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Brief paper

Continuous-time norm-constrained Kalman filtering[☆]

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ABSTRACT

This paper considers continuous-time state estimation when part of the state estimate or the entire state estimate is norm-constrained. In the former case continuous-time state estimation is considered by posing a constrained optimization problem. The optimization problem can be broken up into two separate optimization problems, one which solves for the optimal observer gain associated with the unconstrained state estimates, while the other solves for the optimal observer gain associated with the constrained state estimates. The optimal constrained state estimate is found by projecting the time derivative of an unconstrained estimate onto the tangent space associated with the norm constraint. The special case where the entire state estimate is norm-constrained is briefly discussed. The utility of the filtering results developed are highlighted through a spacecraft attitude estimation example. Numerical simulation results are included.

1. Introduction

The control of a system often relies on an estimate of the system state. Moreover, the majority of real systems are nonlinear. For instance, estimates of position, velocity, attitude, and angular velocity are needed to control spacecraft, aircraft, and ground vehicles. As a result, the development of state estimators that can robustly and reliably provide a state estimate of a nonlinear process is paramount.

Broadly speaking, stochastic estimation methods can be divided into two main categories (Crassidis & Junkins, 2012; Jazwinski, 1970; Simon, 2006): batch methods and sequential methods. Batch methods, such as weighted-least-squares methods, sliding-window filters, and smoothers, use many or all measurements to estimate the state of the system over a range of time. Sequential methods, the most popular being the Kalman filter (Kalman, 1960), provide a state estimate in "one-step-ahead" fashion. Although batch methods can generally provide a better state estimate, for real-time and online applications, one-step-ahead methods are often preferred. Historically, the Kalman filter and its nonlinear variants (e.g., the extended Kalman filter (EKF) Simon, 2006, pp. 400–403, the unscented Kalman filter (UKF) Julier, Uhlmann, & Durrant-Whyte, 2000) have proven to be both computationally efficient and reliable. However, the traditional Kalman filter structure has no means to directly handle state constraints.

Various authors have considered discrete-time Kalman filtering while simultaneously accounting for linear or nonlinear state constraints. Inspiration for the present paper comes from Zanetti, Majji, Bishop, and Mortari (2009) where Kalman filtering in a discrete-time setting directly considering a norm constraint on all or part of the state is considered. The derivation of the discrete-time norm-constrained Kalman filter is accomplished by augmenting the objective function, that being the minimization of the error covariance, with the norm constraint. A particularly interesting result highlighted in Zanetti et al. (2009) is that normalizing the unconstrained estimate is in fact optimal.

Numerous other papers considering linear and nonlinear state constraints appear in the literature. For example, in Alouani and Blair (1993), Gupta (2007), Richards (1995), Tahk and Speyer (1990) and Wang, Chiang, and Chang (2002) linear equality state constraints are incorporated into the Kalman filter as pseudo-measurements. Doing so leads to a measurement noise covariance that is singular, which from a theoretical point of view is not problematic, but numerical issues may arise (Simon, 2010). In Gupta (2007) and Simon and Chia (2002) linear equality state constraints are enforced by projecting the unconstrained state estimate generated by the Kalman filter onto the constraint surface. The work of Simon and Chia (2002) is extended in Yang and

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Blasch (2009) where nonlinear equality constraints are considered. As an alternative to the approach developed in Chen (2010), Ko and Bitmead (2007, 2010) and Simon and Chia (2002) use the linear equality state constraints to formulate a projected system, and then the Kalman filter is applied to the projected system to generate a state estimate. Unscented Kalman filtering accounting nonlinear equality state constraints is considered in Julier and La Viola (2007). The sigma points generated via the unscented transformation are projected onto the constraint surface. After the mean is computed (which does not necessarily satisfy the constraint), the mean is projected onto the constraint surface. For a survey of discrete-time Kalman filtering methods that account for linear and nonlinear state constraints, see Simon (2010).

This paper considers continuous-time Kalman filtering subject to a norm constraint on the state estimates. The main contribution of this work is the derivation of the continuous-time norm-constrained Kalman filter. This has not been previously considered in the literature. Estimating the state when only part of the state estimate is norm constrained and when the entire state estimate is norm constrained is investigated. A subtle feature of the filter presented is that, although a portion or the entire state estimate must satisfy a norm constraint, the true system state does not necessarily have to be constrained in the same way. Additionally, unlike Zanetti et al. (2009) a weight on the norm is incorporated into the filter formulation. Although inspiration for this work comes from Zanetti et al. (2009), the solution presented is different. Following the traditional continuous-time Kalman filter derivation, the time derivative of the error covariance is minimized. However, in order to force the state estimate to satisfy the norm constraint, the objective function is augmented not with the norm constraint directly, but with its time derivative. The solution to the optimization problem posed results in the time derivative of the unconstrained state estimate being projected onto the tangent space of the constraint surface. This projection is not forced upon the filter structure, but rather falls out naturally from the derivation. To showcase the utility of the continuous-time norm-constrained Kalman filter, the filter is used within an extended Kalman filter (EKF) framework to estimate the attitude of a rigid-body spacecraft. Spacecraft attitude estimation has been extensively considered in the literature; see Bar-Itzhack and Oshman (1985), Choukroun, Bar-Itzhack, and Oshman (2006), Shuster (1989) and Shuster and Oh (1981), as well as the survey paper Crassidis, Markley, and Cheng (2007).

The remainder of this paper is as follows. Preliminaries are reviewed in Section 2. Section 3.1 considers state estimation when only part of the state estimate is norm constrained. Norm-constrained Kalman filtering when the entire state estimate is constrained is briefly considered in Section 3.2. The role of a particular matrix, which is in fact a projection matrix, is discussed in Section 3.3. Spacecraft attitude estimation is considered in Section 4. The process and measurement models are presented in Sections 4.1 and 4.2. The EKF form of the estimator, resulting in the continuous-time norm-constrained EKF, is presented in Section 4.3. Numerical simulation results are presented in Section 4.4. The paper is drawn to a close in Section 5.

2. Preliminaries

Consider the continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \boldsymbol{\Gamma}_w(t)\mathbf{w}(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \boldsymbol{\Gamma}_v(t)\mathbf{v}(t), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, $\mathbf{u} \in \mathbb{R}^{n_u}$ is the known control input, $\mathbf{y} \in \mathbb{R}^{n_y}$ is the measurement, $\mathbf{w} \in \mathbb{R}^{n_w}$ is the process noise/disturbance, and $\mathbf{v} \in \mathbb{R}^{n_v}$ is the measurement noise. The time-varying matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{C}(\cdot)$, $\boldsymbol{\Gamma}_w(\cdot)$, and $\boldsymbol{\Gamma}_v(\cdot)$ are of appropriate dimension and piecewise continuous, and $\boldsymbol{\Gamma}_v(\cdot)$ has full row rank. The process and measurement noise are assumed to be zero-mean and white with autocovariances $E\left[\mathbf{w}(t)\mathbf{w}^\mathsf{T}(\tau)\right] = \mathbf{Q}(t)\delta(t-\tau)$ and $E\left[\mathbf{v}(t)\mathbf{v}^\mathsf{T}(\tau)\right] = \mathbf{R}(t)\delta(t-\tau)$, respectively, where $\mathbf{Q}(\cdot) \geq 0$ and $\mathbf{R}(\cdot) > 0$ are piecewise continuous. Additionally, $\mathbf{x}(\cdot)$, $\mathbf{w}(\cdot)$, and $\mathbf{v}(\cdot)$ are assumed to be independent for all time. To be concise, the temporal argument of functions and matrices will no longer be written unless clarity is required.

3. Norm-constrained Kalman filtering

3.1. Norm-constraining part of the state

Consider (1) and (2) partitioned in the following way:

$$\begin{bmatrix}\dot{\mathbf{z}}\\ \dot{\mathbf{q}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A}_{zz} & \mathbf{A}_{zq}\\ \mathbf{A}_{qz} & \mathbf{A}_{qq}\end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix}\mathbf{z}\\ \mathbf{q}\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}\mathbf{B}_{z}\\ \mathbf{B}_{q}\end{bmatrix}}_{\mathbf{B}} \mathbf{u} + \underbrace{\begin{bmatrix}\boldsymbol{\Gamma}_{w,z}\\ \boldsymbol{\Gamma}_{w,q}\end{bmatrix}}_{\boldsymbol{\Gamma}_w} \mathbf{w}, \tag{3}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix}\mathbf{C}_z & \mathbf{C}_q\end{bmatrix}}_{\mathbf{C}} \begin{bmatrix}\mathbf{z}\\ \mathbf{q}\end{bmatrix} + \boldsymbol{\Gamma}_v\mathbf{v}, \tag{4}$$

where $\mathbf{z} \in \mathbb{R}^{n_z}$, $\mathbf{q} \in \mathbb{R}^{n_q}$, and $n = n_z + n_q$. The matrices $\mathbf{A}_{zz}$, $\mathbf{A}_{zq}$, $\mathbf{A}_{qz}$, $\mathbf{A}_{qq}$, $\mathbf{B}_z$, $\mathbf{B}_q$, $\boldsymbol{\Gamma}_{w,z}$, $\boldsymbol{\Gamma}_{w,q}$, $\mathbf{C}_z$, and $\mathbf{C}_q$ are dimensioned appropriately.

