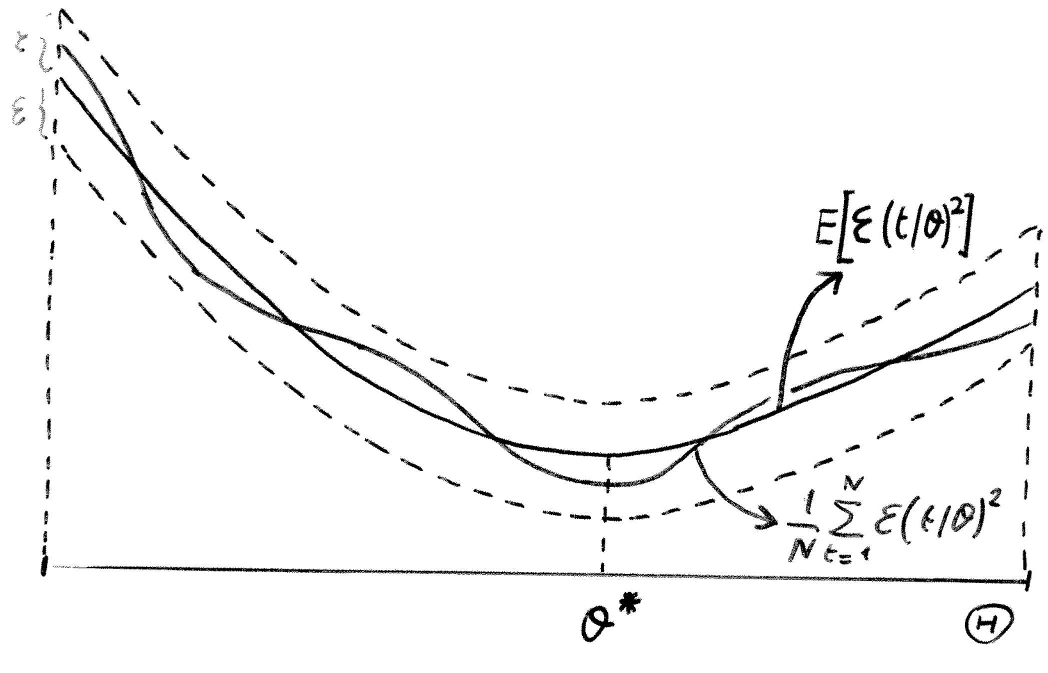
$\widehat{\theta}_N$ minimizes $\displaystyle \frac{1}{N}\sum_{t=1}^{N} \epsilon(t|\theta)^2$

$$\frac{1}{N}\sum_{t=1}^{N} \epsilon(t|\theta)^2 \xrightarrow{?} E\left[\epsilon(t|\theta)^2\right]$$

$$\frac{1}{N}\sum_{t=1}^{N}\epsilon(t|\theta)^2 \longrightarrow E\left[\epsilon(t|\theta)^2\right]$$ uniformly in θ with probability 1

$\epsilon > 0, \qquad N \geq N(\epsilon)$

$$\left|\frac{1}{N}\sum_{t=1}^{N}\epsilon(t|\theta)^2 - E\left[\epsilon(t|\theta)^2\right]\right| \leq \epsilon, \quad \forall\theta$$

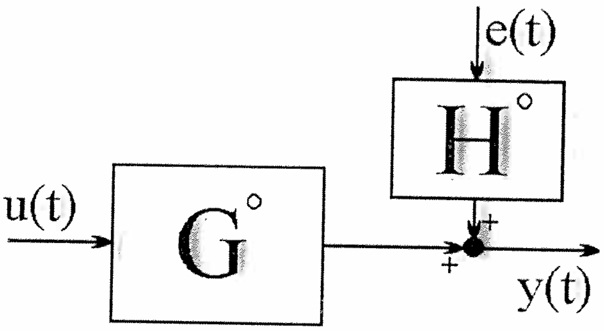


$\widehat{\theta}_N \rightarrow \theta^\star$ with probability 1

$\widehat{\theta}_N$ asymptotically minimizes $E\left[\epsilon(t|\theta)^2\right]$

