

Mathematical Foundations of Data Sciences



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Chapter 1

Shannon Theory

The main reference is [28].

1.1 Analog vs. Discrete Signals

To develop numerical tools and analyze their performances, the mathematical modeling is usually done over a continuous setting. An analog signal is a 1D function $f_0 \in L^2([0, 1])$ where $[0, 1]$ denotes the domain of acquisition, which might for instance be time. An analog image is a 2D function $f_0 \in L^2([0, 1]^2)$ where the unit square $[0, 1]^2$ is the image domain.

Although these notes are focussed on the processing of sounds and natural images, most of the methods extend to multi-dimensional datasets, which are higher dimensional mappings

$$f_0 : [0, 1]^d \rightarrow [0, 1]^s$$

where d is the dimensionality of the input space ($d = 1$ for sound and $d = 2$ for images) whereas s is the dimensionality of the feature space. For instance, gray scale images corresponds to $(d = 2, s = 1)$, videos to $(d = 3, s = 1)$, color images to $(d = 2, s = 3)$ where one has three channels (R, G, B). One can even consider multi-spectral images where $(d = 2, s \gg 3)$ that is made of a large number of channels for different light wavelengths. Figures 1.1 and 1.2 show examples of such data.

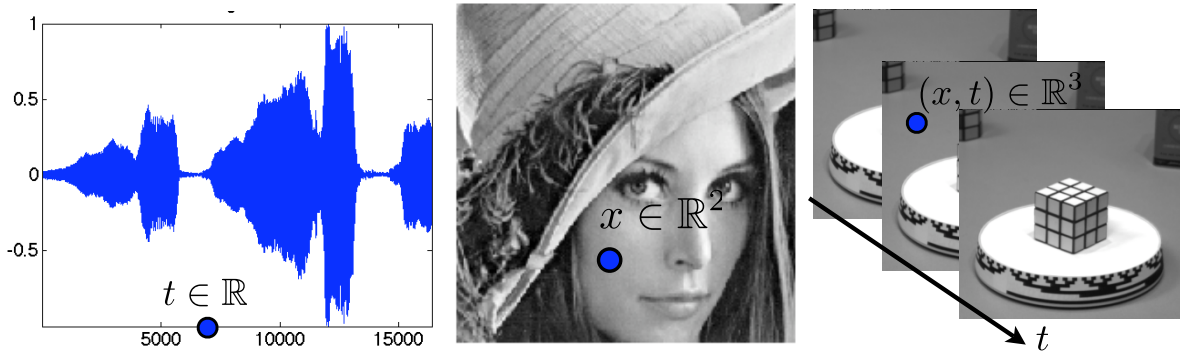


Figure 1.1: Examples of sounds ($d = 1$), image ($d = 2$) and videos ($d = 3$).

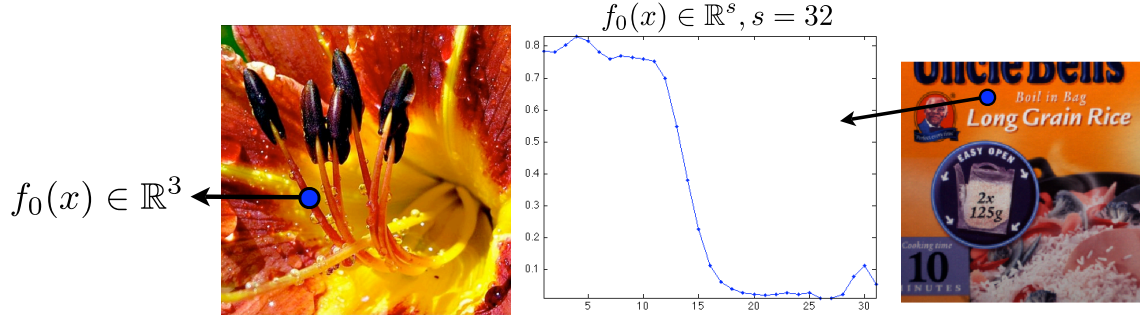


Figure 1.2: Example of color image $s = 3$ and multispectral image ($s = 32$).