Consider the following linear estimator dynamics:

$$\begin{bmatrix}\dot{\hat{\mathbf{z}}}\\ \dot{\hat{\mathbf{q}}}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{zz} & \mathbf{A}_{zq}\\ \mathbf{A}_{qz} & \mathbf{A}_{qq}\end{bmatrix} \underbrace{\begin{bmatrix}\hat{\mathbf{z}}\\ \hat{\mathbf{q}}\end{bmatrix}}_{\hat{\mathbf{x}}} + \begin{bmatrix}\mathbf{B}_{z}\\ \mathbf{B}_{q}\end{bmatrix} \mathbf{u} + \underbrace{\begin{bmatrix}\bar{\mathbf{K}}_{z}\\ \bar{\mathbf{K}}_{q}\end{bmatrix}}_{\bar{\mathbf{K}}} \mathbf{r}, \tag{5}$$

where $\hat{\mathbf{z}} \in \mathbb{R}^{n_z}$ is the estimate of $\mathbf{z}$, $\hat{\mathbf{q}} \in \mathbb{R}^{n_q}$ is the estimate of $\mathbf{q}$, $\mathbf{r} = \mathbf{y} - \hat{\mathbf{y}}$ is the measurement residual, and $\hat{\mathbf{y}} = \mathbf{C}_z\hat{\mathbf{z}} + \mathbf{C}_q\hat{\mathbf{q}}$ is the predicted measurement. The observer gain $\bar{\mathbf{K}} \in \mathbb{R}^{n\times n_y}$ has been partitioned into $\bar{\mathbf{K}}_z \in \mathbb{R}^{n_z\times n_y}$ and $\bar{\mathbf{K}}_q \in \mathbb{R}^{n_q\times n_y}$. The estimate $\hat{\mathbf{z}} \in \mathbb{R}^{n_z}$ is not constrained, however, $\hat{\mathbf{q}} \in \mathbb{R}^{n_q}$ is constrained in the following way:

$$\hat{\mathbf{q}}^\mathsf{T}\mathbf{W}\hat{\mathbf{q}} = \ell, \quad \forall t \in \mathbb{R}^+, \tag{6}$$

where $\mathbf{W} \in \mathbb{R}^{n_q\times n_q}$, $\mathbf{W} = \mathbf{W}^\mathsf{T} > 0$ is a constant weighting matrix. The constraint (6) can be equivalently written as $\left\|\sqrt{\mathbf{W}}\hat{\mathbf{q}}\right\| = \sqrt{\ell}$ where $\sqrt{\mathbf{W}}$ is the square root of the matrix $\mathbf{W}$. Differentiating (6) gives

$$2\hat{\mathbf{q}}^\mathsf{T}\mathbf{W}^\mathsf{T}\dot{\hat{\mathbf{q}}} = 0, \quad \forall t \in \mathbb{R}^+. \tag{7}$$

The initial state estimates are $\hat{\mathbf{z}}(0)$ and $\hat{\mathbf{q}}(0)$ where $\hat{\mathbf{q}}^\mathsf{T}(0)\mathbf{W}\hat{\mathbf{q}}(0) = \ell$. The objective at hand is to find $\bar{\mathbf{K}}$ in an optimal way so that $2\hat{\mathbf{q}}^\mathsf{T}\mathbf{W}^\mathsf{T}\dot{\hat{\mathbf{q}}} = 0, \forall t \in \mathbb{R}^+$, meaning that $\dot{\hat{\mathbf{q}}}$ must be perpendicular to $\mathbf{W}\hat{\mathbf{q}}$ for all time.

It is worth mentioning that although $\hat{\mathbf{q}}$ must satisfy (6) for all time, the true state $\mathbf{q}$ is not required to satisfy $\mathbf{q}^\mathsf{T}\mathbf{W}\mathbf{q} = \ell$. Such a situation may occur when a real system only approximately satisfies $\mathbf{q}^\mathsf{T}\mathbf{W}\mathbf{q} = \ell$ due to physical limitations, inaccuracies, or deliberate simplification of a more complicated process.

The estimation error is defined as $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$. Using (3) and (5), along with the definition of the estimation error, the error dynamics are $\dot{\mathbf{e}} = (\mathbf{A} - \bar{\mathbf{K}}\mathbf{C})\mathbf{e} + \boldsymbol{\Gamma}_w\mathbf{w} - \bar{\mathbf{K}}\boldsymbol{\Gamma}_v\mathbf{v}$. Defining the estimation-error covariance to be $\mathbf{P}(t) = E\left[\mathbf{e}(t)\mathbf{e}^\mathsf{T}(t)\right]$, and assuming that $\bar{\mathbf{K}}$ is non-random, it is straightforward to show that (Crassidis & Junkins, 2012, p. 170)

$$\dot{\mathbf{P}} = (\mathbf{A} - \bar{\mathbf{K}}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A} - \bar{\mathbf{K}}\mathbf{C})^\mathsf{T} + \boldsymbol{\Gamma}_w\mathbf{Q}\boldsymbol{\Gamma}_w^\mathsf{T} + \bar{\mathbf{K}}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^\mathsf{T}\bar{\mathbf{K}}^\mathsf{T}. \tag{8}$$

Shortly it will be shown that $\bar{\mathbf{K}}$ depends on $\hat{\mathbf{x}}$, is therefore random, and hence (8) is strictly speaking not correct. However, following the formulation presented in Zanetti et al. (2009), the dependence of $\bar{\mathbf{K}}$ on $\hat{\mathbf{x}}$ will be neglected. Additionally, although $\mathbf{P}$ is referred to as the error-covarience, technically it is not the error-covarience but rather an error-covariance-like term.

Before finding the optimal observer gains, the error-covariance will be partitioned as follows:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{zz} & \mathbf{P}_{zq} \\ \mathbf{P}_{qz} & \mathbf{P}_{qq} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_{zz} \\ \mathbf{P}_{zq} \end{bmatrix}, \qquad \mathbf{P}_2 = \begin{bmatrix} \mathbf{P}_{zq} \\ \mathbf{P}_{qq} \end{bmatrix}, \tag{10}$$

and $\mathbf{P}_{zz}$, $\mathbf{P}_{zq}$, $\mathbf{P}_{qz}$, and $\mathbf{P}_{qq}$ are of appropriate dimension. In a similar fashion, it will be helpful to partition the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_z \\ \mathbf{A}_q \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{zz} & \mathbf{A}_{zq} \\ \mathbf{A}_{qz} & \mathbf{A}_{qq} \end{bmatrix}, \tag{11}$$

where

$$\mathbf{A}_z = \begin{bmatrix} \mathbf{A}_{zz} & \mathbf{A}_{zq} \end{bmatrix}, \qquad \mathbf{A}_q = \begin{bmatrix} \mathbf{A}_{qz} & \mathbf{A}_{qq} \end{bmatrix}. \tag{12}$$

Using (9)–(12), the time derivative of the error-covariance presented in (8) is partitioned as

$$\dot{\mathbf{P}} = \begin{bmatrix} \dot{\mathbf{P}}_{zz} & \dot{\mathbf{P}}_{zq} \\ \dot{\mathbf{P}}_{qz} & \dot{\mathbf{P}}_{qq} \end{bmatrix},$$

where

$$\begin{aligned} \dot{\mathbf{P}}_{zz} &= (\mathbf{A}_z - \bar{\mathbf{K}}_z\mathbf{C})\mathbf{P}_1 + \mathbf{P}_1^{\mathsf{T}}(\mathbf{A}_z^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}) \\ &\quad + \boldsymbol{\Gamma}_{w,z}\mathbf{Q}\boldsymbol{\Gamma}_{w,z}^{\mathsf{T}} + \bar{\mathbf{K}}_z\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}, \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{\mathbf{P}}_{zq} &= (\mathbf{A}_z - \bar{\mathbf{K}}_z\mathbf{C})\mathbf{P}_2 + \mathbf{P}_1^{\mathsf{T}}(\mathbf{A}_q^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}) \\ &\quad + \boldsymbol{\Gamma}_{w,z}\mathbf{Q}\boldsymbol{\Gamma}_{w,q}^{\mathsf{T}} + \bar{\mathbf{K}}_z\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}, \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{\mathbf{P}}_{qz} &= (\mathbf{A}_q - \bar{\mathbf{K}}_q\mathbf{C})\mathbf{P}_1 + \mathbf{P}_2^{\mathsf{T}}(\mathbf{A}_z^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}) \\ &\quad + \boldsymbol{\Gamma}_{w,q}\mathbf{Q}\boldsymbol{\Gamma}_{w,z}^{\mathsf{T}} + \bar{\mathbf{K}}_q\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}, \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{\mathbf{P}}_{qq} &= (\mathbf{A}_q - \bar{\mathbf{K}}_q\mathbf{C})\mathbf{P}_2 + \mathbf{P}_2^{\mathsf{T}}(\mathbf{A}_q^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}) \\ &\quad + \boldsymbol{\Gamma}_{w,q}\mathbf{Q}\boldsymbol{\Gamma}_{w,q}^{\mathsf{T}} + \bar{\mathbf{K}}_q\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}. \end{aligned} \tag{16}$$

