Lecture 9: Three-Dimensional Geometry

AER1513: State Estimation

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Outline

Lecture 9: Three-Dimensional Geometry
- Motivation
- Rotations
- Poses
- Sensor Models
- Matrix Lie Groups

Motivation

- the estimation tools we have discussed so far are quite generic
- they assume only that the state to be estimated is a vector
- in aerospace and robotics, we typically want to estimate the pose (i.e., position and orientation) of a vehicle operating in three-dimensional space
- unfortunately, pose variables are not vectors, so we need to be careful
- this lecture reviews basic three-dimensional geometry and introduces the concept of matrix Lie groups

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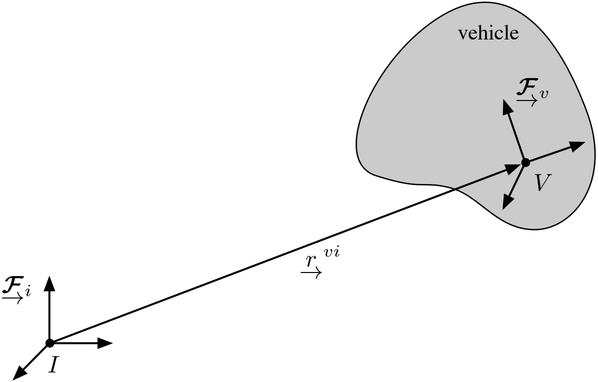
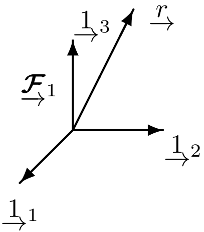
Pose



- the pose of one reference frame with respect to another has six degrees of freedom:
  - three in translation
  - three in rotation
- before we can estimate a pose, we first need some tools

Vectors, reference frames, coordinates



$$\underbrace{\underset{\rightarrow}{r}}_{\text{vector}} = r_1 \underset{\rightarrow}{1}_1 + r_2 \underset{\rightarrow}{1}_2 + r_3 \underset{\rightarrow}{1}_3 = \underbrace{\begin{bmatrix} \underset{\rightarrow}{1}_1 & \underset{\rightarrow}{1}_2 & \underset{\rightarrow}{1}_3 \end{bmatrix}}_{\text{frame}} \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}}_{\text{coords}} = \underset{\rightarrow}{\mathcal{F}}_1^T \mathbf{r} \quad (1)$$

- vectors are quantities that have magnitude and direction, independent of reference frame
- coordinates are the projection of a vector onto the (orthonormal) axes making up a reference frame

Dot product, cross product

- the dot product of two vectors is

$$\underset{\rightarrow}{r} \cdot \underset{\rightarrow}{s} = \mathbf{r}^T \mathbf{s} = r_1 s_1 + r_2 s_2 + r_3 s_3 \tag{2}$$

where the coordinates are in a common frame

- the cross product of two vectors is

$$\underset{\rightarrow}{r} \times \underset{\rightarrow}{s} = \begin{bmatrix} \underset{\rightarrow}{1}_1 & \underset{\rightarrow}{1}_2 & \underset{\rightarrow}{1}_3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}}_{\mathbf{r}^\times} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1^T \mathbf{r}^\times \mathbf{s} \tag{3}$$

where again the coordinates are in a common frame

Skew-symmetric operator

- the skew-symmetric operator is useful for implementing the cross product with coordinates:

$$\mathbf{r}^\times = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{4}$$

- it has some nice properties

$$(\mathbf{r}^\times)^T = -\mathbf{r}^\times, \quad \mathbf{r}^\times\mathbf{r} = \mathbf{0}, \quad \mathbf{r}^\times\mathbf{s} = -\mathbf{s}^\times\mathbf{r} \tag{5}$$

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Rotations

- express a vector using coordinates in two different frames:

$$\underrightarrow{r} = \underrightarrow{\boldsymbol{\mathcal{F}}}_1^T \mathbf{r}_1 = \underrightarrow{\boldsymbol{\mathcal{F}}}_2^T \mathbf{r}_2$$

- the two sets of coordinates are related by a 3×3 rotation matrix:

$$\mathbf{r}_2 = \mathbf{C}_{21} \mathbf{r}_1 \tag{6}$$

- rotation matrices have 9 parameters but only 3 degrees of freedom: 6 constraints come from the fact that they are orthonormal (and have determinant 1)

$$\mathbf{C}^T \mathbf{C} = \mathbf{1} \quad \Rightarrow \quad \mathbf{C}^T = \mathbf{C}^{-1}, \qquad \det \mathbf{C} = 1 \tag{7}$$

- rotations can be compounded, but order matters:

$$\mathbf{C}_{31} = \mathbf{C}_{32} \mathbf{C}_{21} \neq \mathbf{C}_{21} \mathbf{C}_{32} \tag{8}$$

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Principal rotations

– principal rotations are those about one of the coordinate axes
– the associated rotation matrices are

$$\mathbf{C}_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{9}$$

$$\mathbf{C}_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{10}$$

$$\mathbf{C}_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

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Euler angles

- we can compound 3 principal rotations to build any rotation matrix
- for example using a 3-1-3 sequence,

$$\mathbf{C}(\theta, \gamma, \psi) = \mathbf{C}_3(\theta)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi) \tag{12}$$

- the angles, (θ, γ, ψ), are called an Euler sequence or Euler angles
- there are 12 unique Euler sequences that can be built from the 3 principal rotations
- all Euler sequences have singularities: it is not always possible to uniquely work out the angles given the rotation matrix
- for example, for 3-1-3 when the middle angle is zero the other two rotations are about the 3 axis:

$$\mathbf{C}(\theta, 0, \psi) = \mathbf{C}_3(\theta)\mathbf{C}_1(0)\mathbf{C}_3(\psi) = \mathbf{C}_3(\theta + \psi) \tag{13}$$

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Between a rock and a hard place

- rotations have 3 degrees of freedom
- we have seen that 3×3 rotation matrices have 9 parameters and 6 constraints: $\mathbf{C}^T\mathbf{C} = \mathbf{1}$
- we have seen that Euler sequences have exactly 3 parameters but suffer from singularities
- it turns out this dilemma is much more general:

> There is no representation of rotations that has exactly 3 parameters (and therefore no constraints) and is also free of singularities.

- this has big implications, including for state estimation: we have to choose between singularities and constraints

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Some other common rotational representations

- axis-angle: $\mathbf{a}$ (unit-length axis) and ϕ (angle), 4 parameters and 1 constraint ($\mathbf{a}^T\mathbf{a} = 1$):

$$\mathbf{C} = \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times \tag{14}$$

- unit-length quaternions: $\boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{\phi}{2}$ and $\eta = \cos\frac{\phi}{2}$, 4 parameters and 1 constraint ($\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} + \eta^2 = 1$):

$$\mathbf{C} = (\eta^2 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - 2\eta\boldsymbol{\varepsilon}^\times \tag{15}$$

- Gibbs vector: $\mathbf{g} = \mathbf{a}\tan\frac{\phi}{2}$, 3 parameters but a singularity at $\phi = \pi$:

$$\mathbf{C} = \left(\mathbf{1} + \mathbf{g}^\times\right)^{-1}\left(\mathbf{1} - \mathbf{g}^\times\right) \tag{16}$$

Saving grace

- when rotations become 'small', all the Euler sequences look alike
- an infinitesimal rotation is

$$\mathbf{C} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} = \mathbf{1} - \boldsymbol{\theta}^\times \tag{17}$$

  where

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \phi\mathbf{a}$$

  is referred to as a rotation vector
- the intuition behind this is that when rotations are very small, we can start to think of them more like vectors – this will be the key to making state estimation work for these quantities

