Chapter 4 Hybrid Evolutionary Algorithm: A Case Study on Graph Coloring

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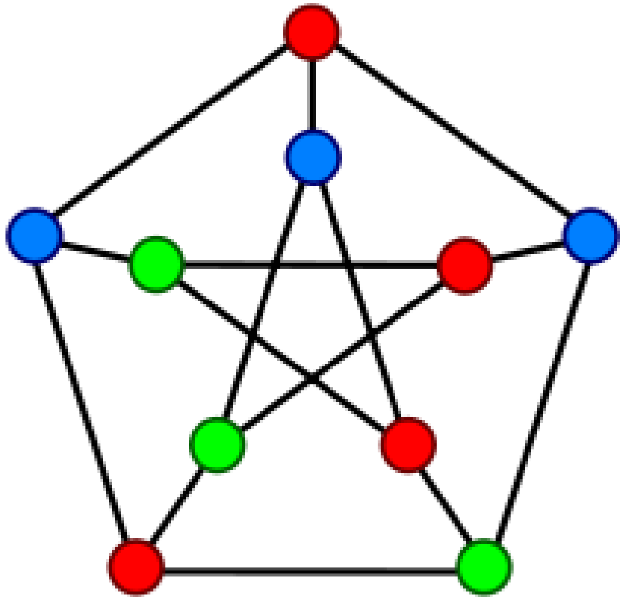
Graph Coloring

- Given an undirected graph G=(V,E), the graph coloring problem (GCP) consists of assigning a color $c_i (1 \le c_i \le k)$ to each vertex such that adjacent vertices receive different colors and the number of colors used k is minimized.



Optimization or Decision?

* GCP—**Optimization** Problem (NP Hard):
  * To find the smallest number of colors k.

* ***k***-Coloring—**Decision** Problem (NP Complete):
  * Given a ***k***, we are asked whether there exists a coloring such that all the adjacent coloring constraints are satisfied.

* The **Optimization** version of GCP can be solved by tackling a series of the **Decision** version of GCP problem with a gradually decreasing ***k***.

* Thus, these two versions are equivalent to each other.

Solution Procedure

* We starts from an initial *k* and solve the *k*-coloring problem. As soon as the *k*-coloring problem is solved, we decrease *k* by setting *k* to *k*-1 and solve again the *k*-coloring problem.
* This process is repeated until no legal *k*-coloring can be found.
* Smaller *k* → harder *k*-coloring problem. Thus, the solution approach just described solves thus a series of *k*-coloring problems of increasing difficulty.
* We only consider the K-coloring problem in this presentation.

ILP Formulation

$$x_{v,c} = \begin{cases} 1 & \text{if vertex } v \text{ is coloured with colour } c \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{c=1}^{k} x_{v,c} = 1 \quad \forall \text{ vertices } v \in V$$

$$x_{u,c} + x_{v,c} \leq 1 \quad \forall \text{ colours } c \in K \quad \forall \text{ edges } \{u,v\} \in E$$

- Given ***k*** colors, $x_{v,c}$ is the decision variables.
- The first constraint requires that each vertex receives only one color.
- The second constraint denotes that adjacent vertices should receive different colors.

Assignment Representation

* A solution of k-coloring problem can be represented as a series of colors that each vertex receives:

* $S = \{c_1, c_2, \dots, c_n\}$ where c_v denotes the color of vertex v. It is required that for any $(u, v) \in E$, $c_u \neq c_v$.

* This representation is **natural**, but not **intuitive** and **essential**.

Grouping Representation

* The feasible solution of k-coloring problem can also be presented as a set of independent sets, where an independent set is a set of non-adjacent vertices.

* $S = I_1 \cup I_2 \cup \cdots \cup I_k$, where
  1. $I_j \cap I_l = \emptyset$ for any j and l
  2. $I_1 \cup I_2 \cup \cdots \cup I_k = V$
  3. I_j is an independent set for any j.

* Thus, the k-coloring problem becomes to partition the N vertices into k independent sets.

