Matemática Contemporânea
https://doi.org/10.1007/s44425-025-00020-9

ORIGINAL ARTICLE

Diameter Reduction via Arc Reversal

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Received: 15 September 2025 / Accepted: 15 September 2025

Abstract

The diameter of a directed graph is the maximum distance between any pair of vertices. We study a problem that generalizes ORIENTED DIAMETER: For a given directed graph and a positive integer d, what is the minimum number of arc reversals required to obtain a graph with diameter at most d? We investigate variants of this problem, considering the number of arc reversals and the target diameter as parameters. We show hardness results under certain parameter restrictions, and give polynomial time algorithms for planar and cactus graphs. This work is partly motivated by the relation between oriented diameter and the volume of directed edge polytopes, which we show to be independent.

Keywords Diameter reduction · Arc reversal · Oriented diameter

Mathematics Subject Classification 05C85 · 68R10

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Published online: 12 November 2025

1 Introduction

Inspired by an application in road traffic control, to make traffic flow more efficient when using only one-way streets, Robbins [1] states that an undirected graph admits a strongly connected orientation, if and only if it is 2-edge-connected[1]. However, what makes an orientation "good" can vary according to the problem description. For example, we might want to minimize the distance from a given vertex to all other vertices or we might seek to minimize the average distance between all pairs of vertices. Here, we consider an *oriented diameter problem* on directed graphs. In this problem our goal is to minimize the maximum distance between the vertices (called the *diameter* of the graph) by changing the direction of as few arcs as possible. One motivation for this problem is the following: Suppose that there are *n* airports, of which some ordered pairs are connected by a flight. For a given integer *d* how many flights do we have to establish to make sure that from each airport one can get to another by changing planes at most *d* times? Other instances where knowing the diameter of a graph is helpful include the analysis of social networks as well as parallel computing, see e.g. [2, 3].

To formalize the problem, we introduce some notation. Given a graph (directed or undirected) $D = (V, A)$, we denote an edge between $u, v \in V$ by uv and an arc from u to v by (u, v). We denote a path P between $u, v \in V$ by $(u, w_1, w_2, \dots, w_\ell, v)$ where $w_0 = u, w_1, \dots, w_\ell, w_{\ell+1} = v$ are the vertices of the path so that $(w_i, w_{i+1}) \in A$ for all $i = 0, \dots, \ell$. The *length* of P is its number of edges (or arcs) and it is denoted by $|P|$. For any $u, v \in V$, the *distance* between them is the length (or weight if the edges are weighted) of the shortest path connecting u and v; by convention, this distance is ∞ if there is no path connecting those vertices. The *diameter* of the graph D, denoted by $\mathrm{diam}(D)$ is the maximum distance among its pairs of vertices. An *arc reversal* of $e = (u, v)$ in a directed graph (V, A) results in a graph $(V, (A \setminus (u, v)) \cup (v, u))$.

In this paper we consider the following problems.

Diameter Reduction In At Most k Steps (k-Reversals)

Input: A digraph $D = (V, A)$ with diameter d_D, and two integers $2 \leq d < d_D$, and $k \geq 0$.

Goal: Decide whether a digraph of diameter at most d can be obtained from D by performing at most k arc reversals.

Weighted k-Reversals

Input: A digraph $D = (V, A)$ with diameter d_D, nonnegative edge weights $w : A \to \mathbb{R}_{\geq 0}$, and two integers $2 \leq d < d_D$, and $k \geq 0$.

Goal: Decide whether a digraph of diameter at most d can be obtained from D by performing arc reversals of weight at most k.

Previous Results. The Oriented Diameter problem takes an undirected simple graph and an integer $d \geq 0$, seeking an orientation with a diameter of at most d. Chvátal and Thomassen [4] first addressed this problem, proving that finding an orientation with a diameter of 2 is NP-hard, and extending their result to any diameter

[1] A graph is *k*-edge-connected (resp. *k*-arc-connected) if at least *k* edges (resp. arcs) have to be removed in order to destroy connectivity.

