Operations Research I. 2025/2026 Fall

1. practicum

Course Structure

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- Midterms: 1. Midterm: Around the 7th week (October 20-22.), 2. Midterm: Second to last week (December 1-5.). Both midterms are 70 points (7 problems), minimum 20 points are required from each. Both are 120 minutes!
  Corrections: At the start of the first week of the exam season (Only one midterm can be corrected).
  Replacement exam (both midterms worth of material in 120 minutes, min 40 points): Second half of the first week of exam season.
- Homeworks: Turn in on moodle for the given point value. Only the problems with a point value are worth points, the others will be discussed during practicum. At most 5 problems can be turned in per week, and a minimum of 5 points must be obtained over the course of the semester. Further points count towards your final grade.
  Weekly deadline: Midnight on Wednesday.
- Grading scheme: 2 midterm points sums + (Homework points).
  Point-grade totals: 40=2, 60=3, 80=4, 100=5

Certificates

1. Give certificates/witness for the following properties, as well as for their refutal (if applicable):

a) G graph bipartite (2-colorable).

b) G graph is planar.

c) D digraph acyclic.

d) G graph contains Hamiltonian path (a simple path that reaches all vertices).

e) D digraph contains an $s-t$ path on 5 vertices.

f) D digraph contains an $s-t$ path on k vertices (k part of the input).

g) A $D=(V,A)$ digraph with a given cost function $c: A \to \mathbb{R}$ doesn't contain a negative cost cycle.

h) G graph does not contain two edge-disjoint spanning trees.

i) (0.5) G graph contains two vertex-disjoint $s-t$ paths for some s,t.

j) (0.5) A graph G has diameter k, where k is part of the input. The diameter of a graph is the maximum length of a shortest path between any pair of vertices.

2. List any previously learned

(a) Algorithms (with input and output);

(b) Min-max theorems;

(c) Connect the two!

3. Given a directed graph with edge costs, we would like to decide if it contains a negative cost cycle. What is wrong with the following algorithm:
List each cycle, and see if there is one with negative total cost.

Reductions

We will often wish to prove statements of the following form: "If we can solve problem A in polynomial time then we can solve problem B in polynomial time as well." Such a statement is a (polynomial time) reduction from problem B to problem A. We can prove such a statement by giving a polynomial time algorithm that uses an algorithm solving problem A as a subroutine to solve problem B.

4. Give a reduction from finding a Hamiltonian cycle to solving the following problems. In other words, decide if a given $G = (V, E)$ graph has a Hamiltonian cycle using one of the following problems as a subroutine:

(a) st-Hamiltonian-path: decides if a graph has a s-t Hamiltonian path. INPUT: $G = (V, E)$ graph, $s, t \in V$ vertices; OUTPUT: yes/no.

(b) (1) Unconstrained shortest path: given arbitrary(!) $c : E \to \mathbb{R}$ edge costs decides if there is a path of cost $\leq K$. INPUT: $G = (V, E)$ graph, $c : E \to \mathbb{R}$ edge costs, $K \in \mathbb{R}$ bound; OUTPUT: yes/no.

5. (1) We are given a subroutine that finds the shortest $s - t$ path in an edge weighted acyclic graph, for some fixed vertices s, t. Using this, give an algorithm to find the shortest path among any two vertices in the digraph. Do the reverse as well, to show the two problems are equivalent up to polynomial time reductions.

Shortest paths and BFS

6. (1) Given an acyclic graph with *vertex* weights $b : V \to \mathbb{R}$, give an algorithm to find the minimal weight s, t paths for some fixed vertices s and t.

7. (1) Why are shortest/longest path problems equivalent on (potentially negative) edge-weighted acyclic graphs?

8. (1) Modify the BFS algorithm to check if a given graph G is bipartite.

9. (1) Given a string of integers, give an algorithm to find the longest non-decreasing substring. For example, the longest such substring of $(5, 2, 7, 3, 5, 7, 3)$ would be $(2, 3, 5, 7)$.

