$\{s,t\}$-Separating Principal Partition Sequence of Submodular Functions

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Abstract

Narayanan and Fujishige showed the existence of the principal partition sequence of a submodular function, a structure with numerous applications in areas such as clustering, fast algorithms, and approximation algorithms. In this work, motivated by two applications, we develop a theory of $\{s,t\}$-separating principal partition sequence of a submodular function. We define this sequence, show its existence, and design a polynomial-time algorithm to construct it. We show two applications: (1) approximation algorithm for the $\{s,t\}$-separating submodular k-partitioning problem for monotone and posimodular functions and (2) polynomial-time algorithm for the hypergraph orientation problem of finding an orientation that simultaneously has strong connectivity at least k and (s,t)-connectivity at least ℓ.

Keywords: Submodular functions, Principal Partition Sequence, Submodular Partitioning, Hypergraph Orientation

Contents

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1 Introduction

A set function $f\colon 2^V \to \mathbb{R}$ is *submodular* if it satisfies the inequality $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$. Submodular functions arise throughout combinatorial optimization and economics; common examples include graph and hypergraph cut function, matroid rank function, and coverage function. The *principal partition sequence* is a central tool in submodular optimization, originating from the study of principal partitions by Kishi and Kajitani [23] in the context of graph tri-partitions defined via maximally distant spanning trees. Initially viewed as decompositions of discrete systems into partially ordered components, these partitions were later recognized to have a natural foundation in submodularity. Building on this foundation, Narayanan [30] developed the theory of the principal partition sequence of submodular functions, while Fujishige [16] provided a comprehensive survey of its development. Over time, the theory has been extended well beyond graphs [36] to matrices [19, 20], matroids [4, 29, 42], and general submodular systems [13, 14, 21, 22, 28], offering a lattice-theoretic decomposition framework for submodular functions.

Principal partition sequence is a well-structured sequence of partitions of the ground set minimizing $\sum_{A\in\mathcal{P}} f(A) - \lambda \cdot |\mathcal{P}|$ as λ varies from $-\infty$ to ∞ (see Section 1.2 for a formal definition). The sequence exhibits a refinement structure for submodular functions. A principal partition sequence of a submodular function given by its evaluation oracle can be found in polynomial time [24, 30, 31]. This is in contrast to minimizing $\sum_{A\in\mathcal{P}} f(A)$ over partitions $\mathcal{P}$ of the ground set with a given number of parts, which is NP-hard. The computational tractability has made the principal partition sequence a powerful structural tool in submodularity with many applications. Examples include Cunningham's network strength measure that is used to quantify network vulnerability [9], the realization of finite state machines [10], recursive ideal tree packing [8], approximation algorithms for graph and submodular partitioning problems [1, 6, 32, 39], graph clustering [27, 37], submodular sparsification [38], and dense subgraph decomposition [5, 13, 18].

In this paper, we extend the principal partition sequence to incorporate separation of two designated terminals – say $s, t \in V$. In particular, we show that the sequence of $\{s,t\}$-separating partitions minimizing $\sum_{A\in\mathcal{P}} f(A) - \lambda \cdot |\mathcal{P}|$ as λ varies from $-\infty$ to ∞ is well-structured and can be found in polynomial-time. The formal definition of $\{s,t\}$-separating principal partition sequence is fairly intricate, so we postpone its definition for now. Here, we discuss two concrete applications of this framework to illustrate its power.

1.1 Applications

The first application concerns approximation algorithms and the second concerns polynomial-time algorithms.

1.1.1 $\{s,t\}$-Separating Submodular k-Partition

Given a submodular function $f : 2^V \to \mathbb{R}$ via its evaluation oracle, the *submodular k-partition* problem (abbreviated Submod-k-Part) asks for a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of the ground set V into k non-empty parts that minimizes $f(\mathcal{P}) \coloneqq \sum_{i=1}^{k} f(V_i)$. Many problems in combinatorial optimization can be formulated as special cases, including partitioning problems over graphs, hypergraphs, matrices, and matroids. Submod-k-Part is NP-hard [17], does not admit a $(2-\epsilon)$-approximation assuming polynomial number of function evaluation queries [40], does not admit a $n^{1/(\log\log n)^c}$-approximation for every constant c assuming the Exponential Time Hypothesis [7], and the best approximation factor that is known is $O(k)$ [35, 43]. Nevertheless, constant factor approximations are known for broad subfamilies of submodular functions.

We recall that a function f is *symmetric* if $f(S) = f(V \setminus S)$ for all $S \subseteq V$, *monotone* if $f(S) \leq f(T)$ for every $S \subseteq T \subseteq V$, and *posimodular* if $f(S) + f(T) \geq f(S \setminus T) + f(T \setminus S)$ for every $S, T \subseteq V$. We observe that symmetric/monotone submodular functions are also posimodular. Prominent examples of symmetric submodular functions include graph and hypergraph cut functions, monotone submodular functions include matroid rank functions and coverage functions, and posimodular functions include positive combinations of symmetric submodular and monotone submodular functions. A well-studied special case of Submod-k-Part for symmetric submodular functions is Graph-k-Cut: the input is a graph and the goal is to find a minimum number of edges to delete so that the resulting graph has at least k components. If k is a fixed constant, then

Graph-k-Cut is polynomial-time solvable [17]. For k part of input, Graph-k-Cut is NP-complete [17] and does not have a polynomial-time $(2-\epsilon)$-approximation for every constant $\epsilon > 0$ under the Small Set Expansion Hypothesis [25]. There are several approaches that achieve a 2-approximation for Graph-k-Cut [1, 26, 39, 41]. One of these approaches is the principal partition sequence, which has been generalized to achieve a 2-approximation for posimodular submodular k-partition and a 4/3-approximation for monotone submodular k-partition [6].

Our first application of $\{s,t\}$-separating principal partition sequence is to the submodular k-partition problem with the additional constraint that the minimizing partition separates two specified terminals. We term this as $\{s,t\}$-Sep-Submod-k-Part and formally define it below. For a ground set V and a pair of terminals $s,t \in V$, a partition $\mathcal{P}$ of V is $\{s,t\}$-*separating* if $|P \cap \{s,t\}| \leq 1$ for every $P \in \mathcal{P}$.

