UNIVERSITY OF TORONTO
INSTITUTE FOR AEROSPACE STUDIES

Name: ____________________
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AER1513: State Estimation

– QUIZ II ON LECTURES 1-7 –

Answer all of the below questions (4) in the space provided or on separate sheets and assemble your responses into a single PDF file with the questions in numerical order. Your responses can be handwritten, typeset, or a mixture. Marks for each question are shown in the left margin in parentheses, totalling (50). You may use any material from the course (book, slides, lecture videos) to help solve the problems but you must work alone. You may not make use of other resources such as the internet or any generative AI models (e.g., ChatGPT) during the quiz. The quiz will need to be submitted within 5 hours, but hopefully you can finish in less time.

1. Short Answer Questions

(2) (a) Describe the additional challenges that come along with nonlinear state estimation as compared to the linear case.

(2) (b) Describe two common strategies for handling outliers in state estimation and explain when each would be preferable.

(2) (c) Explain the advantages of using matrix Lie groups when carrying out state estimation involving rotations.

(2) (d) What is the difference between consistency and unbiasedness in the context of an estimator? Provide an example for each.

(2) (e) What does it mean for a system to be *observable* in the context of state estimation? Why is this property critical?

2. Short Problems

(5) (a) Compute the number of RANSAC iterations k required to achieve a 99.99% probability of selecting 3 inliers, assuming each data point has a 20% chance of being an inlier.

(5) (b) Given the relative transforms in $SE(2)$,

$$\mathbf{T}_{1,2} = \begin{bmatrix} \cos\theta & -\sin\theta & 2 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}_{2,3} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

calculate $\mathbf{T}_{1,3}^{-1}$.

3. Lie Groups and Lie Algebras

The special Euclidean group $SE(2)$ (along with its associated Lie algebra $\mathfrak{se}(2)$) represents rigid-body transformations in 2D space, including rotations and translations. Work through the following steps to establish properties of $SE(2)$:

(3) (a) The $\wedge$ operator maps a vector $\boldsymbol{\xi} = \begin{bmatrix} \rho_x & \rho_y & \phi \end{bmatrix}^\top$ in $\mathbb{R}^3$ to a matrix in $\mathfrak{se}(2)$:

$$\mathbf{T} = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} = \exp(\boldsymbol{\xi}^\wedge).$$

Explain which of the following is the correct $\wedge$ operator for $SE(2)$:

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} 0 & -\phi & x \\ \phi & 0 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\xi}^\wedge = \begin{bmatrix} 0 & \phi & \rho_x \\ -\phi & 0 & \rho_y \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\xi}^\wedge = \begin{bmatrix} 0 & -\phi & \rho_x \\ \phi & 0 & \rho_y \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\xi}^\wedge = \begin{bmatrix} 0 & -\phi & x \\ \phi & 0 & y \\ 0 & 0 & 0 \end{bmatrix},$$

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(5) (b) Show that the Lie bracket of two elements in $\mathfrak{se}(2)$ is also in $\mathfrak{se}(2)$.

(4) (c) How are x, y related to ρ_x, ρ_y?

(3) (d) Prove the closure property of SE(2): if $\mathbf{T}_1, \mathbf{T}_2 \in$ SE(2), then their product $\mathbf{T}_3 = \mathbf{T}_1\mathbf{T}_2$ also belongs to SE(2).

4. Recursive Nonlinear State Estimation Consider a robot moving in the plane with its pose represented by $\mathbf{T}_{iv} \in$ SE(2); this is the inverse transformation from what we normally use so we must account for this in our development. The robot's pose evolves according to odometry measurements, which provide angular velocity ω and linear velocity v. The robot observes the range to landmarks, r, whose positions are known. This is basically the same as Assignment 2 without the bearing measurements, but now we are using what we've learned from Lie groups to represent the state. Develop the Extended Kalman Filter (EKF) for this system as follows:

(4) (a) Write down the motion and observation models:

- The state is represented as $\mathbf{T}_{iv,k} \in$ SE(2).
- The control inputs are the angular and linear velocities coming from odometry, $\boldsymbol{\varpi}_k = \begin{bmatrix} v_k & 0 & \omega_k \end{bmatrix}^\top$, which you can consider to be constants over each time step, τ.
- The known landmark positions are $\mathbf{p}_i = \begin{bmatrix} a & b & 1 \end{bmatrix}^\top$ in homogeneous form, written in the stationary frame.

Starting from the following general forms:

$$\mathbf{T}_{iv,k} = \mathbf{T}_{iv,k-1}\, \delta\mathbf{T}(\boldsymbol{\varpi}_k), \quad \mathbf{y}_k = \mathbf{g}(\mathbf{T}_{iv,k}, \mathbf{p}_i),$$

explicitly write the motion model (state transition) and observation model (measurement function) for this system. Define all terms and noise components.

(6) (b) Linearize the motion and observation models using our perturbation approach but this time do the perturbation on the right since that is the 'vehicle' side of our transformation matrix:

$$\mathbf{T}_{iv,k} = \mathbf{T}_{\mathrm{op},iv,k} \exp(\boldsymbol{\epsilon}_k^\wedge)$$

(5) (c) We will represent the uncertainty on SE(2) according to $\mathcal{N}(\hat{\mathbf{T}}_{iv,k}, \hat{\mathbf{P}}_{iv,k})$ where this means

$$\mathbf{T}_{iv} = \hat{\mathbf{T}}_{iv} \exp(\boldsymbol{\xi}^\wedge), \quad \boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \hat{\mathbf{P}}_{iv}),$$

again on the right since that is the 'vehicle' side of our transformation matrix. Develop the full EKF update equations:

- Express the prediction step, including the state propagation and covariance update.
- Write the correction step, including the Kalman gain, state update, and covariance update.