

Theory of T-norms and fuzzy inference methods

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Abstract: In this paper, the theory of T-norm and T-conorm is reviewed and the T-norm, T-conorm and negation function are defined as a set of T-operators. Some typical T-operators and their mathematical properties are presented. Finally, the T-operators are extended to the conventional fuzzy reasoning methods which are based on the MIN and MAX operators. This extended fuzzy reasoning provides both a general and a flexible method for the design of fuzzy logic controllers and, more generally, for the modelling of any decision-making process.

Keywords: T-norms; T-conorms; T-operators; fuzzy inference; fuzzy logic controller.

1. Introduction

The triangular norm (T-norm) and the triangular conorm (T-conorm) originated from the studies of probabilistic metric spaces [1, 2] in which triangular inequalities were extended using the theory of T-norm and T-conorm. Later, Höhle [3], Alsina et al. [4], etc. introduced the T-norm and the T-conorm into fuzzy set theory and suggested that the T-norm and the T-conorm be used for the intersection and union of fuzzy sets. Since then, many other researchers have presented various types of T-operators for the same purpose [5, 6, 7] and even proposed some methods to generate the variations of these operators [8] which are given in the Appendix.

Zadeh's conventional T-operators, MIN and MAX, have been used in almost every design of fuzzy logic controllers and even in the modelling of other decision-making processes. However, some theoretical and experimental studies seem to indicate that other types of T-operators may work better in some situations, especially in the context of decision-making processes. For example, the product operator may be preferred to the MIN operator [9]. On the other hand, when choosing a set of T-operators for a given decision-making process, one has to consider their properties, the accuracy of the model, their simplicity, computer and hardware implementation, etc. For these and other reasons, it is of interest to use other sets of T-operators in the modelling of decision-making processes, so that multiple options are available for selecting T-operators that may be better suited for given problems.

In this paper we will give a detailed exposition of the theory of T-operators, the various methods of their generations, and possible applications in fuzzy reasoning processes.

2. Definitions of T-operators

T-norm, T-conorm and negation functions are used to calculate the membership values of intersection, union and complement of fuzzy sets, respectively. The definitions of T-operators have been given by many researchers. In this section, however, an attempt is made to give a complete set of definitions to T-operators.

Definition 1. Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$. T is a T-norm, if and only if (iff) for all $x, y, z \in [0, 1]$:

- (1.1) $T(x, y) = T(y, x)$ (commutativity),
- (1.2) $T(x, y) \leq T(x, z)$, if $y \leq z$ (monotonicity),
- (1.3) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
- (1.4) $T(x, 1) = x$.

A T-norm is Archimedean, iff:

- (1.5) $T(x, y)$ is continuous,
- (1.6) $T(x, x) < x \forall x \in (0, 1)$.

An Archimedean T-norm is strict, iff

- (1.7) $T(x', y') < T(x, y)$, if $x' < x, y' < y, \forall x', y', x, y \in (0, 1)$.

Definition 2. Let $T^* : [0, 1] \times [0, 1] \rightarrow [0, 1]$. T^* is a T-conorm, iff for all $x, y, z \in [0, 1]$:

- (2.1) $T^*(x, y) = T^*(y, x)$ (commutativity),
- (2.2) $T^*(x, y) \leq T^*(x, z)$, if $y \leq z$ (monotonicity),
- (2.3) $T^*(x, T^*(y, z)) = T^*(T^*(x, y), z)$ (associativity),
- (2.4) $T^*(x, 0) = x$.

A T-conorm is Archimedean, iff:

- (2.5) T^* is continuous,
- (2.6) $T^*(x, x) > x \forall x \in (0, 1)$.

An Archimedean T-conorm is strict, iff

- (2.7) $T^*(x', y') < T^*(x, y)$, if $x' < x, y' < y, \forall x', y', x, y \in (0, 1)$.

Note that for a T-norm T and a T-conorm T^* ,

$$\begin{aligned} T(0, 0) &= 0, & T(1, 1) &= 1, \\ T^*(0, 0) &= 0, & T^*(1, 1) &= 1. \end{aligned}$$

Definition 3. Let $N : [0, 1] \rightarrow [0, 1]$. N is a negation function, iff:

- (3.1) $N(0) = 1, N(1) = 0$,
- (3.2) $N(x) \leq N(y)$, if $x \geq y$ (monotonicity).

A negation function is strict, iff:

- (3.3) $N(x)$ is continuous,
- (3.4) $N(x) < N(y)$, for $x > y \forall x, y \in [0, 1]$.

A strict negation function is involutive, iff

- (3.5) $N(N(x)) = x, \forall x \in [0, 1]$.

3. Some typical T-operators

In this section, eleven sets of T-operators are given and some of their relevant properties are studied.

Zadeh's T-operators are the most popular ones in the literature, and are defined as follows:

$$T_1(x, y) = \min(x, y), \quad (1a)$$

$$T_1^*(x, y) = \max(x, y), \quad (1b)$$

$$N_1(x) = 1 - x. \quad (1c)$$

Goguen [7], Bandler et al. [12], etc. proposed and studied a set of T-operators which are also called probabilistic operators defined as

$$T_2(x, y) = x \cdot y, \quad (2a)$$

$$T_2^*(x, y) = x + y - xy, \quad (2b)$$

$$N_2(x) = 1 - x. \quad (2c)$$

A set of T-operators given as

$$T_3(x, y) = \max(x + y - 1, 0), \quad (3a)$$

$$T_3^*(x, y) = \min(x + y, 1), \quad (3b)$$

$$N_3(x) = 1 - x, \quad (3c)$$

are called Lukasiewicz logics and have been studied by Giles [11] and others.

Another set of T-operators are defined as

$$T_4(x, y) = \frac{xy}{x + y - xy}, \quad (4a)$$

$$T_4^*(x, y) = \frac{x + y - 2xy}{1 - xy}, \quad (4b)$$

$$N_4(x) = 1 - x. \quad (4c)$$

Weber [7] and others studied a set of T-operators which are given by

$$T_5(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (5a)$$

$$T_5^*(x, y) = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (5b)$$

$$N_5(x) = 1 - x. \quad (5c)$$

This set of operators is the only one which is not continuous.