$\{s,t\}$-Sep-Submod-k-Part
Input: A submodular function $f\colon 2^V \to \mathbb{R}_{\geq 0}$ given by a value oracle, distinct elements $s,t \in V$, and $k \in \mathbb{Z}_{\geq 0}$.
Goal:

$$\min\left\{\sum_{i=1}^{k} f(V_i)\colon \{V_i\}_{i=1}^{k} \text{ is a } \{s,t\}\text{-separating partition of } V \text{ into } k \text{ non-empty parts}\right\}.$$

A concrete special case of $\{s,t\}$-Sep-Submod-k-Part is $\{s,t\}$-Sep-Graph-k-Cut: the input is a graph with two specified terminal vertices s,t and the goal is to find a minimum number of edges to delete so that the resulting graph has at least k components with s and t being in different components. If k is a fixed constant, $\{s,t\}$-Sep-Graph-k-Cut is solvable in polynomial time [2]. Thus, for fixed constant k, the complexity status of $\{s,t\}$-Sep-Graph-k-Cut is identical to that of Graph-k-Cut. This status for $\{s,t\}$-Sep-Graph-k-Cut for constant k inspired us to investigate $\{s,t\}$-Sep-Graph-k-Cut, and more generally, $\{s,t\}$-Sep-Submod-k-Part, when k is part of input. Is it possible to achieve the same approximation results for $\{s,t\}$-Sep-Submod-k-Part as for Submod-k-Part?

We observe that $\{s,t\}$-Sep-Submod-k-Part closely resembles Submod-k-Part, but known approaches for Submod-k-Part do not apply directly. It is easy to see that $\{s,t\}$-Sep-Submod-k-Part is a special case of matroid constrained submodular k-partition and consequently, it admits a 2-approximation for symmetric submodular functions via the Gomory-Hu tree approach [3]. In this work, we exploit $\{s,t\}$-separating principal partition sequence to design approximation algorithms for $\{s,t\}$-Sep-Submod-k-Part with an approximation guarantee that matches that of Submod-k-Part for monotone and posimodular submodular functions.

Theorem 1.1. *$\{s,t\}$-Sep-Submod-k-Part admits a 2-approximation for posimodular submodular functions and a 4/3-approximation for monotone submodular functions.*

We note that the guarantee for posimodular submodular functions implies the same guarantee for symmetric submodular functions. Consequently, we have a 2-approximation for $\{s,t\}$-Sep-Graph-k-Cut.

1.1.2 Hypergraph Orientation

In graph orientation problems, the goal is to orient a given undirected graph to obtain a directed graph with certain properties. Two fundamental properties of interest are (s,t)-connectivity and strong-connectivity. In Graph-(s,t)-ℓ-Conn-Orient, the input is an undirected graph with two specified terminals s and t and an integer $\ell \in \mathbb{Z}_{\geq 0}$. The goal is to verify if there exists an orientation of G that has ℓ arc-disjoint paths from s to t and if so, find one. By Menger's theorem, an undirected graph G has an orientation with ℓ arc-disjoint paths from s to t if and only if there exist ℓ edge-disjoint paths between s and t in the undirected graph G. Thus, Graph-(s,t)-ℓ-Conn-Orient can be solved using an application of max (s,t)-flow. In Graph-k-Conn-Orient, the input is an undirected graph and an integer $k \in \mathbb{Z}_{\geq 0}$. The goal is to verify if there exists a k-arc-connected orientation of the graph. We recall that a digraph is k-arc-connected if for every pair of distinct vertices $u,v \in V$, there exist k arc-disjoint paths from u to v. A classic result of Nash-Williams [33] shows that an undirected graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

We recall that an undirected graph $G = (V, E)$ is $2k$-edge-connected if for every pair of distinct vertices $u, v \in V$, there exist $2k$ edge-disjoint paths between u and v. Nash-Williams' result is also constructive that leads to a polynomial time algorithm to solve Graph-k-Conn-Orient.

Frank, Király, and Király [12] showed an intriguing result concerning a generalization of these two problems that we term as Graph-$(k, (s,t), \ell)$-Conn-Orient. The input here is an undirected graph with two specified terminals s and t and integers $k, \ell \in \mathbb{Z}_{\geq 0}$ (with the interesting case being $k < \ell$). The goal is to verify if there exists an orientation $\overrightarrow{G}$ of G such that $\overrightarrow{G}$ is k-arc-connected and has ℓ arc-disjoint paths from s to t – we will call such an orientation to be $(k, (s,t), \ell)$-arc-connected orientation. They showed a complete characterization for the existence of a $(k, (s,t), \ell)$-arc-connected orientation while also giving a polynomial time algorithm to find such an orientation via Mader's splitting-off result. Our focus is on this generalized orientation problem over hypergraphs instead of graphs.

Frank, Király, and Király [12] considered all three orientation problems mentioned above in hypergraphs as opposed to graphs. We begin by discussing the two constituent problems in hypergraphs. A hypergraph $G = (V, E)$ is specified by a vertex set V and hyperedge set E where each $e \in E$ is a subset of V. An orientation $\overrightarrow{G}$ of a hypergraph $G = (V, E)$ is a directed hypergraph specified by $\overrightarrow{G} = (V, E, \text{head}\colon E \to V)$, where $\text{head}(e) \in e$ for every $e \in E$. In hypergraph orientation problems, the goal is to orient a given hypergraph to obtain a directed hypergraph with certain properties. For vertices a, b, an (a, b)-path in a directed hypergraph is an alternating sequence, without repetition, of vertices and hyperarcs $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ where $v_i \in e_i \setminus \text{head}(e_i)$ and $v_{i+1} = \text{head}(e_i)$. In (s,t)-ℓ-Conn-Orient, the input is a hypergraph with two specified terminals s and t and an integer $\ell \in \mathbb{Z}_{\geq 0}$. The goal is to verify if there exists an orientation of G that has ℓ hyperarc-disjoint paths from s to t and if so, then find one. Menger's theorem extends to hypergraphs and thus, resolves (s,t)-ℓ-Conn-Orient in hypergraphs. In k-Conn-Orient, the input is a hypergraph and an integer $k \in \mathbb{Z}_{\geq 0}$. The goal is to verify if there exists a k-hyperarc-connected orientation of the graph. A directed hypergraph is k-hyperarc-connected if for every pair of distinct vertices $u, v \in V$, there exist k hyperarc-disjoint paths from u to v. Frank, Király, and Király generalized Nash-Williams' result from graphs to hypergraphs by giving a complete characterization for the existence of a k-hyperarc-connected orientation of a hypergraph and moreover, their characterization is constructive, i.e., it leads to a polynomial-time algorithm to verify if a given hypergraph has a k-hyperarc-connected orientation and if so, then find one.

We now address the main application of interest to our work, namely $(k, (s,t), \ell)$-Conn-Orient that is defined as follows (with the interesting case being $k < \ell$).

> $(k, (s,t), \ell)$-Conn-Orient
> **Input:** A hypergraph $G = (V, E)$, vertices $s, t \in V$, and $k, \ell \in \mathbb{Z}_{\geq 0}$.
> **Goal:** Verify if there exists an orientation $\overrightarrow{G}$ of G such that $\overrightarrow{G}$ is k-hyperarc-connected and has ℓ hyperarc-disjoint paths from s to t.