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Rotational kinematics

- derivatives are not the same in different reference frames
- we can relate the derivative of a vector expressed in one frame rotating with respect to another via

$$\dot{\mathbf{r}}_1 = \mathbf{C}_{12}\left(\dot{\mathbf{r}}_2 + \boldsymbol{\omega}_2^{21^\times}\mathbf{r}_2\right) \tag{18}$$

  where $\boldsymbol{\omega}_2^{21^\times}$ is the angular velocity of frame 2 with respect to frame 1, expressed in frame 2
- the relationship between a rotation matrix and angular velocity is the rotational kinematics:

$$\dot{\mathbf{C}}_{21} = -\boldsymbol{\omega}_2^{21^\times}\mathbf{C}_{21} \tag{19}$$

  which is called Poisson's equation

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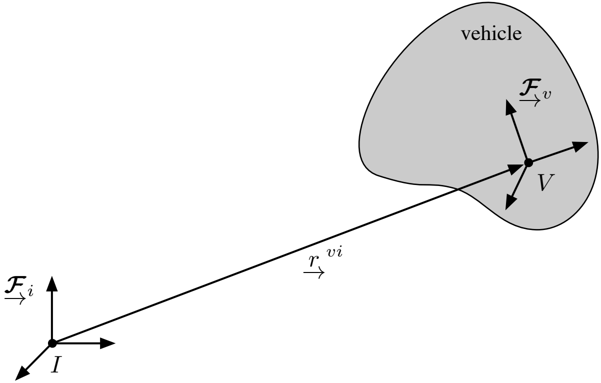
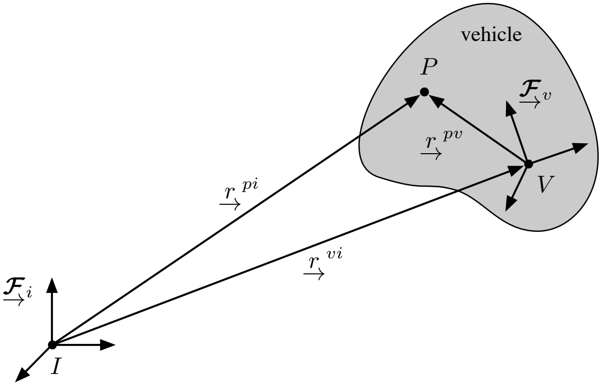
Pose



- the pose of one reference frame with respect to another has six degrees of freedom:
  - three in translation: $\mathbf{r}_i^{vi}$
  - three in rotation: $\mathbf{C}_{iv}$

Transforming points



– if we know the pose, $\left\{\mathbf{r}_i^{vi}, \mathbf{C}_{iv}\right\}$, we can transform the coordinates of a point, P, from one frame to another:

$$\mathbf{r}_i^{pi} = \mathbf{C}_{iv}\mathbf{r}_v^{pv} + \mathbf{r}_i^{vi} \tag{20}$$

Transformation matrices

- we can combine the translation and rotation of a pose into a more convenient form called the (4×4) transformation matrix:

$$\begin{bmatrix} \mathbf{r}_i^{pi} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix}}_{\mathbf{T}_{iv}} \begin{bmatrix} \mathbf{r}_v^{pv} \\ 1 \end{bmatrix} \tag{21}$$

- we see that this allows us to easily transform points from one frame to another in so-called (4×1) homogenous point representation:

$$\begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{22}$$

which just has an extra 1 at the bottom

Transformation matrices

– to transform the coordinates back the other way, we require the inverse of a transformation matrix:

$$\begin{bmatrix} \mathbf{r}_v^{pv} \\ 1 \end{bmatrix} = \mathbf{T}_{iv}^{-1} \begin{bmatrix} \mathbf{r}_i^{pi} \\ 1 \end{bmatrix} \tag{23}$$

where

$$\begin{aligned} \mathbf{T}_{iv}^{-1} = \begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{C}_{iv}^T & -\mathbf{C}_{iv}^T \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{vi} & -\mathbf{r}_v^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ = \begin{bmatrix} \mathbf{C}_{vi} & \mathbf{r}_v^{iv} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{T}_{vi} \end{aligned} \tag{24}$$

and we have used that $\mathbf{r}_v^{iv} = -\mathbf{r}_v^{vi}$, which simply flips the direction of the vector

Transformation matrices

- transformation matrices are 4×4 and always have this special structure:

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{25}$$

- they have 16 parameters but only 6 degrees of freedom and therefore must have 10 constraints
- 6 constraints come from $\mathbf{C}^T\mathbf{C} = \mathbf{1}$ and the other 4 come from the fact that the bottom row is always $(0, 0, 0, 1)$
- we can compound transformation matrices (just like rotation matrices):

$$\mathbf{T}_{iv} = \mathbf{T}_{ia}\mathbf{T}_{ab}\mathbf{T}_{bv} \tag{26}$$

  and the structure always holds (more on this later); just like rotations, order matters

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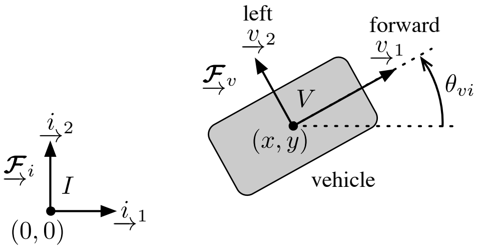
A word on conventions



- consider a planar 'robot' whose position is $\mathbf{r}_i^{vi} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- the heading (i.e., orientation, rotation) is usually taken to be θ_{vi}
- often the pose is then just written as (x, y, θ_{vi})

A word on conventions

– if we want to express the pose, (x, y, θ_{vi}), using a transformation matrix, we might then write

$$\mathbf{T}_{iv} = \begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_{vi} & -\sin\theta_{vi} & 0 & x \\ \sin\theta_{vi} & \cos\theta_{vi} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

which is fine

– however, it is easy to get confused because

$$\mathbf{C}_{iv} = \mathbf{C}_3(-\theta_{vi}) \tag{28}$$

where we need the negative of θ_{vi} since it's θ_{iv} that normally is associated with $\mathbf{C}_{iv}$

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A word on conventions

- imagine someone tells you, “the angle of the rotation is θ”
- from the $\mathbf{T}_{iv}$ perspective, that could mean θ_{iv}
- from the (x, y, θ) perspective, that could mean θ_{vi}
- this is a really common source of error

Be very careful to understand the sign of the angle of rotation!