Hamacher [7] proposed a set of T-operators which are defined as

$$T_6(x, y) = \frac{\lambda xy}{1 - (1 - \lambda)(x + y - xy)}, \quad (6a)$$

$$T_6^*(x, y) = \frac{\lambda(x + y) + xy(1 - 2\lambda)}{\lambda + xy(1 - \lambda)}, \quad (6b)$$

$$N_6(x) = 1 - x. \quad (6c)$$

Table 1. T-operators

N	$T_N(x, y)$	$T_N^*(x, y)$	$N_N(x)$	Comment
1	$\min(x, y)$	$\max(x, y)$	$1 - x$	Zadeh [10]
2	$x \cdot y$	$x + y - xy$	$1 - x$	Goguen [7], Bandler [12], etc
3	$\max(x + y - 1, 0)$	$\min(x + y, 1)$	$1 - x$	Giles [11], etc
4	$\frac{xy}{x + y - xy}$	$\frac{x + y - 2xy}{1 - xy}$	$1 - x$	
5	$\begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$	$1 - x$	Weber [7], etc.
6	$\frac{\lambda xy}{1 - (1 - \lambda)(x + y - xy)}$	$\frac{\lambda(x + y) + xy(1 - 2\lambda)}{\lambda + xy(1 - \lambda)}$	$1 - x$	Hamacher [7] $\lambda \rightarrow 0, \rightarrow T_5$ and T_5^* $\lambda = 1, \rightarrow T_2$ and T_2^* $\lambda \rightarrow \infty, \rightarrow T_4$ and T_4^*
7	$\max(1 - ((1 - x)^p + (1 - y)^p)^{1/p}, 0)$	$\min((x^p + y^p)^{1/p}, 1)$	$1 - x$	Yager [5] $p = 1, \rightarrow T_3$ and T_3^* $p \rightarrow \infty, \rightarrow T_1$ and T_1^*

8	$\frac{1}{1 + \left(\left(\frac{1}{x} - 1 \right)^\lambda + \left(\frac{1}{y} - 1 \right)^\lambda \right)^{\frac{1}{1+\lambda}}}$	$\frac{1}{1 + \left(\left(\frac{1}{x} - 1 \right)^{-\lambda} + \left(\frac{1}{y} - 1 \right)^{-\lambda} \right)^{-\frac{1}{1+\lambda}}}$	Dombi [6] $\lambda \rightarrow 0, \rightarrow T_5 \text{ and } T_5^*$ $\lambda = 1, \rightarrow T_4 \text{ and } T_4^*$ $\lambda \rightarrow \infty, \rightarrow T_1 \text{ and } T_1^*$
9	$\frac{xy}{\max(x, y, \lambda)}$	$1 - \frac{(1-x)(1-y)}{\max(1-x, 1-y, \lambda)}$	Dubois et al. [9] $\lambda = 0, \rightarrow T_1 \text{ and } T_1^*$ $\lambda = 1, \rightarrow T_2 \text{ and } T_2^*$
10	$\max\left(\frac{x+y-1+\lambda xy}{1+\lambda}, 0\right)$	$\min(x+y+\lambda xy, 1)$	Weber [7] $\lambda \rightarrow -1, \rightarrow T_5 \text{ and } T_2^*$ $\lambda = 0, \rightarrow T_3 \text{ and } T_3^*$ $\lambda \rightarrow \infty, \rightarrow T_2 \text{ and } T_5^*$
11	$\max((1+\lambda)(x+y-1) - \lambda xy, 0)$	$\min(x+y+\lambda xy, 1)$	Yu Yandong [14], etc. $\lambda \rightarrow -1, \rightarrow T_2 \text{ and } T_2^*$ $\lambda = 0, \rightarrow T_3 \text{ and } T_3^*$ $\lambda \rightarrow \infty, \rightarrow T_5 \text{ and } T_5^*$

Note the following limiting case for T_6 and T_6^* :

- (i) for $\lambda \rightarrow 0$, $T_6 \rightarrow T_5$, and $T_6^* \rightarrow T_5^*$;
- (ii) for $\lambda = 1$, $T_6 = T_2$, and $T_6^* = T_2^*$; and
- (iii) for $\lambda \rightarrow \infty$, $T_6 = T_4$, and $T_6^* = T_4^*$.

Yager [5] proposed a set of T-operators which are defined as follows:

$$T_7(x, y) = \text{MAX}(1 - ((1 - x)^p + (1 - y)^p)^{1/p}, 0), \quad (7a)$$

$$T_7^*(x, y) = \text{MIN}((x^p + y^p)^{1/p}, 1), \quad (7b)$$

$$N_7(x) = 1 - x. \quad (7c)$$

Again, note the following:

- (i) for $p = 1$, $T_6 = T_3$, and $T_6^* = T_3^*$; and
- (ii) for $p \rightarrow \infty$, $T_6 \rightarrow T_1$, and $T_6^* \rightarrow T_1^*$.

Dombi [6] presented the following set of T-operators:

$$T_8(x, y) = \frac{1}{1 + \left(\left(\frac{1}{x} - 1 \right)^\lambda + \left(\frac{1}{y} - 1 \right)^\lambda \right)^{1/\lambda}}, \quad (8a)$$

$$T_8^*(x, y) = \frac{1}{1 + \left(\left(\frac{1}{x} - 1 \right)^{-\lambda} + \left(\frac{1}{y} - 1 \right)^{-\lambda} \right)^{-1/\lambda}}, \quad (8b)$$

$$N_8(x) = 1 - x. \quad (8c)$$

Note the following:

- (i) for $\lambda \rightarrow 0$, $T_8 \rightarrow T_5$, and $T_8^* \rightarrow T_5^*$;
- (ii) for $\lambda = 1$, $T_8 = T_4$, and $T_8^* = T_4^*$; and
- (iii) for $\lambda \rightarrow \infty$, $T_8 \rightarrow T_1$, and $T_8^* \rightarrow T_1^*$.

Dubois and Prade [9] also gave a set of T-operators defined as

$$T_9(x, y) = \frac{xy}{\text{MAX}(x, y, \lambda)}, \quad (9a)$$

$$T_9^*(x, y) = 1 - \frac{(1 - x)(1 - y)}{\text{MAX}(1 - x, 1 - y, \lambda)}, \quad (9b)$$

$$N_9(x) = 1 - x. \quad (9c)$$

Note the following:

- (i) for $\lambda = 0$, $T_9 = T_1$, and $T_9^* = T_1^*$; and
- (ii) for $\lambda = 1$, $T_9 = T_2$, and $T_9^* = T_2^*$.