Computing $E\left[\epsilon(t|\theta)^2\right]$

$$\mathcal{S} : y(t) = G^\circ u(t) + H^\circ e(t)$$



$$\epsilon(t|\theta) = y(t) - \widehat{y}(t|\theta)$$

$$= y(t) - H^{-1}(\theta)G(\theta)u(t) - [1 - H^{-1}(\theta)]y(t)$$

$$= -H^{-1}(\theta)G(\theta)u(t) + H^{-1}(\theta)y(t)$$

$$= -H^{-1}(\theta)G(\theta)u(t) + H^{-1}(\theta)[G^\circ u(t) + H^\circ e(t)]$$

$$= +H^{-1}(\theta)[G^\circ - G(\theta)]u(t) + H^{-1}(\theta)H^\circ e(t)$$

Open-Loop

$$
\begin{aligned}
& E\left[\epsilon(t|\theta)^2\right] \\
&= E\left[\left(H^{-1}(\theta)[G^\circ - G(\theta)]u(t)\right)^2\right] + E\left[\left(H^{-1}(\theta)H^\circ e(t)\right)^2\right] \\
&= \int_{-\pi}^{\pi} \left\{ \left|[G^\circ(e^{i\omega}) - G(e^{i\omega},\theta)]\right|^2 \frac{1}{|H(e^{i\omega},\theta)|^2}\Phi_u(\omega) \right. \\
&\qquad \left. + \frac{|H^\circ(e^{i\omega})|^2}{|H(e^{i\omega},\theta)|^2}\Phi_e(\omega) \right\} d\omega
\end{aligned}
$$

$$
\epsilon_L(t|\theta) := Ly(t) - L\widehat{y}(t|\theta)
$$

$$
\begin{aligned}
& E\left[\epsilon_L(t|\theta)^2\right] = \\
& \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{|L|^2}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}|L|^2\Phi_e \right\} d\omega
\end{aligned}
$$

... introductory example

$$\widehat{y}(t|\theta) = ay(t-1) + bu(t-1)$$

$$\mathcal{M}(\theta) : y(t) = ay(t-1) + bu(t-1) + e(t)$$

$$A(\theta) = 1 - aq^{-1} \qquad B(\theta) = bq^{-1}$$

$$A(\theta)y(t) = B(\theta)u(t) + e(t)$$

$$y(t) = \frac{B(\theta)}{A(\theta)}u(t) + \frac{1}{A(\theta)}e(t)$$

$$G(\theta) = \frac{B(\theta)}{A(\theta)} \qquad H(\theta) = \frac{1}{A(\theta)}$$

$$\mathcal{S} : y(t) = \frac{B^\circ}{A^\circ}u(t) + e(t)$$

$$E\left[\epsilon(t|\theta)^2\right] =$$

$$= \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}\Phi_e(\omega) \right\} d\omega$$

$$= \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \, |A(\theta)|^2 \Phi_u + |A(\theta)|^2 \Phi_e \right\} d\omega$$

An interpretation in the time domain

$$\mathcal{S} : y(t) = \frac{B^\circ}{A^\circ} u(t) + e(t)$$

$$y(t) = a^\circ y(t-1) + b^\circ u(t-1) + e(t) - a^\circ e(t-1)$$

$$\mathcal{M}(\theta) \Rightarrow H(\theta) \neq 1$$

$\widehat{y}(t|\theta)$ depends on the past $y(t)$'s:

$$\widehat{y}(t|\theta) = ay(t-1) + bu(t-1)$$

$\mathcal{M}(\theta) \Rightarrow H(\theta) = 1$

$\widehat{y}(t|\theta)$ does not depend on the past $y(t)$:

$$\widehat{y}(t|\theta) = \frac{B(\theta)}{A(\theta)} u(t)$$

$$\mathcal{M}(\theta) : y(t) = \frac{B(\theta)}{A(\theta)} u(t) + e(t)$$

$$G(\theta) = \frac{B(\theta)}{A(\theta)} \qquad H(\theta) = 1$$

$$\mathcal{S} : y(t) = \frac{B^\circ}{A^\circ} u(t) + e(t)$$

$$E\left[\epsilon(t|\theta)^2\right] =$$

$$= \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2} \Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2} \Phi_e(\omega) \right\} d\omega$$

$$= \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \Phi_u + \Phi_e \right\} d\omega$$

$$E\left[\epsilon(t|\theta)^2\right]$$

$$=\int_{-\pi}^{\pi}\left\{|[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}\Phi_e\right\}d\omega$$

CASE 1

$$\boxed{G^\circ \in \{G(\theta)\},\ H^\circ \in \{H(\theta)\}}$$

With $G(\theta) = G^\circ$ and $H(\theta) = H^\circ$:

$$E\left[\epsilon(t|\theta)^2\right] = \int_{-\pi}^{\pi} \Phi_e d\omega = var[e(t)]$$

- When is the estimate consistent?

  $u(t)$ sufficiently exciting

$$E\left[\epsilon(t|\theta)^2\right]$$
$$=\int_{-\pi}^{\pi}\left\{|[G^\circ - G(\theta)]|^2\frac{1}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}\Phi_e\right\}d\omega$$

CASE 2

$$\boxed{G^\circ \in \{G(\theta)\},\ H^\circ \notin \{H(\theta)\}}$$

- $\underline{H(\theta) = \bar{H}}$

$$\int_{-\pi}^{\pi}|[G^\circ - G(\theta)]|^2\,\frac{1}{|\bar{H}|^2}\Phi_u d\omega$$

- $\underline{H(\theta) \text{ not fixed}}$

$$\int_{-\pi}^{\pi} \left\{ |[G^{\circ} - G(\theta)]|^2 \frac{1}{|H(\theta)|^2} \Phi_u + \frac{|H^{\circ}|^2}{|H(\theta)|^2} \Phi_e \right\} d\omega$$

frequency weighting for ΔG : $\dfrac{\Phi_u}{|H(\hat{\theta})|^2}$

$$E\left[\epsilon(t|\theta)^2\right]$$
$$=\int_{-\pi}^{\pi}\left\{|[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}\Phi_e\right\}d\omega$$

CASE 3

$$\boxed{G^\circ \notin \{G(\theta)\},\ H^\circ \notin \{H(\theta)\}}$$

$$\int_{-\pi}^{\pi}\left\{|[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2}\Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2}\Phi_e\right\}d\omega$$

frequency weighting for ΔG : $\underline{\dfrac{\Phi_u}{|H(\hat{\theta})|^2}}$

if L is used:
frequency weighting for ΔG : $\underline{\dfrac{\Phi_u}{|H(\hat{\theta})|^2}|L|^2}$

Summary:

MPE features in open loop

- $G^{\circ} \in \{G(\theta)\}$

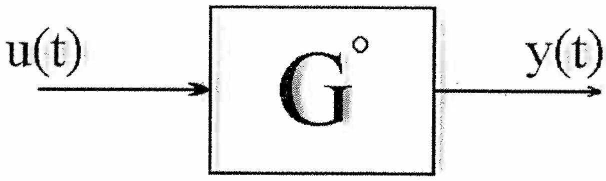
  consistent estimate if $H(\theta) = \bar{H}$