Applications

* Mobile radio frequency assignment
* Timetabling: Education, transportation, sports
* Register allocation
* Crew scheduling
* Printed circuit testing
* Air traffic flow management
* Satellite range scheduling
* Routing and wavelength assignment in WDM networks

Literature Review (1)

Constructive Greedy Algorithms

- The first heuristic approaches to solving the graph coloring problem, which color the vertices of the graph one by one guided by a greedy function.

- They are very fast by nature but their quality is unsatisfactory.

- The best known algorithms in this class are:
  - the *largest saturation degree* heuristic (DSATUR) (D. Brelaz, 1979)
  - *recursive largest first* heuristic (RLF) (F.T. Leighton, 1979)

- Often used to generate initial solutions for advanced algorithms

Literature Review (2)

Local Search Algorithms

- One representative example is the so-called *Tabucol* algorithm which is the first application of Tabu Search to graph coloring (Hertz, de Werra, 1987)
- Other local search metaheuristic methods include:
  - Simulated Annealing (Johnson et al, 1991)
  - Iterated Local Search (Chiarandini and Stutzle, 2002)
  - Reactive Partial Tabu Search (Blochliger, Zufferey, 2008)
  - GRASP (Laguna, Marti, 2001)
  - Variable Neighborhood Search (Avanthay et al, 2003)
  - Variable Space Search (Hertz et al, 2008)
  - *Clustering*-Guided Tabu Search (Porumbel, Hao, 2009)
- Interested readers are referred to [Galinier, Hertz, 2006] for a comprehensive survey of the local search approaches.
- I will describe the famous *Tabucol* algorithm in detail.

Literature Review (3)

Hybrid Evolutionary Algorithms

* One of the most recent and very promising approaches is based upon hybridization that embeds a **local search** algorithm into the framework of an **evolutionary algorithm** in order to achieve a better tradeoff between intensification and diversification, see for examples:
  * Dorne, Hao, 1998
  * Galinier, Hao, 1999
  * Galinier, Hertz, Zufferey, 2008
  * Malaguti, Monaci, Toth, 2008
  * Porumbel, Hao, Kuntz, 2009

Initial Solution——DSATUR

* The heuristic of DSATUR(Degree of Saturation) is to sequentially color vertices according to a DANGER-based heuristic. The main idea of DSATUR is based on least saturation degree.

* At each phase, it consists of two steps: The first is to choose a vertex to color and the other is to choose a color for the chosen vertex.

Initial Solution——DSATUR

* DSATUR starts by assigning color 1 to a vertex of maximal degree.

* Suppose F is a partial coloring of the vertices of G. The degree of saturation of a vertex x, degs(x), is the number of available colors that vertex x can use.

* The vertex to be colored next in the sequential coloring procedure of DSATUR is a vertex x with smallest degs(x), breaking ties by favoring vertex with larger uncolored degree.

* When deciding a color for a chosen vertex the color that is least likely to be required by neighboring vertices is selected.

Improvements for DSATUR

* It is possible to improve DSATUR heuristic by considering more sophisticated information.
* For example, in case of choosing a color for the vertex, it would be better to consider the number of available colors.
* What else heuristics can be inspired in choosing vertex and choosing color?

Local Search

TabuCol

Search Space

* In this paper, we adapt the ***k-fixed penalty strategy*** which is also used by many coloring algorithms.
* For a given graph $G = (V; E)$, the number k of colors is fixed and the search space contains all possible (legal and illegal) k-colorings.
* A k-coloring is represented by $S = \{V1, \ldots, Vk\}$ such that Vi is the set of vertices receiving color i.
* Thus, if for all Vi *are independent sets*, then S is a legal k-coloring. Otherwise, S is an illegal (or conflicting) k-coloring.

Evaluation Function

- The optimization objective is then to minimize the number of conflicting edges (referred to *confict number* hereafter) and find a legal *k*-coloring in the search space.

- Given a *k*-coloring S = {*V*1,... ,*Vk*}, the evaluation function *f* counts the conflict number induced by S such that

$$f(S) = \sum_{\{u,v\}\in E} \delta_{uv}$$

where

$$\delta_{uv} = \begin{cases} 1, \text{ if } u \in V_i,\ v \in V_j \text{ and } i = j, \\ 0, \text{ otherwise.} \end{cases}$$

Initial Coloring

- The initial solution of our algorithm is randomly generated, i.e., each vertex in the graph is randomly assigned a color from 1 to k.
- Other greedy constructive heuristics are possible, like DSATUR, RLF, DANGER, etc.
- However, we observe that strong local search algorithms are not sensitive to the initial solutions.