Weber [7] proposed another set of T-operators which are defined as

$$T_{10}(x, y) = \text{MAX}\left(\frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0\right), \quad (10a)$$

$$T_{10}^*(x, y) = \text{MIN}(x + y + \lambda xy, 1), \quad (10b)$$

$$N_{10}(x) = \frac{1 - x}{1 + \lambda x}. \quad (10c)$$

Note the following:

- (i) for $\lambda = 0$, $T_{10} = T_3$, and $T_{10}^* = T_3^*$;
- (ii) for $\lambda \rightarrow -1$, $T_{10} \rightarrow T_5$, and $T_{10}^* \rightarrow T_2^*$; and
- (iii) For $\lambda \rightarrow \infty$, $T_{10} \rightarrow T_2$, and $T_{10}^* = T_5^*$.

Yu Yandong [14] studied a set of T-operators which are given by

$$T_{11}(x, y) = \max((1 + \lambda)(x + y - 1) - \lambda xy, 0), \quad (11a)$$

$$T_{11}^*(x, y) = \min(x + y + \lambda xy, 1), \quad (11b)$$

$$N_{11}(x) = 1 - x. \quad (11c)$$

Note the following:

- (i) for $\lambda \rightarrow -1$, $T_{11} \rightarrow T_2$, and $T_{11}^* \rightarrow T_2^*$;
- (ii) for $\lambda = 0$, $T_{11} = T_3$, and $T_{11}^* = T_3^*$; and
- (iii) for $\lambda \rightarrow \infty$, $T_{11} \rightarrow T_5$, and $T_{11}^* \rightarrow T_5^*$.

4. Some properties of T-operators

In the following, some important mathematical properties of T-operators are presented. For simplicity, all proofs are omitted.

According to the definitions, T and T^* possess the following two important properties:

P_1 : Commutativity:

$$T(x, y) = T(y, x), \quad (12a)$$

$$T^*(x, y) = T^*(y, x). \quad (12b)$$

P_2 : Associativity:

$$T(x, T(y, z)) = T(T(x, y), z), \quad (13a)$$

$$T^*(x, T(y, z)) = T^*(T^*(x, y), z). \quad (13b)$$

Also, consider the following additional important properties for T-operators.

P_3 : Distributivity:

$$T(x, T^*(y, z)) = T^*(T(x, y), T(x, z)), \quad (14a)$$

$$T^*(x, T(y, z)) = T(T^*(x, y), T^*(x, z)). \quad (14b)$$

P_4 : Absorption:

$$T(T^*(x, y), x) = x, \quad (15a)$$

$$T^*(T(x, y), x) = x. \quad (15b)$$

P_5 : Idempotency:

$$T(x, x) = x, \quad (16a)$$

$$T^*(x, x) = x. \quad (16b)$$

Table 2. Properties of eleven sets of T-operators

T, T^*, N	Distributivity	Idempotency	The Excluded Middle Laws	Absorption	Associativity	Commutativity	De Morgan's Laws
T_1, T_1^*, N_1	+	+	-	+	+	+	+
T_2, T_2^*, N_2	-	-	-	-	+	+	+
T_3, T_3^*, N_3	-	-	+	-	+	+	+
T_4, T_4^*, N_4	-	-	-	-	+	+	+
T_5, T_5^*, N_5	-	-	+	-	+	+	+
T_6, T_6^*, N_6	-	-	-	-	+	+	+
T_7, T_7^*, N_7	-	-	-	-	+	+	+
T_8, T_8^*, N_8	-	-	-	-	+	+	+
T_9, T_9^*, N_9	-	-	-	-	+	+	+
T_{10}, T_{10}^*, N_{10}	-	-	+	-	+	+	+ ^a
T_{11}, T_{11}^*, N_{11}	-	-	+ ^b	-	+	+	+

^a Only when $x + y + \lambda xy \geq 1$ ($\lambda \neq 0$).

^b Only when $\lambda > 0$.

Theorem 1.

$$\text{Distributivity} \Rightarrow \text{Absorption} \Rightarrow \text{Idempotency} \Rightarrow \begin{cases} T = T_1, \\ T^* = T_1^*. \end{cases}$$

According to this theorem, it is clear that all T-norms and T-conorms do not satisfy the properties P_3 , P_4 and P_5 , except for T_1 and T_1^* .

The excluded-middle laws are stated as

$$P_6: \quad T(x, N(x)) = 0, \quad (17a)$$

$$P_7: \quad T^*(x, N(x)) = 1. \quad (17b)$$

Theorem 2. *If T and T^* satisfy P_6 and P_7 , then they do not fulfil P_3 , P_4 and P_5 .*

(T_3, T_3^*, N_3) , (T_5, T_5^*, N_5) , $(T_{10}, T_{10}^*, N_{10})$ and $(T_{11}, T_{11}^*, N_{11})$ are the only ones in Table 1 which satisfy P_6 and P_7 and, therefore, do not have properties P_3 , P_4 and P_5 . Note that $(T_{11}, T_{11}^*, N_{11})$ has P_6 and P_7 only when $\lambda > 0$.

The well known De Morgan laws for T-operators are stated as follows:

$$P_8: \quad N(T(x, y)) = T^*(N(x), N(y)), \quad (18a)$$

$$P_9: \quad N(T^*(x, y)) = T(N(x), N(y)). \quad (18b)$$

Theorem 3. *If $N(x)$ is involutive, Eqs. (18.a) and (18.b) are equivalent, and the following equations are also true:*

$$P_{10}: \quad T(x, y) = N(T^*(N(x), N(y))), \quad (19a)$$

$$P_{11}: \quad T^*(x, y) = N(T(N(x), N(y))). \quad (19b)$$

All eleven sets of T-operators satisfy De Morgan's laws, although $(T_{10}, T_{10}^*, N_{10})$ is needed to satisfy the requirement $x + y + \lambda xy \geq 1$ ($\lambda \neq 0$).

