Decisions Over Time

Problem 1

DISCOUNT PRICES

A local department store sells products at a given initial price, and every week a product goes unsold, its price is discounted by 25% of the original price. If it is not sold after four weeks, it is sent back to the regional warehouse. A set of kitchen knives was just put out for Rs. 200. Your willingness to pay for the knives (your Rupee value) is Rs. 180, so if you buy them at a price P, your payoff is $u = 180 - P$. If you don't buy the knives, the chances that they will be sold to someone else, conditional on not having been sold the week before, are as follows:

Week 1	0.2
Week 2	0.4
Week 3	0.6
Week 4	0.8

For example, if you do not buy the knives during the first two weeks, the likelihood that they will be available at the beginning of the third week is the likelihood that they do not sell in either week 1 or week 2, which is $0.8 \times 0.6 = 0.48$.

(a) Draw your decision tree for the four weeks after the knives are put out for sale.
(b) At the beginning of which week, if any, should you run to buy the knives?
(c) Find a willingness to pay for the knives that would make it optimal to buy at the beginning of the first week.
(d) Find a willingness to pay that would make it optimal to buy at the beginning of the fourth week.

KEY LEARNINGS

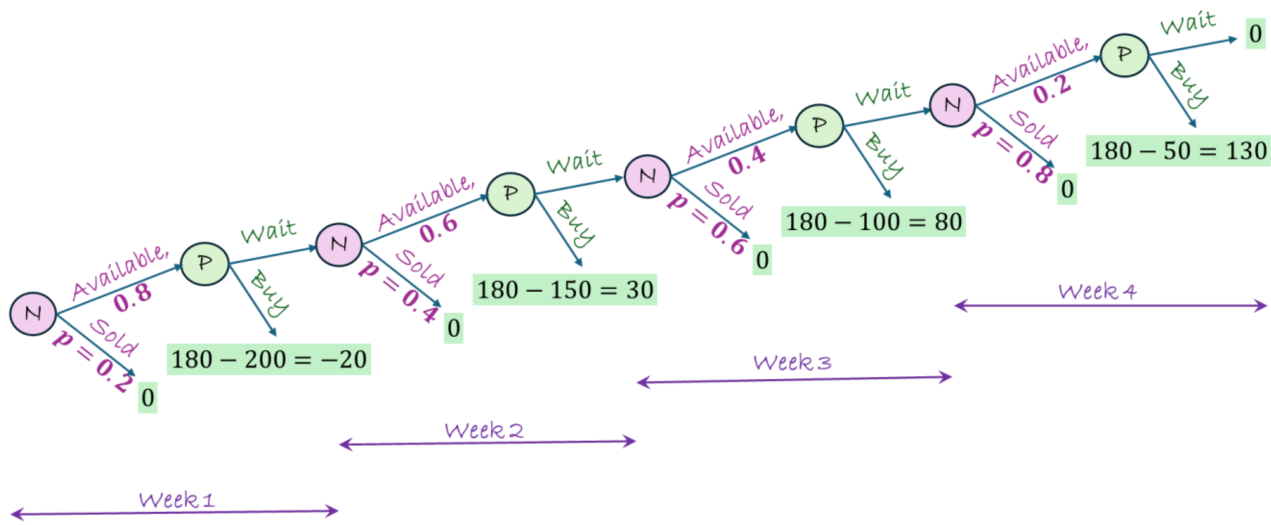
- Backward induction
  - Reduce Nature nodes using expected payoffs ('average action')
  - Reduce Player nodes using rationality ('best action')
- Decision tree with multiple layers
- Cumbersome process
- Calculating payoffs based on problem-parameters (willingness to pay, w)

INTERPRETATION

Think of each week as a single point in time. In each week, there are two possibilities: (1) knives are available for buying, (2) knives are not available for buying by the decision-maker. These two possibilities are outside the control of the decision-maker and are hence actions that must be assigned to Nature. The probabilities for the second possibility (knives not available for buying) are listed in the table for the four weeks.

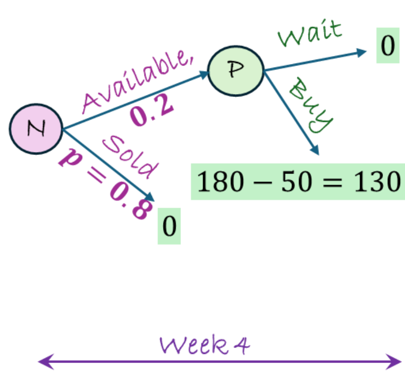
Each week, Nature picks between available and not available. If Nature picks available, then the player decides between buying and waiting. Each terminal node will have a payoff associated with it that will depend on the price of the knives in that week.