Drawing inspiration from the derivation of the unconstrained continuous-time Kalman filter (which, for completeness, is presented in the Appendix), to find the optimal observer gain consider the following optimization problem:

$$\min J(\bar{\mathbf{K}}) \quad \text{subject to} \quad 2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}} = 0, \tag{17}$$

where $J(\bar{\mathbf{K}}) = \operatorname{tr}\left[\dot{\mathbf{P}}\right]$. Notice that the objective function can be written as $J(\bar{\mathbf{K}}) = \operatorname{tr}\left[\dot{\mathbf{P}}\right] = \operatorname{tr}\left[\dot{\mathbf{P}}_{zz}\right] + \operatorname{tr}\left[\dot{\mathbf{P}}_{qq}\right]$, and from (13) and (16), it can be seen that $\dot{\mathbf{P}}_{zz}$ depends only on $\bar{\mathbf{K}}_z$ while $\dot{\mathbf{P}}_{qq}$ depends only on $\bar{\mathbf{K}}_q$. As a result, the optimal observer gains $\bar{\mathbf{K}}_z$ and $\bar{\mathbf{K}}_q$ can be found independently.

To find $\bar{\mathbf{K}}_z$ the following optimization must be solved: $\min J_z(\bar{\mathbf{K}}_z)$ where $J_z(\bar{\mathbf{K}}_z) = \operatorname{tr}\left[\dot{\mathbf{P}}_{zz}\right]$ and, by using (13), can be written as:

$$\begin{aligned} J_z(\bar{\mathbf{K}}_z) &= \operatorname{tr}\Big[(\mathbf{A}_z - \bar{\mathbf{K}}_z\mathbf{C})\mathbf{P}_1 + \mathbf{P}_1^{\mathsf{T}}(\mathbf{A}_z^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}) \\ &\qquad + \boldsymbol{\Gamma}_{w,z}\mathbf{Q}\boldsymbol{\Gamma}_{w,z}^{\mathsf{T}} + \bar{\mathbf{K}}_z\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_z^{\mathsf{T}}\Big]. \end{aligned}$$

Taking the derivative of $J_z(\cdot)$ with respect to $\bar{\mathbf{K}}_z$ and setting the result to zero gives

$$\frac{\partial J_z(\bar{\mathbf{K}}_z)}{\partial \bar{\mathbf{K}}_z} = 2\left(-\mathbf{P}_1^{\mathsf{T}}\mathbf{C}^{\mathsf{T}} + \bar{\mathbf{K}}_z\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right) = \mathbf{0},$$

the first-order necessary condition for optimality. Solving for $\bar{\mathbf{K}}_z$ leads to

$$\bar{\mathbf{K}}_z = (\mathbf{C}\mathbf{P}_1)^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}. \tag{18}$$

Therefore, from (5), $\dot{\hat{\mathbf{z}}} = \mathbf{A}_{zz}\hat{\mathbf{z}} + \mathbf{A}_{zq}\hat{\mathbf{q}} + \mathbf{B}_z\mathbf{u} + \bar{\mathbf{K}}_z\mathbf{r}$. If $\bar{\mathbf{K}}_z$ in (18) is compared to the unconstrained gain presented in (A.3) of Appendix they are very similar. This is because the gain $\bar{\mathbf{K}}_z$ is not enforcing any sort of constraint on the state estimate $\hat{\mathbf{z}}$.

In order to find $\bar{\mathbf{K}}_q$ a solution to the following optimization problem must be found:

$$\min J_q(\bar{\mathbf{K}}_q) \quad \text{subject to} \quad 2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}} = 0,$$

where $J_q(\bar{\mathbf{K}}_q) = \operatorname{tr}\left[\dot{\mathbf{P}}_{qq}\right]$ and $\bar{\mathbf{K}}_q$ is the observer gain that accounts for the constraint $2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}} = 0$. Using the estimator dynamics given in (5) the constraint $2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}} = 0$ can be written as:

$$\begin{aligned} 0 &= 2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\left(\mathbf{A}_{qz}\hat{\mathbf{z}} + \mathbf{A}_{qq}\hat{\mathbf{q}} + \mathbf{B}_q\mathbf{u} + \bar{\mathbf{K}}_q\mathbf{r}\right) \\ &= 2\operatorname{tr}\left[\hat{\mathbf{z}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{A}_{qz} + \hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{A}_{qq} + \mathbf{u}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{B}_q + \mathbf{W}\hat{\mathbf{q}}\mathbf{r}^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}\right]. \end{aligned} \tag{19}$$

Using a Lagrange multiplier, λ, to augment the objective function $J_q(\bar{\mathbf{K}}_q) = \operatorname{tr}\left[\dot{\mathbf{P}}_{qq}\right]$ with the constraint, the associated Lagrangian is

$$\begin{aligned} \hat{J}_q(\bar{\mathbf{K}}_q) &= \operatorname{tr}\Big[(\mathbf{A}_q - \bar{\mathbf{K}}_q\mathbf{C})\mathbf{P}_2 + \mathbf{P}_2^{\mathsf{T}}(\mathbf{A}_q^{\mathsf{T}} - \mathbf{C}^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}) \\ &\qquad + \boldsymbol{\Gamma}_{w,q}\mathbf{Q}\boldsymbol{\Gamma}_{w,q}^{\mathsf{T}} + \bar{\mathbf{K}}_q\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}\Big] \\ &\qquad + 2\lambda\operatorname{tr}\Big[\hat{\mathbf{z}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{A}_{qz} + \hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{A}_{qq} \\ &\qquad + \mathbf{u}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{B}_q + \mathbf{W}\hat{\mathbf{q}}\mathbf{r}^{\mathsf{T}}\bar{\mathbf{K}}_q^{\mathsf{T}}\Big], \end{aligned}$$

where the expression for $\dot{\mathbf{P}}_{qq}$ given in (16) has been used. Taking the derivative of $\hat{J}_q(\cdot)$ with respect to $\bar{\mathbf{K}}_q$ and setting the result to zero gives

$$\frac{\partial \hat{J}_q(\bar{\mathbf{K}}_q)}{\partial \bar{\mathbf{K}}_q} = 2\left(-\mathbf{P}_2^{\mathsf{T}}\mathbf{C}^{\mathsf{T}} + \bar{\mathbf{K}}_q\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}} + \lambda\mathbf{W}\hat{\mathbf{q}}\mathbf{r}^{\mathsf{T}}\right) = \mathbf{0}.$$

Solving for $\bar{\mathbf{K}}_q$ results in

$$\bar{\mathbf{K}}_q = \underbrace{(\mathbf{C}\mathbf{P}_2)^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}}_{\mathbf{K}_q} - \lambda\mathbf{W}\hat{\mathbf{q}}\mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}, \tag{20}$$

where the definition of the unconstrained gain, $\mathbf{K}_q = (\mathbf{C}\mathbf{P}_2)^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}$, has been used (see (A.3) in the Appendix). Now $\bar{\mathbf{K}}_q$ will be substituted into the constraint equation shown in (19), and λ will be solved for:

$$\begin{aligned} 0 &= \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\left(\mathbf{A}_{qz}\hat{\mathbf{z}} + \mathbf{A}_{qq}\hat{\mathbf{q}} + \mathbf{B}_q\mathbf{u} + \mathbf{K}_q\mathbf{r}\right) \\ &\quad - \lambda\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}\mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}\mathbf{r}, \end{aligned}$$

resulting in

$$\lambda = \frac{\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}_u}{r\varkappa}, \tag{21}$$

where

$$\left.\begin{aligned} \dot{\hat{\mathbf{q}}}_u &= \mathbf{A}_{qz}\hat{\mathbf{z}} + \mathbf{A}_{qq}\hat{\mathbf{q}} + \mathbf{B}_q\mathbf{u} + \mathbf{K}_q\mathbf{r}, \\ \varkappa &= \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}, \\ r &= \mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}\mathbf{r}, \end{aligned}\right\} \tag{22}$$

and it is assumed that $\mathbf{r} \neq \mathbf{0}$ (which is true with probability equal to one). Note that $\dot{\hat{\mathbf{q}}}_u$ is not itself a variable; $\dot{\hat{\mathbf{q}}}_u$ is used to indicate what $\dot{\hat{\mathbf{q}}}$ would be if the constraint were not enforced. Additionally, notice that there is only one solution for λ, unlike in the discrete-time case where the Lagrange multiplier may take on two different values (Zanetti et al., 2009).