There are some other important properties which are described by the following inequalities:

$$P_{12}: \quad T_5(x, y) \leq T(x, y) \leq T_1(x, y), \quad (20a)$$

$$P_{13}: \quad T_1^*(x, y) \leq T^*(x, y) \leq T_5^*(x, y), \quad (20b)$$

$$P_{14}: \quad T_5(x, y) < T_3(x, y) < T_2(x, y) < T_4(x, y) < T_1(x, y), \quad (21a)$$

$$P_{15}: \quad T_1^*(x, y) < T_4^*(x, y) < T_2^*(x, y) < T_3^*(x, y) < T_5^*(x, y). \quad (21b)$$

These conclusions are also demonstrated in Figure 1 for triangular fuzzy numbers.

5. Fuzzy inference methods based on T-operators

In this section, a generalized fuzzy reasoning method is proposed by extending T-operators to conventional fuzzy reasoning algorithms in which MIN and MAX operators are widely being used. Then, by using Mamdani's implication function,

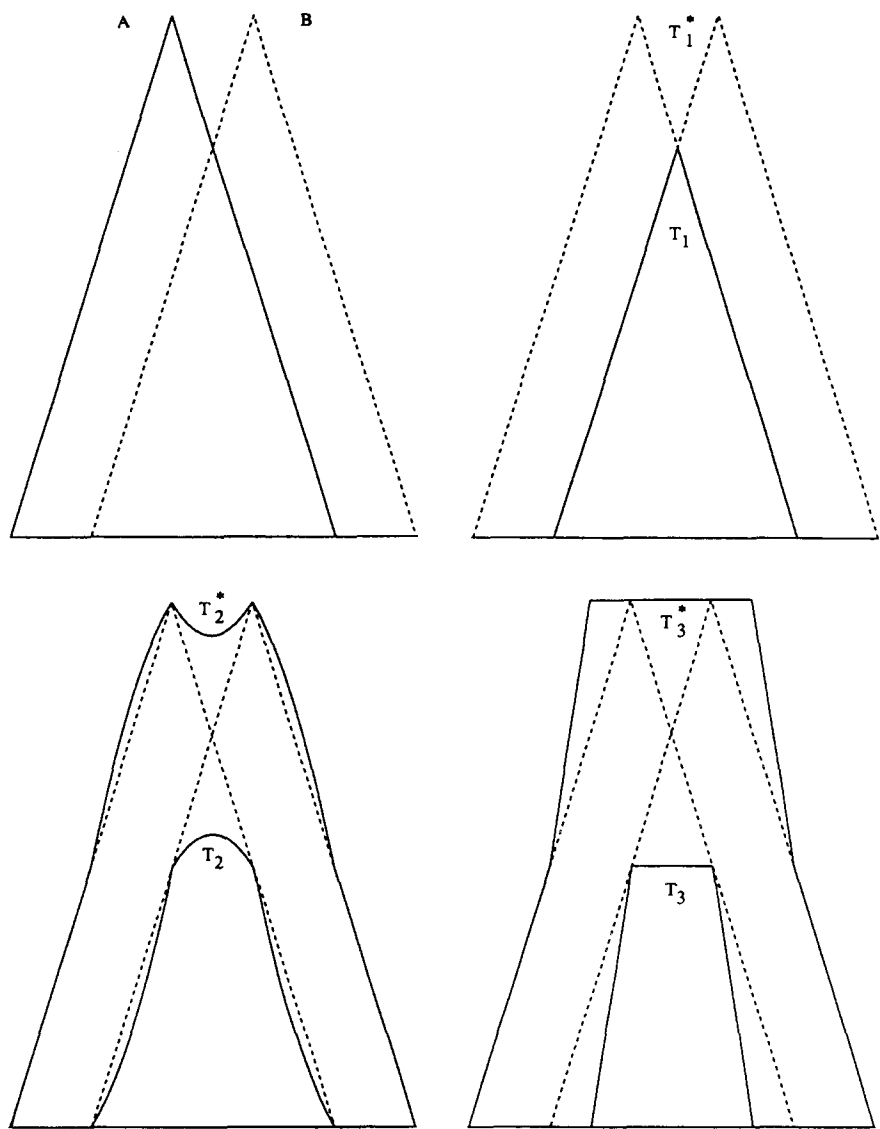


Fig. 1. T-Norms and T-conorms.

which is simple and easy to implement, a series of examples are given based on the proposed fuzzy reasoning methods.

As discussed before, theoretical and experimental studies have indicated that some T-operators work better in some situations, especially in the context of decision-making processes, than \min and \max operators which have been widely used for the same purpose [9]. In fact, the choice of an operator is always a matter of context, and it mostly depends on the real-world problem which is to be modelled. It is appropriate, therefore, to use the general concept of T-operators

