WHAT IS A BASIS (OR ORDERED BASIS) GOOD FOR?

1. Small vs Huge Vector spaces

If V is a vector space that is not "too big" we can find

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

that is a *basis for V*. The **definition** is that B is a finite set of vectors in V so that

(1) the span of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ is V, and
(2) the $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ are linearly independent.

So to show vectors form a basis for some vector space, you show:

(1) every vector in the space equals some linear combination of the supposed basis vectors.
(2) the only linear combination of the supposed basis vectors that equals zero is the one with all zero scalars.

Buy why would you even want a basis?

First of all, really big vector spaces, like $C[0,1]$ don't have a basis. The ones that have a basis are called *finite dimensional* and certainly that sounds like something that will make life easier.

The real point is that once you have a basis for V, you can turn all questions about V into questions about $\mathbb{R}^n$. A basis will get you out of the abstract and back to dealing with columns of numbers.

2. A Basis needs an Order

A basis on its own is a set of n vectors, and really we want to fix an order for the vectors. That way, we are not dealing with

$$2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 5\begin{bmatrix}1\\0\\0\\2\end{bmatrix}$$

one second and the next staring at

$$9\begin{bmatrix}1\\0\\0\\2\end{bmatrix} + 11\begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

That's like working with a system

$$\begin{aligned} x - y &= 1 \\ y + 2x &= 2 \end{aligned}$$

and wondering why you are making errors trying to add the equations.

An ordered basis is just a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ plus an agreement that $\mathbf{b}_1$ is to come first in formulas, and so on. Leon uses the notation

$$\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$$

(If you know about ordered pairs and ordered n-tuples, you have to wonder.)

So you've checked that a set spans of V, you've checked it has linear independence, and you've agreed on an order in which the $\mathbf{b}_k$ should be appear. Congratulation, you have an ordered basis.

Here's why you care:

Theorem 1. *If $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$ is an ordered basis for V, then:*

(1) *given any abstract vector $\mathbf{v}$ in V, there is a unique list of n scalars β_k to solve*

$$\mathbf{v} = \beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2 + \cdots + \beta_n\mathbf{b}_n.$$

We call

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{B}$.

(2) *Every "linear algebra" question about vectors in V can be settled by asking the same question about the corresponding coordinate vectors in $\mathbb{R}^n$.*

I need to be vague, not defining "linear algebra questions," but basically this theorem is telling you:

(1) A finite dimensional vector space V acts just like $\mathbb{R}^n$,
(2) Any ordered basis you find gives you a coordinate system for V. A coordinate system gives you a way to translate back and forth between V and $\mathbb{R}^n$.

Most books prove the following, but students seem to miss the significance:

Theorem 2. *If a vector space V has a basis with n vectors, then any other basis of V will have n vectors. The* dimension of V *is n.*

If V behaves like $\mathbb{R}^3$, it cannot also behave like $\mathbb{R}^2$. If I report to my boss there are "three degrees of freedom" in the solution to a problem, no new hire is going to correctly tell the boss "I've found a way to get four degrees of freedom."

3. One telling example (I hope)

You've noticed that solving

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = r\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + s\begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}$$

is not so different from solving

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \end{bmatrix},$$

and not so different from solving

$$x^3 + 2x^2 + 3x + 4 = r\left(x^3 + 2x^2 + x + 2\right) + s\,(4x + 4)\,.$$

The point is that the ordered basis

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

gives coordinates to the 2-by-2 matrices so that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

has coordinate vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Likewise, the ordered basis

$$\left[x^3, x^2, x^1, x^0\right]$$

gives coordinates to $\mathcal{P}_3$ so that

$$x^3 + 2x^2 + 3x + 4$$

has coordinate vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$