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It is worth mentioning that the objective function $J_q(\cdot)$ is quadratic in $\bar{\mathbf{K}}_q$, and hence strictly convex. Additionally, the constraint is affine in $\bar{\mathbf{K}}_q$, and thus the constraint set is convex also. Therefore, the optimization problem is convex. As such, the solution to the problem is a unique global minimum (Boyd & Vandenberghe, 2004, pp. 136–140).

The gain $\bar{\mathbf{K}}_q$ can now be written as follows:

$$\bar{\mathbf{K}}_q = \mathbf{K}_q - \frac{1}{r}\left(\frac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{q}}}_u\mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}. \tag{23}$$

Substituting (23) into the expression for $\dot{\hat{\mathbf{q}}}$ given in (5) and simplifying using the expressions for $\dot{\hat{\mathbf{q}}}_u$ and r given in (22) allows $\dot{\hat{\mathbf{q}}}$ to be written as

$$\dot{\hat{\mathbf{q}}} = \left(\mathbf{1} - \frac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{q}}}_u, \tag{24}$$

where $\mathbf{1}$ is the identity matrix (with appropriate dimension). The significance of the matrix $(\mathbf{1} - \mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}/\varkappa)$ in (24) will be discussed in Section 3.3.

To find an expression for $\dot{\mathbf{P}}$, and expressions for $\dot{\mathbf{P}}_{zz}$, $\dot{\mathbf{P}}_{zq}$, $\dot{\mathbf{P}}_{qz}$, and $\dot{\mathbf{P}}_{qq}$, the expressions for $\bar{\mathbf{K}}_z$ and $\bar{\mathbf{K}}_q$ given in (18) and (23), respectively, must be substituted into (13), (14), (15), and (16). Alternatively, a more concise expression for $\dot{\mathbf{P}}$ can be found, as discussed next. Define the unconstrained Kalman gain as $\mathbf{K} = \left[\bar{\mathbf{K}}_z^{\mathsf{T}}\ \mathbf{K}_q^{\mathsf{T}}\right]^{\mathsf{T}}$. Note that the unconstrained gain $\mathbf{K}$ may be written concisely as (A.3) in the Appendix. Then, the gain for the constrained problem may be written as

$$\bar{\mathbf{K}} = \mathbf{K} + \Delta\mathbf{K}, \tag{25}$$

where

$$\Delta\mathbf{K} = \begin{bmatrix} \mathbf{0} \\ -\dfrac{1}{r}\left(\dfrac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{q}}}_u\mathbf{r}^{\mathsf{T}} \end{bmatrix}(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}})^{-1}.$$

It follows by substitution of (25) into (8), that the time derivative of the error covariance $\dot{\mathbf{P}}$ can be written as the nominal unconstrained case, together with an additional term,

$$\dot{\mathbf{P}} = \dot{\mathbf{P}}_u + \Delta\mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\Delta\mathbf{K}, \tag{26}$$

where

$$\dot{\mathbf{P}}_u = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K}\mathbf{C})^{\mathsf{T}} + \boldsymbol{\Gamma}_w\mathbf{Q}\boldsymbol{\Gamma}_w^{\mathsf{T}} + \mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\mathbf{K}^{\mathsf{T}}, \tag{27}$$

and

$$\Delta\mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\Delta\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dfrac{1}{r}\left(\dfrac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{q}}}_u\dot{\hat{\mathbf{q}}}_u^{\mathsf{T}}\left(\dfrac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right) \end{bmatrix}. \tag{28}$$

Notice that $\mathbf{P}_u$ is not a variable; the notation $\dot{\mathbf{P}}_u$ is used to indicate what $\dot{\mathbf{P}}$ would reduce to if the constraint were not enforced.

Note that the gain $\bar{\mathbf{K}}$ is optimal in the sense that it solves the optimization problem given in (17). However, it cannot be concluded that $\bar{\mathbf{K}}$ provides an optimal estimate in the minimum variance sense, which is the case in the standard Kalman filter. This is because the assumed form of the covariance propagation in (8) is not strictly speaking correct, because of the dependence of $\bar{\mathbf{K}}$ on $\hat{\mathbf{x}}$ (specifically, $\bar{\mathbf{K}}_q$ on $\hat{\mathbf{q}}$).

3.2. Norm-constraining the entire state

Consider the system (1) and (2) once again, along with the following linear estimator dynamics:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \bar{\mathbf{K}}\mathbf{r} \tag{29}$$

where $\hat{\mathbf{x}} \in \mathbb{R}^n$ is the state estimate, $\bar{\mathbf{K}} \in \mathbb{R}^{n\times n_y}$ is an observer gain that we seek to determine optimally, and $\mathbf{r}$ is the measurement residual. The estimate of the state must satisfy

$$\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{x}} = \ell, \quad \forall t \in \mathbb{R}^+, \tag{30}$$

which can alternatively be written as:

$$2\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{x}}} = 0, \quad \forall t \in \mathbb{R}^+.$$

Again, although $\hat{\mathbf{x}}$ must satisfy (30), the true state $\mathbf{x}$ does not have to satisfy $\mathbf{x}^{\mathsf{T}}\mathbf{W}\mathbf{x} = \ell$. To find $\bar{\mathbf{K}}$ consider the following optimization problem:

$$\min J(\bar{\mathbf{K}}) \quad \text{subject to} \quad 2\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{x}}} = 0,$$

where $J(\bar{\mathbf{K}}) = \operatorname{tr}\left[\dot{\mathbf{P}}\right]$ and $\dot{\mathbf{P}}$ is given in (8). Following a similar procedure to the procedure outlined in Section 3.1 where only part of the state was constrained, the gain $\bar{\mathbf{K}}$ is readily found to be

$$\bar{\mathbf{K}} = \mathbf{K} - \frac{1}{r}\left(\frac{\mathbf{W}\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{x}}}_u\mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}, \tag{31}$$

where $\mathbf{K} = \mathbf{P}\mathbf{C}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}$ is the unconstrained observer gain (which should be compared to (A.3) in the Appendix), and

$$\left.\begin{aligned} \dot{\hat{\mathbf{x}}}_u &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}\mathbf{r}, \\ \varkappa &= \hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{x}}, \\ r &= \mathbf{r}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}\mathbf{r}. \end{aligned}\right\} \tag{32}$$

Again, $\hat{\mathbf{x}}_u$ is not itself a variable, and $\dot{\hat{\mathbf{x}}}_u$ is used to indicate what $\dot{\hat{\mathbf{x}}}$ would be should the constraint not be enforced. Substituting (31) into (29) and (8) and simplifying allows the estimator dynamics and $\dot{\mathbf{P}}$ to be written as

$$\dot{\hat{\mathbf{x}}} = \left(\mathbf{1} - \frac{\mathbf{W}\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{x}}}_u$$

and

$$\dot{\mathbf{P}} = \dot{\mathbf{P}}_u + \Delta\mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\Delta\mathbf{K}^{\mathsf{T}}, \tag{33}$$

where $\dot{\mathbf{P}}_u$ is given in (27),

$$\Delta\mathbf{K} = -\frac{1}{r}\left(\frac{\mathbf{W}\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{x}}}_u\mathbf{r}^{\mathsf{T}}(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}})^{-1}, \tag{34}$$

and hence

$$\Delta\mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\Delta\mathbf{K}^{\mathsf{T}} = \frac{1}{r}\left(\frac{\mathbf{W}\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{x}}}_u\dot{\hat{\mathbf{x}}}_u^{\mathsf{T}}\left(\frac{\mathbf{W}\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right). \tag{35}$$

The form of (33) parallels the discrete-time case (Zanetti et al., 2009). The definition of $\Delta\mathbf{K}$ in (34) allows $\bar{\mathbf{K}}$ in (31) to be equivalently written as

$$\bar{\mathbf{K}} = \mathbf{K} + \Delta\mathbf{K}. \tag{36}$$

Notice that Eq. (33) can be obtained by directly substituting (36) into (8) and simplifying, as is expected.