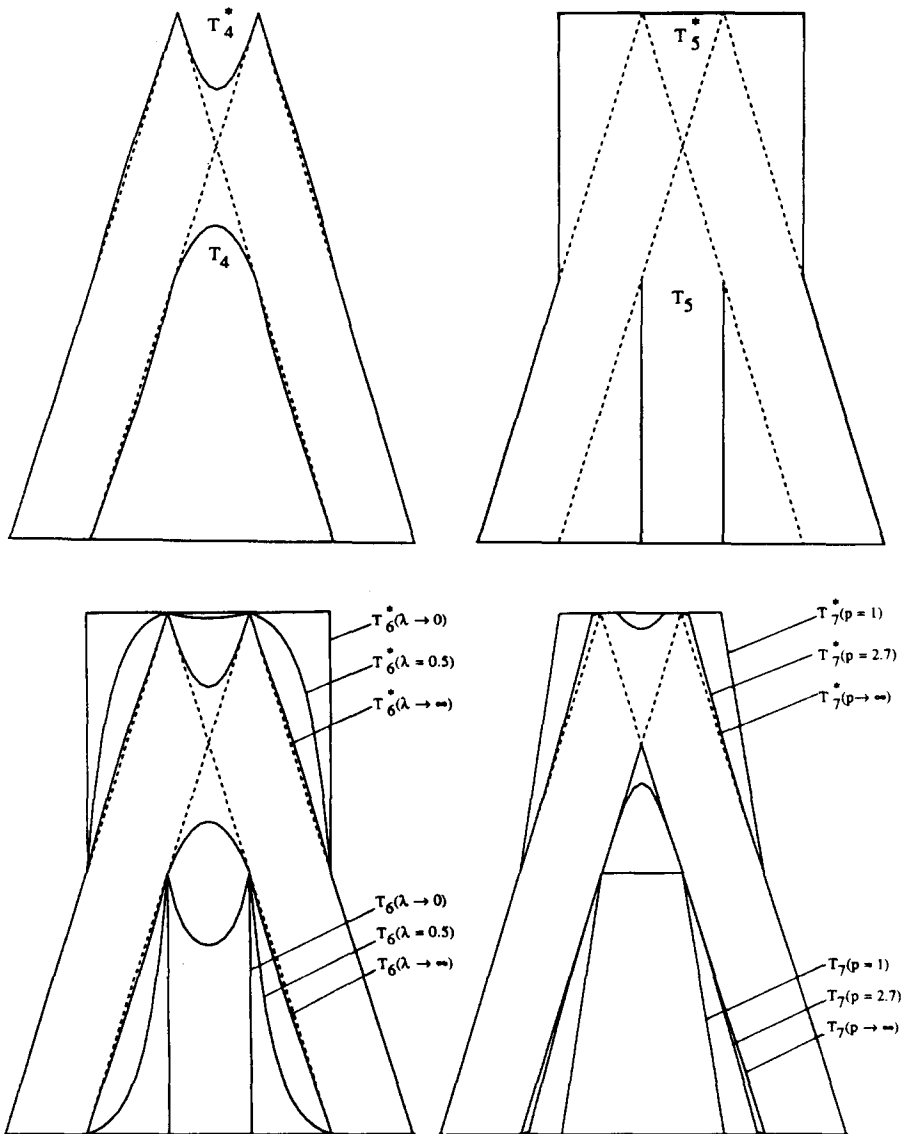


Fig. 1 (continued).

in the modelling of decision-making process, in addition to the MIN and MAX operators, so as to provide more options and flexibility for the selection of T-operators that may be better suited for a given problem.

In decision-making environments, the human brain tends to make inferences in which fuzzy expressions are often involved. These types of inferences cannot be satisfactorily modelled by using classical two-valued logic. A fuzzy inference means deducing new conclusions from the given information in the form of 'IF-THEN' rules in which antecedents and consequents are fuzzy sets. The

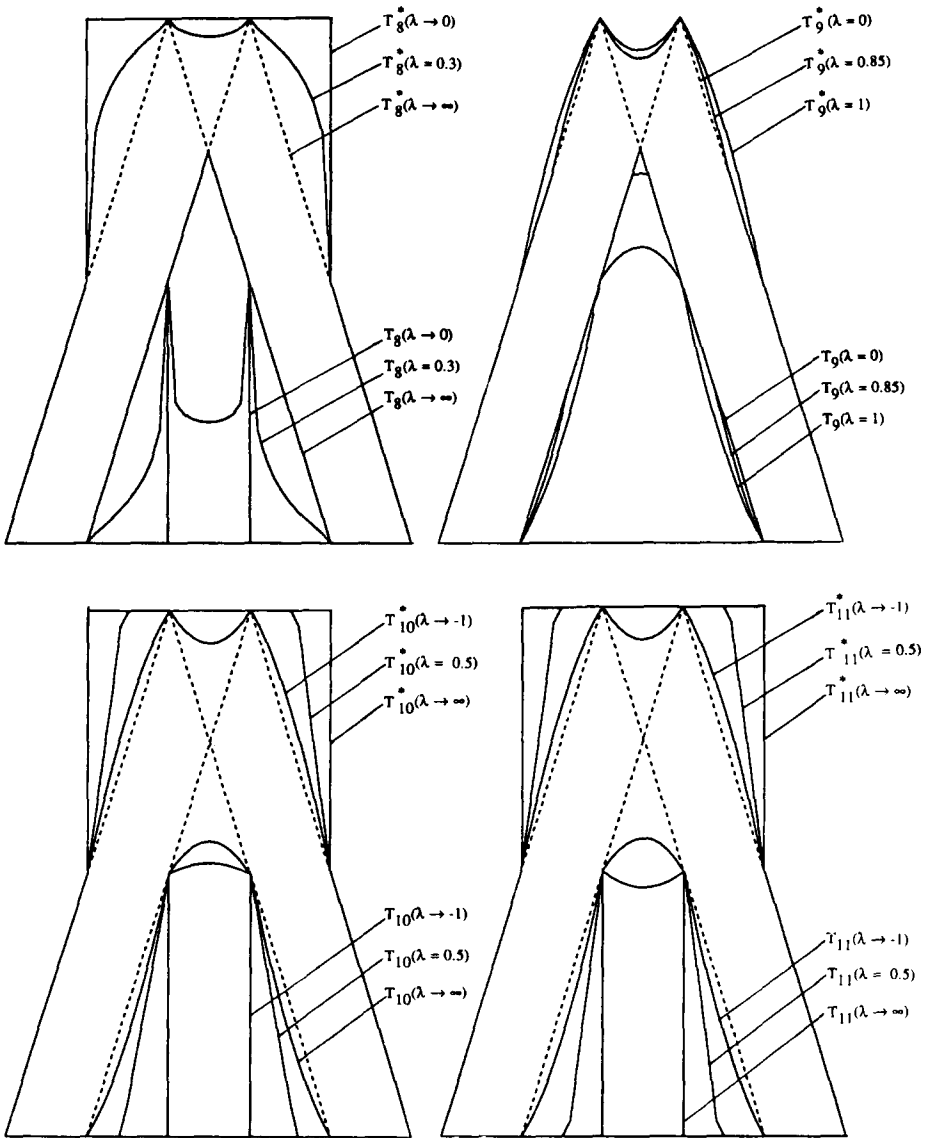


Fig. 1 (continued)

following is the form of fuzzy inferences:

Implication: if x is A , then y is B
Premise: x is A'

Conclusion: y is B'

where x and y are linguistic variables, such as position, velocity, pressure and temperature, etc., and A , A' , B and B' are fuzzy sets representing linguistic

labels over the universes of discourse X , X , Y and Y respectively, such as SMALL, MEDIUM, HIGH and VERY LOW, etc.

This type of inference is called generalized modus ponens which reduces to classical modus ponens for $A' = A$ and $B' = B$.

The following is an example of the above form of inference which may be used in a temperature control system.

Implication: if temperature is high then fuel input is low
Premise: the temperature is very high

Conclusion: the fuel input is very low

Consider another form of inference,

Implication: if x is A , then y is B
Premise: y is B'

Conclusion: x is A'

Similarly, if $A' = \bar{A}$ and $B' = \bar{B}$, this inference becomes modus tollens. Therefore, it is called the generalized modus tollens.