We will call a feasible orientation for $(k, (s,t), \ell)$-Conn-Orient to be a $(k, (s,t), \ell)$-hyperarc-connected orientation. Frank, Király, and Király gave a characterization for the existence of a $(k, (s,t), \ell)$-connected orientation of a hypergraph. However, their proof of the characterization relies on certain steps which did not seem constructive. Consequently, it was unclear whether there exists a polynomial-time algorithm to verify whether a given hypergraph has a $(k, (s,t), \ell)$-hyperarc-connected orientation and if so, then find it. In this work, we exploit algorithmic aspects of $\{s,t\}$-separating principal partition sequence of a submodular function to design a polynomial time algorithm for this problem.

Theorem 1.2. *There exists a polynomial time algorithm to verify whether a given hypergraph admits a $(k, (s,t), \ell)$-hyperarc-connected orientation and if so, then find one.*

We also consider optimization variants of the above problem where ℓ (or k) is given as part of input and the goal is to maximize k (or ℓ respectively) so that there exists a $(k, (s,t), \ell)$-connected orientation, present min-max relations, and show that the corresponding minimization problem is solvable in polynomial time via our results on $\{s,t\}$-separating principal partition sequence.

1.2 {s,t}-Separating Principal Partition Sequence

In this section, we present our definition and state our results concerning $\{s,t\}$-separating principal partition sequence. We begin by discussing principal partition sequence and known results to provide background and context. For a set function $f\colon 2^V \to \mathbb{R}$ and a collection $\mathcal{P}$ of subsets of V, we write $f(\mathcal{P}) \coloneqq \sum_{P\in\mathcal{P}} f(P)$. A *partition* of a set $S \subseteq V$ is a collection of pairwise disjoint non-empty subsets of S whose union is S.

Principal Partition Sequence. Let $f\colon 2^V \to \mathbb{R}$ be a set function and $\mathcal{P}$ be a partition of the ground set V. We define functions $g_{f,\mathcal{P}} : \mathbb{R} \to \mathbb{R}$ and $g_f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} g_{f,\mathcal{P}}(\lambda) &\coloneqq f(\mathcal{P}) - \lambda \cdot |\mathcal{P}|, \\ g_f(\lambda) &\coloneqq \min\{g_{f,\mathcal{P}}(\lambda) : \mathcal{P} \text{ is a partition of } V\}. \end{aligned}$$

We drop the subscript f and simply write $g_{\mathcal{P}}$ and g, respectively, if the function f is clear from context. A *breakpoint* of a piecewise linear function is a point at which the slope of f changes, or equivalently, a point where f is continuous but not differentiable. By definition, the function g_f is piecewise linear and has at most $|V|$ slopes and hence, at most $|V| - 1$ breakpoints.

Narayanan [29] and Fujishige [13] showed that if $f\colon 2^V \to \mathbb{R}$ is submodular, then the partitions attaining the values of the function g_f at its breakpoints exhibit a refinement structure, which is captured by the principal partition sequence.

Definition 1.3 (Refinement)**.** Let $\mathcal{P}$ and $\mathcal{Q}$ be distinct partitions of V.

1. $\mathcal{Q}$ is a *refinement of* $\mathcal{P}$ if for all $B \in \mathcal{Q}$ there exists $A \in \mathcal{P}$ such that $B \subseteq A$.
2. $\mathcal{Q}$ is a refinement of $\mathcal{P}$ *up to one set* if
   (a) $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and
   (b) there exists $P \in \mathcal{P}$ such that for all $B \in \mathcal{Q}$, either $B \subsetneq P$ or $B \in \mathcal{P}$.

See Figure 2a for an example of a refinement. We observe that if $\mathcal{P}$ and $\mathcal{Q}$ are distinct partitions and $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $|\mathcal{Q}| > |\mathcal{P}|$. Using this definition of refinements, a principal partition sequence is defined as follows.

Definition 1.4 (Principal partition sequence)**.** Let $f\colon 2^V \to \mathbb{R}$ be a set function. A sequence $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of partitions of V is a *principal partition sequence of* f if there exist $\lambda_1, \ldots, \lambda_{\ell-1} \in \mathbb{R}$ such that

1. $\lambda_1 \le \ldots \le \lambda_{\ell-1}$,
2. $g(\lambda) = \begin{cases} g_{\mathcal{P}_1}(\lambda) & \text{for } \lambda \in (-\infty, \lambda_1], \\ g_{\mathcal{P}_{j+1}}(\lambda) & \text{for } \lambda \in [\lambda_j, \lambda_{j+1}] \text{ and } j \in [\ell-2], \\ g_{\mathcal{P}_\ell}(\lambda) & \text{for } \lambda \in [\lambda_{\ell-1}, \infty), \end{cases}$
3. $\mathcal{P}_1 = \{V\}$ and $\mathcal{P}_\ell = \{\{v\}\colon v \in V\}$, and
4. for every $j \in [\ell-1]$, $\mathcal{P}_{j+1}$ is a refinement of $\mathcal{P}_j$ up to one set.

The values $\lambda_1, \ldots, \lambda_{\ell-1}$ are called the *critical values associated with the principal partition sequence.*

We observe that properties (1), (2), and (3) in the definition of a principal partition sequence are easy to achieve for every set function. The essence of principal partition sequences lies in the fact that, for submodular functions, property (4) can also be achieved. This was shown by Narayanan [29] and Fujishige [13]. They also designed a polynomial-time algorithm to find such a sequence.

Theorem 1.5 (Narayanan, Fujishige)**.** *Let $f : 2^V \to \mathbb{R}$ be a submodular function. Then, there exists a principal partition sequence of f. Moreover, given oracle access to a submodular function $f\colon 2^V \to \mathbb{R}$, there exists a polynomial-time algorithm to compute a principal partition sequence of f along with its associated critical values.*

$\{s,t\}$-Refinement and $\{s,t\}$-Separating Principal Partition Sequence. Let $f\colon 2^V \to \mathbb{R}$ be a submodular set function on ground set V and $s,t \in V$ be distinct elements. A partition $\mathcal{P}$ of V is an $\{s,t\}$-separating partition if $|\{s,t\} \cap P| \le 1$ for every $P \in \mathcal{P}$. We recall that for every partition $\mathcal{P}$ of V and every $\lambda \in \mathbb{R}$, we defined $g_{f,\mathcal{P}}(\lambda) = f(\mathcal{P}) - \lambda \cdot |\mathcal{P}|$. As a natural $\{s,t\}$-separating counterpart, we define the function $g_f^{s,t}\colon \mathbb{R} \to \mathbb{R}$ by setting

$$g_f^{s,t}(\lambda) \coloneqq \min\{g_{f,\mathcal{P}}(\lambda)\colon \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition of } V\}.$$

We drop the subscript f and simply write $g^{s,t}$ if the function f is clear from context. The function $g^{s,t}$ is piecewise linear with at most $|V|-2$ breakpoints (see Lemma A.1 in appendix).