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3.3. The projection matrix

Recall Eq. (24); the matrix $(\mathbf{1}-\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}/\varkappa)$ in (24) is a projection matrix. Specifically, $(\mathbf{1}-\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}/\varkappa)$ projects any n dimensional vector onto span $\left\{\mathbf{W}\hat{\mathbf{q}}\right\}^{\perp}$. This can be seen by first considering the following:

$$\mathbf{v}_{\perp}=\left(\mathbf{1}-\frac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\mathbf{v}, \tag{37}$$

where $\mathbf{v}\in\mathbb{R}^n$. By direct computation, $\mathbf{v}_{\perp}$ is perpendicular to $\mathbf{W}\hat{\mathbf{q}}$:

$$\begin{aligned}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{v}_{\perp}&=\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\left(\mathbf{1}-\frac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\mathbf{v}\\&=\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{v}-\left(\frac{\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}}{\varkappa}\right)\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{v}=0,\end{aligned}$$

where $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}=\varkappa$ has been used to simplify. Thus, $\mathbf{v}_{\perp}\in$ span $\left\{\mathbf{W}\hat{\mathbf{q}}\right\}^{\perp}$. Moreover, by defining the surface $f(\hat{\mathbf{q}})=\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}-\ell=0$, the gradient of $f(\hat{\mathbf{q}})$ is $\nabla f(\hat{\mathbf{q}})=2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}$. Clearly any $\mathbf{v}_{\perp}$ generated via (37) is perpendicular to $\nabla f(\hat{\mathbf{x}})$. As such, any $\mathbf{v}_{\perp}$ generated via (37) is tangent to the surface $f(\hat{\mathbf{q}})=0$, that being the surface $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{x}}=\ell$. Hence, $(\mathbf{1}-\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}/\varkappa)$ projects any n dimensional vector onto the tangent space of the surface $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}=\ell$.

In the context of Eq. (24), the time derivative of the constrained state estimate, $\dot{\hat{\mathbf{q}}}$, is computed by projecting the time derivative of the unconstrained estimate, $\dot{\hat{\mathbf{q}}}_u$, onto the tangent space of the surface $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}=\ell$, $\forall t\in\mathbb{R}^{+}$. Also, substitution of (24) into $2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}$ yields

$$\begin{aligned}2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}&=2\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\left(\mathbf{1}-\frac{\mathbf{W}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}}{\varkappa}\right)\dot{\hat{\mathbf{q}}}_u\\&=2\left(\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}_u-\left(\frac{\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}}{\varkappa}\right)\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}_u\right)=0,\end{aligned}$$

as expected.

It should be emphasized that the projection of $\dot{\hat{\mathbf{q}}}_u$ onto the tangent space of the surface $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{W}\hat{\mathbf{q}}=\ell$ yielding $\dot{\hat{\mathbf{q}}}$ naturally falls out of the filter derivation; the projection is not forced upon the filter in an ad hoc way. These continuous-time results can be compared with the discrete-time results (Zanetti et al., 2009) where the norm-constrained state estimate is shown to be the unconstrained state estimate normalized in order to satisfy the norm constraint, which is the least-squares optimal projection of the unconstrained estimate onto the sphere of radius $\sqrt{\ell}$. The projection presented herein, and the normalization presented in Zanetti et al. (2009), are the result of derivations.

4. Application to spacecraft attitude estimation

The estimation of a rigid-body spacecraft's attitude will now be considered. First the process and measurement models will be presented, then linearization and application of the continuous-time norm-constrained Kalman filter in an extended manner will be discussed.

4.1. Process model

The attitude kinematics are parameterized in terms of the unit-length quaternion, $\mathbf{q}^{\mathsf{T}}=[\boldsymbol{\epsilon}^{\mathsf{T}}\ \eta]$, which must satisfy $\mathbf{q}^{\mathsf{T}}\mathbf{q}=1$. The relationship between $\dot{\boldsymbol{\epsilon}}$, $\dot{\eta}$, $\boldsymbol{\epsilon}$, η, and the body's angular velocity $\boldsymbol{\omega}$ (expressed in the spacecraft body frame) is Hughes (2004, p. 26)

$$\dot{\mathbf{q}}=\begin{bmatrix}\dot{\boldsymbol{\epsilon}}\\\dot{\eta}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\eta\mathbf{1}+\boldsymbol{\epsilon}^{\times}\\-\boldsymbol{\epsilon}^{\mathsf{T}}\end{bmatrix}\boldsymbol{\omega}, \tag{38}$$

where for any $\mathbf{s}=[s_1\ \ s_2\ \ s_3]^{\mathsf{T}}\in\mathbb{R}^3$ $\mathbf{s}^{\times}\in\mathbb{R}^{3\times3}$ is $\mathbf{s}^{\times}=[\mathbf{s}_1\ \mathbf{s}_2\ \mathbf{s}_3]$, $\mathbf{s}_1=[0\ s_3\ -s_2]^{\mathsf{T}}$, $\mathbf{s}_2=[-s_3\ 0\ s_1]^{\mathsf{T}}$, $\mathbf{s}_3=[s_2\ -s_1\ 0]^{\mathsf{T}}$. The spacecraft attitude dynamics are described by Hughes (2004, p. 284)

$$\mathbf{I}\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega}=\mathbf{u}+\mathbf{w}, \tag{39}$$

where $\mathbf{u}$ is the control input and $\mathbf{w}$ is a disturbance. Augmenting (38) and (39) gives

$$\dot{\mathbf{x}}=\begin{bmatrix}\dot{\boldsymbol{\omega}}\\\dot{\boldsymbol{\epsilon}}\\\dot{\eta}\end{bmatrix}=\underbrace{\begin{bmatrix}\mathbf{I}^{-1}(-\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega}+\mathbf{u}+\mathbf{w})\\\dfrac{1}{2}(\eta\mathbf{1}+\boldsymbol{\epsilon}^{\times})\boldsymbol{\omega}\\-\dfrac{1}{2}\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\omega}\end{bmatrix}}_{\mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{w})}=\begin{bmatrix}\mathbf{f}_{\omega}(\mathbf{x},\mathbf{u},\mathbf{w})\\\mathbf{f}_{q}(\mathbf{x},\mathbf{u},\mathbf{w})\end{bmatrix}, \tag{40}$$

which is the continuous-time process model.

4.2. Measurement model

The spacecraft is assumed to be endowed with m sensors that each provide a vector measurement in a continuous fashion. Specifically, each measurement can be described by

$$\mathbf{s}_b^j=\mathbf{C}_{ba}\mathbf{s}_a^j+\mathbf{v}^j,\quad j=1\ldots m, \tag{41}$$

where $\mathbf{s}_b^j$ is the jth vector measurement expressed in the spacecraft body frame, $\mathbf{s}_a^j$ is the corresponding unit-length reference vector expressed in the inertial frame, $\mathbf{C}_{ba}$ is the rotation matrix corresponding to the quaternion $\mathbf{q}$, and $\mathbf{v}^j$ is zero mean white noise with autocovariance $E\left[\mathbf{v}^j(t)\mathbf{v}^{j\,\mathsf{T}}(\tau)\right]=\mathbf{R}^j(t)\delta(t-\tau)$ where $\mathbf{R}^j(\cdot)>0$. All noise processes, including the process disturbance, are uncorrelated. Augmenting all vector measurements, the measurement equation is then

$$\mathbf{y}=\begin{bmatrix}\mathbf{C}_{ba}\mathbf{s}_a^1\\\vdots\\\mathbf{C}_{ba}\mathbf{s}_a^m\end{bmatrix}+\underbrace{\begin{bmatrix}\mathbf{v}^1\\\vdots\\\mathbf{v}^m\end{bmatrix}}_{\mathbf{v}}=\underbrace{\begin{bmatrix}\mathbf{0}&\mathbf{Y}(\mathbf{s}_a^1,\mathbf{q})\\\vdots&\vdots\\\mathbf{0}&\mathbf{Y}(\mathbf{s}_a^m,\mathbf{q})\end{bmatrix}\mathbf{x}+\mathbf{v}}_{\mathbf{h}(\mathbf{x},\mathbf{v})} \tag{42}$$

where (Leung & Damaren, 2004)

$$\mathbf{Y}(\mathbf{s}_a^j,\mathbf{q})\mathbf{q}=\underbrace{\begin{bmatrix}\eta\mathbf{1}-\boldsymbol{\epsilon}^{\times}&-\boldsymbol{\epsilon}\end{bmatrix}\begin{bmatrix}\mathbf{s}_a^{j\,\times}&\mathbf{s}_a^j\\-\mathbf{s}_a^{j\,\mathsf{T}}&0\end{bmatrix}}_{\mathbf{Y}(\hat{\mathbf{a}}_a,\mathbf{q})}\begin{bmatrix}\boldsymbol{\epsilon}\\\eta\end{bmatrix}. \tag{43}$$

In deriving (43) $\mathbf{C}_{ba}=(\eta^2-\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\epsilon})\mathbf{1}+2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathsf{T}}-2\eta\boldsymbol{\epsilon}^{\times}$ and the identity $-\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times}=\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\epsilon}\mathbf{1}-\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathsf{T}}$ have been employed. Also, note that $E\left[\mathbf{v}(t)\mathbf{v}^{\mathsf{T}}(\tau)\right]=\mathbf{R}(t)\delta(t-\tau)=\mathrm{diag}_{j=1\ldots m}\left\{\mathbf{R}^j(t)\right\}\delta(t-\tau)$.