ANSWER (A). DECISION TREE



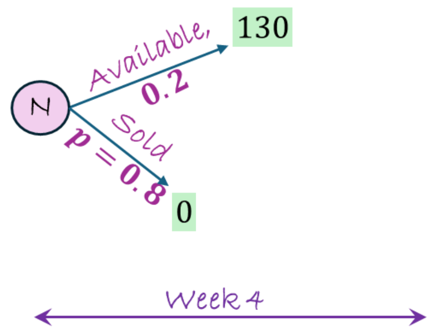
ANSWER (B). WEEK TO BUY THE KNIVES

WEEK 4 (PLAYER DECISION NODE)



Let's start by analyzing week 4. After Nature has chosen between available and sold, the action set for the decision-maker is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of 0 (irrespective of whether the knives are sold to someone or returned to the warehouse). **Choosing to buy gives a payoff of 130. The rational choice, then, is to buy the knives in week 4 for a payoff of 130.**

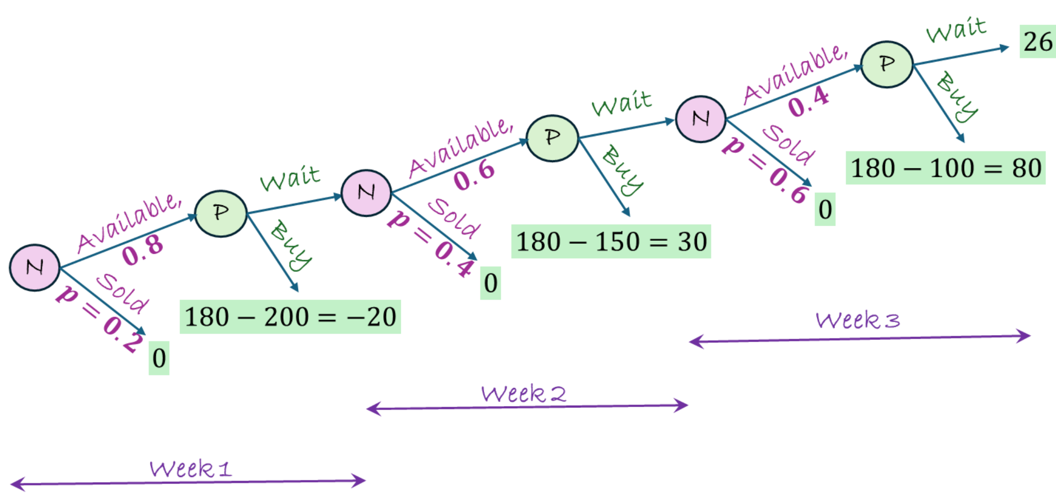
Week 4 (Nature Decision Node)



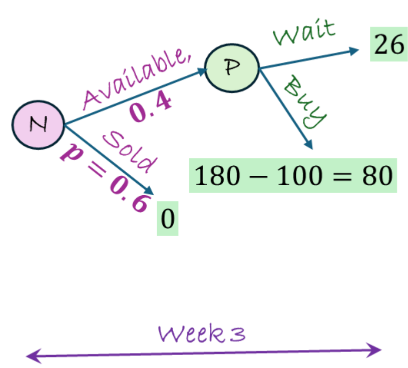
We've replaced the Player decision node with the payoff for the player's best action: 130. Now, it's Nature's turn. Nature picks not the best action but an average of the two actions. The expected payoff is given by—

$$E[N] = 0.2 \times 130 + 0.8 \times 0 = 26$$

After replacing the Nature node with this expected payoff, the reduced decision tree now looks like this—

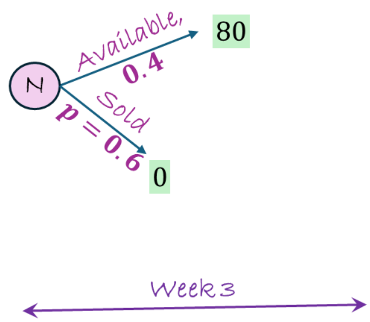


Week 3 (Player Decision Node)