We observe that for every set function $f\colon 2^V \to \mathbb{R}$, one can find a sequence $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of partitions and $\lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$ satisfying properties (1)-(3) in Definition 1.4. Submodularity becomes crucial, however, in order to obtain the refinement property (4). A key difference between principal partition and the $\{s,t\}$-separating partition sequences arises precisely in this property: minimizers for the breakpoints of $g^{s,t}$ do not necessarily satisfy the refinement property; an example is given in Figure 1.

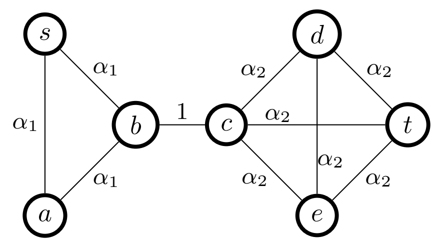


Figure 1: An edge-weighted graph whose cut function f is such that minimizers for the breakpoints of $g_f^{s,t}$ do not necessarily satisfy the refinement property. Here $0 < \varepsilon < 1/24$ is a small constant, and $\alpha_1 = 1/2 + \varepsilon$, $\alpha_2 = 1/3 + 2\varepsilon$. The sequence of minimizers of $g_f^{s,t}$ is unique and is given by $\mathcal{P}_1 = \{\{s,a,b\},\{t,c,d,e\}\}, \mathcal{P}_2 = \{\{s\},\{a\},\{b,t,c,d,e\}\}$, $\mathcal{P}_3 \coloneqq \{\{s,a,b,c\},\{d\},\{e\},\{t\}\}$, $\mathcal{P}_4 = \{\{s\},\{a\},\{b,c\},\{d\},\{e\},\{t\}\}$, and $\mathcal{P}_5 = \{\{s\},\{a\},\{b\},\{c\},\{d\},\{e\},\{t\}\}$.

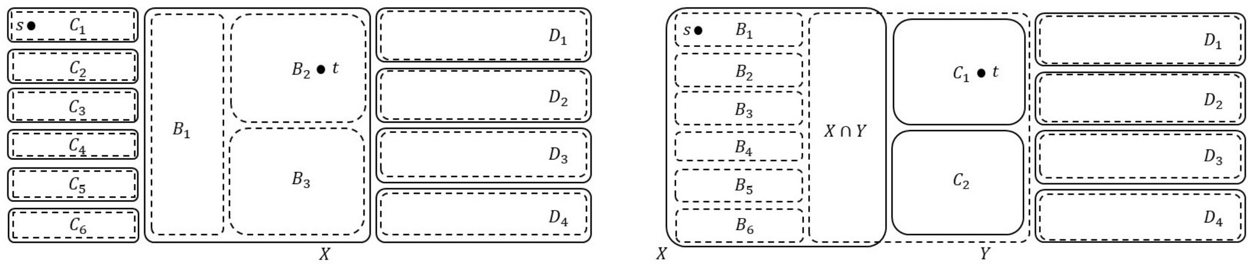
Nevertheless, this sequence still exhibits a more intricate form of refinement. We now formalize this intricate form of refinement. Two sets $X \subseteq V$ and $Y \subseteq V$ are called *intersecting* if all of $X \cap Y$, $X \setminus Y$, and $Y \setminus X$ are nonempty. A pair of intersecting sets $X \subseteq V$ and $Y \subseteq V$ is called $\{s,t\}$-*uncrossable* if they do not separate s and t, meaning that either $|\{s,t\} \cap (X \setminus Y)| \ne 1$ or $|\{s,t\} \cap (Y \setminus X)| \ne 1$.

Definition 1.6 ($\{s,t\}$-refinements)**.** Let $\mathcal{P}$ and $\mathcal{Q}$ be distinct $\{s,t\}$-separating partitions of V.

1. $\mathcal{Q}$ is a $\{s,t\}$-*refinement of* $\mathcal{P}$ *along* (X,Y) if $X \in \mathcal{P} \setminus \mathcal{Q}$, $Y \in \mathcal{Q} \setminus \mathcal{P}$, and the following hold:
   (a) X and Y are intersecting but they are *not* $\{s,t\}$-uncrossable,
   (b) for all $B \in \mathcal{Q} \setminus Y$ there exists $A \in \mathcal{P}$ such that $B \subseteq A$,
   (c) for all $A \in \mathcal{P} \setminus X$, either $A \cap Y = \emptyset$ or $A \subseteq Y$, and
   (d) $|\{A \in \mathcal{P}\colon A \subseteq Y\}| \le |\{B \in \mathcal{Q}\colon B \subseteq X\}|$.

2. $\mathcal{Q}$ is a $\{s,t\}$-*refinement of* $\mathcal{P}$ if there exists $X \in \mathcal{P} \setminus \mathcal{Q}$ and $Y \in \mathcal{Q} \setminus \mathcal{P}$ such that $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$ along (X,Y).

3. $\mathcal{Q}$ is a $\{s,t\}$-*refinement of* $\mathcal{P}$ *up to two sets* if there exists $X \in \mathcal{P} \setminus \mathcal{Q}$ and $Y \in \mathcal{Q} \setminus \mathcal{P}$ such that
   (a) $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$ along (X,Y) and
   (b) $\{P \in \mathcal{P}\colon P \subseteq V \setminus (X \cup Y)\} \subseteq \mathcal{Q}$.



(a) An example where $\mathcal{Q}$ is a refinement of $\mathcal{P}$, where $\mathcal{P} = \{X, C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4\}$ (solid lines) and $\mathcal{Q} = \{B_1, B_2, B_3, C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4\}$ (dashed lines).

(b) An example where $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$ up to X and Y, where $\mathcal{P} = \{X, C_1, C_2, D_1, D_2, D_3, D_4\}$ (solid lines) and $\mathcal{Q} = \{Y, B_1, B_2, B_3, B_4, B_5, B_6, D_1, D_2, D_3, D_4\}$ (dashed lines).

Figure 2: Examples of refinement and $\{s,t\}$-refinement.

See Figures 2a and 2b for examples of refinements and $\{s,t\}$-refinements. We observe that an $\{s,t\}$-refinement allows exactly one pair of intersecting sets in $\mathcal{P} \cup \mathcal{Q}$, and these sets cannot be $\{s,t\}$-uncrossable. Moreover, if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $|\mathcal{Q}| > |\mathcal{P}|$, whereas if $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$, then $|\mathcal{Q}| \geq |\mathcal{P}|$.

Based on the notion of refinements and $\{s,t\}$-refinements, we define an $\{s,t\}$-separating principal partition sequence as follows.