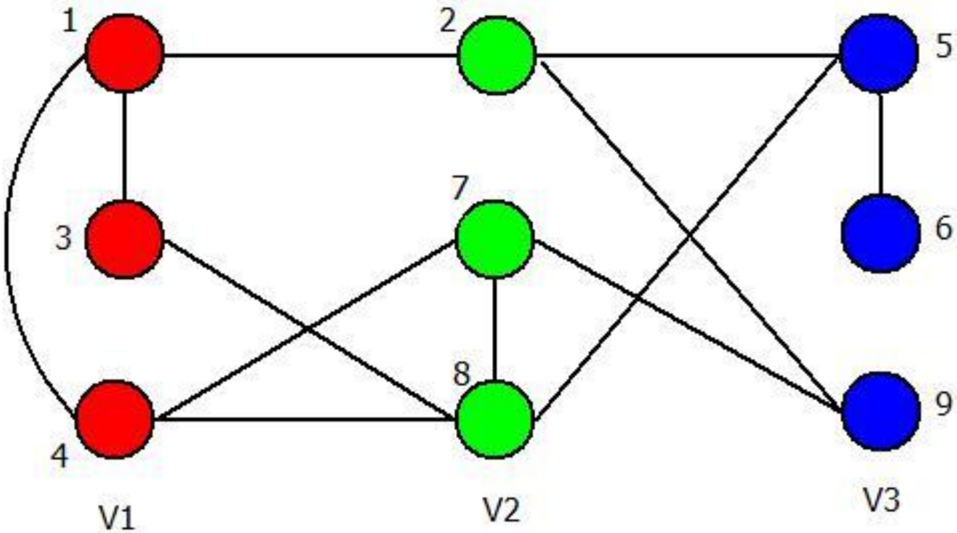
Neighborhood Moves

- A neighborhood of a given k-coloring is obtained by moving a ***conflicting*** vertex u from its original color class V_i to another color class V_j (denoted by $<u, i, j>$), called "critical one-move" neighborhood.

- Therefore, for a k-coloring S with cost $f(S)$, the size of this neighborhood is bounded by $O(f(S) \times k)$.

An Example

* f = 4
* Conflicting pairs: (1,3), (1,4), (7,8), (5,6)
* Critical One-Move: Only considers vertices 1, 3, 4, 7, 8, 5, 6. Totally 7*2=14 moves.



Neighborhood Evaluation

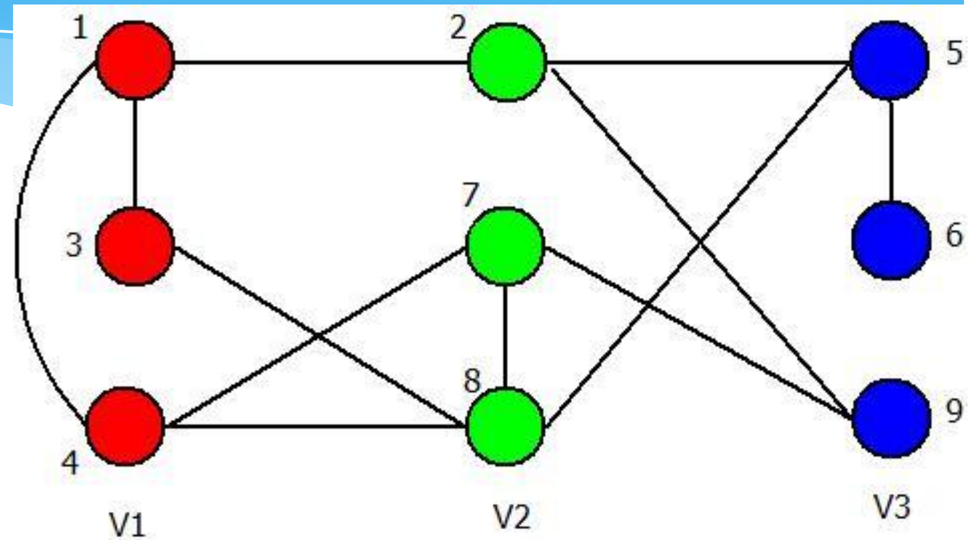
- In order to evaluate the neighborhood efficiently, we employ an ***incremental evaluation*** technique.