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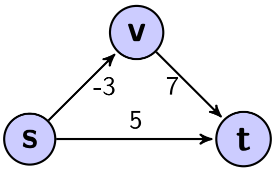
Potential functions, Gallai's theorem, and Bellman-Ford

Feasible potentials: a potential π is c-feasible if $c_\pi \leq c$, that is for all $uv \in E$, $\pi(v) - \pi(u) \leq c(uv)$.
Gallai's Theorem: For $D = (V, A)$ digraph, an edge cost c is conservative if and only if there exists a c-feasible potential function.

1. Consider the follow digraph:



(a) Give any feasible potential!

(b) Give one maximizing $\pi(t) - \pi(s)$, or show why it doesn't exist.

(c) Give one minimizing $\pi(t) - \pi(s)$, or show why it doesn't exist.

2. Given a digraph with potential function π and vertices s, t, prove that any st-path has the same c_π cost.

3. Given a digraph with edge costs c and a c-feasible potential π, give a bound for the length of a st-path. When will this bound be tight?

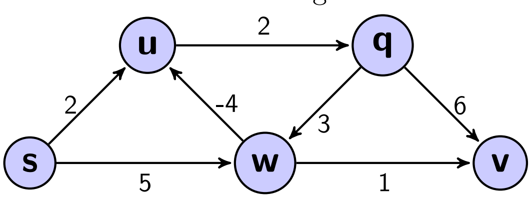
4. (1) Given a conservative cost function c and a c-feasible potential π, use *only Dijkstra's algorithm* to find a shortest st-path. Use problem 2 to prove correctness.

5. Let $D = (V, A)$ be a digraph with edge costs c. Prove that if there is a negative cost path leading into each vertex, then c is not conservative.

(a) (0,5) Argue why the following proof is incorrect:
For each $v \in V$, let P_v be the negative cost path leading into v. Start with an arbitrary vertex v_0, and let v_1 be the first vertex of P_{v_0}, v_2 the first vertex of P_{v_1}, and so on. At some point, because the graph is finite, there will be a path containing some vertex u that we have already seen before. Following these negative cost paths between the two occurrences of v, we create a negative cost cycle, showing that c is not conservative.

(b) (1) Give a correct proof of this claim! (you may use Gallai's theorem)

6. Run the Bellman-Ford algorithm on the following digraph.



7. (1) Given a digraph $D = (V, A)$ with a conservative weight function c, consider the following "auxiliary" graph $D' = (V', A')$. The vertex set of D' has n copies of V numbered $V_1, V_2, \ldots, V_n$, and for each edge $uv \in A$ there is an edge $u_i v_{i+1} \in A'$ for each $i = 1 \ldots n-1$. Show that running the algorithm for shortest path on DAGs on D' is the same as running Bellman-Ford on D.

8. In the following, we are given a digraph and a cost function that is *not necessarily* conservative.

(a) (0.5) Give an algorithm to find the minimum cost walk of length (number of edges) exactly k.

(b) (1) Give an algorithm to find the minimum average cost cycle, where the average cost a cycle C is $c(C)/|C|$.

9. (0.5) You are working at a bank that supports multiple currencies, and offers exchange rates between any pair of currencies. Give an algorithm to check if there is a significant mistake in these conversion rates, namely some sequence of conversions that results in more of a given currency than at the beginning!

10. (2) Given a digraph $D = (V, A)$ and real value $0 < \lambda < 1/2$, decide if there is a cycle C in D for which less than $\lambda|C|$ edges are going in some direction.