Moving backwards to week 3. After Nature has chosen between available and sold, the action set for the decision-maker is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of 26 (the expected payoff from buying the knives in week 4). Choosing to buy in week 3 gives a payoff of 80. **The rational choice, then, is to buy the knives in week 3 for a payoff of 80.**

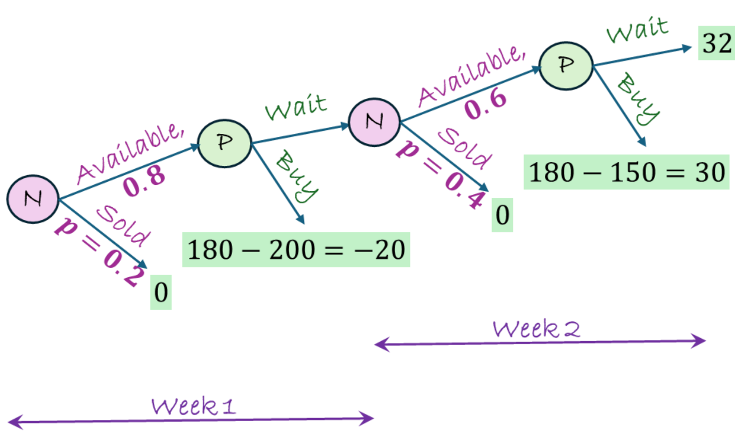
Week 3 (Nature Decision Node)



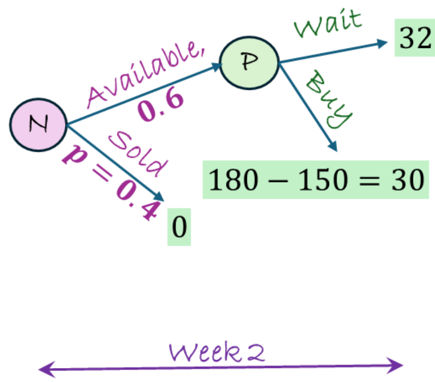
We've replaced the Player decision node with the payoff for the player's best action: 80. Now, it's Nature's turn. Nature picks not the best action but an average of the two actions. The expected payoff is given by—

$$E[N] = 0.4 \times 80 + 0.6 \times 0 = 32$$

After replacing the Nature node with this expected payoff, the reduced decision tree now looks like this—



WEEK 2 (PLAYER DECISION NODE)

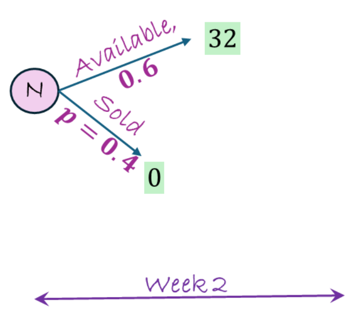


Moving backwards to week 2. After Nature has chosen between available and sold, the action set for the decision-maker is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of 32 (the expected payoff from buying the knives in week 3). Choosing to buy in week 2 gives a payoff of 30. **The rational choice, then, is to wait in week 2 and buy the knives in week 3 for a payoff of** 32**.**

Thus, the answer to the question 'At the beginning of which week, if any, should you run to buy the knives?' is week 3.

For completeness, we'll continue with backward induction below—

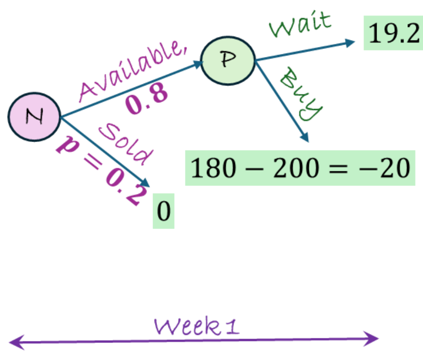
WEEK 2 (NATURE DECISION NODE)