Definition 1.7 ($\{s,t\}$-separating principal partition sequence)**.** Let $f\colon 2^V \to \mathbb{R}$ be a set function. A sequence $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of $\{s,t\}$-separating partitions of V is a $\{s,t\}$*-separating principal partition sequence* of f if there exist $\lambda_1, \ldots, \lambda_{\ell-1} \in \mathbb{R}$ such that

1. $\lambda_1 \leq \ldots \leq \lambda_{\ell-1}$,
2. $g^{s,t}(\lambda) = \begin{cases} g_{\mathcal{P}_1}(\lambda) & \text{for } \lambda \in (-\infty, \lambda_1], \\ g_{\mathcal{P}_{j+1}}(\lambda) & \text{for } \lambda \in [\lambda_j, \lambda_{j+1}] \text{ and } j \in [\ell-2], \\ g_{\mathcal{P}_\ell}(\lambda) & \text{for } \lambda \in [\lambda_{\ell-1}, \infty). \end{cases}$
3. $\mathcal{P}_1$ is $\{s,t\}$-separating partition with two parts such that $f(\mathcal{P}_1) \leq f(A) + f(V \setminus A)$ for every $s \in A \subseteq V - t$, and $\mathcal{P}_\ell = \{\{v\} \colon v \in V\}$,
4. for every $j \in [\ell-1]$, either $\mathcal{P}_{j+1}$ is a refinement of $\mathcal{P}_j$ up to one set or $\mathcal{P}_{j+1}$ is an $\{s,t\}$-refinement of $\mathcal{P}_j$ up to two sets, and moreover, $|\mathcal{P}_j| < |\mathcal{P}_{j+1}|$.

The values $\lambda_1, \ldots, \lambda_{\ell-1}$ are called the *critical values associated with the* $\{s,t\}$*-separating principal partition sequence.*

We encourage the reader to compare and contrast Definitions 1.4 and 1.7. A principal partition sequence can be viewed as a prefix of non-$\{s,t\}$-separating partition subsequence followed by a suffix of $\{s,t\}$-separating partition subsequence. However, the suffix by itself is not necessarily a $\{s,t\}$-separating principal partition sequence. We now state our main result regarding $\{s,t\}$-separating principal partition sequence of a submodular function.

Theorem 1.8. *Let $f\colon 2^V \to \mathbb{R}$ be a submodular function and $s,t \in V$. Then, there exists an $\{s,t\}$-separating principal partition sequence of f. Moreover, given oracle access to f, such a sequence and its critical values can be computed in polynomial time.*

For ease of exposition, we will assume that submodular functions of interest to this work are *strictly submodular* on intersecting pairs throughout, i.e., $f(A) + f(B) > f(A \cap B) + f(A \cup B)$ for every $A, B \subseteq V$ for which $A \cap B, A \setminus B, B \setminus A$ are non-empty (see Lemma A.2 for a justification). This assumption can be relaxed with additional complications in the proof.

2 $\{s,t\}$-Separating Principal Partition Sequence

In this section, we prove Theorem 1.8, i.e., we prove the existence of an $\{s,t\}$-separating principal partition sequence of a submodular function and give a polynomial-time algorithm to construct it. Section 2 provides the formal definition. In Section 2.1, we establish a refinement property of $\{s,t\}$-separating partitions that plays a key role in proving the existence of the sequence (Theorem 2.1). Building on this, Section 2.2 shows that an $\{s,t\}$-separating principal partition sequence always exists (Theorem 2.10), and Section 2.3 presents a polynomial-time algorithm to compute one.

2.1 Refinement Property

We first show a refinement property that will be useful for proving the existence of an $\{s,t\}$-separating principal partition sequence. The main result is the following.

Theorem 2.1. *Let $f\colon 2^V \to \mathbb{R}$ be a submodular function and let $s,t \in V$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are distinct $\{s,t\}$-separating partitions such that at least one of the following holds:*

1. *There exists $\lambda \in \mathbb{R}$ such that*
$$\mathcal{P} \in \arg\min\{|\pi|\colon \pi \textit{ is an } \{s,t\}\textit{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\},$$
$$\mathcal{Q} \in \arg\max\{|\pi|\colon \pi \textit{ is an } \{s,t\}\textit{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}.$$
2. *There exists $\alpha < \lambda$ such that $g^{s,t}(\alpha) = g_{\mathcal{P}}(\alpha)$ and $g^{s,t}(\lambda) = g_{\mathcal{P}}(\lambda) = g_{\mathcal{Q}}(\lambda)$.*

Then, $\mathcal{Q}$ is either a refinement or an $\{s,t\}$-refinement of $\mathcal{P}$.

We begin with a few technical lemmas used in the proof of the theorem. The next lemma shows that every $\{s,t\}$-separating set system that covers each element exactly twice and contains more than two intersecting sets admits an $\{s,t\}$-uncrossable pair of intersecting sets.

Lemma 2.2. *Let $\mathcal{S} \subseteq 2^V$ be a set system such that*

1. *each $v \in V$ is contained in exactly two sets in $\mathcal{S}$, and*
2. *$|\{s,t\} \cap A| \le 1$ for all $A \in \mathcal{S}$.*

Suppose that $\mathcal{S}$ has more than two intersecting pairs of sets. Then, $\mathcal{S}$ has a pair of intersecting sets that are $\{s,t\}$-uncrossable.

Proof. Suppose none of the intersecting pairs in $\mathcal{S}$ are $\{s,t\}$-uncrossable, and let $X_1, X_2 \in \mathcal{S}$ be an intersecting pair. Then $|\{s,t\} \cap (X_1 \setminus X_2)| = 1 = |\{s,t\} \cap (X_2 \setminus X_1)|$. Without loss of generality let $s \in X_1 \setminus X_2$ and $t \in X_2 \setminus X_1$. Since $\mathcal{S}$ has more than two intersecting pairs, consider another such pair $X_3, X_4 \in \mathcal{S}$ with $\{X_1, X_2\} \neq \{X_3, X_4\}$. Similarly as before, we have $|\{s,t\} \cap (X_3 \setminus X_4)| = 1 = |\{s,t\} \cap (X_4 \setminus X_3)|$. We consider two cases.

Case 1. $\{X_1, X_2\} \cap \{X_3, X_4\} = \emptyset$.

Suppose that $s \in X_3 \setminus X_4$ and $t \in X_4 \setminus X_3$; the other case is symmetric, and an analogous argument applies. Then $s \in X_1 \cap X_3$ and $t \notin X_1 \cup X_3$. Since X_1 and X_3 are not $\{s,t\}$-uncrossable, it follows that either $X_1 \subseteq X_3$ or $X_3 \subseteq X_1$, in which case elements of $X_1 \cap X_2$ or $X_3 \cap X_4$ belong to at least three sets – X_1, X_2, X_3 or X_1, X_3, X_4, respectively – yielding a contradiction.