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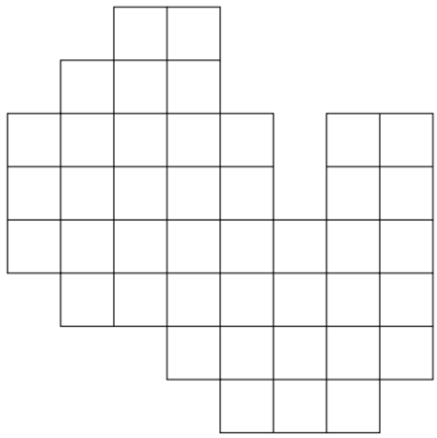
Matchings
Review: Matching, vertex covers, alternating paths, Kőnig's theorem, Hall's theorem

1. Give a witness showing that a bipartite graph has no perfect matching.

2. (0,5) Can the following shape be tiled with 2x1 and 1x2 dominoes?



3. (1) Say a bipartite graph G has exactly one perfect matching. Show that there must then be some vertex with degree 1.

4. (2) We are given a digraph $D = (V, A)$ with terminal sets $S, T \subseteq V$ for which $|S| = |T|$, and $S \cap T = \emptyset$. Reduce the problem of finding $|S|$ vertex-disjoint paths from vertices in S to those in T to finding a perfect matching in a bipartite graph.

5. (1) Let M be a maximal matching in a (not necessarily bipartite) graph G. Show that the size of a maximum matching in G is at most $2|M|$.

6. (1) Given a bipartite graph G such that the degree of each vertex is $\geq k$ and has at least one perfect matching, show that G has at least $k!$ perfect matchings.

7. A *latin square* is an $n \times n$ square grid of integers $1 \ldots n$ such that no row nor column have repeated numbers.

(a) (0,5) Show that for any n, there is at least one $n \times n$ latin square.

(b) (0,5) Given a partially completed $n \times n$ grid, use matchings to decide if it can be completed into a latin square.

(c) (0,5) For $m < n$, a *latin rectangle* is a $m \times n$ grid of integers $1 \ldots n$ such that no row nor column have repeated numbers. Show that any latin rectangle can be extended to a latin square!

(d) (0,5) Show that there are at least $n!$ latin squares.

8. (2) Given a partially ordered set $(S, \leq)$, use Kőnig's theorem to prove that the size of a maximum antichain is equal to the minimum number of chains needed to cover S. A chain in a poset is a set $\{x_1, \ldots, x_k\} \subseteq S$ such that $x_1 \leq x_2 \leq \ldots \leq x_k$; and an antichain is a set of mutually incomparable elements, that is a $\{y_1, \ldots, y_k\} \subseteq S$ such that neither $y_i \leq y_j$ nor $y_j \leq y_i$ for any $1 \leq i < j \leq k$.

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Weighted Matchings

Weighted covers: A function $\pi : V \to \mathbb{R}$ such that $\pi(u) + \pi(v) \geq c(u, v)$ for all edges $uv \in E$.
Tight edge (for a given edge weights c and weighted cover π): an edge for which $\pi(u) + \pi(v) = c(u, v)$.
Egerváry's theorem: For a bipartite graph $G = (S, T, E)$ with a perfect matching and edge weights $c : E \to \mathbb{R}_+$, the total weight of the maximum weighted cover is equal to the minimum total value of a weighted cover. In other words,

$$\min_{\pi} \sum_{v\in V} \pi(v) = \max_{M} \sum_{e\in M} c(e)$$

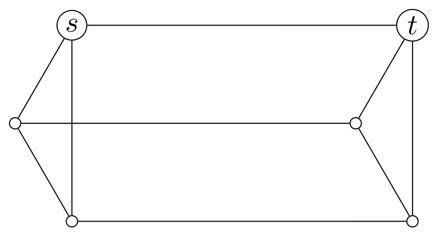
1. Do all weighted covers have the same set of tight edges? What edges are tight in all weighted covers?

2. (1) Give an example where the Hungarian algorithm decreases the number of tight edges when modifying the weighted cover.

3. (1) Prove the following: for a given bipartite graph $G = (S, T; E)$, if there is a max weight matching M_1 covering some $A \subseteq S$, and a max weight matching M_2 covering some $B \subseteq S$, then there is a max weight matching M covering $A \cup B$ as well.