- also, if someones just gives you $\mathbf{T}$ that could mean $\mathbf{T}_{iv}$ or $\mathbf{T}_{vi}$ so you better really understand which one it is

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Pose kinematics

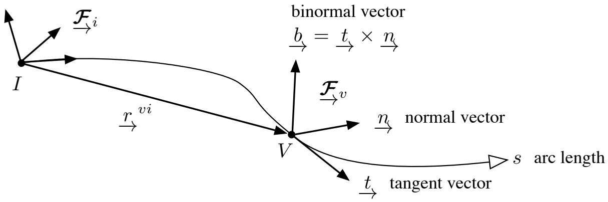
– similar to the rotational kinematics, pose kinematics can be written as

$$\dot{\mathbf{T}}_{vi} = \begin{bmatrix} -\boldsymbol{\omega}_v^{vi^\times} & -\boldsymbol{\nu}_v^{vi} \\ \mathbf{0}^T & 0 \end{bmatrix} \mathbf{T}_{vi} \tag{29}$$

where

$$\boldsymbol{\varpi}_v^{vi} = \begin{bmatrix} \boldsymbol{\nu}_v^{vi} \\ \boldsymbol{\omega}_v^{vi} \end{bmatrix} \tag{30}$$

is a generalized six-degree-of-freedom velocity vector (expressed in the moving frame)



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Pose kinematics

– constraining the velocity vector and initial pose to be

$$\boldsymbol{\varpi}_v^{vi} = \begin{bmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega \end{bmatrix}, \quad \mathbf{T}_{vi}(0) = \begin{bmatrix} \cos\theta_{vi}(0) & -\sin\theta_{vi}(0) & 0 & x(0) \\ \sin\theta_{vi}(0) & \cos\theta_{vi}(0) & 0 & y(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \tag{31}$$

the kinematics collapse to the planar case

$$\dot{x} = v\cos\theta \tag{32a}$$

$$\dot{y} = v\sin\theta \tag{32b}$$

$$\dot{\theta} = \omega \tag{32c}$$

where $\theta = \theta_{vi}$; this is sometimes called the unicycle model

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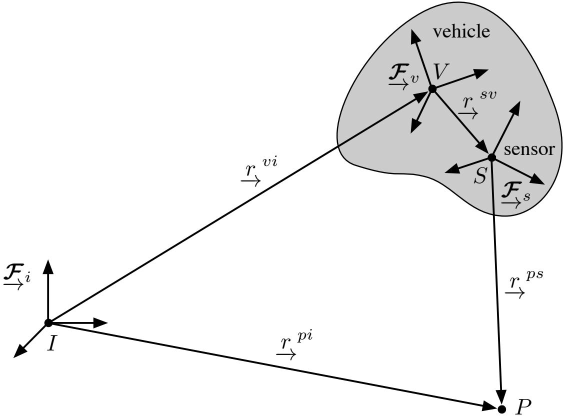
Sensors



- things get more complicated when we add a sensor to our vehicle
- often the sensor frame is not the same as the vehicle frame
- the offset, $\mathbf{T}_{sv}$, must be determined in advance by a calibration technique (or folded into the state estimation)

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Sensors

- we will cover a few common generative models of sensors
- in other words, we want a mathematical model of our sensor that allows us to compute what the sensor 'sees', given the state of the world (including the pose of our vehicle)
- some common sensor types are
  - perspective camera
  - stereo camera
  - range-azimuth-elevation (e.g., lidar)
- we will assume that point P has already been expressed in the sensor frame as $\boldsymbol{\rho}$ (by transforming it from the world frame using the vehicle pose)

$$\begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{T}_{sv} \mathbf{T}_{vi} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} \tag{33}$$

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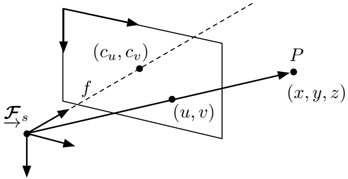
Perspective camera



- the perspective camera model can be written as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{s}(\boldsymbol{\rho}) = \mathbf{P}\,\mathbf{K}\,\frac{1}{z}\boldsymbol{\rho} \tag{34}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} f_u & 0 & c_u \\ 0 & f_v & c_v \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{35}$$

- what we get from the sensor are the pixel coordinates, (u, v)

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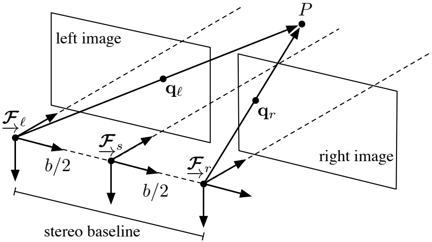
Stereo camera



– the stereo camera model is

$$\begin{bmatrix} u_\ell \\ v_\ell \\ u_r \\ v_r \end{bmatrix} = \mathbf{s}(\boldsymbol{\rho}) = \underbrace{\begin{bmatrix} f_u & 0 & c_u & f_u \frac{b}{2} \\ 0 & f_v & c_v & 0 \\ f_u & 0 & c_u & -f_u \frac{b}{2} \\ 0 & f_v & c_v & 0 \end{bmatrix}}_{\mathbf{M}} \frac{1}{z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{36}$$

where $\mathbf{M}$ is a now a combined parameter matrix for the stereo rig

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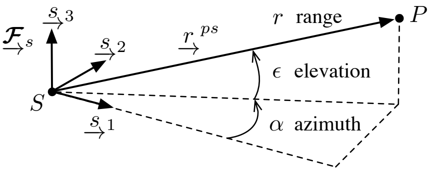
Range-azimuth-elevation



- the range-azimuth-elevation (RAE) sensor model is

$$\begin{bmatrix} r \\ \alpha \\ \epsilon \end{bmatrix} = \mathbf{s}(\boldsymbol{\rho}) = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ \tan^{-1}(y/x) \\ \sin^{-1}\left(z/\sqrt{x^2 + y^2 + z^2}\right) \end{bmatrix} \tag{37}$$

- this collapses to the range-bearing model when $z = 0$

$$\begin{bmatrix} r \\ \alpha \end{bmatrix} = \mathbf{s}(\boldsymbol{\rho}) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}(y/x) \end{bmatrix} \tag{38}$$

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Matrix Lie groups

- we have seen the role of rotation and transformation matrices in our kinematic (i.e., motion) and sensor (i.e., observation) models
- we have also learned that rotations don't behave like vectors, yet all of our estimation tools assume the state is a vector
- poses have similar issues since rotation is embedded
- it turns out that the sets of rotations and poses are not vectorspaces, but another type of mathematical object called matrix Lie groups
- we will use the rest of this lecture to learn about this in the hope that it will guide us in the estimation of rotations and poses

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Special orthogonal group

- the set of rotations is called the special orthogonal group:

$$SO(3) = \left\{ \mathbf{C} \in \mathbb{R}^{3\times3} | \mathbf{C}\mathbf{C}^T = \mathbf{1}, \det \mathbf{C} = 1 \right\} \tag{39}$$

- the $\mathbf{C}\mathbf{C}^T = \mathbf{1}$ orthogonality condition is needed to impose 6 constraints on the 9-parameter rotation matrix, reducing the degrees of freedom to 3
- noticing that

$$(\det \mathbf{C})^2 = \det \left(\mathbf{C}\mathbf{C}^T \right) = \det \mathbf{1} = 1 \tag{40}$$

  we have that $\det \mathbf{C} = \pm 1$, allowing for two possibilities
- choosing $\det \mathbf{C} = 1$ ensures that we have a proper rotation
- the other case, $\det \mathbf{C} = -1$, corresponds to a rotary reflection

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Special Euclidean group

- the set of transformation matrices representing poses is called the special Euclidean group:

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \mathbf{C} \in SO(3),\ \mathbf{r} \in \mathbb{R}^3 \right\} \tag{41}$$