- in general:

  frequency weighting for ΔG:

  $$\frac{\Phi_u}{|H(\hat{\theta})|^2}|L|^2$$

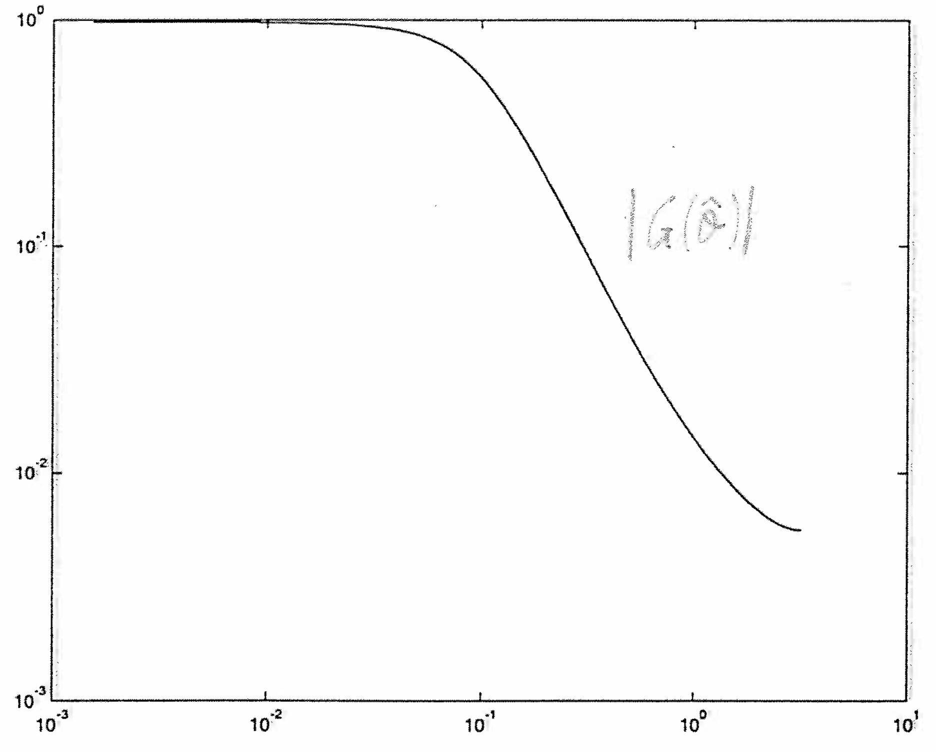
Example



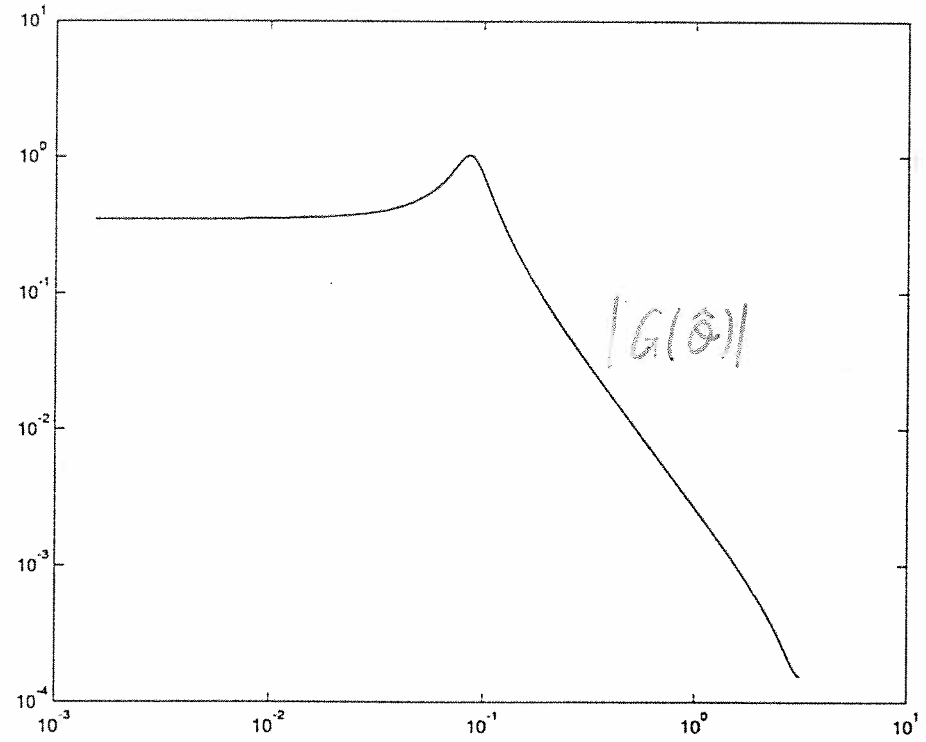
$$G^{\circ} = \frac{0.001q^{-1}\left(1 + 0.43q^{-1} + 0.056q^{-2} + 0.0023q^{-3}\right)}{1 - 3.1q^{-1} + 3.5801q^{-2} - 1.8248q^{-3} + 0.3462q^{-4}}$$

