# AER1513 Quiz 2

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# Question 1

(a) 1° The optimization problem may not be convex, which will bring the result to local minima instead of the optimal one.

2° In state estimation, we assume most of the noise as Gaussian. With linear model, the property of Gaussian still can be used. In nonlinear system, we need either linearize the model, or approximate the new distribution to fit some variance of Gaussian, making the problem more complex, and more errors to deal with.

(b) RANSAC is often used before the main state estimatin process.

M- estimation with robust cost function can be used as a part of the process since it refines the estimation by re-weighing residual from estimations.

We can either use one of them, or use both at the same time.

(c) Matrix Lie groups offer a nice way to carry out unconstrained optimization for rotation and poses.
The main idea is that, we are using perturbation in the Lie algebra, where we don't need to worry above constraints. With optimal perturbations, we apply it them to the inital guess, stored in the Lie group, then we don't need to worry about singularities

(d) unbiased estimator: we would like $E[\hat{e}_k] = 0$ over many trials. It ensures that there is no systematic error.

consist estimator: we would like $E[\hat{e}_k^2 / \hat{P}_K] = 1$. if we ovestimate the velocity of the robot by $\delta v$. As long as $\delta v$ does not change, the estimation still can work, but the bias remain.

(e) observable means we can have one unique solution based on the number of measurements we have. "Observable" ensures we have sufficient information to infer the full system with or without biases.

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# Question 2

(a) each point has 0.2 being an inlier.

$$1-0.9999 = (1-0.2^3)^k$$

$$k = \frac{\ln(1-0.9999)}{\ln(1-0.2^3)} \approx 1146.68$$

since $k$ is integer, $k = 1147$

(b) First we compute $T_{13}$

$$T_{13} = T_{12}\cdot T_{23} = \begin{bmatrix}\cos\theta & -\sin\theta & 2\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-\sin\theta & -\cos\theta & \cos\theta+2\\ \cos\theta & -\sin\theta & \sin\theta\\ 0 & 0 & 1\end{bmatrix}$$

since $T_{12}, T_{23} \in SE(2)$, Then $T_{13} \in SE(2)$ too.

Thus $R_{13} = \begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix} \in SO(2)$

$$R_{13}^T = \begin{bmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{bmatrix} \qquad r = \begin{bmatrix}\cos\theta+2\\ \sin\theta\end{bmatrix}$$

$$R_{13}^T\cdot r = \begin{bmatrix}-\cancel{\cos\theta\sin\theta} - 2\sin\theta + \cancel{\cos\theta\sin\theta}\\ -\cos^2\theta - 2\cos\theta - \sin^2\theta\end{bmatrix} = \begin{bmatrix}-2\sin\theta\\ -2\cos\theta-1\end{bmatrix}$$

Thus $T_{1,3}^{-} = \begin{bmatrix}R_{13} & -R_{13}^T\cdot r\\ 0 & 1\end{bmatrix}$

$$= \begin{bmatrix}-\sin\theta & \cos\theta & 2\sin\theta\\ -\cos\theta & -\sin\theta & 2\cos\theta+1\\ 0 & 0 & 1\end{bmatrix}$$

Check:

$$T_{13}\cdot T_{13}^{-} = \begin{bmatrix}-\sin\theta & -\cos\theta & \cos\theta+2\\ \cos\theta & -\sin\theta & \sin\theta\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\sin\theta & \cos\theta & 2\sin\theta\\ -\cos\theta & -\sin\theta & 2\cos\theta+1\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Question 3

(a) $\varepsilon = \begin{bmatrix} \rho_x \\ \rho_y \\ \phi \end{bmatrix}$ $\quad T = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} = \exp(\varepsilon^\wedge)$

Then $\varepsilon^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} \in \mathbb{R}^{3\times3}$

$$= \begin{bmatrix} 0 & \phi & \rho_x \\ -\phi & 0 & \rho_y \\ 0 & 0 & 0 \end{bmatrix}$$

where $\phi^\wedge = \begin{bmatrix} 0 & \phi \\ -\phi & 0 \end{bmatrix}$ $\quad -\phi^\wedge = (\phi^\wedge)^T$

(b) $se(2) = \{ \Xi = \varepsilon^\wedge \in \mathbb{R}^{3\times3} \mid \varepsilon \in \mathbb{R}^3 \}$