1.1.1 Acquisition and Sampling

Signal acquisition is a low dimensional projection of the continuous signal performed by some hardware device. This is for instance the case for a microphone that acquires 1D samples or a digital camera that acquires 2D pixel samples. The sampling operation thus corresponds to mapping from the set of continuous functions to a discrete finite dimensional vector with N entries.

$$f_0 \in L^2([0, 1]^d) \mapsto f \in \mathbb{C}^N$$

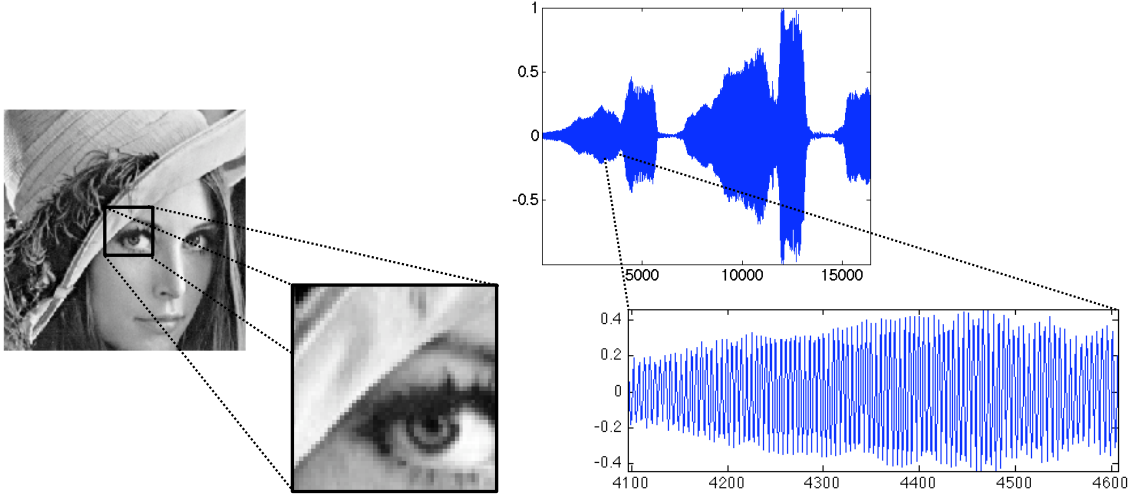


Figure 1.3: Image and sound discretization.

Figure 1.3 shows examples of discretized signals.

1.1.2 Linear Translation Invariant Sampler

A translation invariant sampler performs the acquisition as an inner product between the continuous signal and a constant impulse response h translated at the sample location

$$f[n] = \int_{-S/2}^{S/2} f_0(x) h(n/N - x) dx = f_0 \star h(n/N). \quad (1.1)$$

The precise shape of $h(x)$ depends on the sampling device, and is usually a smooth low pass function that is maximal around $x = 0$. The size S of the sampler determines the precision of the sampling device, and is usually of the order of $1/N$ to avoid blurring (if S is too large) or aliasing (if S is too small).

Section ?? details how to reverse the sampling operation in the case where the function is smooth.

1.2 Shannon Sampling Theorem

Reminders about Fourier transform. For $f \in L^1(\mathbb{R})$, its Fourier transform is defined as

$$\forall \omega \in \mathbb{R}, \quad \hat{f}(\omega) \stackrel{\text{def.}}{=} \int_{\mathbb{R}} f(x) e^{-ix\omega} dx. \quad (1.2)$$

One has $\|\hat{f}\|^2 = (2\pi)^{-1} \|f\|^2$, so that $f \mapsto \hat{f}$ can be extended by continuity to $L^2(\mathbb{R})$, which corresponds to computing \hat{f} as a limit when $T \rightarrow +\infty$ of $\int_{-T}^T f(x) e^{-ix\omega} dx$. When $\hat{f} \in L^1(\mathbb{R})$, one can invert the Fourier transform so that

$$f(x) = \int_{\mathbb{R}} \hat{f}(\omega) e^{ix\omega} d\omega, \quad (1.3)$$

which shows in particular that f is continuous with vanishing limits at $\pm\infty$.