Table 1 Complexity of the problems. Planar graphs is abbreviated as PG, while cactus graphs as CG. NP-hard, Weakly NP-hard, and W[2]-hard are abbreviated by NPh, WNPh and W[2]-h, respectively.

Problem	Parameter	Fixed	Complexity
k-REVERSALS			NPh – Thm. 1
k-REVERSALS		d	NPh – Thm. 1
k-REVERSALS		k	P – Thm. 4
k-REVERSALS	k	d	W[2]h – Thm. 2
k-REVERSALS (PG)			Open
k-REVERSALS (PG)	d		FPT – Sec. 3.2
k-REVERSALS (PG)	k		Open
k-REVERSALS (CG)			P – Thm. 5
WEIGHTEDk-REVERSALS (CG)			WNPh – Thm. 3
ORIENTED DIAMETER (PG)			Open
ORIENTED DIAMETER (PG)	d		FPT –[6]

$d \geq 4$. The case for $d = 3$ remains unresolved. The hardness of the problem for planar graphs is also unknown, despite several attempts to explore it. Bensmail, Duvignau, and Kirgizov [5] recently explored orientations in terms of the so-called *weak diameter*. A digraph D has d-weak (resp. d-strong) diameter if the maximum weak (resp. strong) distance between any two vertices u and v, i.e. the minimum (resp. maximum) of the distance from u to v and that from v to u, is d. They proved that deciding if an undirected graph has a d-weak orientation is NP-complete for $d \geq 2$ and conjectured that the same is true for d-strong orientations. This conjecture would complete the results of Chvátal and Thomassen for $d = 3$. Eggemann and Noble [6, 7] developed an FPT algorithm for ORIENTED DIAMETER in planar graphs, parameterized by treewidth. Mondal, Parthiban, and Rajasingh [8] focused on triangular grid graphs, presenting a polynomial-time algorithm for this case, while also demonstrating that the weighted version of ORIENTED DIAMETER is weakly NP-complete for planar graphs with bounded pathwidth. Additionally, Fomin, Matamala, and Rapaport [9] examined chordal graphs, providing an approximation algorithm for ORIENTED DIAMETER and proving that determining if a chordal graph can achieve diameter k is NP-complete. Ito et al. [10] studied the problem of finding a graph orientation that maximizes arc-connectivity[2]. Hoppenot and Szigeti [11] explored digraphs with k-arc-connectivity at most $\lfloor (k + 1)/2 \rfloor$, showing that if reversing a subset of arcs F can achieve k-arc-connectivity, then reversing only one arc of F does not reduce overall arc-connectivity. Additional problems related to arc-connectivity and reversals are discussed by Bang-Jensen, Hörsch, and Kriesell in [12], while Bang-Jensen, Costa Ferreira da Silva and Havet [13] consider the minimum number of reversals needed to make a digraph acyclic.

Our results. We summarize our results supplemented with the remaining (already solved or still open) cases in Table 1. Note that if k-REVERSALS is solvable on planar graphs, this implies that ORIENTED DIAMETER on planar graphs is also solvable (see

[2] Arc-connectivity is the maximum integer λ such that every non-empty subset $X \subsetneq V$ has at least λ arcs leaving X.

Section 2). On the other hand, if Oriented Diameter is NP-hard for planar graphs, then k-Reversals is also NP-hard on planar graphs.

We organize the remainder of the paper as follows. In Section 2, we explore some hardness results for k-Reversals with some parameters, while in Section 3 we present polynomial algorithms for special cases. Additionally, the results in Section 3.3 offer counter-examples to a conjecture regarding directed edge polytopes. We defer the preliminaries section and some of our results and proofs to the full version of this paper [14]; statements with deferred proofs are marked with the symbol $\star$.