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Matrix Lie groups

- both $SO(3)$ and $SE(3)$ are matrix Lie groups
- to be a group they must have an operator to combine elements that satisfies 4 properties: closure, associativity, identity, invertibility
- to be a Lie group the operator must be 'smooth'
- to be a matrix Lie group the elements must be matrices and the operator matrix multiplication

property	$SO(3)$	$SE(3)$
closure	$\mathbf{C}_1, \mathbf{C}_2 \in SO(3) \Rightarrow \mathbf{C}_1\mathbf{C}_2 \in SO(3)$	$\mathbf{T}_1, \mathbf{T}_2 \in SE(3) \Rightarrow \mathbf{T}_1\mathbf{T}_2 \in SE(3)$
associativity	$\mathbf{C}_1(\mathbf{C}_2\mathbf{C}_3) = (\mathbf{C}_1\mathbf{C}_2)\mathbf{C}_3 = \mathbf{C}_1\mathbf{C}_2\mathbf{C}_3$	$\mathbf{T}_1(\mathbf{T}_2\mathbf{T}_3) = (\mathbf{T}_1\mathbf{T}_2)\mathbf{T}_3 = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$
identity	$\mathbf{C}, \mathbf{1} \in SO(3) \Rightarrow \mathbf{C}\mathbf{1} = \mathbf{1}\mathbf{C} = \mathbf{C}$	$\mathbf{T}, \mathbf{1} \in SE(3) \Rightarrow \mathbf{T}\mathbf{1} = \mathbf{1}\mathbf{T} = \mathbf{T}$
invertibility	$\mathbf{C} \in SO(3) \Rightarrow \mathbf{C}^{-1} \in SO(3)$	$\mathbf{T} \in SE(3) \Rightarrow \mathbf{T}^{-1} \in SE(3)$

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Example property proof

- we won't prove all of the properties but let's do one example: closure of $SO(3)$
- assume we have two rotation matrices that are elements of $SO(3)$:

$$\mathbf{C}_1, \mathbf{C}_2 \in SO(3) \tag{42}$$

- we need to show that the compounding is also in $SO(3)$:

$$\mathbf{C} = \mathbf{C}_1\mathbf{C}_2 \in SO(3) \quad \Leftrightarrow \quad \mathbf{C}\mathbf{C}^T = \mathbf{1}, \quad \det \mathbf{C} = 1 \tag{43}$$

- proof:

$$\mathbf{C}\mathbf{C}^T = (\mathbf{C}_1\mathbf{C}_2)(\mathbf{C}_1\mathbf{C}_2)^T = \mathbf{C}_1 \underbrace{\mathbf{C}_2\mathbf{C}_2^T}_{\mathbf{1}} \mathbf{C}_1^T = \underbrace{\mathbf{C}_1\mathbf{C}_1^T}_{\mathbf{1}} = \mathbf{1} \tag{44}$$

$$\det(\mathbf{C}) = \det(\mathbf{C}_1\mathbf{C}_2) = \underbrace{\det(\mathbf{C}_1)}_{1}\underbrace{\det(\mathbf{C}_2)}_{1} = 1 \tag{45}$$

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Lie algebras

– to every matrix Lie group there is associated a Lie algebra, which consists of a vectorspace, $\mathbb{V}$, over some field, $\mathbb{F}$, together with a binary operation, $[\cdot,\cdot]$, called the Lie bracket (of the algebra) that satisfies four properties:

$$
\begin{aligned}
\text{closure:} \quad & [\mathbf{X},\mathbf{Y}] \in \mathbb{V} \\
\text{bilinearity:} \quad & [a\mathbf{X}+b\mathbf{Y},\mathbf{Z}] = a[\mathbf{X},\mathbf{Z}] + b[\mathbf{Y},\mathbf{Z}], \\
& \qquad [\mathbf{Z},a\mathbf{X}+b\mathbf{Y}] = a[\mathbf{Z},\mathbf{X}] + b[\mathbf{Z},\mathbf{Y}] \\
\text{alternating:} \quad & [\mathbf{X},\mathbf{X}] = \mathbf{0} \\
\text{Jacobi identity:} \quad & [\mathbf{X},[\mathbf{Y},\mathbf{Z}]] + [\mathbf{Z},[\mathbf{Y},\mathbf{X}]] + [\mathbf{Y},[\mathbf{Z},\mathbf{X}]] = \mathbf{0}
\end{aligned}
$$

for all $\mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathbb{V}$ and $a,b \in \mathbb{F}$

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Lie algebra: rotations

- the Lie algebra associated with $SO(3)$ is given by

$$\begin{aligned}\text{vectorspace:} \quad & \mathfrak{so}(3) = \left\{ \mathbf{\Phi} = \boldsymbol{\phi}^\wedge \in \mathbb{R}^{3\times 3} | \boldsymbol{\phi} \in \mathbb{R}^3, \right\} \\ \text{field:} \quad & \mathbb{R} \\ \text{Lie bracket:} \quad & [\mathbf{\Phi}_1, \mathbf{\Phi}_2] = \mathbf{\Phi}_1\mathbf{\Phi}_2 - \mathbf{\Phi}_2\mathbf{\Phi}_1\end{aligned}$$

  where

$$\boldsymbol{\phi}^\wedge = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{\phi} \in \mathbb{R}^3 \qquad (46)$$

- we already saw this linear, skew-symmetric operator when we introduced cross products, only then we used the symbol $(\cdot)^\times$ instead of $(\cdot)^\wedge$

Lie algebra: poses

- the Lie algebra associated with $SE(3)$ is given by

$$\begin{aligned} \text{vectorspace:} \quad & \mathfrak{se}(3) = \left\{ \boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \in \mathbb{R}^{4\times 4} | \boldsymbol{\xi} \in \mathbb{R}^6 \right\} \\ \text{field:} \quad & \mathbb{R} \\ \text{Lie bracket:} \quad & [\boldsymbol{\Xi}_1, \boldsymbol{\Xi}_2] = \boldsymbol{\Xi}_1\boldsymbol{\Xi}_2 - \boldsymbol{\Xi}_2\boldsymbol{\Xi}_1 \end{aligned}$$

  where

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \quad \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3 \tag{47}$$

- this is an overloading of the $(\cdot)^\wedge$ operator from before to take elements of $\mathbb{R}^6$ and turn them into elements of $\mathbb{R}^{4\times 4}$; it is still linear

This is getting exponentially more complicated

- ok, so the sets of rotation and transformation matrices are matrix Lie groups: $SO(3)$ and $SE(3)$
- each one has an associated Lie algebra: $\mathfrak{so}(3)$ and $\mathfrak{se}(3)$
- so what, where is this all going?!
- to get to the next level of understanding, we need a connection between the Lie group and Lie algebra
- that connection is the exponential map
- the matrix exponential is given by

$$\exp(\mathbf{A}) = \mathbf{1} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \qquad (48)$$

where $\mathbf{A} \in \mathbb{R}^{M \times M}$ is a square matrix

Exponential rotations

- for rotations, we can relate elements of $SO(3)$ to elements of $\mathfrak{so}(3)$ through the exponential map:

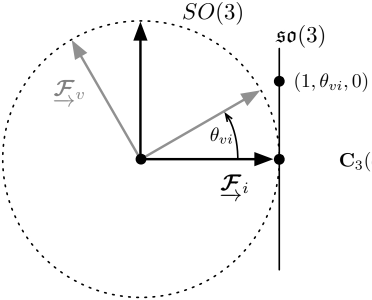
$$\mathbf{C} = \exp\left(\boldsymbol{\phi}^{\wedge}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\boldsymbol{\phi}^{\wedge}\right)^{n} \tag{49}$$

  where $\mathbf{C} \in SO(3)$ and $\boldsymbol{\phi} \in \mathbb{R}^3$ (and hence $\boldsymbol{\phi}^{\wedge} \in \mathfrak{so}(3)$)
- we can also go in the other direction (but not uniquely) using

$$\boldsymbol{\phi} = \ln\left(\mathbf{C}\right)^{\vee} \tag{50}$$

- the mapping is surjective (or onto), meaning every element of $SO(3)$ can be generated by at least one element of $\mathfrak{so}(3)$
- the non-unique inverse mapping is precisely the idea of singularities discussed earlier – in this case $\phi + 2\pi m$ with m any integer produces the same $\mathbf{C}$