Lie bracket: $[\Xi_1, \Xi_2] = \Xi_1\Xi_2 - \Xi_2\Xi_1$

$$\Xi_1 = \varepsilon_1^\wedge = \begin{bmatrix} 0 & \phi_1 & \rho_{x_1} \\ -\phi_1 & 0 & \rho_{y_1} \\ 0 & 0 & 0 \end{bmatrix} \quad \Xi_2 = \varepsilon_2^\wedge = \begin{bmatrix} 0 & \phi_2 & \rho_{x_2} \\ -\phi_2 & 0 & \rho_{y_2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_1^\wedge\varepsilon_2^\wedge = \begin{bmatrix} -\phi_1\phi_2 & 0 & \phi_1\rho_{y_2} \\ 0 & -\phi_1\phi_2 & -\phi_1\rho_{x_2} \\ 0 & 0 & 0 \end{bmatrix} \quad \varepsilon_2^\wedge\varepsilon_1^\wedge = \begin{bmatrix} -\phi_1\phi_2 & 0 & \phi_2\rho_{y_1} \\ 0 & -\phi_1\phi_2 & -\phi_2\rho_{x_1} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $[\varepsilon_1^\wedge, \varepsilon_2^\wedge] = \varepsilon_1^\wedge\varepsilon_2^\wedge - \varepsilon_2^\wedge\varepsilon_1^\wedge = \begin{bmatrix} 0 & 0 & \phi_1\rho_{y_2} - \phi_2\rho_{y_1} \\ 0 & 0 & -\phi_1\rho_{x_2} + \phi_2\rho_{x_1} \\ 0 & 0 & 0 \end{bmatrix}$

let $\rho = \begin{bmatrix} \phi_1\rho_{y_2} - \phi_2\rho_{y_1} \\ -\phi_1\rho_{x_2} + \phi_2\rho_{x_1} \end{bmatrix}$ $\quad \varepsilon^\wedge = \begin{bmatrix} \rho \\ \underline{\phi} \end{bmatrix} = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} = [\varepsilon_1^\wedge, \varepsilon_2^\wedge]$

$\phi = 0$

Thus lie bracket of two elements in se(2) is also se(2).

(c) $T = \exp(\xi^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!}(\xi^\wedge)^n$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix} \phi^\wedge & \rho \\ 0^T & n \end{bmatrix}^n$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}(\phi^\wedge)^n & \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n\right)\rho \\ 0^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C & r \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = J \begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix}$

where $J = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n$

$$= \frac{\sin\phi}{\phi}\mathbf{1} + (1 - \frac{\sin\phi}{\phi})aa^T + \frac{1-\cos\phi}{\phi}a^\wedge$$

(d) $T_1, T_2 \in SE(2)$

$T_1 = \begin{bmatrix} C_1 & r_1 \\ 0^T & 1 \end{bmatrix}$ $\quad T_2 = \begin{bmatrix} C_2 & r_2 \\ 0^T & 1 \end{bmatrix}$

Then $C_1, C_2 \in SO(2)$ and $r_1, r_2 = \mathbb{R}^2$

$$T_3 = T_1 T_2 = \begin{bmatrix} C_1 C_2 & C_1 r_2 + r_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} C_3 & r_3 \\ 0^T & 1 \end{bmatrix}$$

Need to show $C_3 \in SO(3)$

$$C_3 C_3^T = C_1 C_2 \cdot (C_1 C_2)^T = C_1 C_2 C_2^T C_1^T = 1$$

Thus $\det(C_3) = 1$ when $[\det(C_3)]^2 = \det(C_3 C_3^T)$
and choose $\det(C_3) = 1$ only.

Also $C_1 r_2 + r_1 \in \mathbb{R}^3 \Rightarrow r_3 \in \mathbb{R}^3$

Thus $T_3 = \begin{bmatrix} C_3 & r_3 \\ 0^T & 1 \end{bmatrix} \in SE(3)$ where $C_3 \in SO(3)$
and $r_3 \in \mathbb{R}^3$

Question 4

(a) Based on Assignment 2, we have

process noise $w_k \sim N(0, Q_k)$

and measurement noise $n_{j,k} \sim N(0, R_{jk})$

For motion model, we seperate it into two parts

let $\varepsilon_k = T\bar{\varpi}_k^\wedge$

nominal kinematics: $\bar{T}_{iv,k} = \bar{T}_{iv,k-1} \exp(T\bar{\varpi}_k^\wedge)$

perturbation kinematics: $\delta\varepsilon_k = \delta\varepsilon_{k-1} \exp(T\bar{\varpi}_k^\curlywedge)\delta\varepsilon_{k-1} + w_k$

where $T = \bar{T}\exp(\delta\varepsilon^\wedge)$

For measurement model

$$y_{j,k} = g(T_{iv,k}, p_{i,j}) = D^T T_{iv,k}^{-1} p_{i,j} + n_{j,k}$$

$$= \begin{bmatrix} C_k & r_k \\ 0^T & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_{i,j} \\ 1 \end{bmatrix}$$

where $D^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Then $y_{j,k} \in \mathbb{R}^2$