$u(t)$ white

$$\hat{y}(t|\theta) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}} u(t) \qquad \textbf{(OE)}$$



$$\widehat{y}(t|\theta) = a_1 y(t-1) + a_2 y(t-2)$$
$$b_1 u(t-1) + b_2 u(t-2) \quad (\textbf{ARX})$$



Which estimate can we rely on?

(OE)

$$\mathcal{M}(\theta) : y(t) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}} u(t) + e(t)$$

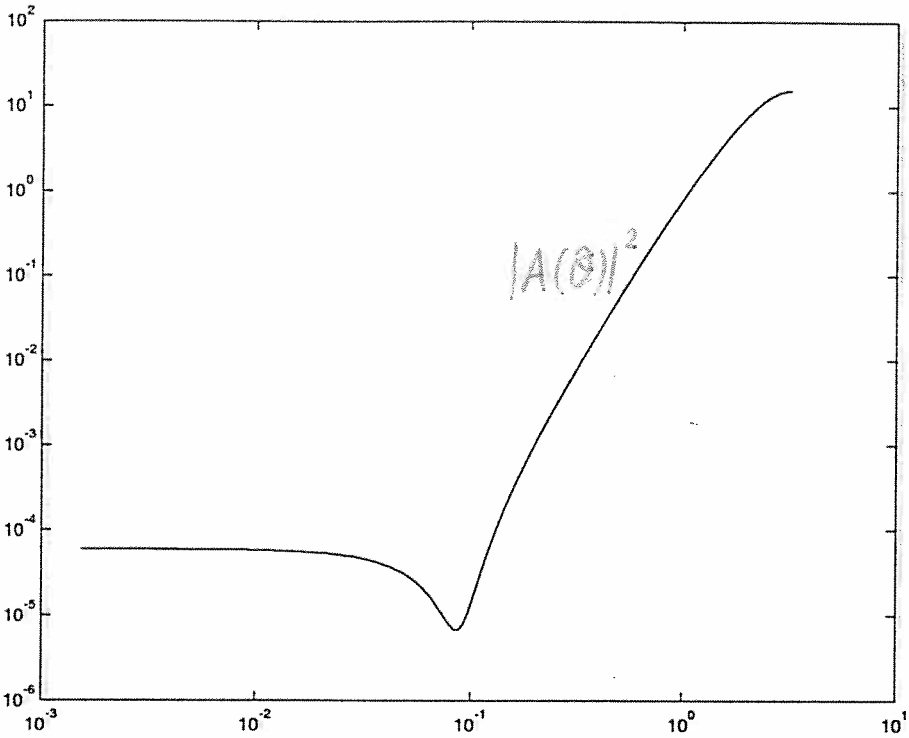
frequency weighting for ΔG : $\dfrac{\Phi_u}{|H(\hat{\theta})|^2} = 1$

$$E\left[\epsilon(t|\theta)^2\right] = \int_{-\pi}^{\pi} |[G^\circ - G(\theta)]|^2 \, d\omega$$