Tangent space



$$\mathbf{C}_3(\theta_{vi})\,\mathbf{1}_1 = \exp\left(\begin{bmatrix} 0 \\ 0 \\ \theta_{vi} \end{bmatrix}^\wedge\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\approx \left(\mathbf{1} + \begin{bmatrix} 0 \\ 0 \\ \theta_{vi} \end{bmatrix}^\wedge\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \theta_{vi} \\ 0 \end{bmatrix}$$

– the vectorspace of a Lie algebra is the tangent space of the associated Lie group at the identity element of the group, and it completely captures the local structure of the group

Rotation forward mapping

- how do we actually compute a rotation matrix using the exponential map?

$$\mathbf{C} = \exp\left(\boldsymbol{\phi}^\wedge\right) \tag{51}$$

- it turns out $\boldsymbol{\phi} = \phi\mathbf{a}$ is the rotation vector we introduced earlier, where the word vector now makes sense since $\boldsymbol{\phi}$ is an element of the vectorspace of the Lie algebra
- we then have

$$\begin{aligned}
\exp(\boldsymbol{\phi}^\wedge) &= \exp(\phi\,\mathbf{a}^\wedge) \\
&= \underbrace{\mathbf{1}}_{\mathbf{a}\mathbf{a}^T - \mathbf{a}^\wedge\mathbf{a}^\wedge} + \phi\mathbf{a}^\wedge + \frac{1}{2!}\phi^2\mathbf{a}^\wedge\mathbf{a}^\wedge + \frac{1}{3!}\phi^3\underbrace{\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge}_{-\mathbf{a}^\wedge} + \frac{1}{4!}\phi^4\underbrace{\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge\mathbf{a}^\wedge}_{-\mathbf{a}^\wedge\mathbf{a}^\wedge} - \cdots \\
&= \mathbf{a}\mathbf{a}^T + \underbrace{\left(\phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \cdots\right)}_{\sin\phi}\mathbf{a}^\wedge - \underbrace{\left(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \cdots\right)}_{\cos\phi}\underbrace{\mathbf{a}^\wedge\mathbf{a}^\wedge}_{-\mathbf{1}+\mathbf{a}\mathbf{a}^T} \\
&= \underbrace{\cos\phi\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi\mathbf{a}^\wedge}_{\mathbf{C}}
\end{aligned} \tag{52}$$

Rotation inverse mapping

- we've seen that

$$\boldsymbol{\phi} = \phi \mathbf{a} = \ln\left(\mathbf{C}\right)^{\vee} \tag{53}$$

  is the inverse mapping from $SO(3)$ to $\mathfrak{so}(3)$, but how to do this?
- a rotation matrix applied to its own axis does not alter the axis,

$$\mathbf{C}\mathbf{a} = \mathbf{a} \tag{54}$$

  which implies that $\mathbf{a}$ is a (unit-length) eigenvector of $\mathbf{C}$
- the angle can be found by exploiting the trace of a rotation matrix:

$$\begin{aligned} \text{tr}(\mathbf{C}) &= \text{tr}\left(\cos\phi\,\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi\,\mathbf{a}^\wedge\right) \\ &= \cos\phi\,\underbrace{\text{tr}(\mathbf{1})}_{3} + (1-\cos\phi)\,\underbrace{\text{tr}\left(\mathbf{a}\mathbf{a}^T\right)}_{\mathbf{a}^T\mathbf{a}=1} + \sin\phi\,\underbrace{\text{tr}\left(\mathbf{a}^\wedge\right)}_{0} = 2\cos\phi + 1 \end{aligned} \tag{55}$$

- solving we have many solutions for ϕ:

$$\phi = \cos^{-1}\left(\frac{\text{tr}(\mathbf{C}) - 1}{2}\right) + 2\pi m \tag{56}$$

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Exponential poses

- for poses, we can relate elements of $SE(3)$ to elements of $\mathfrak{se}(3)$, again through the exponential map:

$$\mathbf{T} = \exp\left(\boldsymbol{\xi}^\wedge\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\boldsymbol{\xi}^\wedge\right)^n \tag{57}$$

  where $\mathbf{T} \in SE(3)$ and $\boldsymbol{\xi} \in \mathbb{R}^6$ (and hence $\boldsymbol{\xi}^\wedge \in \mathfrak{se}(3)$)
- we can also go in the other direction (again, not uniquely) using

$$\boldsymbol{\xi} = \ln\left(\mathbf{T}\right)^\vee \tag{58}$$

- the exponential map from $\mathfrak{se}(3)$ to $SE(3)$ is also surjective: every $\boldsymbol{\xi} \in \mathbb{R}^6$ maps to some $\mathbf{T} \in SE(3)$ (many-to-one) and every $\mathbf{T} \in SE(3)$ can be generated by at least one $\boldsymbol{\xi} \in \mathbb{R}^6$

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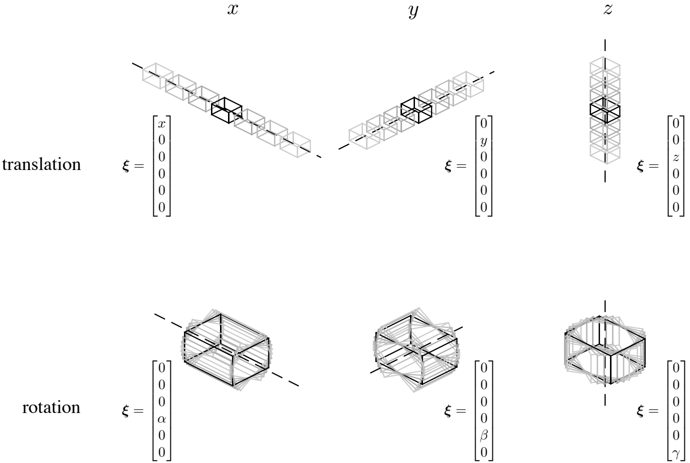
Pose change



– varying each component of $\boldsymbol{\xi}$ then using $\mathbf{T} = \exp\left(\boldsymbol{\xi}^\wedge\right)$ to transform the points comprising the corners of a rectangular prism

Pose forward mapping

– we can compute the exponential map for poses in closed form, too:

$$\begin{aligned}
\exp\left(\boldsymbol{\xi}^\wedge\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\boldsymbol{\xi}^\wedge\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\phi}\end{bmatrix}^\wedge\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix}\boldsymbol{\phi}^\wedge & \boldsymbol{\rho}\\ \mathbf{0}^T & 0\end{bmatrix}^n \\
&= \begin{bmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n & \left(\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\boldsymbol{\phi}^\wedge\right)^n\right)\boldsymbol{\rho}\\ \mathbf{0}^T & 1\end{bmatrix} \\
&= \underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{r}\\ \mathbf{0}^T & 1\end{bmatrix}}_{\mathbf{T}} \in SE(3)
\end{aligned} \tag{59}$$

where $\mathbf{r} = \mathbf{J}\boldsymbol{\rho} \in \mathbb{R}^3$ and

$$\mathbf{J} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\boldsymbol{\phi}^\wedge\right)^n = \frac{\sin\phi}{\phi}\mathbf{1} + \left(1 - \frac{\sin\phi}{\phi}\right)\mathbf{a}\mathbf{a}^T + \frac{1-\cos\phi}{\phi}\mathbf{a}^\wedge \tag{60}$$

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Pose inverse mapping

- we can also compute the inverse mapping from $\mathbf{T}$ to $\boldsymbol{\xi}$ in closed form
- starting from