4.3. Continuous-time norm-constrained extended Kalman filter formulation

In order to use the norm-constrained Kalman filter formulation presented in this paper, the process and measurement models must be linearized and implemented in an EKF framework. Following the EKF development presented in Simon (2006, pp. 400–403), consider a Taylor series expansion in $\mathbf{x}$, $\mathbf{w}$, and $\mathbf{v}$ about some reference trajectory and reference measurement:

$$\dot{\bar{\mathbf{x}}}=\mathbf{f}(\bar{\mathbf{x}},\mathbf{u},\bar{\mathbf{w}}),\qquad\bar{\mathbf{y}}=\mathbf{h}(\bar{\mathbf{x}},\bar{\mathbf{v}}),$$

where $\bar{\mathbf{x}}$, $\bar{\mathbf{w}}$, and $\bar{\mathbf{v}}$ are the reference state trajectory, disturbance, and measurement noise, respectively. Specifically, $\mathbf{x}=\bar{\mathbf{x}}+\delta\mathbf{x}$ where $\bar{\mathbf{x}}$ is the reference state trajectory and $\delta\mathbf{x}$ is a perturbation,

$\mathbf{w} = \mathbf{0} + \delta\mathbf{w}$ where $\bar{\mathbf{w}} = \mathbf{0}$ is the reference disturbance and $\delta\mathbf{w}$ is a perturbation, and $\mathbf{v} = \mathbf{0} + \delta\mathbf{v}$ where $\bar{\mathbf{v}} = \mathbf{0}$ is the reference measurement noise and $\delta\mathbf{v}$ is a perturbation. Substitution of $\mathbf{x} = \bar{\mathbf{x}} + \delta\mathbf{x}$ and $\mathbf{w} = \mathbf{0} + \delta\mathbf{w}$ into (40) and neglecting products of $\delta\boldsymbol{\omega}$, $\delta\mathbf{q}$, and $\delta\mathbf{w}$ yields

$$\dot{\mathbf{x}} \doteq \mathbf{f}(\bar{\mathbf{x}}, \mathbf{u}, \mathbf{0}) + \underbrace{\left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}, \mathbf{u}, \mathbf{0}}}_{\mathbf{A}} \delta\mathbf{x} + \underbrace{\left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{w}}\right|_{\bar{\mathbf{x}}, \mathbf{u}, \mathbf{0}}}_{\boldsymbol{\Gamma}_w} \delta\mathbf{w}$$

$$= \mathbf{A}\mathbf{x} + (\mathbf{f}(\bar{\mathbf{x}}, \mathbf{u}, \mathbf{0}) - \mathbf{A}\bar{\mathbf{x}}) + \boldsymbol{\Gamma}_w \mathbf{w}, \tag{44}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}^{-1}(-\bar{\boldsymbol{\omega}}^{\times}\mathbf{I} + (\mathbf{I}\bar{\boldsymbol{\omega}})^{\times}) & \mathbf{0} & \mathbf{0} \\ \frac{1}{2}(\bar{\eta}\mathbf{1} + \bar{\boldsymbol{\epsilon}}^{\times}) & -\frac{1}{2}\bar{\boldsymbol{\omega}}^{\times} & \frac{1}{2}\bar{\boldsymbol{\omega}} \\ -\frac{1}{2}\bar{\boldsymbol{\epsilon}}^{\mathsf{T}} & -\frac{1}{2}\bar{\boldsymbol{\omega}}^{\mathsf{T}} & 0 \end{bmatrix},$$

$$\boldsymbol{\Gamma}_w = \begin{bmatrix} \mathbf{I}^{-1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

In a similar fashion, substitution of $\mathbf{x} = \bar{\mathbf{x}} + \delta\mathbf{x}$ and $\mathbf{v} = \mathbf{0} + \delta\mathbf{v}$ into (42) and neglecting products of $\delta\boldsymbol{\omega}$, $\delta\mathbf{q}$, and $\delta\mathbf{v}$ gives

$$\mathbf{y} \doteq \mathbf{h}(\bar{\mathbf{x}}, \mathbf{0}) + \underbrace{\left.\frac{\partial \mathbf{h}(\mathbf{x}, \mathbf{v})}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}, \mathbf{0}}}_{\mathbf{C}} \delta\mathbf{x} + \underbrace{\left.\frac{\partial \mathbf{h}(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v}}\right|_{\bar{\mathbf{x}}, \mathbf{0}}}_{\boldsymbol{\Gamma}_v} \delta\mathbf{x}$$

$$= \mathbf{C}\mathbf{x} + (\mathbf{h}(\bar{\mathbf{x}}, \mathbf{0}) - \mathbf{C}\bar{\mathbf{x}}) + \boldsymbol{\Gamma}_v \mathbf{v}, \tag{45}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{Y}}(\mathbf{s}_a^1, \bar{\mathbf{q}}) \\ \vdots & \vdots \\ \mathbf{0} & \bar{\mathbf{Y}}(\mathbf{s}_a^m, \bar{\mathbf{q}}) \end{bmatrix}, \qquad \boldsymbol{\Gamma}_v = \mathbf{1},$$

and

$$\bar{\mathbf{Y}}(\mathbf{s}_a^j, \bar{\mathbf{q}}) = \mathbf{Y}(\mathbf{s}_a^j, \bar{\mathbf{q}}) + \left[\left((\mathbf{s}_a^{j\times}\bar{\boldsymbol{\epsilon}} + \mathbf{s}_a^j\bar{\eta})^{\times} + \mathbf{s}_a^{j\mathsf{T}}\bar{\boldsymbol{\epsilon}}\,\mathbf{1} \right) \ (\mathbf{s}_a^{j\times}\bar{\boldsymbol{\epsilon}} + \mathbf{s}_a^j\bar{\eta}) \right].$$

Using (44) and (45), the continuous-time norm-constrained EKF can be formulated. Following Simon (2006, p. 400), applying (5) to the linearized equations in (44) and (45), the state estimate becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \left(\mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{0}) - \mathbf{A}\hat{\mathbf{x}}\right) + \bar{\mathbf{K}}\mathbf{r} = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{0}) + \bar{\mathbf{K}}\mathbf{r}$$

where $\mathbf{r} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \mathbf{h}(\hat{\mathbf{x}}, \mathbf{0})$ is the residual. The constraint on the state estimate is $\hat{\mathbf{q}}^{\mathsf{T}}\hat{\mathbf{q}} = 1$; therefore, from (6), it follows that $\mathbf{W} = \mathbf{1}$ and $\ell = 1$. From (18) and (23) the optimal gain is

$$\bar{\mathbf{K}} = \begin{bmatrix} \bar{\mathbf{K}}_\omega \\ \bar{\mathbf{K}}_q \end{bmatrix} = \begin{bmatrix} (\mathbf{C}\mathbf{P}_1)^{\mathsf{T}} \left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1} \\ \mathbf{K}_q - \frac{1}{r}\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\dot{\hat{\mathbf{q}}}_u\mathbf{r}^{\mathsf{T}} \left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1} \end{bmatrix},$$

where $\varkappa = 1$, $\mathbf{K}_q$ is given in (20), r is given in (22), and $\mathbf{P}$ is partitioned as in (9). The time-rate-of-change of $\hat{\boldsymbol{\omega}}$ is

$$\dot{\hat{\boldsymbol{\omega}}} = \mathbf{f}_\omega(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{0}) + \bar{\mathbf{K}}_\omega\mathbf{r},$$

while the time-rate-of-change of $\hat{\mathbf{q}}$ is

$$\dot{\hat{\mathbf{q}}} = \left(\mathbf{1} - \hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathsf{T}}\right)\dot{\hat{\mathbf{q}}}_u,$$

where the time derivative of the unconstrained quaternion estimate, $\dot{\hat{\mathbf{q}}}_u$, is generated via

$$\dot{\hat{\mathbf{q}}}_u = \mathbf{f}_q(\hat{\mathbf{x}}, \mathbf{u}, \mathbf{0}) + \mathbf{K}_q\mathbf{r},$$

and $\mathbf{f}_\omega(\cdot,\cdot,\cdot)$ and $\mathbf{f}_q(\cdot,\cdot,\cdot)$ are defined in (40). Finally, the time derivative of $\mathbf{P}$ is

$$\dot{\mathbf{P}} = \begin{bmatrix} \dot{\mathbf{P}}_{\omega\omega} & \dot{\mathbf{P}}_{\omega q} \\ \dot{\mathbf{P}}_{q\omega} & \dot{\mathbf{P}}_{qq} \end{bmatrix}$$

where $\dot{\mathbf{P}}_{\omega\omega}$, $\dot{\mathbf{P}}_{\omega q}$, $\dot{\mathbf{P}}_{q\omega}$, and $\dot{\mathbf{P}}_{qq}$ are given in Eqs. (13), (14), (15), (16) where the subscript zz, zq, and qz are replaced with subscripts $\omega\omega$, ωq, and $q\omega$.