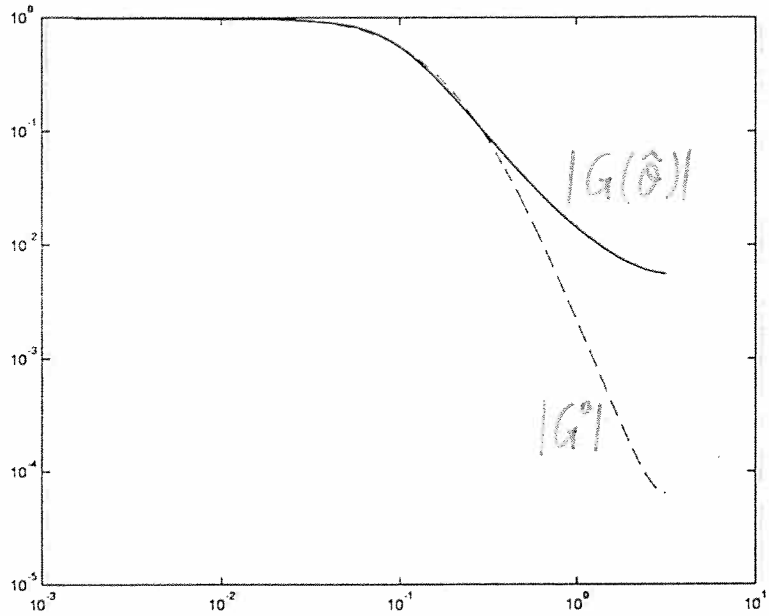
(ARX)

$$\mathcal{M}(\theta) : \quad y(t) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}} u(t) + \frac{1}{1 - a_1 q^{-1} - a_2 q^{-2}} e(t)$$

frequency weighting for ΔG: $\dfrac{\Phi_u}{|H(\hat{\theta})|^2} = |A(\hat{\theta})|^2$

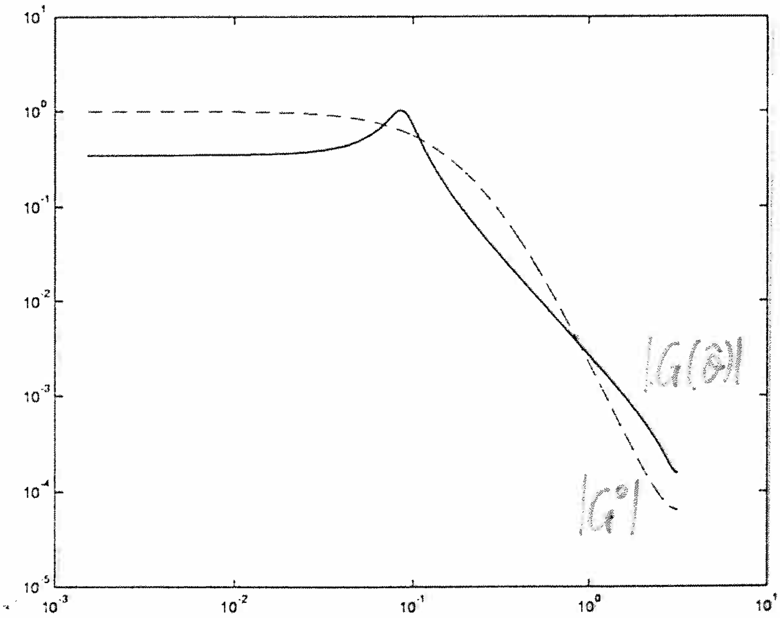

$|A(\delta)|^2$
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}

OE



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
$|G(\hat{\theta})|$
$|G^0|$