Case 2: $|\{X_1, X_2\} \cap \{X_3, X_4\}| = 1$.

Without loss of generality, assume $X_1 = X_4$. Since $s \in X_1 = X_4$ and $|\{s,t\} \cap (X_3 \setminus X_1)| = 1$, it follows that $t \in (X_2 \setminus X_1) \cap (X_3 \setminus X_1)$. If X_2 and X_3 are intersecting, they would be $\{s,t\}$-uncrossable, since $(X_2 \setminus X_3) \cap \{s,t\} = \emptyset$. Therefore, either $X_2 \subseteq X_3$ or $X_3 \subseteq X_2$, and consequently $X_1 \cap X_2 \cap X_3 \neq \emptyset$, contradicting the assumption that each element is contained in exactly two sets in $\mathcal{S}$. $\square$

The following lemma shows that a pair of $g^{s,t}$-minimizing partitions cannot contain an $\{s,t\}$-uncrossable pair of intersecting sets. The proof crucially relies on the strict submodularity of the set function f on intersecting pairs.

Lemma 2.3. *Let $\lambda \in \mathbb{R}$, and let $\mathcal{P}$ and $\mathcal{Q}$ be $\{s,t\}$-separating partitions such that $g^{s,t}(\lambda) = g_{\mathcal{P}}(\lambda) = g_{\mathcal{Q}}(\lambda)$. Then $\mathcal{P} \cup \mathcal{Q}$ contains no $\{s,t\}$-uncrossable pair of intersecting sets.*

Proof. For a contradiction, suppose $\mathcal{P} \cup \mathcal{Q}$ contains an $\{s,t\}$-uncrossable pair of intersecting sets. As long as such a pair $A, B \in \mathcal{P} \cup \mathcal{Q}$ exists, replace it with $A \cap B$ and $A \cup B$. By strict submodularity, each uncrossing strictly decreases the sum of the function values, so this procedure terminates in finitely many steps. Let $\mathcal{S}$ denote the resulting family.

By the indirect assumption, we have $f(\mathcal{S}) < f(\mathcal{P}) + f(\mathcal{Q})$, while $|\mathcal{S}| = |\mathcal{P}| + |\mathcal{Q}|$ and each element remains in exactly two sets. We claim that $\mathcal{S}$ is the disjoint union of two $\{s,t\}$-separating partitions.

Claim 2.4. *None of the sets in $\mathcal{S}$ contains both s and t.*

Proof. We show that each uncrossing step preserves the property that no set contains both s and t. Initially, both $\mathcal{P}$ and $\mathcal{Q}$ are $\{s,t\}$-separating, so the claim holds. For the sake of contradiction, suppose the property holds before uncrossing a pair of intersecting sets A, B but fails after the uncrossing.

If $\{s,t\} \subseteq A \cap B$, then $\{s,t\} \subseteq A$, contradicting the property before uncrossing. Therefore, we must have $\{s,t\} \subseteq A \cup B$. Since neither A nor B contains both s and t, it follows that $|\{s,t\} \cap A| = 1 = |\{s,t\} \cap B|$, which would imply that A and B were not $\{s,t\}$-uncrossable. But then the procedure would not have uncrossed A and B, a contradiction. $\square$

Now we show that $\mathcal{S}$ decomposes into two partitions.

Claim 2.5. *$\mathcal{S}$ is the disjoint union of two partitions.*

Proof. By Lemma 2.2, $\mathcal{S}$ contains at most one pair of intersecting sets X and Y that are not $\{s,t\}$-uncrossable. If there is such a pair, then let $U \coloneqq X \cup Y$, otherwise let $U \coloneqq \emptyset$ for the rest of the proof of the claim. Let

$$\mathcal{P}_1 \coloneqq \{X\} \cup \{A \in \mathcal{S} \colon A \subsetneq Y\}$$
$$\mathcal{Q}_1 \coloneqq \{Y\} \cup \{B \in \mathcal{S} \colon B \subsetneq X\}.$$

Observe that $\mathcal{P}_1$ and $\mathcal{Q}_1$ are partitions of U, since X and Y form the only intersecting pair in $\mathcal{S}$. The remainder, $\mathcal{L} \coloneqq \mathcal{S} \setminus (\mathcal{P}_1 \cup \mathcal{Q}_1)$, is a laminar system covering each element exactly twice, as it contains no intersecting sets. Let $\mathcal{P}_2$ and $\mathcal{Q}_2$ be the inclusion-wise maximal and minimal sets in $\mathcal{L}$, respectively, with any duplicates distributed between them. Then $\mathcal{P}_2$ and $\mathcal{Q}_2$ are partitions of $V \setminus U$. Setting $\mathcal{P}' \coloneqq \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{Q}' \coloneqq \mathcal{Q}_1 \cup \mathcal{Q}_2$ thus yields two $\{s,t\}$-separating partitions of V with $\mathcal{S} = \mathcal{P}' \cup \mathcal{Q}'$. $\square$

Let $\mathcal{P}'$ and $\mathcal{Q}'$ be the partitions provided by Claim 2.5. By Claim 2.4, $\mathcal{P}'$ and $\mathcal{Q}'$ are $\{s,t\}$-separating partitions. Recall that $f(\mathcal{P}') + f(\mathcal{Q}') = f(\mathcal{S}) < f(\mathcal{P}) + f(\mathcal{Q})$ and $|\mathcal{P}'| + |\mathcal{Q}'| = |\mathcal{S}| = |\mathcal{P}| + |\mathcal{Q}|$. Moreover, $g^{s,t}(\lambda) = g_{\mathcal{P}}(\lambda) = g_{\mathcal{Q}}(\lambda)$ by assumption and $g^{s,t}(\lambda) \leq g_{\mathcal{P}'}(\lambda)$ and $g^{s,t}(\lambda) \leq g_{\mathcal{Q}'}(\lambda)$ by definition. Combining these observations, we obtain

$$\begin{aligned} 2g^{s,t}(\lambda) &= g_{\mathcal{P}}(\lambda) + g_{\mathcal{Q}}(\lambda) \\ &= f(\mathcal{P}) + f(\mathcal{Q}) - \lambda(|\mathcal{P}| + |\mathcal{Q}|) \\ &> f(\mathcal{P}') + f(\mathcal{Q}') - \lambda(|\mathcal{P}'| + |\mathcal{Q}'|) \\ &= g_{\mathcal{P}'}(\lambda) + g_{\mathcal{Q}'}(\lambda) \\ &\geq 2g^{s,t}(\lambda), \end{aligned}$$

a contradiction. This concludes the proof of the lemma. $\square$

The next lemma shows that under the hypothesis of Theorem 2.1, the parts of the partitions $\mathcal{P}$ and $\mathcal{Q}$ cannot be rearranged to create two other $\{s,t\}$-separating partitions $\mathcal{P}'$ and $\mathcal{Q}'$ with $|\mathcal{Q}'| > |\mathcal{Q}|$. The proof of the lemma in fact does not rely on submodularity of the set function f.