4. (1) If the bipartite graph doesn't have a perfect matching, prove that there is no minimum weighted cover under any edge weight.

5. (1) Give an edge weighting of the following graph where the following claim is false: If every edge of a perfect matching M is contained in some max weight matching, then M itself is a maximum weight matching.



Flows, Circulations

Feasible flow: Given a directed graph $D = (V, A)$ with edge capacities $c : A \to \mathbb{R}_+$ and two vertices $s, t \in V$, a function $f : A \to \mathbb{R}_+$ such that $f(e) \leq c(e)$ for all edges $uv \in A$ and $\rho_f(v) := \sum_{uv\in A} f(uv) = \sum_{vw\in A} f(vw) =: \delta_f(v)$ for all vertices $v \in V \setminus \{s, t\}$.
Feasible circulation: Given a directed graph $D = (V, A)$ with edge capacities $c : A \to \mathbb{R}_+$, a function $f : A \to \mathbb{R}_+$ such that $f(e) \leq c(e)$ for all edges $uv \in A$ and $\sum_{uv\in A} f(uv) = \sum_{vw\in A} f(vw)$ for all vertices $v \in V$.

6. Give a feasible s, t flow for the graph from problem 5 with unit capacities.

7. Prove the following:

a) (0,5) Every $\{0, 1\}$-valued circulation is the sum (as incidence vectors) of edge-disjoint directed cycles.

b) (0,5) Every nonnegative circulation can be written as a nonnegative linear combination of the incidence vectors of directed cycles.

8. (1) Prove that every feasible flow f can be written as the sum of a feasible circulation and some s, t paths.

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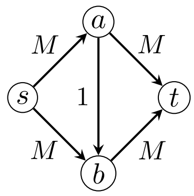
Ford-Fulkerson, Max Flow Min Cut

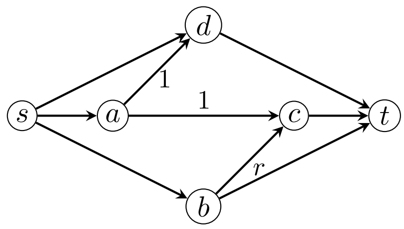
Feasible flow: Given a directed graph $D = (V, A)$ with edge capacities $c : A \to \mathbb{R}_+$ and two vertices $s, t \in V$, a function $f : A \to \mathbb{R}_+$ such that $f(e) \leq c(e)$ for all edges $uv \in A$ and $\rho_f(v) := \sum_{uv \in A} f(uv) = \sum_{vw \in A} f(vw) =: \delta_f(v)$ for all vertices $v \in V \setminus \{s, t\}$.
Max Flow Min Cut Theorem: Given a directed graph $D = (V, A)$ with edge capacities $c : A \to \mathbb{R}_+$ and two vertices $s, t \in V$, the weight of the maximum flow is equal to the weight of the minimum cut, ie

$$\max_{f \text{ flow}} \rho_f(t) = \min_{s \in S \subseteq V \setminus \{t\}} c(\delta(S))$$

1. Run the Ford-Fulkerson on the following graph. If you always choose the augmenting path of length 3, what is the running time of the algorithm? Here $M \geq 1$ is an arbitrary large value.



2. (2) Let $r = \frac{1-\sqrt{5}}{2}$ in the graph below. Show that under some path choice the Ford-Fulkerson algorithm can run indefinitely on the following graph, where unlabeled edges have some large constant capacity.



3. Connect the duality on flows to some duality in matchings:

a) (0,5) Use the Ford-Fulkerson algorithm to find a maximum matching in an unweighted bipartite graph.

b) (1) What problem corresponds to the minimum cut in this reduction? Deduce a min-max theorem about matchings from max flow min cut.