2 Hardness Results

We show that k-Reversals is NP-hard. First, recall that Oriented Diameter has as input a *simple* undirected graph $G = (V, E)$ and $d \in \mathbb{Z}_{\geq 0}$, and the goal is to find an orientation such that the diameter is at most d. Chvátal and Thomassen [4] showed that it is NP-hard to decide whether an undirected graph admits an orientation of diameter 2. The authors also showed that for every $d \geq 4$, it is NP-hard to determine if an undirected graph can be oriented with diameter d.

Theorem 1 *The problem k-Reversals is NP-hard.*

Proof A solution to k-Reversals with parameters $k = |A|$ and d also yields a solution to Oriented Diameter with parameter d. By the above result of Chvátal and Thomassen, k-Reversals is NP-hard. □

For the rest of the section, we distinguish between weakly NP-hard problems, which are hard only with binary input, and strongly NP-hard problems, which remain hard with unary input (see e.g. [15]). For FPT algorithms, intractability often relies on reductions to complete problems in the $W[t]$ hierarchy.

2.1 W[2]-hardness for Number of Arc Reversals

Theorem 2 *($\star$) k-Reversals is* W[2]*-hard when parameterized by k.*

Proof (Proof sketch.) We reduce from Dominating Set, a W[2]-complete problem (e.g. [15]). In this problem, we are given an undirected graph $G = (V, E)$ and an integer $\ell \geq 0$, the goal is to determine if there is a dominating set of size at most ℓ. The reduction from an instance (G, ℓ) of Dominating Set is a digraph H with diameter four. The generated instance $(H, 3, \ell)$ has an arc reversal if and only if (G, ℓ) has a dominating set. □

2.2 Weak NP-hardness for Weighted Cactus Graphs

We consider k-Reversals for cactus graphs which is weakly NP-hard. Later, in Section 3.1 we show a dynamic programming algorithm. Recall that cactus graphs are

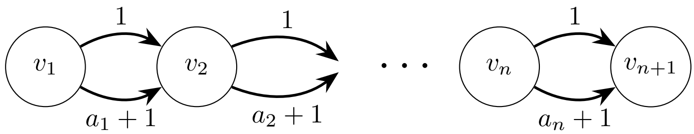

Fig. 1 Instance of Weighted *k*-Reversals on cactus graphs generated by an instance of Partition.

connected graphs where any two simple cycles share at most one vertex. We denote $[n] = \{1, 2, \ldots, n\}$ for an integer $n \geq 1$.

Theorem 3 *The problem Weighted k-Reversals is weakly NP-hard for cactus graphs.*

Proof We make a reduction from Partition which is known to be weakly NP-hard. Recall that in Partition, we are given *n* positive integers $A = \{a_1, a_2, \ldots, a_n\}$ with sum 2*b*, and the goal is to determine if there is a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = b$.

Given an input (*A*, *b*) for Partition, we construct a weighted cactus graph formed by two paths on the same vertex set– one weighted by *A* and the other with unit weights. Let $G = (V, E)$ be a graph with vertices $v_1, v_2, \ldots, v_{n+1}$. For all $i \in [n]$ we have two arcs from v_i to v_{i+1} denoted by e_i and f_i. Set the weights $w(e_i) = 1$ and $w(f_i) = (a_i + 1)$. Finally, we set $k = b + n$ and $d = (b + n)$. This is our input for Weighted *k*-Reversals on a cactus graph, as shown in Figure 1. Note that each arc is oriented either clockwise (for all arcs e_i) or counterclockwise (for all arcs f_i). This graph has diameter $\operatorname{diam}(G) = \infty$.

If there is an orientation with diameter $(b + n)$, the distance of v_1 and v_{n+1} are such that the arcs directed clockwise sum to at most $(b + n)$, and the arcs directed counterclockwise sum to at most $(b + n)$. As the total weight of the arcs is $2(b + n)$, it follows that both the distance from v_1 to v_n and the distance from v_n to v_1 must each be exactly $b + n$. If $S \subseteq [n]$ is the set of indices *i* such that arc f_i is directed clockwise, we have

$$\sum_{i \in S}(a_i + 1) + \sum_{i \in A \setminus S} 1 = \sum_{i \in A \setminus S} (a_i + 1) + \sum_{i \in S} 1$$
$$\sum_{i \in S} a_i + n = \sum_{i \in A \setminus S} a_i + n$$
$$\sum_{i \in S} a_i = \sum_{i \in A \setminus S} a_i = b.$$