- The effect of each move on the objective function can be quickly calculated by a special data structure.

- Each time a move is carried out, only the move values affected by this move are updated accordingly.

Adjacent-Color Table

Vertex	Red V1	Green V2	Blue V3
1	**2**	1	0
2	1	**0**	1
3	**1**	1	0
4	**1**	2	0
5	0	2	1
6	0	0	**1**
7	1	**1**	1
8	2	**1**	1
9	0	2	**0**

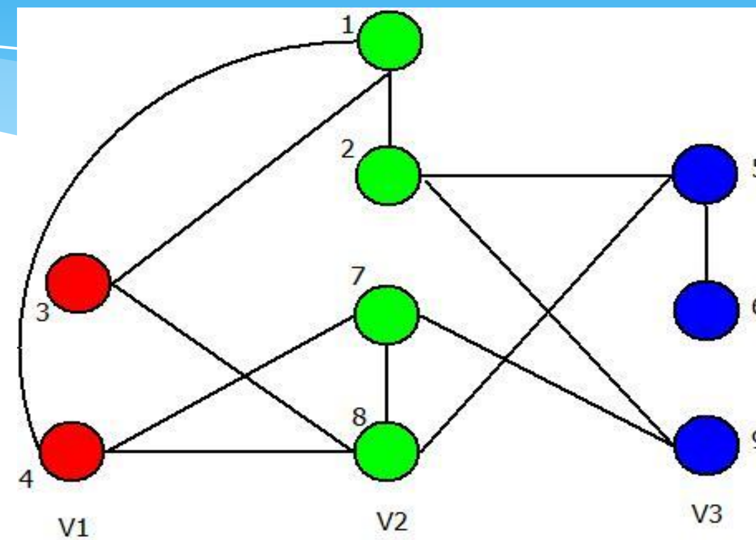


- This matrix M[u][i] (N*k) measures the number of adjacent vertices if vertex ***u*** receives color ***i***.

- Thus, the incremental move value of a move <u, i, j> can be quickly calculated as: $\Delta(u,i,j) = M[u][j] - M[u][i]$

Updating of Adjacent-Color Table

Vertex	Red V1	Green V2	Blue V3
1	2	**1**	0
2	1-1=0	**0+1=1**	1
3	**1-1=0**	1+1=2	0
4	**1-1=0**	2+1=3	0
5	0	2	1
6	0	0	**1**
7	1	**1**	1
8	2	**1**	1
9	0	2	**0**



* Move (1, v1, v2):
* Only its adjacent vertices 2, 3 and 4 are affected, and only the v1 and v2 columns need to be updated.
* All old color (v1) columns decrease by 1.
* All new color (v2) columns increase by 1.

Simple Local Search

1. Generate initial solution S, Calculate f(S)
2. Initialize the adjacent-color table M.
3. While {there exist improving moves}

3.1 Construct the neighborhood of S, denoted by N(S)

3.2 Calculate the Δ values of all critical one-moves

3.3 Find the best move with the least Δ value

3.4 Perform the best move: $f' = f + \Delta_{best}$

3.5 Update the adjacent-color table M

End

Tabu Search
Escaping from Local Optimum

- Tabu Search incorporates a tabu list as a “recency-based” memory structure to assure that solutions visited within a certain span of iterations, called tabu tenure, will not be revisited.