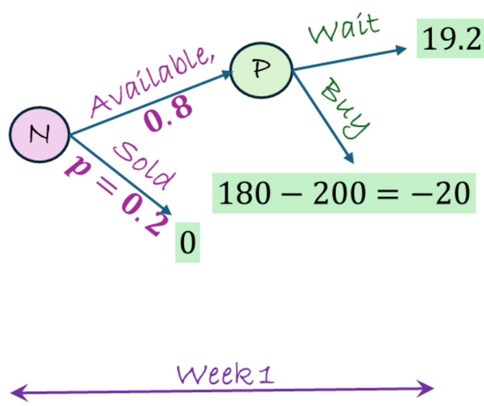
We've replaced the Player decision node with the payoff for the player's best action: 32. Now, it's Nature's turn. Nature picks not the best action but an average of the two actions. The expected payoff is given by—

$$E[N] = 0.6 \times 32 + 0.4 \times 0 = 19.2$$

After replacing the Nature node with this expected payoff, the reduced decision tree now looks like this—

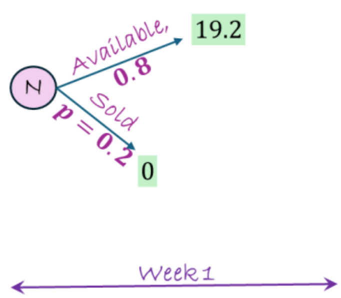


Week 1 (Player Decision Node)



Moving backwards to week 1. After Nature has chosen between available and sold, the action set for the decision-maker is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of 19.2 (the expected payoff from buying the knives in week 2). Choosing to buy in week 1 gives a payoff of -20. The rational choice, then, is to wait in week 1 and buy the knives in week 2 for a payoff of 19.2.

Week 2 (Nature Decision Node)



We've replaced the Player decision node with the payoff for the player's best action: 19.2. Now, it's Nature's turn. Nature picks not the best action but an average of the two actions. The expected payoff is given by—

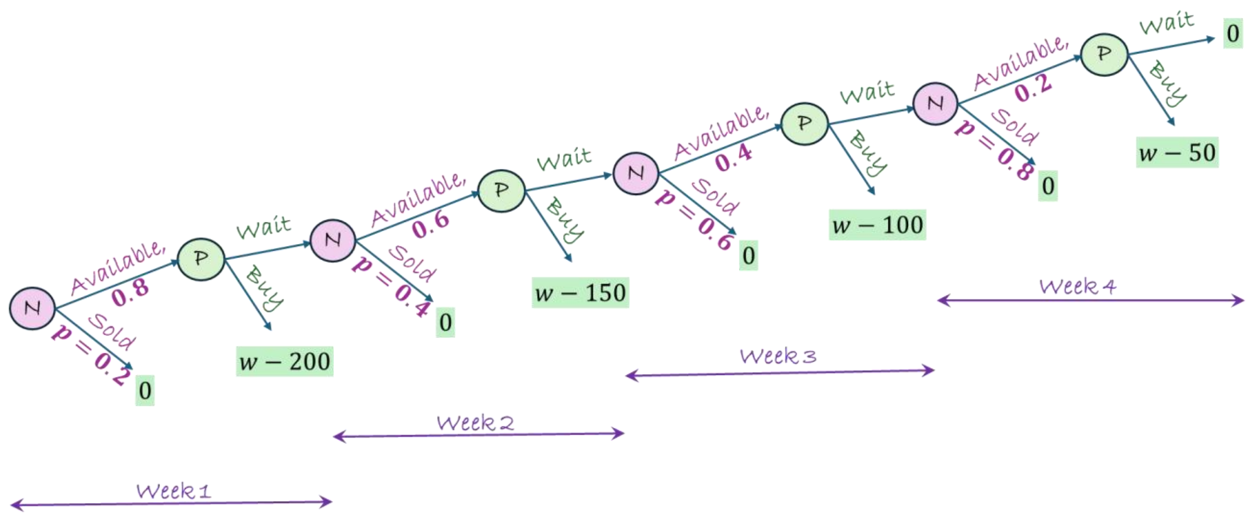
$$E[N] = 0.8 \times 19.2 + 0.2 \times 0 = 15.36$$

This is the expected payoff from the knives.

ANSWER (C). WILLINGNESS (VALUE) OF KNIVES FOR BUYING IN WEEK 1

Instead of the numerical value of 180 for the willingness (Rupee value) of knives given in the problem statement, let this value be denoted by w. The solution needs to be worked out, once again, by backward induction just like in part (c).

This is the decision tree with the willingness set to w (instead of 180)—



WEEK 4 (PLAYER NODE)

Starting with week 4. The action set for the player is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of 0. Choosing to buy gives a payoff of $w - 50$. We want w to be such that the rational choice will be to buy the knives in week 4, i.e., $w - 50 > 0$.