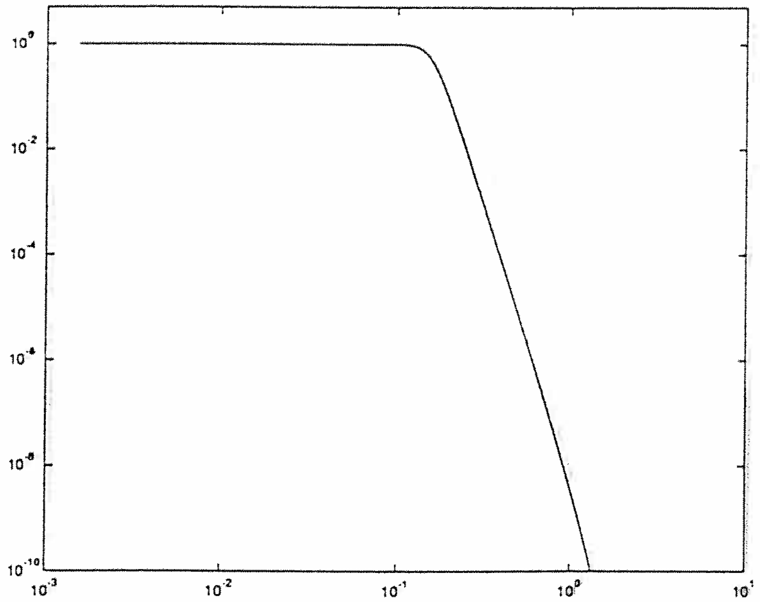
ARX



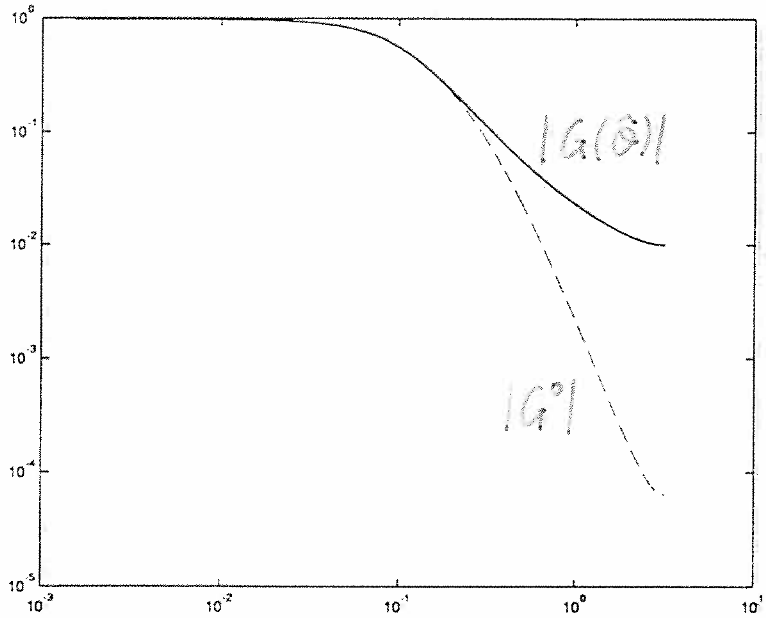
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
$|G(\hat{\theta})|$
$|G^0|$

The use of L

L = butterworth filter (5^{th} order)
$\bar{\omega} = 0.1$

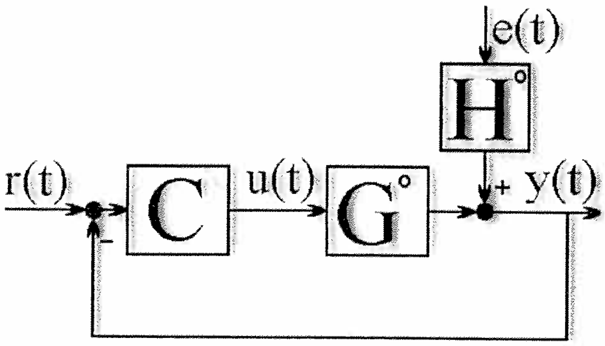


(ARX)