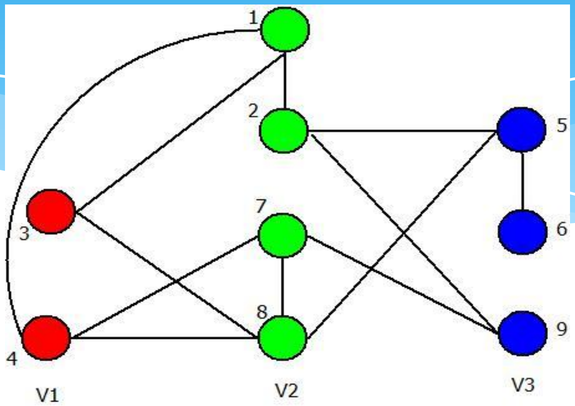
- TS then restricts consideration to moves not forbidden by the tabu list, and selects a move that produces the best move value to perform.

Tabu Tenure

* For the tabu list, once move <*u*, *i*, *j*> is performed, vertex *u* is forbidden to move back to color class *Vi* for the next *tt* iterations.

TabuTenure Table

Vertex	Red V1	Green V2	Blue V3
1	9	0	0
2	0	10	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0



- At the begining of the search, the TabuTenure table is initialized to be zero.
- once move <*u*, *i*, *j*> is performed, the value of Table[u][i]=TabuTenure.
- Once the search progresses, the non-zero value of the table is decreased by one at each time.
- In the following search, we can decide if a move <*u*, *i*, *j*> is tabu by checking if Table[u][j]>0.

Aspiration

* If one move can override the best found solution found so far, it is accepted even if it is in tabu status.

TS Algorithm

1. Generate initial solution S, Calculate f(S)
2. Initialize the adjacent-color table M.
3. While {stop condition is not met}