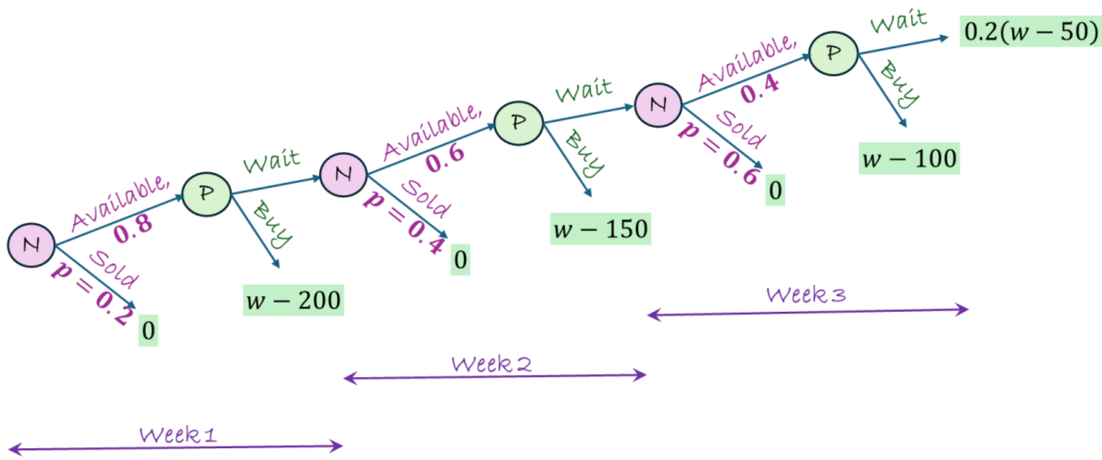
We replace the Player decision node with the best payoff of $w - 50$.

WEEK 4 (NATURE NODE)

Nature has two actions to choose from: available or sold. The payoff of the action 'available' is $w - 50$. The payoff of sold is 0. The probability of available is 0.2. The probability of sold is 0.8. The expected payoff is—

$$E[N] = 0.2 \times (w - 50) + 0.8 \times 0 = 0.2(w - 50)$$

This is what the reduced decision tree looks like—



Week 3 (Player Node)

Moving backwards to week 3. The action set for the player is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of $0.2(w-50)$. Choosing to buy gives a payoff of $w-100$. We want w to be such that the rational choice will be to buy the knives in week 3, i.e., $w-100 > 0.2(w-50)$.

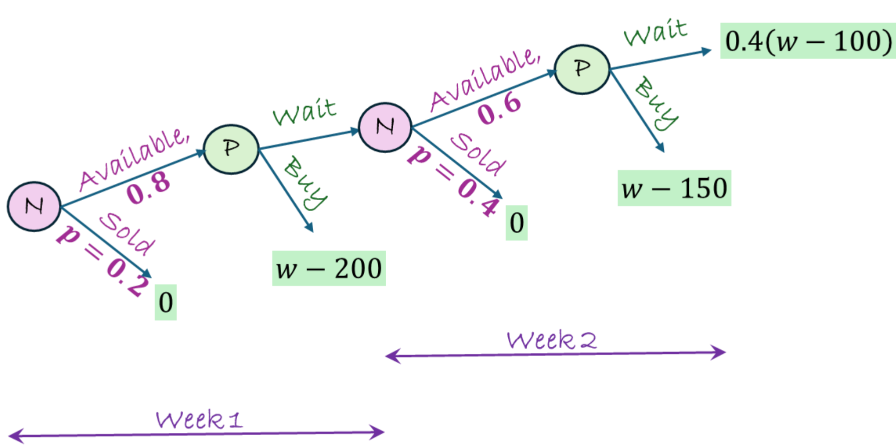
We replace the Player decision node with the best payoff of $w-100$.

Week 3 (Nature Node)

Nature has two actions to choose from: available or sold. The payoff of the action 'available' is $(w-100)$. The payoff of sold is 0. The probability of available is 0.4. The probability of sold is 0.6. The expected payoff is—

$$E[N] = 0.4(w-100) + 0.6 \times 0 = 0.4(w-100)$$

The reduced decision tree looks like this—



Week 2 (Player Node)

Moving backwards to week 2. The action set for the player is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of $0.4(w-100)$. Choosing to buy gives a payoff of $(w-150)$. We want w to be such that the rational choice will be to buy the knives in week 2, i.e., $w-150 > 0.4(w-100)$.