In the design of fuzzy logic controllers, the fuzzy inference with the form of generalized modus ponens is used as shown in the following:

Implication: if control condition A , then control action B
Premise: control condition A'

Conclusion: control action B'

Consider a fuzzy logic controller as an example of a fuzzy reasoning process. Suppose that an experienced human operator provides verbal descriptions of his expert knowledge about the process to be controlled in the form of IF-THEN rules as follows:

Rule 1: If x is A_1 , then y is B_1
 Rule 2: If x is A_2 , then y is B_2
 ...
 Rule i : If x is A_i , then y is B_i
 ...
 Rule N : If x is A_N , then y is B_N

or, this set of control rules can also be written as an ensemble of IF-THEN rules:

$$\bigcup_{i=1}^N \text{If } x \text{ is } A_i, \text{ then } y \text{ is } B_i$$

where x and y are linguistic variables, and A_i and B_i are fuzzy sets over a universe of discourse X and Y .

To implement the above decision rules, an implication function is required. If the fuzzy relation between A_i and B_i is represented by $R_{A_i \rightarrow B_i}$ on the universe of discourse $X \times Y$, then its membership function is given in terms of T-norms as

follows:

$$\mu_{R_{A_i \rightarrow B_i}}(x, y) = T(\mu_{A_i}(x), \mu_{B_i}(y)), \quad x \in X, y \in Y. \quad (22)$$

This is in fact the generalized Mamdani implication function. If Zadeh's implication methods are used, the following conclusions can be drawn:

$$(1) \quad R_{A_i \rightarrow B_i} = (A_i \times B_i) \cup (\bar{A}_i \times Y), \quad (23a)$$

$$\mu_{R_{A_i \rightarrow B_i}}(x, y) = T^*(T(\mu_{A_i}(x), \mu_{B_i}(y)), N(\mu_{A_i}(x))), \quad (23b)$$

$$(2) \quad R_{A_i \rightarrow B_i} = (X \times B_i) \cup (\bar{A}_i \times Y), \quad (24a)$$

$$\mu_{R_{A_i \rightarrow B_i}}(x, y) = T^*(\mu_{B_i}(y), N(\mu_{A_i}(x))). \quad (24b)$$

Similar extensions can be made to other implication methods which may be found in [13]. It is up to the user to choose a particular implication method for a given decision process. A general representation of implication functions is defined by $f_{\rightarrow}(\cdot, \cdot)$. We have then,

$$\mu_{A_i \rightarrow B_i}(x, y) = f_{\rightarrow}(\mu_{A_i}(x), \mu_{B_i}(y)). \quad (25)$$

The overall fuzzy relation R is then given by

$$\mu_R(x, y) = \bigwedge_{i=1}^N (\mu_{R_{A_i \rightarrow B_i}}(x, y)). \quad (26)$$

Given an antecedent A' (control condition) and the fuzzy relation R (expert's knowledge), the consequent B' (control action) is inferred through the generalized modus ponens which is shown in Figure 2.

The consequent B' is calculated from the antecedent A' and the fuzzy relation R by the compositional rule of inferences as follows:

$$B' = A' \circ R, \quad (27a)$$

$$\mu_{B'}(y) = \sup_x T(\mu_{A'}(x), \mu_{R_{A_i \rightarrow B_i}}(x, y)). \quad (27b)$$

Based on the Equations (25), (26) and (27b), a generalized fuzzy reasoning algorithm is given by

$$\mu_{B'}(y) = \sup_x T\left(\mu_{A'}(x), \bigwedge_{i=1}^N (f_{\rightarrow}(\mu_{A_i}(x), \mu_{B_i}(y)))\right). \quad (28)$$

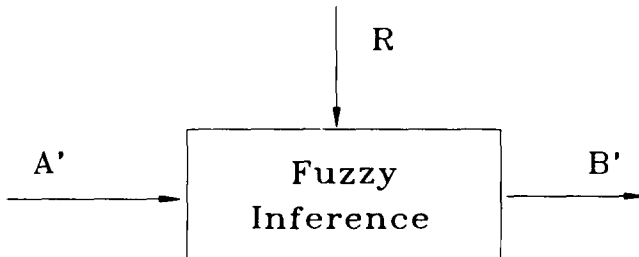


Fig. 2. Fuzzy inference.

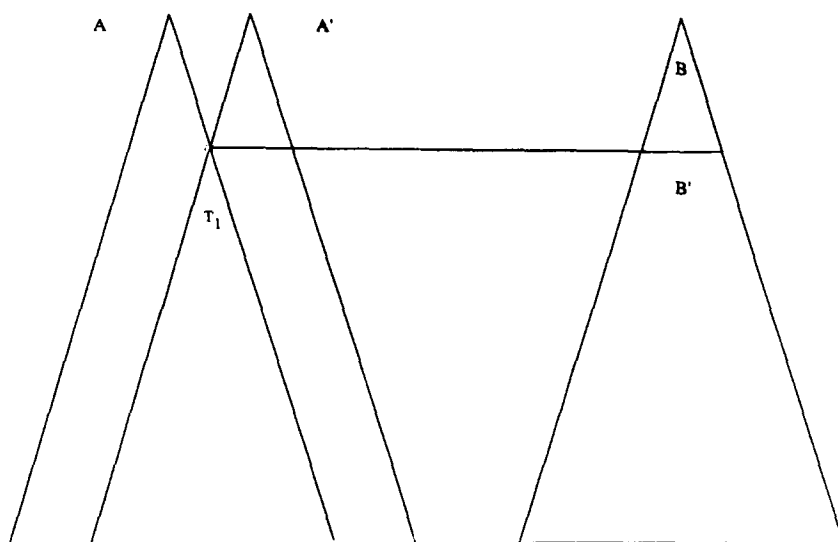


Fig. 3. Fuzzy inference methods.

If $N = 1$ and $f_{\rightarrow}(\cdot, \cdot) = (22)$, then (28) can be further simplified as

$$\mu_{B'}(y) = T(\alpha, \mu_B(y)) \quad (29)$$

where $\alpha = \sup_x T(\mu_{A'}(x), \mu_{A_1}(x))$.