The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ exchanges regularity and decay. For instance, if $f \in C^p(\mathbb{R})$ with an integrable Fourier transform, then $\mathcal{F}(f^{(p)})(\omega) = (i\omega)^{-p} \hat{f}(\omega)$ so that $|\hat{f}(\omega)| = O(1/|\omega|^p)$. Conversely,

$$\int_{\mathbb{R}} (1 + |\omega|)^{-p} |\hat{f}(\omega)| d\omega < +\infty \implies f \in C^p(\mathbb{R}). \quad (1.4)$$

Reminders about Fourier series. We denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the torus. A function $f \in L^2(\mathbb{T})$ is 2π -periodic, and can be viewed as a function $f \in L^2([0, 1])$ (beware that this means that the boundary points are glued together), and its Fourier coefficients are

$$\forall n \in \mathbb{Z}, \quad \hat{f}_n \stackrel{\text{def.}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

This formula is equivalent to the computation of an inner-product $\hat{f}_n = \langle f, e_n \rangle$ for the inner-product $\langle f, g \rangle \stackrel{\text{def.}}{=} \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \bar{g}(x) dx$. For this inner product, $(e_n)_n$ is orthonormal and is actually an Hilbert basis, meaning that one reconstruct with the following converging series

$$f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n \quad (1.5)$$

which means $\|f - \sum_{n=-N}^N \langle f, e_n \rangle e_n\|_{L^2(\mathbb{T})} \rightarrow 0$ for $N \rightarrow +\infty$. The pointwise convergence of (1.5), and is ensured (and there is normal convergence) when for instance $f \in C^3(\mathbb{T})$.

Poisson formula. The poisson formula connects the Fourier transform and the Fourier series to sampling and periodization operators. For some function $\hat{f}(\omega)$ defined on \mathbb{R} , its periodization reads

$$\hat{f}_P(\omega) \stackrel{\text{def.}}{=} \sum_n f(\omega - 2\pi n). \quad (1.6)$$

This formula makes sense if $\hat{f} \in L^1(\mathbb{R})$, and in this case $\|\hat{f}_P\|_{L^1(\mathbb{T})} \leq \|\hat{f}\|_1$. The Poisson formula, state in Proposition 1 bellow, corresponds to proving that the following diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{\mathcal{F}} & \hat{f}(\omega) \\ \text{sampling} \downarrow & & \downarrow \text{periodization} \\ (f(n))_n & \xrightarrow{\text{Fourier serie}} & \sum_n f(n) e^{-i\omega n} \end{array}$$

is actually commutative.

Proposition 1 (Poisson formula). Assume that \hat{f} has compact support and that $|f(x)| \leq C(1 + |x|)^{-3}$ for some C . Then one has

$$\forall \omega \in \mathbb{R}, \quad \sum_n f(n) e^{-i\omega n} = \hat{f}_P(\omega). \quad (1.7)$$

Proof. Since \hat{f} is compactly supported, \hat{f}_P is well defined (it involves only a finite sum) and since f has fast decay, using (1.4), \hat{f}_P is C^1 . It is thus the sum of its Fourier transform

$$\hat{f}_P(\omega) = \sum_k c_k e^{ik\omega}, \quad (1.8)$$

where

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}_P(\omega) e^{-ik\omega} d\omega = \frac{1}{2\pi} \int_0^{2\pi} \sum_n f(x - 2\pi n) e^{-ik\omega} d\omega.$$