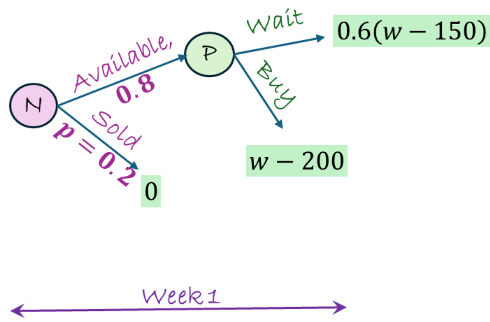
We replace the Player decision node with the best payoff of $w - 150$.

WEEK 2 (NATURE NODE)

Nature has two actions to choose from: available or sold. The payoff of the action 'available' is $(w - 150)$. The payoff of sold is 0. The probability of available is 0.6. The probability of sold is 0.4. The expected payoff is—

$$E[N] = 0.6 \times (w - 150) + 0.4 \times 0 = 0.6(w - 150)$$

This is the reduced decision tree—



WEEK 1 (PLAYER NODE)

Moving backwards to week 1. The action set for the player is $A \in \{wait, buy\}$. Choosing to wait gives a payoff of $0.6(w - 150)$. Choosing to buy gives a payoff of $(w - 200)$. We want w to be such that the rational choice will be to buy the knives in week 1, i.e., $w - 200 > 0.6(w - 150)$.

We replace the Player decision node with the best payoff of $(w - 200)$.

WEEK 1 (NATURE NODE)

Nature has two actions to choose from: available or sold. The payoff of the action 'available' is $0.8(w - 200)$. The payoff of sold is 0. The probability of available is 0.8. The probability of sold is 0.2. The expected payoff is—

$$E[N] = 0.8(w - 200) + 0.2 \times 0 = 0.8(w - 200)$$

This is the expected payoff of the knives in terms of their rupee value, w.

For it to be optimal to buy at the beginning of week 1, we have the condition—

$$w - 200 > 0.6(w - 150)$$

$$\therefore w - 200 > 0.6w - 90$$

$$\therefore 0.4w > 110$$

$$\therefore w > 275$$

So, if the knives are worth Rs. 275 or greater, the decision-maker should purchase them at the start of week 1.

Answer (D). Willingness (Value) of Knives for Buying in Week 4

For it to be optimal to buy at the start of week 4 (in week 4, since each week is a 'point' in time), it should be advisable to wait instead of buy in weeks 1, 2, and 3. From part (c), we know that—

Week 1

$$v(wait) = 0.6(w - 150)$$

$$v(buy) = w - 200$$

For it to be optimal to wait in week 1—

$$v(wait) > v(buy)$$

$$\therefore 0.6(w - 150) > w - 200$$

$$\therefore w - 200 < 0.6(w - 150)$$

$$\therefore w - 200 < 0.6w - 90$$

$$\therefore 0.4w < 110$$

$$\therefore w < 275$$

So, it's worthwhile to wait in week 1 if the decision-maker values the knives at $w < 275$.

Week 2

$$v(wait) = 0.4(w - 100)$$

$$v(buy) = w - 150$$

For it to be optimal to wait in week 2—

$$v(wait) > v(buy)$$

$$\therefore 0.4(w - 100) > w - 150$$

$$\therefore w - 150 < 0.4(w - 100)$$

$$\therefore w - 150 < 0.4w - 40$$

$$\therefore 0.6w < 110$$

$$\therefore w < 183.33$$

So, it's worthwhile to wait in week 2 if the decision-maker values the knives at $w < 183.33$.

Week 3

$$v(wait) = 0.2(w - 50)$$

$$v(buy) = w - 100$$

For it to be optimal to wait in week 3—

$$v(wait) > v(buy)$$

$$\therefore 0.2(w-50) > w-100$$

$$\therefore w-100 < 0.2(w-50)$$

$$\therefore w-100 < 0.2w-10$$

$$\therefore 0.8w < 90$$

$$\therefore w < 112.5$$

So, it’s worthwhile to wait in week 3 if the decision-maker values the knives at $w < 112.5$.

Thus, it would be optimal to buy at the beginning of the fourth week if the willingness to pay is $w < 112.5$.