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{61}$$

  we extra $\mathbf{C}$ and $\mathbf{r}$
- use $\mathbf{C}$ to compute $\boldsymbol{\phi}$ using the rotation inverse mapping (this step is not unique)
- use $\boldsymbol{\phi}$ to compute $\mathbf{J}$ from the last slide
- compute $\boldsymbol{\rho} = \mathbf{J}^{-1}\mathbf{r}$
- assemble

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix} \tag{62}$$

Adjoint poses

- there is a 6×6 version of the transformation matrix, $\boldsymbol{\mathcal{T}}$, that can be constructed directly from the components of the 4×4 transformation matrix so that $(\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge = \mathbf{T}\mathbf{x}^\wedge\mathbf{T}^{-1}$
- we call this the adjoint of an element of $SE(3)$:

$$\boldsymbol{\mathcal{T}} = \mathrm{Ad}(\mathbf{T}) = \mathrm{Ad}\left(\begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{C} & \mathbf{r}^\wedge\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \tag{63}$$

- we will abuse notation a bit and say that the set of adjoints of all the elements of $SE(3)$ is denoted

$$\mathrm{Ad}(SE(3)) = \{\boldsymbol{\mathcal{T}} = \mathrm{Ad}(\mathbf{T}) | \mathbf{T} \in SE(3)\} \tag{64}$$

- it turns out that $\mathrm{Ad}(SE(3))$ is also a matrix Lie group

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Adjoint poses

- we can also talk about the adjoint of an element of $\mathfrak{se}(3)$
- let $\boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \in \mathfrak{se}(3)$; then the adjoint of this element is

$$\text{ad}(\boldsymbol{\Xi}) = \text{ad}\left(\boldsymbol{\xi}^\wedge\right) = \boldsymbol{\xi}^\curlywedge \tag{65}$$

where

$$\boldsymbol{\xi}^\curlywedge = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}^\curlywedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho}^\wedge \\ \mathbf{0} & \boldsymbol{\phi}^\wedge \end{bmatrix} \in \mathbb{R}^{6\times 6}, \quad \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3 \tag{66}$$

- note that we have used uppercase, $\text{Ad}(\cdot)$, for the adjoint of $SE(3)$ and lowercase, $\text{ad}(\cdot)$, for the adjoint of $\mathfrak{se}(3)$

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Adjoint poses

- the Lie algebra associated with Ad$(SE(3))$ is given by

vectorspace:	$\text{ad}(\mathfrak{se}(3)) = \left\{ \boldsymbol{\Psi} = \text{ad}(\boldsymbol{\Xi}) \in \mathbb{R}^{6\times 6} \middle| \boldsymbol{\Xi} \in \mathfrak{se}(3), \right\}$
field:	$\mathbb{R}$
Lie bracket:	$[\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2] = \boldsymbol{\Psi}_1\boldsymbol{\Psi}_2 - \boldsymbol{\Psi}_2\boldsymbol{\Psi}_1$

- the relationship between Ad$(SE(3))$ and ad$(\mathfrak{se}(3))$ is again the exponential map:

$$\boldsymbol{\mathcal{T}} = \exp\left(\boldsymbol{\xi}^{\curlywedge}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\boldsymbol{\xi}^{\curlywedge}\right)^n \tag{67}$$

where $\boldsymbol{\mathcal{T}} \in$ Ad$(SE(3))$ and $\boldsymbol{\xi} \in \mathbb{R}^6$ (and hence $\boldsymbol{\xi}^{\curlywedge} \in$ ad$(\mathfrak{se}(3))$)

- we can go in the other direction using

$$\boldsymbol{\xi} = \ln\left(\boldsymbol{\mathcal{T}}\right)^{\curlyvee} \tag{68}$$

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Pose relationships

– there is a nice commutative relationship between the various Lie groups and algebras associated with poses:

$$
\begin{array}{ccccc}
 & \text{Lie algebra} & & \text{Lie group} \\
4 \times 4 & \boldsymbol{\xi}^\wedge \in \mathfrak{se}(3) & \xrightarrow{\exp} & \mathbf{T} \in SE(3) \\
 & \Big\downarrow \text{ad} & & \Big\downarrow \text{Ad} \\
6 \times 6 & \boldsymbol{\xi}^\curlywedge \in \text{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \boldsymbol{\mathcal{T}} \in \text{Ad}(SE(3))
\end{array}
$$

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Rotation identities

$SO(3)$ Identities and Approximations

Lie Algebra

$$
\begin{gathered}
\mathbf{u}^\wedge = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \\
(\alpha \mathbf{u} + \beta \mathbf{v})^\wedge \equiv \alpha \mathbf{u}^\wedge + \beta \mathbf{v}^\wedge \\
\mathbf{u}^{\wedge^T} \equiv -\mathbf{u}^\wedge \\
\mathbf{u}^\wedge \mathbf{v} \equiv -\mathbf{v}^\wedge \mathbf{u} \\
\mathbf{u}^\wedge \mathbf{u} \equiv \mathbf{0} \\
(\mathbf{W}\mathbf{u})^\wedge \equiv \mathbf{u}^\wedge \left(\operatorname{tr}(\mathbf{W})\,\mathbf{1} - \mathbf{W}\right) - \mathbf{W}^T \mathbf{u}^\wedge \\
\mathbf{u}^\wedge \mathbf{v}^\wedge \equiv -(\mathbf{u}^T \mathbf{v})\,\mathbf{1} + \mathbf{v}\mathbf{u}^T \\
\mathbf{u}^\wedge \mathbf{W}\,\mathbf{v}^\wedge \equiv -\left(-\operatorname{tr}(\mathbf{v}\mathbf{u}^T)\,\mathbf{1} + \mathbf{v}\mathbf{u}^T\right) \\
\times \left(-\operatorname{tr}(\mathbf{W})\,\mathbf{1} + \mathbf{W}^T\right) \\
+ \operatorname{tr}(\mathbf{W}^T \mathbf{v}\mathbf{u}^T)\,\mathbf{1} - \mathbf{W}^T \mathbf{v}\mathbf{u}^T \\
\mathbf{u}^\wedge \mathbf{v}^\wedge \mathbf{u}^\wedge \equiv \mathbf{u}^\wedge \mathbf{u}^\wedge \mathbf{v}^\wedge + \mathbf{v}^\wedge \mathbf{u}^\wedge \mathbf{u}^\wedge + (\mathbf{u}^T \mathbf{u})\,\mathbf{v}^\wedge \\
\mathbf{u}^\wedge \mathbf{u}^\wedge \mathbf{u}^\wedge \equiv -(\mathbf{u}^T \mathbf{u})\,\mathbf{u}^\wedge \\
\mathbf{u}^\wedge \mathbf{v}^\wedge \mathbf{v}^\wedge - \mathbf{v}^\wedge \mathbf{v}^\wedge \mathbf{u}^\wedge \equiv (\mathbf{v}^\wedge \mathbf{u}^\wedge \mathbf{v})^\wedge \\
[\mathbf{u}^\wedge, \mathbf{v}^\wedge] \equiv \mathbf{u}^\wedge \mathbf{v}^\wedge - \mathbf{v}^\wedge \mathbf{u}^\wedge \equiv (\mathbf{u}^\wedge \mathbf{v})^\wedge \\
\underbrace{[\mathbf{u}^\wedge, [\mathbf{u}^\wedge, \ldots [\mathbf{u}^\wedge,}_{n} \mathbf{v}^\wedge] \ldots]] \equiv \left((\mathbf{u}^\wedge)^n \mathbf{v}\right)^\wedge
\end{gathered}
$$