4. Let $D = (V, A)$ be a directed graph with fixed vertices s, t.

a) (0,5) Use the Ford-Fulkerson algorithm to decide if there are k edge-disjoint paths between s and t.

b) (1) Use max flow min cut to prove the following version of Menger's theorem: There exist k edge disjoint $s \to t$ paths in D if and only if $\delta(U) \geq k$ for all $s \in U \subseteq V \setminus \{t\}$.

5. (1) Design a polynomial-time algorithm that decides, for a given flow problem, whether there is a maximum flow f that has $f(e) = c(e)$ for a given edge e.

6. (1) We are given a cost function $c : V \to \mathbb{R}$ on the vertices of a digraph. Design a polynomial-time algorithm to find a maximum weight set with no edges entering it.

7. (1) A strongly connected digraph D is Eulerian is the indegree of every vertex is equal to the outdegree. Design a polynomial-time algorithm to find the minimum number of edge flips needed to make a given graph Eulerian.

8. (1) Give a polynomial-time algorithm to find a minimum cost perfect matching in an edge-weighted bipartite graph with integer weights.

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Practice Midterm 1

1. Given a directed graph $D = (V, A)$ with a conservative weight function $c : A \to \mathbb{R}$, show that a directed cycle with total weight 0 consists of only tight edges under any feasible potential function.

2. Given a directed graph $D = (V, A)$ with two fixed vertices s, t and a conservative weight function $c : A \to \mathbb{R}$, give a polynomial-time algorithm to decide if there are two edge-disjoint minimum cost $s \to t$ paths.

3. Let $G = (A, B; E)$ be a bipartite graph with edge weights $c : E \to \mathbb{R}$, and $M \subseteq E$ a perfect matching of G. Orient G by directing the edges of M from A to B, and edges of $E \setminus M$ towards A to get a directed graph G. Let c' be such that edges of M have weight c, and edges of $E \setminus M$ have weight $-c$. Prove that M is a maximum c-weight perfect matching if and only if c' is a conservative weighting of D.

4. Let $G = (A, B; E)$ be a k-regular bipartite graph. Prove that G is the disjoint union of k perfect matchings.

5. Let π_1, π_2 be weighted covers of a bipartite graph $G = (A, B; E)$ with edge weights c. Prove that $(\pi_1 + \pi_2)/2$ is also a weighted cover, showing that the weighted covers form a convex set.

6. We are given a $D = (V, A)$ directed graph with a function $m : V \to \mathbb{Z}$ on the vertices. A function $x : A \to \mathbb{Z}$ is a m-circulation, if $\rho_x(v) - \delta_x(v) = m(v)$ for all $v \in V$. Reduce the problem of finding a nonnegative m-circulation to a flow problem.

7. Given a digraph $D = (V, A)$ with capacities $c : A \to \mathbb{R}$, say flows f_1 and f_2 are both maximal $s - t$ flows. Prove that there exists a maximal flow f' which has the same flow as f_1 on edges adjacent to t, and the same flow as f_2 on edges adjacent to s.

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Polyhedrons and cones

Review: Polyhedron, polytope, cone, polyhedral cone, finitely generated cone, dimension, face, vertex, facet.

1. (1) Prove the following:

a) The intersection of two polyhedral cones is a polyhedral cone.
b) The Minkowski sum of two finitely generated cones is a finitely generated cone.
c) The intersection of two convex sets is convex.
d) The intersection of two polyhedra is a polyhedron.
e) The Minkowski sum of two polytopes is a polytope.

2. (1) Let $P = \{x \in \mathbb{R}^n | Qx \leq b\}$ be a nonempty polyhedron. Group the following items into equivalence classes, with reasoning:

a) P is bounded.
b) P is not bounded.
c) P contains a line.
d) P contains a ray.
e) P has at least one vertex.
f) P is a polytope.
g) P is a polytope.
h) $Qx = 0$ has a nonzero solution.
i) $Qx \leq 0$ has a nonzero solution.
j) Q has rank n.

