

Mathematical Foundations of Data Sciences



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Chapter 15

Compressed Sensing

Main ref: [29, 23, 38]
TODO.

15.1 Motivation and Potential Applications

15.2 Dual Certificate Theory and Non-Uniform Guarantees

Polytopes, descent cone and Gaussian width.

15.3 RIP Theory for Uniform Guarantees

15.3.1 RIP Constants

The RIP constant δ_s of a matrix $\Phi \in \mathbb{R}^{P \times N}$ is defined as

$$\forall z \in \mathbb{R}^N, \quad \|z\|_0 \leq s \implies (1 - \delta_s)\|z\|^2 \leq \|\Phi z\|^2 \leq (1 + \delta_s)\|z\|^2 \quad (15.1)$$

15.3.2 RIP implies stable recovery

RIP implies dual certificate guarantees.

In the following, we fix a vector $x \in \mathbb{R}^N$ and denote $y = \Phi x + w$ the measurement, with $\|w\| \leq \varepsilon$. We denote $x_s \in \mathbb{R}^N$ the best s -term approximation of x , obtained by only keeping the s largest coefficients in magnitude from x and setting the others to 0.

We consider a solution x^* of

$$\min_{\|\Phi \tilde{x} - y\| \leq \varepsilon} \|\tilde{x}\|_1.$$

This note recall the proof from [7] of the following theorem

Theorem 45 ([7]). *If $\delta_{2s} \leq \sqrt{2} - 1$ then there exists C_0, C_1 such that*

$$\|x^* - x\|_1 \leq \frac{C_0}{\sqrt{s}} \|x_s - x\| + C_1 \varepsilon.$$

The remaining of this section is devoted to proving this theorem.

Notations. We denote in the following $h = x^* - x$ and denote T_0 the largest s coefficients of x in magnitude (so that $x_s = x_{T_0}$), T_1 the s largest coefficients of $h_{T_0^c}$, T_2 the following s largest coefficients of $h_{T_0^c}$ and so on. We denote $T = T_0 \cup T_1$ which is an index set of size $2s$.

Lemma 4. *One has*

$$\sum_{j \geq 2} \|h_{T_j}\| \leq \frac{1}{\sqrt{s}} \|h_{T_0^c}\|_1$$

Proof. By the definition of T_j for $j \geq 2$, one has, for all $j \geq 2$

$$\forall i \in T_{j-1}, \quad \|h_{T_j}\|_\infty \leq h_i,$$

and hence

$$\|h_{T_j}\|_\infty \leq \frac{1}{s} \|h_{T_{j-1}}\|_1.$$

This proves that

$$\|h_{T_j}\| \leq \sqrt{s} \|h_{T_j}\|_\infty \leq \frac{1}{\sqrt{s}} \|h_{T_{j-1}}\|_1$$

and thus

$$\sum_{j \geq 2} \|h_{T_j}\| \leq \frac{1}{\sqrt{s}} \sum_{j \geq 1} \|h_{T_j}\|_1 = \frac{1}{\sqrt{s}} \|h_{T_0^c}\|_1.$$

□

Lemma 5. *One has*

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

Proof. One has

$$\begin{aligned} \|x\|_1 &\geq \|x + h\| && \text{because } x^* \text{ is a minimizer} \\ &= \|(x + h)_{T_0}\|_1 + \|(x + h)_{T_0^c}\|_1 \\ &\geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1 && \text{using the triangular inequality.} \end{aligned}$$

Decomposing the left hand side $\|x\|_1 = \|x_{T_0}\|_1 + \|x_{T_0^c}\|_1$, one obtains the result. □

Lemma 6. *If z and z' have disjoint supports and $\|z\| \leq s$ and $\|z'\|_0 \leq s$,*

$$|\langle \Phi z, \Phi z' \rangle| \leq \delta_{2s} \|z\| \|z'\|.$$

Proof. Using the RIP (15.1) since $z \pm z'$ has support of size $2s$ and the fact that $\|z \pm z'\|^2 = \|z\|^2 + \|z'\|^2$, one has

$$(1 - \delta_{2s}) (\|z\|^2 + \|z'\|^2) \leq \|\Phi z \pm \Phi z'\|^2 \leq (1 + \delta_{2s}) (\|z\|^2 + \|z'\|^2).$$

One thus has using the parallelogram equality

$$|\langle \Phi z, \Phi z' \rangle| = \frac{1}{4} |\|\Phi z + \Phi z'\|^2 - \|\Phi z - \Phi z'\|^2| \leq \delta_{2s} \|z\| \|z'\|.$$

□

Theorem 45 requires bounding $\|h\|$. We bound separately $\|h_T\|$ and $\|h_{T^c}\|$.