If A' and A_1 are finite fuzzy sets, then we have

$$\alpha = \bigvee_x T(\mu_{A'}(x), \mu_{A_1}(x)).$$

Suppose that $A(=A_1)$, A' , and B are triangular fuzzy numbers which are usually the cases for fuzzy logic controllers. Substituting T by T_1 to T_5 in (29), five different types of fuzzy reasoning methods are obtained which are illustrated in Figure 3.

6. Conclusions

The T-operators presented in this paper are flexible tools for designing fuzzy logic controllers. More generally, these T-operators can be used for modelling decision-making processes where Zadeh's MIN and MAX operators are commonly applied. The broad range of T-operators that are available will enable designers to select the best one for their particular applications. The general fuzzy reasoning method discussed above is only one of the many possible approaches. Other implication functions and other operators can also be employed to produce similar methods in fuzzy reasoning. At present, further research is underway towards implementing this fuzzy reasoning method in control systems applications.

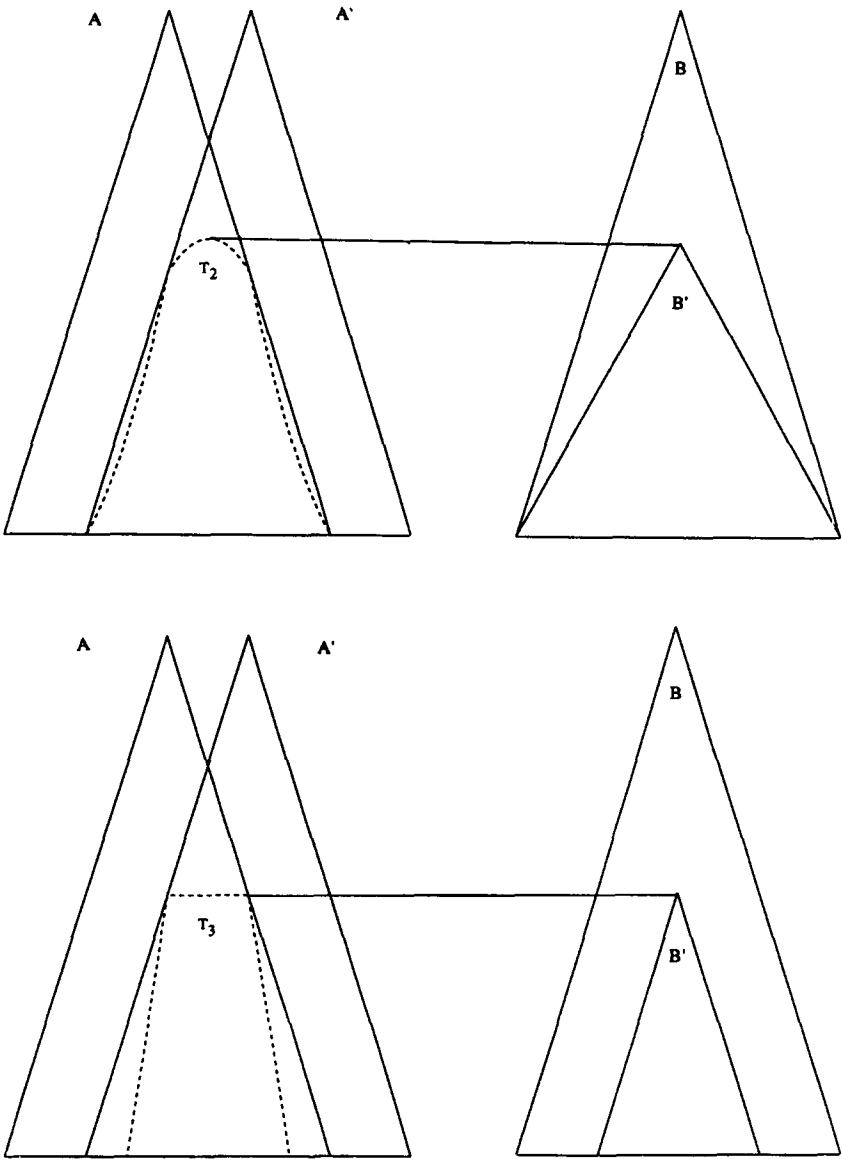


Fig. 3 (continued).

Appendix

In the following, two major methods of generating T-norms and T-conorms are given [7, 8]. The difference between the two methods is that the second method generates a new T-norm (or T-conorm) based on a given T-norm (or T-conorm). Some other methods may be found in [8].