4.4. Numerical simulation results

Consider a rigid-body spacecraft in a circular orbit. The orbit inclination and altitude are 97.6° and 600 (km), respectively. The true spacecraft inertia matrix is $\mathbf{I} = \mathrm{diag}\{27, 17, 25\}$ (kg m^2). A gravity-gradient and magnetic disturbance are included in the truth model, but not in the estimator dynamics. The gravity-gradient disturbance is $\mathbf{w}^{gg} = (3\mu/r_b^5)\mathbf{r}_b^{\times}\mathbf{I}\mathbf{r}_b$ where $\mathbf{r}_b$ is the position of the spacecraft expressed in the spacecraft body frame, $r_b = \|\mathbf{r}_b\|$, and μ is the gravitational constant of the Earth (Hughes, 2004, p. 238). The magnetic disturbance is $\mathbf{w}^m = \mathbf{m}^{\times}\mathbf{b}$ where $\mathbf{b}$ is the magnetic field vector of the Earth expressed in the spacecraft body frame (Hughes, 2004, p. 264), and $\mathbf{m} = [1 \quad 1 \quad 1]^{\mathsf{T}}$ (A m^2) is the spacecraft's magnetic dipole. Moreover, because the true inertia of the spacecraft is never known exactly, the estimator uses an inertia matrix equal to $\mathbf{I}' = \mathbf{C}_{b'b}\,(\frac{4}{5}\mathbf{I})\mathbf{C}_{b'b}^{\mathsf{T}}$ where $\mathbf{C}_{b'b} = \mathbf{C}_1(7.5°)\mathbf{C}_2(-5°)\mathbf{C}_3(10°)$, and $\mathbf{C}_\alpha$, $\alpha = 1, 2, 3$ are principal rotation matrices (Hughes, 2004, p. 15). The spacecraft is endowed with two vector measurements. The first vector measurement is given by a sun sensor, while the second is given by a magnetometer. Both measurements are normalized. Zero mean white noise corrupts the measurements; the standard deviation of the noise is $\sigma_s = 0.005$ and $\sigma_m = 0.01$ for the sun sensor and magnetometer, respectively. The filter uses $\mathbf{Q} = \mathrm{diag}\left\{\sigma_w^2, \sigma_w^2, \sigma_w^2\right\}$ where $\sigma_w = 0.5$ (N m) and $\mathbf{R} = \mathrm{diag}\left\{\sigma_s^2, \sigma_s^2, \sigma_s^2, \sigma_m^2, \sigma_m^2, \sigma_m^2\right\}$. All simulation results presented use initial angular velocity and attitude estimates of $\hat{\boldsymbol{\omega}}(0) = \mathbf{0}$ (s^{-1}) and $\hat{\mathbf{q}}(0) = [0 \quad 0 \quad 0 \quad 1]^{\mathsf{T}}$, and an initial error covariance of $\mathbf{P}(0) = \frac{1}{10}\mathbf{1}$. In a real mission scenario a simple algorithm such as the TRIAD algorithm (Shuster & Oh, 1981) would be used to generate an initial attitude estimate that would be used to initialize the EKF. Additionally, in a real mission scenario the spacecraft would be detumbled and eventually three-axis stabilized; here $\mathbf{u} = \mathbf{0}$ for all time and the spacecraft continues to tumble, representing a harder estimation problem.

First consider the case where the initial angular velocity and attitude are $\boldsymbol{\omega}(0) = [0.02 \quad -0.02 \quad 0.02]^{\mathsf{T}}$ (s^{-1}) and $\mathbf{q}(0) = [\sin(\phi(0)/2)\mathbf{a}^{\mathsf{T}}(0) \quad \cos(\phi(0)/2)]^{\mathsf{T}}$ where $\mathbf{a}(0) = (1/\sqrt{14})[2 \ 3 \ -1]^{\mathsf{T}}$ and $\phi(0) = 60°$. These initial conditions are quite severe. The angular velocity error, $\mathbf{e}_\omega = \boldsymbol{\omega} - \hat{\boldsymbol{\omega}}$, is plotted versus time in Fig. 1. The angular velocity error is small indicating that $\hat{\boldsymbol{\omega}}$ matches $\boldsymbol{\omega}$ closely. It is worth emphasizing that angular velocity estimate is not generated using a rate gyro; rather, it is created using the process model given in (40) that is based on kinematic and dynamic principles. To assess the attitude error, the multiplicative error quaternion, denoted $\delta\mathbf{q} = [\delta\epsilon_1 \ \delta\epsilon_2 \ \delta\epsilon_3 \ \delta\eta]$, is computed from $\mathbf{q}$ and $\hat{\mathbf{q}}$ (Crassidis & Junkins, 2012, p. 452). The vector part of the multiplicative error quaternion versus time is plotted in Fig. 2. As with the angular velocity error, the attitude error is small indicating the filter is performing quite well.

Monte Carlo results will now be presented. The mean and standard deviation of the angular velocity error and the vector part of the multiplicative error quaternion are numerically computed from 300 simulations. The initial angular velocity and attitude are randomly generated via $\boldsymbol{\omega}(0) \sim \mathcal{N}(\mathbf{0}, (0.02)^2\,\mathbf{1})$ and $\mathbf{q}(0) = [\sin(\phi(0)/2)\mathbf{a}^{\mathsf{T}}(0) \ \cos(\phi(0)/2)]^{\mathsf{T}}$ where $\mathbf{a}'(0) \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$, $\mathbf{a}(0) = \mathbf{a}'(0)/\left\|\mathbf{a}'(0)\right\|$ and $\phi(0) \sim \mathcal{N}(0, (30°)^2)$. Again, these initial conditions are quite severe. Figs. 3 and 4 show the 300 runs between

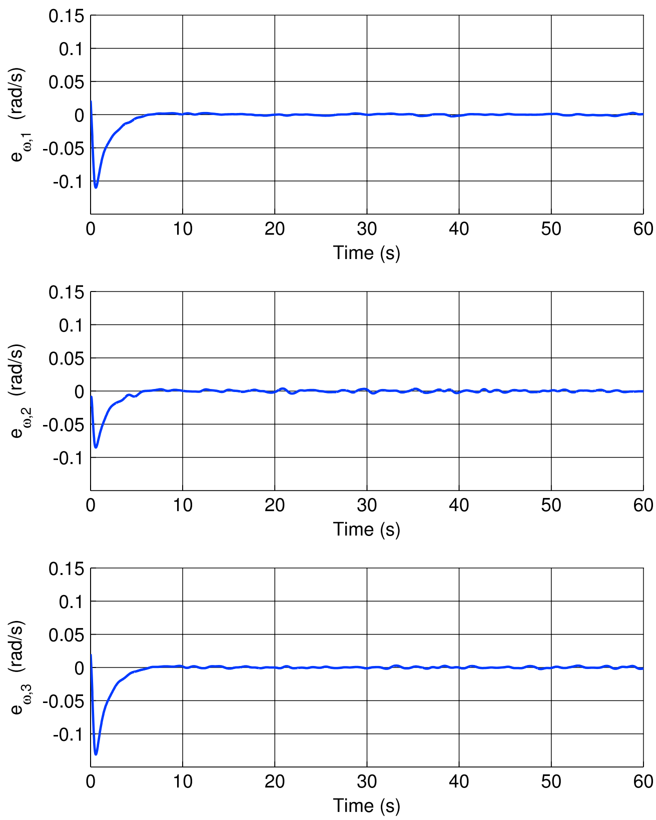

Fig. 1. Angular velocity error versus time.

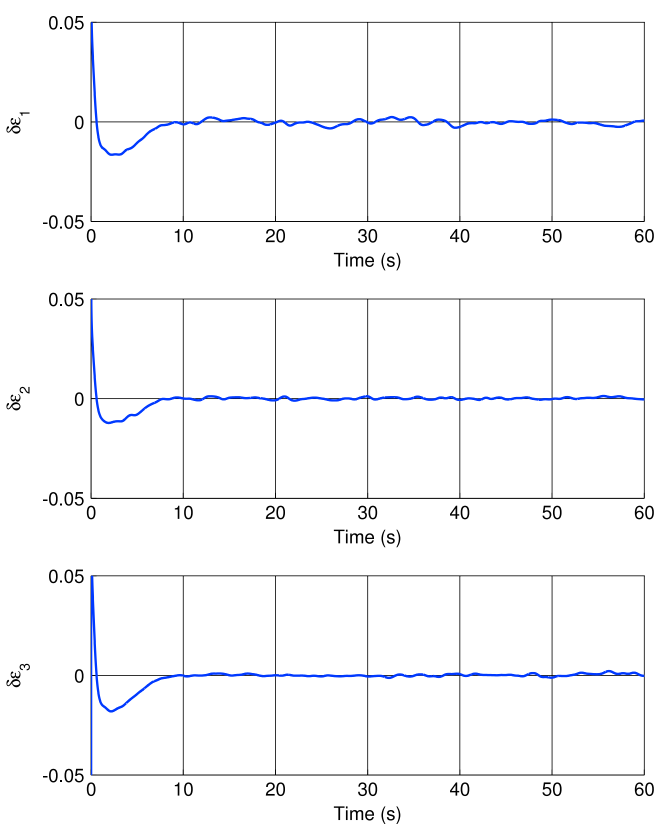

Fig. 2. Multiplicative error quaternion versus time.