3. (0,5) Prove that a polyhedron $P = \{x : Ax \leq b\}$ contains a line in the direction of a vector q passing through the point x_0 if and only if $Aq = 0$. (What is the counterpart of this statement for rays?)

4. (0,5) Consider $Q = \text{conv}(e_1, e_2, \ldots, e_n, -e_1, -e_2, \ldots, -e_n)$, where e_i is ith elementary vector (only 1 at i) and conv() is the convex hull. Write Q as a polyhedron!

5. (0,5) Is it true that if $\max cx,\ Ax \leq b, x \geq 0$ is unbounded then there is an elementary vector e_k such that $\max e_k x,\ Ax \leq b, x \geq 0$ is unbounded?

6. (1) Find a polyhedron P such that the convex hull of integer points within P is not a polyhedron.

Oracles for LP problems

7. (0,5) Say we had an oracle (subroutine) that tells us if a polyhedron $Qx \leq b$ is empty. Using this, decide if a polyhedron of the form $Ax = b, x \geq 0$ is empty.

8. (1) Say we had an oracle (subroutine) that tells us if a polyhedron $Ax = b, x \geq 0$ is empty. Using this, decide if a polyhedron of the form $Qx \leq b, x \geq 0$ is empty.

9. (2) Again, say we had an oracle (subroutine) that tells us if a polyhedron $Qx \leq b, x \geq 0$ is empty. Using *only a single call* to this subroutine, decide if a polyhedron of the form $Qx \leq b$ is empty.

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Farkas Lemma

Farkas Lemma, standard form: $\nexists x \in \mathbb{R}^n$ ($\mathbb{Q}^n$): $Ax = b$, $x \geq 0$ if and only if $\exists y^T \in \mathbb{R}^m$ ($\mathbb{Q}^m$): $yA \geq 0$, $yb < 0$.

1. (0,5) Use the Farkas Lemma to show the following polyhedron has a solution:

$$\begin{aligned} x + y + z &= 5 \\ x - y + v &= -5 \\ y - z + w &= -5 \\ x, y, z, v, w &\geq 0 \end{aligned}$$

2. (1) Prove the following form of the Farkas Lemma using the above form:

$$\exists x : Qx \leq b \iff \nexists y \geq 0 : yQ = 0, yb < 0.$$

3. (1) Prove Gallai's theorem on conservative edge weightings using the Farkas Lemma!

4. (0,5) Let $P = \{x \colon Qx \leq b\}$ be a nonempty polyhedron. Prove that the first inequality in $Qx \leq b$ is implicit (true with equality for every $x \in P$) if $\exists y \geq 0 \colon yQ = 0, yb \leq 0, y_1 > 0$.

Polyhedrons and Polytopes

5. Remember Lajos and his Halloween party? (Problem sheet 6, problem 1) Let P be the polytope of Lajos's solutions. What are the faces of P? The facets? What are the active rows at the point with 5 beers and 5 sausages? And at the point (3 beers, 3 sausages)?

6. (0,5) Prove that every facet is the intersection of some faces.

7. (0,5) In lecture we saw every polyhedron P can be written as the sum of a polytope P' and its characteristic cone K. Using this, prove that if for some $x_0 \in P$ the ray $x_0 + \lambda q \in P$ for all $\lambda \geq 0$, then for any $x \in P$, the ray $x + \lambda q \in P$ for all $\lambda \geq 0$.

8. (0,5) Given a nonempty polyhedron $P = \{x \colon Ax = b, x \geq 0\}$, prove that P has a vertex.

9. (1) Let $C \subset \mathbb{R}^n$ be a generated cone that does not contain some $b \in \mathbb{R}^n$. Then show that there exists some homogeneous halfspace H such that $C \subseteq H$, but $b \notin H$. Show the same for a polytope P. How does this relate to the Farkas Lemma?

10. (1,5) Given an integer polytope $P = \{x \colon Ax \leq b\}$ (the vertices are integers) and a point $x \in P$, give a polynomial time algorithm to find an integer point in P.