Lie Group

$$
\begin{gathered}
\boldsymbol{\phi} = \phi \mathbf{a} \\
\mathbf{a}^T \mathbf{a} \equiv 1 \\
\mathbf{C}^T \mathbf{C} \equiv \mathbf{1} \equiv \mathbf{C}\mathbf{C}^T \\
\operatorname{tr}(\mathbf{C}) \equiv 2\cos\phi + 1 \\
\det(\mathbf{C}) \equiv 1 \\
\mathbf{C} = \exp\left(\boldsymbol{\phi}^\wedge\right) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left(\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} + \boldsymbol{\phi}^\wedge \\
\mathbf{C} \equiv \cos\phi \mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi \mathbf{a}^\wedge \\
\mathbf{C}^{-1} \equiv \mathbf{C}^T \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\phi}^\wedge \\
\mathbf{C}\mathbf{a} \equiv \mathbf{a} \\
\mathbf{C}\boldsymbol{\phi} = \boldsymbol{\phi} \\
\mathbf{C}\mathbf{a}^\wedge \equiv \mathbf{a}^\wedge \mathbf{C} \\
\mathbf{C}\boldsymbol{\phi}^\wedge \equiv \boldsymbol{\phi}^\wedge \mathbf{C} \\
(\mathbf{C}\mathbf{u})^\wedge \equiv \mathbf{C}\mathbf{u}^\wedge \mathbf{C}^T \\
\exp\left((\mathbf{C}\mathbf{u})^\wedge\right) \equiv \mathbf{C} \exp\left(\mathbf{u}^\wedge\right) \mathbf{C}^T
\end{gathered}
$$

(left) Jacobian

$$
\begin{gathered}
\mathbf{J} = \int_0^1 \mathbf{C}^\alpha \, d\alpha \equiv \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} + \frac{1}{2}\boldsymbol{\phi}^\wedge \\
\mathbf{J} \equiv \frac{\sin\phi}{\phi}\mathbf{1} + \left(1 - \frac{\sin\phi}{\phi}\right)\mathbf{a}\mathbf{a}^T + \frac{1-\cos\phi}{\phi}\mathbf{a}^\wedge \\
\mathbf{J}^{-1} \equiv \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} - \frac{1}{2}\boldsymbol{\phi}^\wedge \\
\mathbf{J}^{-1} \equiv \frac{\phi}{2}\cot\frac{\phi}{2}\mathbf{1} + \left(1 - \frac{\phi}{2}\cot\frac{\phi}{2}\right)\mathbf{a}\mathbf{a}^T - \frac{\phi}{2}\mathbf{a}^\wedge \\
\exp\left((\boldsymbol{\phi} + \delta\boldsymbol{\phi})^\wedge\right) \approx \exp\left((\mathbf{J}\,\delta\boldsymbol{\phi})^\wedge\right) \exp\left(\boldsymbol{\phi}^\wedge\right) \\
\mathbf{C} \equiv \mathbf{1} + \boldsymbol{\phi}^\wedge \mathbf{J} \\
\mathbf{J}(\boldsymbol{\phi}) \equiv \mathbf{C}\,\mathbf{J}(-\boldsymbol{\phi}) \\
\\
\left(\exp\left(\delta\boldsymbol{\phi}^\wedge\right)\mathbf{C}\right)^\alpha \approx \left(\mathbf{1} + \left(\mathbf{A}(\alpha, \boldsymbol{\phi})\,\delta\boldsymbol{\phi}\right)^\wedge\right)\mathbf{C}^\alpha \\
\mathbf{A}(\alpha, \boldsymbol{\phi}) = \alpha\,\mathbf{J}(\alpha\boldsymbol{\phi})\mathbf{J}(\boldsymbol{\phi})^{-1} = \sum_{n=0}^{\infty} \frac{F_n(\alpha)}{n!} \left(\boldsymbol{\phi}^\wedge\right)^n
\end{gathered}
$$

$$\alpha, \beta \in \mathbb{R},\ \mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{W}, \mathbf{A}, \mathbf{J} \in \mathbb{R}^{3\times 3},\ \mathbf{C} \in SO(3)$$

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Pose identities

$SE(3)$ Identities and Approximations

Lie Algebra

$$\mathbf{x}^\wedge = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}^\wedge = \begin{bmatrix}\mathbf{v}^\wedge & \mathbf{u}\\ \mathbf{0}^T & 0\end{bmatrix}$$
$$\mathbf{x}^\curlywedge = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}^\curlywedge = \begin{bmatrix}\mathbf{v}^\wedge & \mathbf{u}^\wedge\\ \mathbf{0} & \mathbf{v}^\wedge\end{bmatrix}$$
$$(\alpha\mathbf{x} + \beta\mathbf{y})^\wedge \equiv \alpha\mathbf{x}^\wedge + \beta\mathbf{y}^\wedge$$
$$(\alpha\mathbf{x} + \beta\mathbf{y})^\curlywedge \equiv \alpha\mathbf{x}^\curlywedge + \beta\mathbf{y}^\curlywedge$$
$$\mathbf{x}^\curlywedge\mathbf{y} \equiv -\mathbf{y}^\curlywedge\mathbf{x}$$
$$\mathbf{x}^\curlywedge\mathbf{x} \equiv \mathbf{0}$$
$$[\mathbf{x}^\wedge, \mathbf{y}^\wedge] \equiv \mathbf{x}^\wedge\mathbf{y}^\wedge - \mathbf{y}^\wedge\mathbf{x}^\wedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\wedge$$
$$[\mathbf{x}^\curlywedge, \mathbf{y}^\curlywedge] \equiv \mathbf{x}^\curlywedge\mathbf{y}^\curlywedge - \mathbf{y}^\curlywedge\mathbf{x}^\curlywedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\curlywedge$$
$$\underbrace{[\mathbf{x}^\wedge, [\mathbf{x}^\wedge, \dots [\mathbf{x}^\wedge,}_{n} \mathbf{y}^\wedge] \dots]] \equiv ((\mathbf{x}^\curlywedge)^n\,\mathbf{y})^\wedge$$
$$\underbrace{[\mathbf{x}^\curlywedge, [\mathbf{x}^\curlywedge, \dots [\mathbf{x}^\curlywedge,}_{n} \mathbf{y}^\curlywedge] \dots]] \equiv ((\mathbf{x}^\curlywedge)^n\,\mathbf{y})^\curlywedge$$

$$\mathbf{p}^\odot = \begin{bmatrix}\boldsymbol{\varepsilon}\\ \eta\end{bmatrix}^\odot = \begin{bmatrix}\eta\mathbf{1} & -\boldsymbol{\varepsilon}^\wedge\\ \mathbf{0}^T & \mathbf{0}^T\end{bmatrix}$$
$$\mathbf{p}^\circledcirc = \begin{bmatrix}\boldsymbol{\varepsilon}\\ \eta\end{bmatrix}^\circledcirc = \begin{bmatrix}\mathbf{0} & \boldsymbol{\varepsilon}\\ -\boldsymbol{\varepsilon}^\wedge & \mathbf{0}\end{bmatrix}$$
$$\mathbf{x}^\wedge\mathbf{p} \equiv \mathbf{p}^\odot\mathbf{x}$$
$$\mathbf{p}^T\mathbf{x}^\wedge \equiv \mathbf{x}^T\mathbf{p}^\circledcirc$$