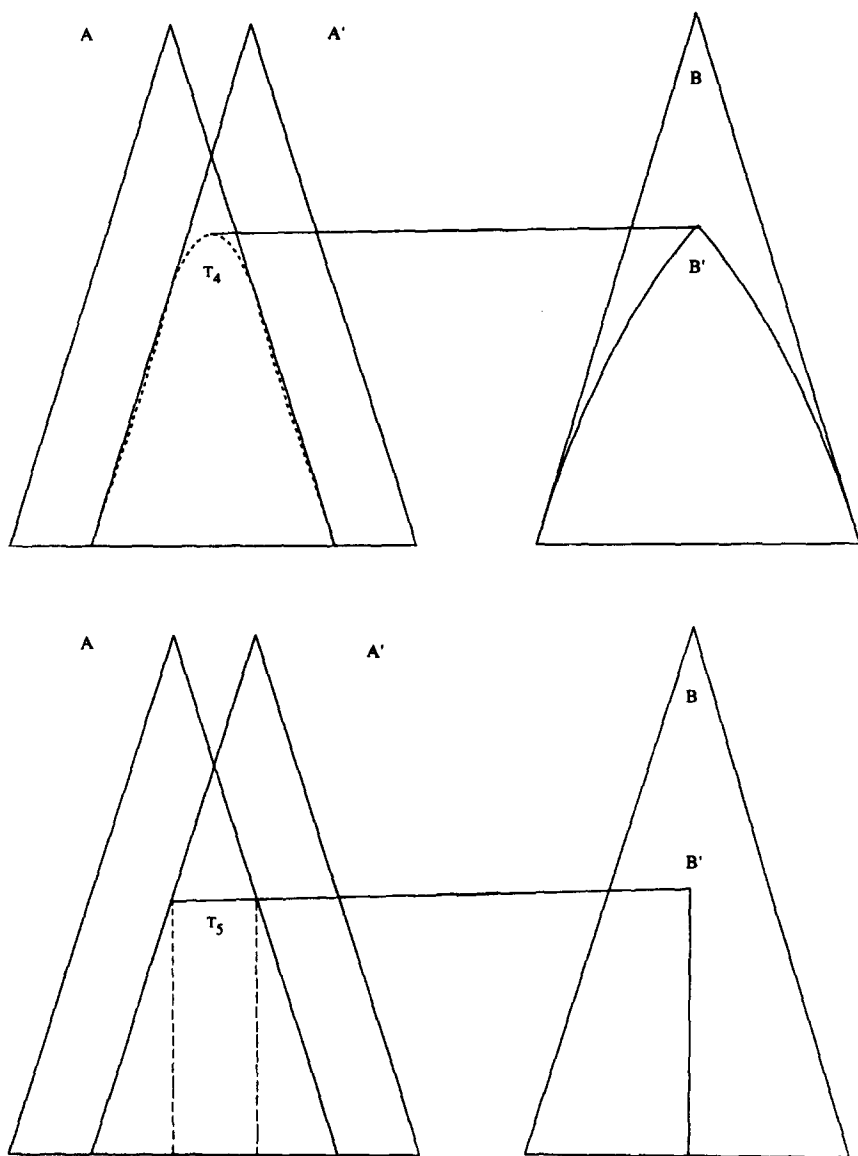


Fig. 3 (continued).

Method 1. Let $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$. If there exists a decreasing and continuous function $f: [0, 1] \rightarrow [0, \infty]$ with $f(1) = 0$, then

$$T(x, y) = f^{(-1)}(f(x) + f(y)), \quad x, y \in [0, 1],$$

is a T-norm, and $f^{(-1)}$ is the pseudo-inverse of f , and is defined by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{for } x \in [0, f(0)], \\ 0 & \text{for } x \in [f(0), \infty]. \end{cases}$$

Note that T is an Archimedean T-norm and if $f(0) \rightarrow \infty$, T is strict.

Let $T^*: [0, 1] \times [0, 1] \rightarrow [0, 1]$. If there exists an increasing and continuous function $g: [0, 1] \rightarrow [0, \infty]$ with $g(0) = 0$, then

$$T^*(x, y) = g^{(-1)}(g(x) + g(y)), \quad x, y \in [0, 1],$$

is a T-conorm, and where $g^{(-1)}$ is the pseudo inverse of g , and is defined by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{for } x \in [0, g(1)], \\ 1 & \text{for } x \in [g(1), \infty]. \end{cases}$$

Similarly, T^* is an Archimedean T-conorm and if $g(1) \rightarrow \infty$, T^* is strict.

Two examples are given in the following:

Example 1.

$$f(x) = \left(\frac{1}{x} - 1\right)^\lambda, \quad g(x) = \left(\frac{1}{x} - 1\right)^{-\lambda}, \quad \lambda > 0 \quad \text{and} \quad x \in [0, 1],$$

$$f^{(-1)}(x) = \frac{1}{1 + x^{1/\lambda}}, \quad g^{(-1)}(x) = \frac{1}{1 + x^{-1/\lambda}},$$

$$\begin{aligned} T(x, y) &= f^{(-1)}\left(\left(\frac{1}{x} - 1\right)^\lambda + \left(\frac{1}{y} - 1\right)^\lambda\right) \\ &= \frac{1}{1 + \left(\left(\frac{1}{x} - 1\right)^\lambda + \left(\frac{1}{y} - 1\right)^\lambda\right)^{-1/\lambda}}. \end{aligned}$$

Similarly,

$$T^*(x, y) = \frac{1}{1 + \left(\left(\frac{1}{x} - 1\right)^{-\lambda} + \left(\frac{1}{y} - 1\right)^{-\lambda}\right)^{-1/\lambda}}.$$

They are T_8 and T_8^* in Table 1. Because $f(0) \rightarrow \infty$ and $g(1) \rightarrow \infty$, they are strict Archimedean T-norm and T-conorm.

Example 2.

$$f(x) = 1 - x, \quad g(x) = x, \quad x \in [0, 1],$$

$$f^{(-1)}(x) = \begin{cases} 1 - x & \text{for } x \in [0, 1], \\ 0 & \text{for } x \in [1, \infty], \end{cases} \quad g^{(-1)}(x) = \begin{cases} x & \text{for } x \in [0, 1], \\ 1 & \text{for } x \in [1, \infty], \end{cases}$$

$$\begin{aligned} T(x, y) &= f^{(-1)}(f(x) + f(y)) = f^{(-1)}(2 - x - y) \\ &= \begin{cases} x + y - 1, & x + y - 1 \geq 0, \\ 0, & x + y - 1 \leq 0 \end{cases} \\ &= \max(x + y - 1, 0). \end{aligned}$$

Similarly,

$$T^*(x, y) = \min(x + y, 0).$$

They are T_3 and T_3^* . Because $f(0) = 1$ and $g(1) = 1$, they are non-strict Archimedean T-norm and T-conorm.

Method 2. Let $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$. If T' is a T-norm and $f(x)$ is strictly monotonic in a segment of R with $f(1) = 1$, then

$$T(x, y) = f^{-1}(T'(f(x), f(y)))$$

is a T-norm

Let $T^*: [0, 1] \times [0, 1] \rightarrow [0, 1]$. If T'^* is a T-conorm and $g(x)$ is strictly monotonic in a segment of R with $g(0) = 0$, then

$$T^*(x, y) = g^{-1}(T'^*(g(x), g(y)))$$

is a T-conorm.

Two examples are given:

Example 3.

$$T'(x, y) = xy, \quad f(x) = \frac{1}{x}, \quad f^{-1}(x) = \frac{1}{x}.$$

$$T(x, y) = f^{-1}\left(\frac{1}{x} \cdot \frac{1}{y}\right) = \frac{1}{\frac{1}{x} \cdot \frac{1}{y}} = xy.$$

This is T_2 and it generates itself.

Example 4.

$$T'^* = x + y - xy, \quad g(x) = x^2, \quad x \in [0, 1],$$

$$g^{-1}(x) = x^{1/2},$$

$$\begin{aligned} T^*(x, y) &= g^{-1}(x^2 + y^2 - x^2y^2) \\ &= (x^2 + y^2 - x^2y^2)^{1/2}. \end{aligned}$$

This is a T-conorm.

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