One has

$$\int_0^{2\pi} \sum_n |f(x - 2\pi n) e^{-ik\omega}| d\omega = \int_{\mathbb{R}} |f|$$

which is bounded because $\hat{f} \in L^1(\mathbb{R})$ (it has a compact support and is C^1), so one can exchange the sum and integral

$$c_k = \sum_n \frac{1}{2\pi} \int_0^{2\pi} f(x - 2\pi n) e^{-ik\omega} d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-ik\omega} d\omega = f(-k)$$

where we used the inverse Fourier transform formula (1.3), which is legit because $\hat{f} \in L^1(\mathbb{R})$. \square

Shannon theorem. Shannon sampling theorem state a sufficient condition ensuring that the sampling operator $f \mapsto (f(ns))_n$ is invertible for some sampling step size $s > 0$. It require that $\text{supp}(\hat{f}) \subset [-\pi/s, \pi/s]$, which, thanks to formula (1.3), implies that \hat{f} is C^∞ (in fact it is even analytic).

Theorem 1. If $|f(x)| \leq C(1 + |x|)^{-3}$ for some C and $\text{supp}(\hat{f}) \subset [-\pi/s, \pi/s]$, then one has

$$\forall x \in \mathbb{R}, \quad f(x) = \sum_n f(ns) \text{sinc}(x/s - n) \quad \text{where} \quad \text{sinc}(u) = \frac{\sin(\pi u)}{\pi u} \quad (1.9)$$

with uniform convergence.

Proof. The change of variable $g = f(s \cdot)$ results in $\hat{g} = s\hat{f}(s \cdot)$ so that we can restrict our attention to $s = 1$. The compact support hypothesis implies $\hat{f}(\omega) = 1_{[-\pi, \pi]}(\omega) \hat{f}_P(\omega)$. Combining the inversion formula (1.3) with Poisson formula (1.8)

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_P(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_n f(n) e^{i\omega(x-n)} d\omega.$$

Since f has fast decay, $\int_{-\pi}^{\pi} \sum_n |f(n) e^{i\omega(x-n)}| d\omega = \sum_n |f(n)| < +\infty$, so that one can exchange summation and integration and obtain

$$f(x) = \sum_n f(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(x-n)} d\omega = \sum_n f(n) \text{sinc}(x - n).$$

\square

1.3 Shannon Source Coding Theorem

We consider an alphabet (x_1, \dots, x_K) of K symbols, and assume at our disposal some probability distribution over this alphabet, which is just an histogram $p = (p_1, \dots, p_K) \in \mathbb{R}_+^K$ in the simplex, i.e. $\sum_k p_k = 1$.

The entropy of such an histogram is

$$H(p) \stackrel{\text{def.}}{=} - \sum_k p_k \log_2(p_k)$$

with the convention $0 \log_2(0) = 0$.

Lemma 1. *One has*

$$0 \leq H(p) \leq \log_2(K).$$

Proof. We consider the following constrained optimization problem

$$\min_p \left\{ f(p) ; g(p) = \sum_k p_k = 1 \right\}$$

where $f = -H$. According to the linked extrema theorem, at an optimum p^* , $\nabla f(p^*) = \lambda \nabla g(p^*)$ for some $\lambda \in \mathbb{R}$, so that here $\log(p_k^*) + 1 = \lambda$, i.e. $p_k^* = c$ is constant, and since $\sum_k p_k^* = 1$, one has $p_k^* = 1/K$ and thus $H(p) = \log_2(K)$. \square

A code $c_k = c(x_k)$ associate to each symbol x_k a code word $c_k \in \{0, 1\}^{\mathbb{N}}$ with a varying length $|c_k| \in \mathbb{N}^*$. We denote the average length associated to this code as

$$L(c) \stackrel{\text{def.}}{=} \sum_k p_k |c_k|.$$

A prefix code $c_k = c(x_k)$ is such that no word c_k is the beginning of another word c'_k . This is equivalent to be able to embed the $(c_k)_k$ as leaves of a binary tree T , with the code being output of a traversal from root to leaves (with a convention that going to a left (resp. right) child output a 0 (resp. a 1). We denote $c = \text{Leaves}(T)$ such prefix property. The following fundamental lemma describes the set of prefix code using an inequality.