Lie Group

$$\boldsymbol{\xi} = \begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\phi}\end{bmatrix}$$
$$\mathbf{T} = \exp\left(\boldsymbol{\xi}^\wedge\right) \equiv \sum_{n=1}^{\infty} \tfrac{1}{n!}\left(\boldsymbol{\xi}^\wedge\right)^n \approx \mathbf{1} + \boldsymbol{\xi}^\wedge$$
$$\mathbf{T} \equiv \begin{bmatrix}\mathbf{C} & \mathbf{J}\boldsymbol{\rho}\\ \mathbf{0}^T & 1\end{bmatrix}$$
$$\boldsymbol{\mathcal{T}} = \exp\left(\boldsymbol{\xi}^\curlywedge\right) \equiv \sum_{n=1}^{\infty} \tfrac{1}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} + \boldsymbol{\xi}^\curlywedge$$
$$\boldsymbol{\mathcal{T}} = \mathrm{Ad}\left(\mathbf{T}\right) \equiv \begin{bmatrix}\mathbf{C} & (\mathbf{J}\boldsymbol{\rho})^\wedge\mathbf{C}\\ \mathbf{0} & \mathbf{C}\end{bmatrix}$$
$$\mathrm{tr}(\mathbf{T}) \equiv 2\cos\phi + 2$$
$$\det(\mathbf{T}) \equiv 1$$
$$\mathrm{Ad}\left(\mathbf{T}_1\mathbf{T}_2\right) = \mathrm{Ad}\left(\mathbf{T}_1\right)\mathrm{Ad}\left(\mathbf{T}_2\right)$$
$$\mathbf{T}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\wedge\right) \equiv \sum_{n=1}^{\infty} \tfrac{1}{n!}\left(-\boldsymbol{\xi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\wedge$$
$$\mathbf{T}^{-1} \equiv \begin{bmatrix}\mathbf{C}^T & -\mathbf{C}^T\mathbf{r}\\ \mathbf{0}^T & 1\end{bmatrix}$$
$$\boldsymbol{\mathcal{T}}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\curlywedge\right) \equiv \sum_{n=1}^{\infty} \tfrac{1}{n!}\left(-\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\curlywedge$$
$$\boldsymbol{\mathcal{T}}^{-1} \equiv \begin{bmatrix}\mathbf{C}^T & -\mathbf{C}^T(\mathbf{J}\boldsymbol{\rho})^\wedge\\ \mathbf{0} & \mathbf{C}^T\end{bmatrix}$$
$$\boldsymbol{\mathcal{T}}\boldsymbol{\xi} \equiv \boldsymbol{\xi}$$
$$\mathbf{T}\boldsymbol{\xi}^\wedge \equiv \boldsymbol{\xi}^\wedge\mathbf{T}$$
$$\boldsymbol{\mathcal{T}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{T}}$$
$$(\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge \equiv \mathbf{T}\mathbf{x}^\wedge\mathbf{T}^{-1}$$
$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge\right) \equiv \mathbf{T}\exp\left(\mathbf{x}^\wedge\right)\mathbf{T}^{-1}$$
$$(\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge \equiv \boldsymbol{\mathcal{T}}\mathbf{x}^\curlywedge\boldsymbol{\mathcal{T}}^{-1}$$
$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge\right) \equiv \boldsymbol{\mathcal{T}}\exp\left(\mathbf{x}^\curlywedge\right)\boldsymbol{\mathcal{T}}^{-1}$$
$$\mathrm{Ad}\left(\mathbf{x}^\wedge\mathbf{T}\right) \equiv \mathbf{x}^\curlywedge\boldsymbol{\mathcal{T}}$$
$$(\mathbf{T}\mathbf{p})^\odot \equiv \mathbf{T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$
$$(\mathbf{T}\mathbf{p})^{\odot^T}(\mathbf{T}\mathbf{p})^\odot \equiv \boldsymbol{\mathcal{T}}^{-T}\mathbf{p}^{\odot^T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$

(left) Jacobian

$$\boldsymbol{\mathcal{J}} = \int_0^1 \boldsymbol{\mathcal{T}}^\alpha\, d\alpha \equiv \sum_{n=0}^{\infty} \tfrac{1}{(n+1)!}\left(\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} + \tfrac{1}{2}\boldsymbol{\xi}^\curlywedge$$
$$\boldsymbol{\mathcal{J}} \equiv \begin{bmatrix}\mathbf{J} & \mathbf{Q}\\ \mathbf{0} & \mathbf{J}\end{bmatrix}$$
$$\boldsymbol{\mathcal{J}}^{-1} \equiv \sum_{n=0}^{\infty} \tfrac{B_n}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\xi}^\curlywedge$$
$$\boldsymbol{\mathcal{J}}^{-1} \equiv \begin{bmatrix}\mathbf{J}^{-1} & -\mathbf{J}^{-1}\mathbf{Q}\mathbf{J}^{-1}\\ \mathbf{0} & \mathbf{J}^{-1}\end{bmatrix}$$
$$\begin{aligned}\mathbf{Q} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tfrac{1}{(n+m+2)!}\left(\boldsymbol{\phi}^\wedge\right)^n\boldsymbol{\rho}^\wedge\left(\boldsymbol{\phi}^\wedge\right)^m \\ &\equiv \tfrac{1}{2}\boldsymbol{\rho}^\wedge + \tfrac{\phi - \sin\phi}{\phi^3}\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ &\quad - \tfrac{1 - \frac{\phi^2}{2} - \cos\phi}{\phi^4}\left(\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge - 3\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ &\quad - \tfrac{1}{2}\left(\tfrac{1 - \frac{\phi^2}{2} - \cos\phi}{\phi^4} - 3\tfrac{\phi - \sin\phi - \frac{\phi^3}{6}}{\phi^5}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge \right. \\ &\quad \left. + \boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)\end{aligned}$$
$$\exp\left((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\wedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)$$
$$\exp\left((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\curlywedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\curlywedge\right)\exp\left(\boldsymbol{\xi}^\curlywedge\right)$$
$$\boldsymbol{\mathcal{T}} \equiv \mathbf{1} + \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$
$$\boldsymbol{\mathcal{J}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$
$$\boldsymbol{\mathcal{J}}(\boldsymbol{\xi}) \equiv \boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{J}}(-\boldsymbol{\xi})$$

$$\left(\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\mathbf{T}\right)^\alpha \approx \left(\mathbf{1} + \left(\boldsymbol{\mathcal{A}}(\alpha, \boldsymbol{\xi})\,\delta\boldsymbol{\xi}\right)^\wedge\right)\mathbf{T}^\alpha$$
$$\boldsymbol{\mathcal{A}}(\alpha, \boldsymbol{\xi}) = \alpha\,\boldsymbol{\mathcal{J}}(\alpha\boldsymbol{\xi})\boldsymbol{\mathcal{J}}(\boldsymbol{\xi})^{-1} = \sum_{n=0}^{\infty} \tfrac{F_n(\alpha)}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$\alpha, \beta \in \mathbb{R},\ \mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{p} \in \mathbb{R}^4,\ \mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \delta\boldsymbol{\xi} \in \mathbb{R}^6,\ \mathbf{C} \in SO(3),\ \mathbf{J}, \mathbf{Q} \in \mathbb{R}^{3\times3},\ \mathbf{T}, \mathbf{T}_1, \mathbf{T}_2 \in SE(3),\ \boldsymbol{\mathcal{T}} \in \mathrm{Ad}(SE(3)),\ \boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{A}} \in \mathbb{R}^{6\times6}$

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Summary

- we began by reviewing basic concepts from three-dimensional geometry: vectors, reference frames, coordinates, rotations, translations, poses
- we showed how rotations and poses were important in expressing motion (i.e., kinematic) and observation (i.e., sensor) models in three-dimensional space
- we then looked more deeply into the mathematical nature of rotations and poses by learning about matrix Lie groups
- next time we will use our new understanding of matrix Lie groups to start seeing how to apply our estimation techniques to rotations and poses

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