Closed-Loop

$$\epsilon(t|\theta) = H^{-1}(\theta)[G^{\circ} - G(\theta)]u(t) + H^{-1}(\theta)H^{\circ}e(t)$$



$$u(t) = CS^{\circ}r(t) - H^{\circ}CS^{\circ}e(t)$$

$$S^{\circ} = \frac{1}{1 + CG^{\circ}}$$

$$\epsilon(t|\theta) = H^{-1}(\theta)[G^{\circ} - G(\theta)]CS^{\circ}r(t)$$

$$+H^{-1}(\theta)H^{\circ}S^{-1}(\theta)S^{\circ}e(t)$$

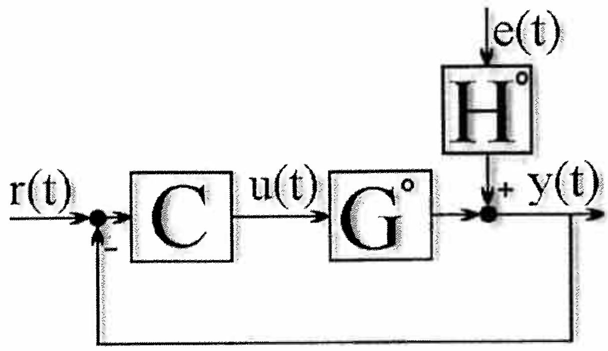
$$S(\theta) = \frac{1}{1 + CG(\theta)}$$

$$E\left[\epsilon(t|\theta)^2\right]$$
$$= \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{|CS^\circ|^2}{|H(\theta)|^2} \Phi_r \right.$$
$$\left. + \frac{|H^\circ|^2 |S^\circ|^2}{|H(\theta)|^2 |S(\theta)|^2} \Phi_e \right\} d\omega$$

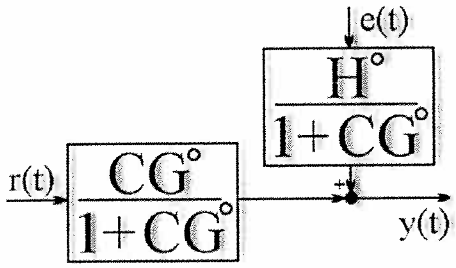
- $G^\circ \in \{G(\theta)\}$
  even if $H(\theta) = \bar{H}$, the estimate is not consistent

- in general:
  frequency weighting for ΔG : $\dfrac{|CS^\circ|^2}{|H(\hat{\theta})|^2} \Phi_r$
  unknown even a-posteriori !

Two possible remedies

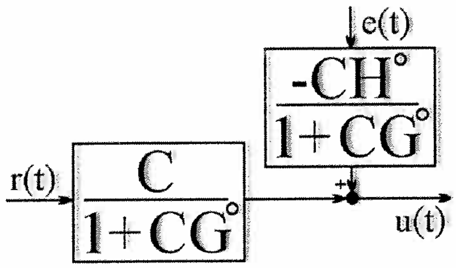


• Estimate

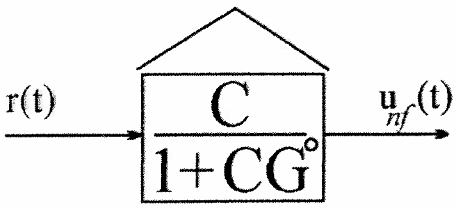


find G° from the estimated $\dfrac{CG^\circ}{1+CG^\circ}$

• Estimate



generate $u_{NF}(t)$:



Conclusions

- If N is large enough then $\widehat{\theta}_N$ minimizes $E\left[\epsilon(t|\theta)^2\right]$
- open-loop:

$$E\left[\epsilon(t|\theta)^2\right] = \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{1}{|H(\theta)|^2} \Phi_u + \frac{|H^\circ|^2}{|H(\theta)|^2} \Phi_e \right\} d\omega$$

- closed-loop:

$$E\left[\epsilon(t|\theta)^2\right] = \int_{-\pi}^{\pi} \left\{ |[G^\circ - G(\theta)]|^2 \frac{|CS^\circ|^2}{|H(\theta)|^2} \Phi_r + \frac{|H^\circ|^2 |S^\circ|^2}{|H(\theta)|^2 |S(\theta)|^2} \Phi_e \right\} d\omega$$