Lemma 2 (Kraft inequality). *(i) For a code c , if there exists a tree T such that $c = \text{Leaves}(T)$ then*

$$\sum_k 2^{-|c_k|} \leq 1. \tag{1.10}$$

(ii) Conversely, if $(\ell_k)_k$ are such that

$$\sum_k 2^{-\ell_k} \leq 1 \tag{1.11}$$

then there exists a code $c = \text{Leaves}(T)$ such that $|c_k| = \ell_k$.

Proof. \Rightarrow We suppose $c = \text{Leaves}(T)$. We denote $m = \max_k |c_k|$ and consider the full binary tree. Bellow each c_k , one has a sub-tree of height $m - |c_k|$. This sub-tree has $2^{m-|c_k|}$ leaves. Since all these sub-trees do not overlap, the total number of leaf do not exceed the total number of leaves 2^m of the full binary tree, hence

$$\sum_k 2^{m-|c_k|} \leq 2^m,$$

hence (1.10).

\Leftarrow Conversely, we assume (1.10) holds. Without loss of generality, we assume that $|c_1| \leq \dots \leq |c_K|$. We start by putting a sub-tree of height $2^{m-|c_1|}$. Since the second tree is smaller, one can put it immediately aside, and continue this way. Since $\sum_k 2^{m-|c_k|} \leq 2^m$, this ensure that we can stack side-by-side all these sub-tree, and this defines a proper sub-tree of the full binary tree. \square

We now are ready to state and prove Shannon theory for entropic coding.

Theorem 2. (i) If $c = \text{Leaves}(T)$ for some tree T , then

$$L(c) \geq H(p).$$

(ii) Conversely, there exists a code c with $c = \text{Leaves}(T)$ such that

$$L(c) \leq H(p) + 1.$$

Proof. First, we consider the following optimization problem

$$\min_{\ell=(\ell_k)_k} \left\{ f(\ell) \stackrel{\text{def.}}{=} \sum_k \ell_k p_k ; g(\ell) \stackrel{\text{def.}}{=} \sum_k 2^{-\ell_k} \leq 1 \right\}. \quad (1.12)$$

We first show that at an optimal ℓ^* , the constraint is saturated, i.e. $g(\ell^*) = 1$. Indeed, if $g(\ell^*) = 2^{-u} < 1$, with $u > 0$, we define $\ell'_k \stackrel{\text{def.}}{=} \ell_k^* - u$, which satisfies $g(\ell') = 1$ and also $f(\ell') = \sum_k (\ell_k^* - u) p_k < f(\ell^*)$, which is a contradiction. So we can restrict in (1.12) the constraint to $g(\ell) = 1$ and apply the linked extra theorem, which shows that necessarily, there exists $\lambda \in \mathbb{R}$ with $\nabla f(\ell^*) = \nabla g(\ell^*)$, i.e. $(p_k)_k = -\lambda \ln(2)(2^{-\ell_k^*})_k$. Since $\sum_k p_k = \sum_k 2^{-\ell_k^*} = 1$, we deduce that $\ell_k^* = -\log(p_k)$.

(i) If $c = \text{Leaves}(T)$, then by Kraft inequality (1.10), necessarily $\ell_k = |c_k|$ satisfy the constraints of (1.12), and thus $H(p) = f(\ell^*) \leq f(\ell) = L(\ell)$.

(ii) We define $\ell_k \stackrel{\text{def.}}{=} \lceil -\log_2(p_k) \rceil \in \mathbb{N}^*$. Then $\sum_k 2^{-\ell_k} \leq \sum_k 2^{\log_2(p_k)} = 1$, so that these lengths satisfy (1.11). Thanks to Proposition 2 (ii), there thus exists a prefix code c with $|c_k| = \lceil -\log_2(p_k) \rceil$. Furthermore

$$L(c) = \sum_k p_k \lceil -\log_2(p_k) \rceil \leq \sum_k p_k (-\log_2(p_k) + 1) = H(p) + 1.$$

□

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