Similarly, if $S \subseteq [n]$ is a set of indices with $\sum_{i \in S} a_i = b$, then reverse the arcs e_i if $i \in S$ and reverse f_i if $i \in A \setminus S$. We made $2n$ arc-reversals of weight $\sum_{i \in S} 1 + \sum_{i \in A \setminus S}(a_i + 1) = (b + n)$. For all $i \in [n]$, v_i and v_{i+1} have two arcs in both directions. Thus, the unique path from v_1 to v_2 uses arcs e_i if $i \in A \setminus S$ and arcs

f_i if $i \in S$, which has weight $b + n$. The rest of the arcs form a path from v_n to v_1 with weight $b + n$, implying $\mathrm{diam}(G) = b + n$ as required. □

3 Algorithms

In this section we give positive results for some special cases of the problem *k*-Reversals. Our first observation is that in case *k* is fixed, examining all sets of size at most *k* yields a polynomial time algorithm.

Theorem 4 *(⋆) k-Reversals is polynomially solvable if k is fixed.*

3.1 Dynamic Programming Algorithm for Cactus Graphs

First, note that the hardness result for weighted cactus graphs does not apply here as it only showed weak NP-hardness. For unweighted cactus graphs we can construct a unary weighted cactus graph by contracting paths and weighting them by their lengths. The key difference is that the orientations of a cactus graph are simpler to handle, as each cycle must be oriented either clockwise or counterclockwise.

Theorem 5 *(⋆) k-Reversals is solvable in polynomial time for unweighted cactus graphs.*

3.2 FPT Algorithm on Diameter for Planar Graphs

k-Reversals problem can be solved in polynomial time on planar graphs using the results of Eggeman and Noble [6, Thm. 2.5], by adding a extra factor of *k*, as shown in last section. See the details in the full version [14, Sec. C1].

3.3 Directed Edge Polytopes

A simple digraph $D = (V, A)$ is *symmetric* if for every arc $(u, v) \in A$ its reverse (v, u) is also contained in A. Given a symmetric digraph $D = (V, A)$ its *symmetric edge polytope* is defined as the convex hull of the vectors $e_v - e_w \in \mathbb{R}^V$ where $(w, v) \in A$. In recent years, symmetric edge polytopes have become a popular object of study. It is natural to extend the definition to general directed graphs, leading to the notion of a *directed edge polytope*, see e.g. [16]. Studies of these objects are still limited, but observations suggested an inverse relationship between the diameter of a directed graph and the volume of the associated directed edge polytope. Cactus graphs provide a family of counter-examples to this claim.

Theorem 6 *(⋆) The diameter of the graph does not depend on the volume of its associated directed edge polytope.*

Acknowledgements This work began at the 14th Emléktábla Workshop. We thank Lilla Tóthmérész for introducing the problem and for her insightful discussions, as well as Florian Hörsch for his valuable contributions. We thank the anonymous reviewers for their helpful feedback.

Author Contributions Not applicable.

Funding Open access funding provided by Eötvös Loránd University. Panna Gehér and Mirabel Mendoza-Cadena were supported by the Lendület Programme of the Hungarian Academy of Sciences – grant number LP2021-1/2021. Mirabel Mendoza-Cadena and Daniel P. Szabo were supported by the Hungarian National Research, Development and Innovation Office – NKFIH, grant number FK128673; by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, financed under the ELTE TKP 2021-NKTA-62 funding scheme. Mirabel Mendoza-Cadena was supported by CONAHCyT: CBF 2023-2024-552.

Data Availability Not applicable.

Declarations

Conflict of interest Not applicable.

Ethics Approval and Consent to Participate Not applicable.

Consent for Publication Not applicable.

Materials Availability Not applicable.

Code Availability Not applicable.

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