Lemma 2.6. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are distinct $\{s,t\}$-separating partitions such that one of the following holds:*

1. *There exists $\lambda \in \mathbb{R}$ such that*

$$\mathcal{P} \in \arg\min\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\},$$
$$\mathcal{Q} \in \arg\max\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}.$$

2. *There exists $\alpha < \lambda$ such that $g^{s,t}(\alpha) = g_{\mathcal{P}}(\alpha)$ and $g^{s,t}(\lambda) = g_{\mathcal{P}}(\lambda) = g_{\mathcal{Q}}(\lambda)$.*

Then no $\{s,t\}$-separating partitions $\mathcal{P}'$ and $\mathcal{Q}'$ exist with $\mathcal{P}' \cup \mathcal{Q}' = \mathcal{P} \cup \mathcal{Q}$ and $|\mathcal{Q}'| > |\mathcal{Q}|$.

Proof. For the sake of contradiction, suppose such partitions $\mathcal{P}'$ and $\mathcal{Q}'$ exist. Considering the two assumptions in the lemma, we arrive at a contradiction in both cases.

Suppose first that (1) holds. Then,

$$\begin{aligned}
2g^{s,t}(\lambda) &= g_{\mathcal{P}}(\lambda) + g_{\mathcal{Q}}(\lambda)\\
&= f(\mathcal{P}) + f(\mathcal{Q}) - \lambda(|\mathcal{P}| + |\mathcal{Q}|)\\
&= f(\mathcal{P}') + f(\mathcal{Q}') - \lambda(|\mathcal{P}'| + |\mathcal{Q}'|)\\
&= g_{\mathcal{P}'}(\lambda) + g_{\mathcal{Q}'}(\lambda)\\
&\geq 2g^{s,t}(\lambda).
\end{aligned}$$

Consequently, we have $g_{\mathcal{P}'}(\lambda) = g_{\mathcal{Q}'}(\lambda) = g^{s,t}(\lambda)$, contradicting the choice of $\mathcal{Q}$.

Now suppose that (2) holds. Then,

$$\begin{aligned}
g_{\mathcal{P}}(\alpha) + g_{\mathcal{Q}}(\lambda) &= g_{\mathcal{P}}(\alpha) + g_{\mathcal{Q}}(\alpha) - (\lambda - \alpha)|\mathcal{Q}|\\
&= f(\mathcal{P}) + f(\mathcal{Q}) - \alpha|\mathcal{P}| - \alpha|\mathcal{Q}| - (\lambda - \alpha)|\mathcal{Q}|\\
&= f(\mathcal{P}') + f(\mathcal{Q}') - \alpha|\mathcal{P}| - \alpha|\mathcal{Q}| - (\lambda - \alpha)|\mathcal{Q}|\\
&= g_{\mathcal{P}'}(\alpha) + g_{\mathcal{Q}'}(\alpha) - (\lambda - \alpha)|\mathcal{Q}|\\
&= g_{\mathcal{P}'}(\alpha) + g_{\mathcal{Q}'}(\lambda) + (\lambda - \alpha)(|\mathcal{Q}'| - |\mathcal{Q}|)\\
&\geq g_{\mathcal{P}}(\alpha) + g_{\mathcal{Q}'}(\lambda) + (\lambda - \alpha)(|\mathcal{Q}'| - |\mathcal{Q}|).
\end{aligned}$$

Hence,

$$g_{\mathcal{Q}}(\lambda) \geq g_{\mathcal{Q}'}(\lambda) + (\lambda - \alpha)(|\mathcal{Q}'| - |\mathcal{Q}|).$$

Since $\lambda > \alpha$ and $|\mathcal{Q}'| > |\mathcal{Q}|$, we have $g_{\mathcal{Q}}(\lambda) > g_{\mathcal{Q}'}(\lambda) \geq g^{s,t}(\lambda) = g_{\mathcal{Q}}(\lambda)$, a contradiction. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We consider two cases.

Case 1. There is no pair of intersecting sets in $\mathcal{P} \cup \mathcal{Q}$.

Note that $\mathcal{P} \cup \mathcal{Q}$ is laminar. We show that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. For the sake of contradiction, suppose $\mathcal{Q}$ is not a refinement of $\mathcal{P}$. Then there exists $B \in \mathcal{Q}$ such that $B \not\subseteq A$ for all $A \in \mathcal{P}$. Since $\mathcal{P} \cup \mathcal{Q}$ is laminar, at least two sets of $\mathcal{P}$ are contained in B. Let $\mathcal{P}' := \{B\} \cup A \in \mathcal{P} \colon A \cap B = \emptyset$ and $\mathcal{Q}' := A \in \mathcal{P} \colon A \subseteq B \cup (\mathcal{Q} \setminus B)$. Then $\mathcal{P}'$ and $\mathcal{Q}'$ are $\{s,t\}$-separating partitions with $\mathcal{P}' \cup \mathcal{Q}' = \mathcal{P} \cup \mathcal{Q}$ and $|\mathcal{Q}'| > |\mathcal{Q}|$, contradicting Lemma 2.6.

Case 2. There exists a pair of intersecting sets in $\mathcal{P} \cup \mathcal{Q}$.

By Lemmas 2.2 and 2.3, there is exactly one intersecting pair $X, Y \in \mathcal{P} \cup \mathcal{Q}$. Since both $\mathcal{P}$ and $\mathcal{Q}$ are partitions, X and Y must lie in distinct partitions; thus, without loss of generality, assume that $X \in \mathcal{P} \setminus \mathcal{Q}$ and $Y \in \mathcal{Q} \setminus \mathcal{P}$. We complete the proof by showing that $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$ along (X, Y). Since X and Y are not $\{s,t\}$-uncrossable, we have that $|\{s,t\} \cap (X \setminus Y)| = 1 = |\{s,t\} \cap (Y \setminus X)|$. The next three claims complete the proof of the remaining properties.

Claim 2.7. *Every set B in $\mathcal{Q} \setminus Y$ is contained in some set A of $\mathcal{P}$.*

Proof. For the sake of contradiction, let $B \in \mathcal{Q} \setminus \{Y\}$ be such that $B \not\subseteq A$ for all $A \in \mathcal{P}$. The set B cannot be intersecting with any of the sets in $\mathcal{P} \cup \mathcal{Q}$, since X and Y are the only intersecting pair in $\mathcal{P} \cup \mathcal{Q}$. Hence, for every $A \in \mathcal{P}$, either $B \cap A = \emptyset$ or $A \subseteq B$. Moreover, since $B \setminus A \neq \emptyset$ for all $A \in \mathcal{P}$, it follows that there exist at least two sets of $\mathcal{P}$ that are contained in B.