Part 1: bounding $\|h_T\|$. One has

$$\begin{aligned}
\|h_{T^c}\| &= \left\| \sum_{j \geq 2} h_{T_j} \right\| \leq \sum_{j \geq 2} \|h_{T_j}\| && \text{using the triangular inequality} \\
&\leq \frac{1}{\sqrt{s}} \|h_{T_0^c}\|_1 && \text{using Lemma 4} \\
&\leq \frac{1}{\sqrt{s}} \|h_{T_0}\|_1 + \frac{2}{\sqrt{s}} \|x_{T_0^c}\|_1 && \text{using Lemma 5} \\
&\leq \frac{1}{\sqrt{s}} \|h_{T_0}\|_1 + 2e_0 && \text{denoting } e_0 = \frac{1}{\sqrt{s}} \|x_{T_0^c}\|_1 \\
&\leq \|h_{T_0}\| + 2e_0 && \text{using Cauchy-Schwartz} \\
&\leq \|h_T\| + 2e_0 && \text{because } T_0 \subset T.
\end{aligned}$$

The final bound reads

$$\|h_{T^c}\| \leq \|h_T\| + 2e_0. \quad (15.2)$$

Part 2: bounding $\|h_{T^c}\|$. One has

$$\begin{aligned}
\|h_T\|^2 &\leq \frac{1}{1 - \delta_{2s}} \|\Phi h_T\|^2 && \text{using the RIP (15.1)} \\
&= \frac{A - B}{1 - \delta_{2s}} && \text{using } \Phi h_T = \Phi h - \sum_{j \geq 2} \Phi h_{T_j},
\end{aligned}$$

where we have introduced

$$A = \langle \Phi h_T, \Phi h \rangle \quad \text{and} \quad B = \langle \Phi h_T, \sum_{j \geq 2} \Phi h_{T_j} \rangle.$$

One has

$$\begin{aligned}
|A| &\leq \|\Phi h_T\| \|\Phi h\| && \text{using Cauchy-Schwartz} \\
&\leq \sqrt{1 + \delta_{2s}} \|h_T\| \|\Phi h\| && \text{using the RIP (15.1)} \\
&\leq \sqrt{1 + \delta_{2s}} \|h_T\| 2\varepsilon && \text{using } \|\Phi h\| \leq \|\Phi x - y\| + \|\Phi x^* - y\| \leq 2\varepsilon
\end{aligned}$$

The final bound reads

$$|A| \leq 2\varepsilon \sqrt{1 + \delta_{2s}} \|h_T\| \quad (15.3)$$

One has

$$\begin{aligned}
|B| &\leq |\langle \Phi h_{T_0}, \sum_{j \geq 2} \Phi h_{T_j} \rangle| + |\langle \Phi h_{T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle| && \text{using the triangular inequality} \\
&\leq \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| && \text{using the triangular inequality} \\
&\leq \sum_{j \geq 2} \delta_{2s} \|h_{T_0}\| \|h_{T_j}\| + \delta_{2s} \|h_{T_1}\| \|h_{T_j}\| && \text{using Lemma 6} \\
&= \delta_{2s} (\|h_{T_0}\| + \|h_{T_1}\|) \sum_{j \geq 2} \|h_{T_j}\| \\
&\leq \delta_{2s} \sqrt{2} \|h_T\| \sum_{j \geq 2} \|h_{T_j}\| && T_0 \text{ and } T_1 \text{ are disjoint} \\
&\leq \frac{\sqrt{2} \delta_{2s}}{\sqrt{s}} \|h_T\| \|h_{T_0^c}\|_1 && \text{using Lemma 4}
\end{aligned}$$

The final bound reads

$$|B| \leq \frac{\sqrt{2} \delta_{2s}}{\sqrt{s}} \|h_T\| \|h_{T_0^c}\|_1. \quad (15.4)$$

Putting together (15.3) and (15.4) one obtains

$$\|h_T\|^2 \leq \frac{\|h_T\|}{1 - \delta_{2s}} \left(\sqrt{1 + \delta_{2s}} 2\varepsilon + \frac{2}{\sqrt{s}} \delta_{2s} \|h_{T_0^c}\|_1 \right)$$

thus

$$\begin{aligned} \|h_T\| &\leq \alpha\varepsilon + \frac{\rho}{\sqrt{s}} \|h_{T_0^c}\|_1 && \text{denoting } \begin{cases} \alpha = 2\frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \\ \rho = \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}. \end{cases} \\ &\leq \alpha\varepsilon + \frac{\rho}{\sqrt{s}} \|h_{T_0}\|_1 + \frac{2\rho}{\sqrt{s}} \|x_{T_0^c}\|_1 && \text{using Lemma 5} \\ &\leq \alpha\varepsilon + \rho \|h_{T_0}\| + 2\rho e_0 && \text{using Cauchy-Schwartz} \\ &\leq \alpha\varepsilon + \rho \|h_T\| + 2\rho e_0 && \text{because } T_0 \subset T. \end{aligned}$$

Note that since $\delta_{2s} < \sqrt{2} - 1$, one has $\rho < 1$. This implies

$$\|h_T\| \leq \frac{\alpha}{1 - \rho} \varepsilon + \frac{2\rho}{1 - \rho} e_0. \quad (15.5)$$

Conclusion. One has

$$\begin{aligned} \|h\| &\leq \|h_T\| + \|h_{T^c}\| && \text{using the triangular inequality} \\ &\leq 2\|h_T\| + 2e_0 && \text{using (15.2)} \\ &\leq \frac{2\alpha}{1 - \rho} \varepsilon + 2\frac{1 + \rho}{1 - \rho} e_0 && \text{using (15.5)} \end{aligned}$$

which proves the theorem.

15.3.3 Gaussian Matrices RIP

15.3.4 Fourier sampling RIP

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