40 (s) and 60 (s). This smaller time window is plotted because after 40 (s) transients associated with aggressive initial conditions have

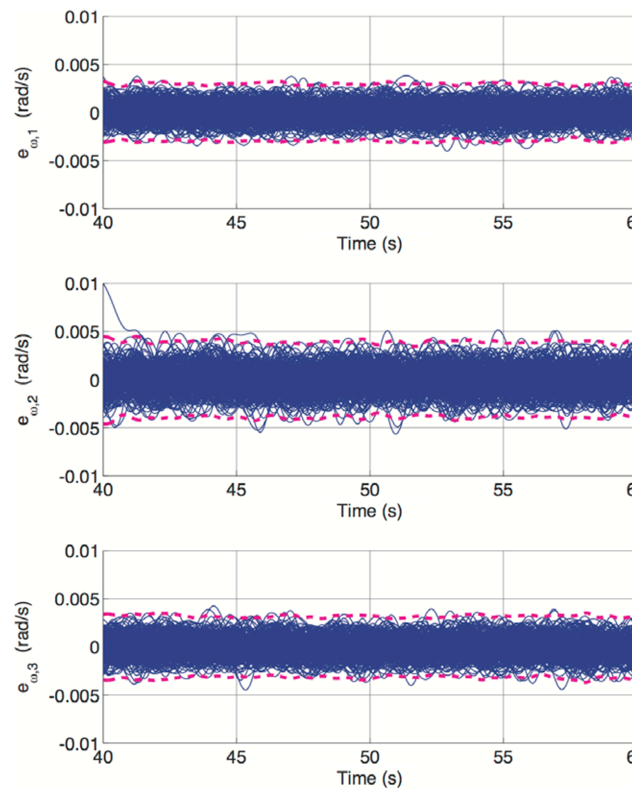

Fig. 3. Monte Carlo simulations results of angular velocity error (solid lines) versus time with $\pm 3\sigma$ bounds (dashed lines).

died out, and the steady-state characteristics of the filter can be observed. Specifically, Fig. 3 shows the angular velocity error versus time along with $\pm 3\sigma$ bounds, and Fig. 4 displays the multiplicative error quaternion versus time with $\pm 3\sigma$ bounds. The $\pm 3\sigma$ bounds are numerically computed from the Monte Carlo runs. The $\pm 3\sigma$ bounds almost always (i.e., essentially 99.7% of the time) capture the angular velocity and attitude error.

5. Closing remarks

The primary contribution of this work is the development of the continuous-time norm-constrained Kalman filter. Inspiration comes from the discrete-time norm-constrained Kalman filter. The motivation for such a filter stems from the need to estimate the angular velocity and attitude of a rigid body, such as a spacecraft, where the quaternion representing the body's attitude must satisfy a unit-length constraint. In this paper, two state estimation scenarios have been considered. The first scenario is when only part of the state estimate is norm constrained; the second is when the entire state estimate must conform to a norm constraint. The solution to the state estimation problems posed are found by solving constrained optimization problems. Of interest is the fact that the time derivative of the unconstrained state estimate is projected onto the tangent space of the norm constraint. To highlight the utility of the filter developed, estimation of the angular velocity and attitude of a spacecraft is considered. Owing to the nonlinear nature of the problem, an extended form of the filter is used yielding the continuous-time norm-constrained EKF. Numerical results indicate the filter performs well.

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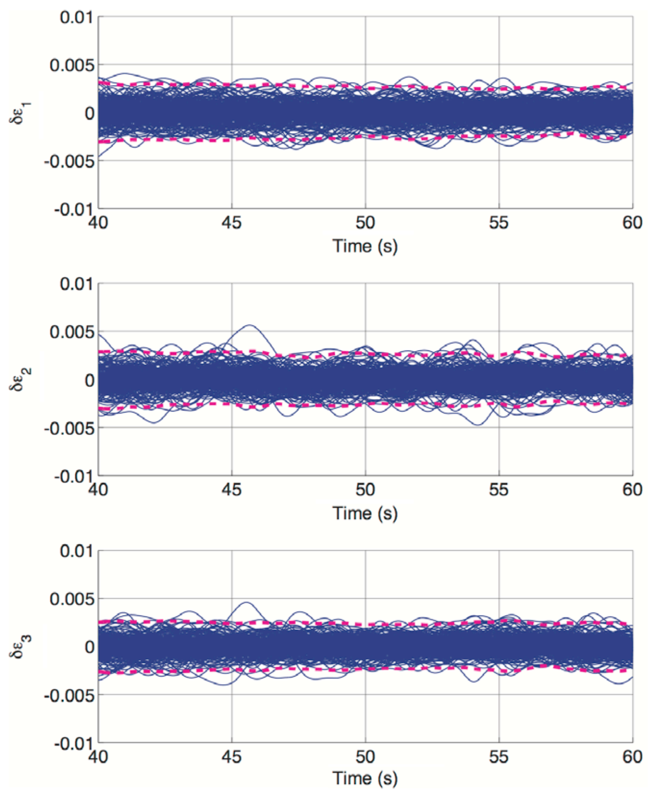

Fig. 4. Monte Carlo simulations results of multiplicative error quaternion (solid lines) versus time with $\pm 3\sigma$ bounds (dashed lines).

Appendix. Unconstrained continuous-time Kalman filtering

In this appendix the traditional continuous-time state estimation will be reviewed. Specifically, the continuous-time Kalman filter will be derived following the procedure outlined in Crassidis and Junkins (2012, pp. 168–170). Consider the linear estimator dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}\mathbf{r}, \tag{A.1}$$

where $\hat{\mathbf{x}} \in \mathbb{R}^n$ is the state estimate, $\mathbf{K} \in \mathbb{R}^{n\times n_y}$ is an observer gain that we seek to determine optimally, $\mathbf{r} = \mathbf{y} - \hat{\mathbf{y}}$ is the measurement residual (also called the innovation), and $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$ is the predicted measurement. In order to find an optimal estimate of the state, the observer gain must be found optimally. To this end, define the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$. Using (1), (A.1), and the definition of the estimation error, the error dynamics are $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \boldsymbol{\Gamma}_w\mathbf{w} - \mathbf{K}\mathbf{r} = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e} + \boldsymbol{\Gamma}_w\mathbf{w} - \mathbf{K}\boldsymbol{\Gamma}_v\mathbf{v}$. Defining the estimation-error covariance to be $\mathbf{P}(t) = E\left[\mathbf{e}(t)\mathbf{e}^{\mathsf{T}}(t)\right]$ it can be shown that

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K}\mathbf{C})^{\mathsf{T}} + \boldsymbol{\Gamma}_w\mathbf{Q}\boldsymbol{\Gamma}_w^{\mathsf{T}} + \mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\mathbf{K}^{\mathsf{T}}. \tag{A.2}$$

To find the optimal observer gain, consider the following optimization problem: $\min J(\mathbf{K})$ where

$$J(\mathbf{K}) = \operatorname{tr}\left[\dot{\mathbf{P}}\right] = \operatorname{tr}\left[(\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{K}\mathbf{C})^{\mathsf{T}} + \boldsymbol{\Gamma}_w\mathbf{Q}\boldsymbol{\Gamma}_w^{\mathsf{T}} + \mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\mathbf{K}^{\mathsf{T}}\right].$$

Taking the derivative of $J(\cdot)$ with respect to $\mathbf{K}$ and setting the result to zero gives

$$\frac{\partial J(\mathbf{K})}{\partial \mathbf{K}} = 2\left(-\mathbf{P}\mathbf{C}^{\mathsf{T}} + \mathbf{K}\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right) = \mathbf{0},$$

the first-order necessary condition for optimality. Isolating $\mathbf{K}$ yields the optimal observer gain,

$$\mathbf{K} = \mathbf{P}\mathbf{C}^{\mathsf{T}}\left(\boldsymbol{\Gamma}_v\mathbf{R}\boldsymbol{\Gamma}_v^{\mathsf{T}}\right)^{-1}. \tag{A.3}$$

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