Part (b) on Next Page →

(b) linearize motion model

$$T_{iv,k} = \bar{T}_{iv,k} \exp(\delta \varepsilon_k^\wedge)$$
$$= \bar{T}_{iv,k} (I + \varepsilon_k^\wedge)$$

linearize measurement model

$$y_{j,k} = \bar{y}_{j,k} + \delta y_{jk} = D^T T_{iv,k}^{-1} p_{i,j} + n_{jk}$$

$$T_{iv,k}^{-1} = \left[\bar{T}_{iv,k} \exp(\delta \varepsilon_k^\wedge)\right]^{-1} \qquad (AB)^{-1} = B^{-1} A^{-1}$$

$$= \exp(\delta \varepsilon_k^\wedge)^{-1} \cdot \bar{T}_{iv,k}^{-1} = \exp(-\delta \varepsilon_k^\wedge) \cdot \bar{T}_{iv,k}$$
$$= (1 - \delta \varepsilon_k^\wedge) \bar{T}_{iv,k}^{-1}$$

Then $y_{j,k} = D^T (1 - \delta \varepsilon_k^\wedge) \bar{T}_{iv,k}^{-1} p_{i,j} + n_{jk}$

$$\bar{y}_{j,k} = D^T \bar{T}_{iv,k}^{-1} p_{i,j} \qquad \delta y_{j,k} = -D^T \left(\bar{T}_{iv,k}^{-1} p_{i,j}\right)^{\odot} \delta \varepsilon_k + n_{j,k}$$

(c) predict the mean through nominal kinematics

$$\check{T}_{i,k} = \check{T}_{i,k-1} \exp(T \varpi_k^\wedge)$$

$$\delta \check{\varepsilon}_k = \delta \hat{\varepsilon}_{k-1} \exp(T \varpi_k^\curlywedge) + w_k$$

where $\exp(T \varpi_k^\curlywedge) = F_{k-1}$

$$\check{P}_{i,k} = E[\delta \check{\varepsilon}_k \delta \check{\varepsilon}_k^T]$$

since $E[w_k] = 0$

$$E[\delta \check{\varepsilon}_k \delta \check{\varepsilon}_k^T] = E[\delta \hat{\varepsilon}_{k-1} F_{k-1} F_{k-1}^T \delta \hat{\varepsilon}_{k-1}^T] + E[w_k w_k^T]$$

$$= F_{k-1} \hat{P}_{k-1} F_{k-1} + Q_k$$

* since $F_{k-1}$ only depends on $\Upsilon$ and $\varpi_k$, so we can move it outside the expectation

Recall: we have $\delta y_{j,k} = \underbrace{-D^T(\bar{T}_{iv,k}^{-1} P_{i,j})^{\odot}}_{G_k} d\varepsilon_k + n_{j,k}$

Let $G_k = -D^T(\bar{T}_{iv,k}^{-1} P_{i,j})^{\odot}$

With the assumption we have $m$ landmarks.

$$y_k = \begin{bmatrix} y_{1k} \\ \vdots \\ y_{mk} \end{bmatrix} \quad G_k = \begin{bmatrix} G_{1k} \\ \vdots \\ G_{mk} \end{bmatrix} \quad R_k = \mathrm{diag}(R_{1k} \dots R_{mk})$$

Then the kalman gain will be

$$K_k = \check{P}_k G_k^T (G_k \check{P}_k G_k^T + R_k)^{-1}$$

$$\hat{P}_k = (I - K_k G_k)\check{P}_k$$

for the final estimator:

$$\varepsilon_k = \ln(\hat{T}_{i,k} \check{T}_{i,k}^{-1})^{\vee} = K_k(\underbrace{y_k - \bar{y}_k}_{\delta y_k})$$

where $\bar{y}_k = \begin{bmatrix} \bar{y}_{1k} \\ \vdots \\ \bar{y}_{mk} \end{bmatrix}$ $\quad \bar{y}_{j,k} = D^T \bar{T}_{iv,k}^{-1} P_{i,j}$

Thus we can get $\hat{T}_{i,k} = \check{T}_{i,k} \exp(K_k(y_k - \check{y}_k))$

In summary, we have

predictor: $\check{P}_k = F_{k-1} \hat{P}_{k-1} F_{k-1} + Q_k$ where $F_{k-1} = \exp(\tau \varpi_k^{\curlywedge})$

$\check{T}_{i,k} = \check{T}_{i,k-1} \exp(\tau \varpi_k^{\wedge})$

kalman gain: $\underline{K}_k = \check{P}_k G_k^T (G_k \check{P}_k G_k^T + R_k)^{-1}$ where $G_k = -D^T (\bar{T}_{iv,k}^{-1} P_{i,j})^{\odot}$

$\hat{P}_k = (1 - \underline{K}_k G_k) \check{P}_k$

Corrector: $\hat{T}_{i,k} = \hat{T}_{i,k-1} \exp\left( \left( \underline{K} (y_k - \bar{y}_k) \right)^{\wedge} \right)$

$\blacksquare$

Thank you ! :)