3.1 Construct the neighborhood of S, denoted by N(S)

3.2 Calculate the Δ values of all critical one-moves

3.3 Find the best tabu and non-tabu moves with the least Δ value

3.4 If {the aspiration condition is satisfied}

perform the best tabu move,

else

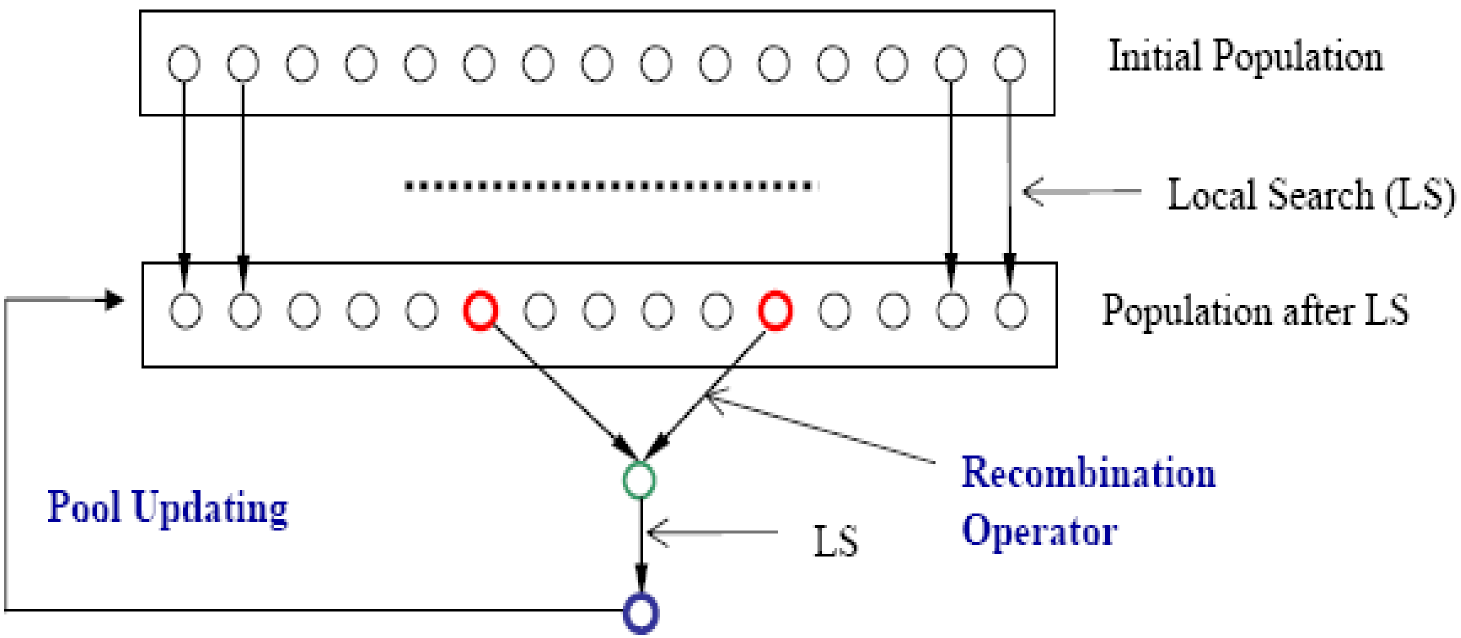
perform the best non-tabu move

3.5 Update *f* and the adjacent-color table M

End

Hybrid Evolutionary Algorithm

Main Scheme



Hybrid Evolutionary Algorithm (LS+EA)

Hybrid Evolutionary Algorithm

The hybrid coloring algorithm

Data : *graph* $G = (V, E)$, *integer k*
Result : *the best configuration found*
begin
    P=*InitPopulation*($|P|$)
    while *not Stop-Condition ()* **do**
        (s1,s2)=*ChooseParents*(P)
        s=*Crossover*(s1,s2)
        s=*LocalSearch*(s, L)
        P=*UpdatePopulation*(P,s)
end

Crossover Operator (1)

Table 2. The crossover algorithm: an example.

parent s_1 →	A B C	D E F G	H I J	$V_1 := \{D, E, F, G\}$	A B C		H I J
parent s_2	C D E G	A F I	B H J	remove D,E,F and G	C	A I	B H J
offspring s					D E F G		
parent s_1	A B C		H I J	$V_2 := \{B, H, J\}$	A C		I
parent s_2 →	C	A I	B H J	remove B,H and J	C	A I	
offspring s	D E F G				D E F G	B H J	
parent s_1 →	A C		I	$V_3 := \{A, C\}$			I
parent s_2	C	A I		remove A and C		I	
offspring s	D E F G	B H J			D E F G	B H J	A C

Crossover Operator (2)

* A legal *k*-coloring is a collection of *k* independent sets.
* With this point of view, if we could maximize the size of the independent sets by a crossover operator as far as possible, it will in turn help to push those left vertices into independent sets.
* In other words, the more vertices are transmitted from parent individuals to the offspring within *k* steps, the less vertices are left unassigned.
* In this way, the obtained offspring individual has more possibility to become a legal coloring.

Crossover Operator (3)

The GPX crossover algorithm

Data : *configurations* $s_1 = \{V_1^1, \dots, V_k^1\}$ *and* $s_2 = \{V_1^2, \dots, V_k^2\}$
Result : *configuration* $s = \{V_1, \dots, V_k\}$
begin
  for $l (1 \leq l \leq k)$ **do**
    if l is odd, then $A := 1$, else $A := 2$
    choose i such that V_i^A has a maximum cardinality
    $V_l := V_i^A$
    remove the vertices of V_l from s_1 and s_2
  Assign randomly the vertices of $V - (V_1 \cup \dots \cup V_k)$
end

HEA Algorithm Scheme

Algorithm 1 Pseudcode of the Hybrid Evolutionary Algorithm for k-Coloring

Input: Graph G
Output: The best solution S^* found so far
$\{S_1, \ldots, S_p\} \leftarrow$ Initial Population
for $i = \{1, \ldots, p\}$ **do**
    $S_i \leftarrow$ Tabu Search(S_i)
end for
$S^* = arg\ min\{f(S_i), i = 1, \ldots, p\}$
repeat
    Randomly choose two parent solutions $\{S_{i1}, S_{i2}\}$
    $S_0 \leftarrow$ Crossover_Operator (S_{i1}, S_{i2})
    $S_0 \leftarrow$ Tabu Search(S_0)
    if $f(S_0) < f(S^*)$ **then**
        $S^* = S_0$
    end if
    $\{S_1, \ldots, S_p\} \leftarrow$ Pool Updating$(S_0, S_1, \ldots, S_p)$
until Stop condition met

Thank You!