We now claim that $B \subseteq V \setminus (X \cup Y)$. To prove this, recall that $B \cap Y = \emptyset$ since B and Y are parts of the partition $\mathcal{Q}$ and hence disjoint. It remains to show that $X \cap B = \emptyset$. Suppose, for the sake of contradiction, that $X \cap B \neq \emptyset$. Then, since for every $A \in \mathcal{P}$ we have either $B \cap A = \emptyset$ or $A \subseteq B$, it follows that $X \subseteq B$. Consequently, $B \cap Y \supseteq X \cap Y \neq \emptyset$, a contradiction.

Consider the family $\mathcal{L} \coloneqq \{S \in \mathcal{P} \cup \mathcal{Q} \colon S \cap X = \emptyset = S \cap Y\}$. Since X and Y are the only intersecting sets in $\mathcal{P} \cup \mathcal{Q}$, the family $\mathcal{L}$ is laminar. Each element of $V \setminus (X \cup Y)$ is contained in exactly two sets of $\mathcal{L}$. Let $\mathcal{P}_1$ and $\mathcal{Q}_1$ be the inclusion-wise maximal and minimal sets in $\mathcal{L}$, respectively, with any duplicates distributed between the two. Let

$$\mathcal{P}' \coloneqq \{X\} \cup \{A \in \mathcal{P} \colon A \subseteq Y\} \cup \mathcal{P}_1$$
$$\mathcal{Q}' \coloneqq \{Y\} \cup \{C \in \mathcal{Q} \colon C \subseteq X\} \cup \mathcal{Q}_1.$$

Then both $\mathcal{P}'$ and $\mathcal{Q}'$ are $\{s,t\}$-separating partitions of V with $\mathcal{P}' \cup \mathcal{Q}' = \mathcal{P} \cup \mathcal{Q}$. Since $B \in \mathcal{P}_1$ and there are at least two sets of $\mathcal{L}$ contained in B, we get $|\mathcal{Q}'| > |\mathcal{Q}|$, contradicting Lemma 2.6. ☐

Claim 2.8. *Every set A in $\mathcal{P} \setminus X$ is either disjoint from Y or contained in Y.*

Proof. For the sake of contradiction, let $A \in \mathcal{P} \setminus \{X\}$ satisfy $A \cap Y \neq \emptyset$ and $A \setminus Y \neq \emptyset$. The sets Y and A cannot be intersecting, since X and Y are the only intersecting pair in $\mathcal{P} \cup \mathcal{Q}$. Thus $Y \setminus A = \emptyset$, which implies $Y \subseteq A$. Then $A \cap X \supseteq Y \cap X \neq \emptyset$, contradicting the fact that A and X are both in the partition $\mathcal{P}$ and therefore disjoint. ☐

Claim 2.9. $|\{A \in \mathcal{P} \colon A \subseteq Y\}| \leq |\{B \in \mathcal{Q} \colon B \subseteq X\}|$.

Proof. Suppose indirectly that the claim fails. Consider

$$\mathcal{P}' \coloneqq \{B \in \mathcal{Q} \colon B \subseteq X \cup Y\} \cup \{A \in \mathcal{P} \colon A \subseteq V \setminus (X \cup Y)\}$$
$$\mathcal{Q}' \coloneqq \{A \in \mathcal{P} \colon A \subseteq X \cup Y\} \cup \{B \in \mathcal{Q} \colon A \subseteq V \setminus (X \cup Y)\}.$$

Then $\mathcal{P}'$ and $\mathcal{Q}'$ are $\{s,t\}$-separating partitions with $\mathcal{P}' \cup \mathcal{Q}' = \mathcal{P} \cup \mathcal{Q}$. By the indirect assumption, we get $|\mathcal{Q}'| > |\mathcal{Q}|$, contradicting Lemma 2.6. ☐

Thus we get that properties (b), (c) and (d) of Definition 1.6(1) hold by Claims 2.7, 2.8 and 2.9, respectively. That is, $\mathcal{Q}$ is an $\{s,t\}$-refinement of $\mathcal{P}$ along (X, Y), concluding the proof of the theorem. ☐

2.2 Existence

The goal of this section is to show that every submodular function admits an $\{s,t\}$-separating principal partition sequence.

Theorem 2.10. *Let $f \colon 2^V \to \mathbb{R}$ be a submodular function and let $s, t \in V$. Then, there exists an $\{s,t\}$-separating principal partition sequence of f.*

The proof of Theorem 2.10 relies on three lemmas that characterize how the partitions minimizing $g^{s,t}(\lambda)$ behave between breakpoints and at a breakpoint.

Lemma 2.11. *Let $\lambda_a < \lambda_b$ be adjacent breakpoints of $g^{s,t}$ and let*

$$\mathcal{P}_a \in \arg\max\{|\pi|\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_a) = g^{s,t}(\lambda_a)\},$$
$$\mathcal{P}_b \in \arg\min\{|\pi|\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_b) = g^{s,t}(\lambda_b)\}.$$

Then, $g_{\mathcal{P}_a}(\lambda) = g^{s,t}(\lambda) = g_{\mathcal{P}_b}(\lambda)$ for all $\lambda \in [\lambda_a, \lambda_b]$, and moreover $|\mathcal{P}_a| = |\mathcal{P}_b|$.

Proof. Since λ_a and λ_b are adjacent breakpoints of $g^{s,t}$, there exists an $\{s,t\}$-separating partition $\mathcal{R}$ such that $g_{\mathcal{R}}(\lambda) = g^{s,t}(\lambda)$ for every $\lambda \in [\lambda_a, \lambda_b]$.

We first show that $g_{\mathcal{P}_a}(\lambda) = g^{s,t}(\lambda)$ for every $\lambda \in [\lambda_a, \lambda_b]$. By the definition of $\mathcal{P}_a$, it follows that $|\mathcal{R}| \le |\mathcal{P}_a|$. Applying Theorem 2.1 with $\mathcal{P} = \mathcal{P}_a$, $\mathcal{Q} = \mathcal{R}$, $\alpha = \lambda_a$, and $\lambda = \lambda_b$, we obtain that $\mathcal{R}$ is either a refinement or an $\{s,t\}$-refinement of $\mathcal{P}$, and consequently $|\mathcal{R}| \ge |\mathcal{P}_a|$. Thus $|\mathcal{R}| = |\mathcal{P}_a|$. Since $f(\mathcal{R}) - \lambda_a|\mathcal{R}| = f(\mathcal{P}_a) - \lambda_a|\mathcal{P}_a|$ and $|\mathcal{R}| = |\mathcal{P}_a|$, it follows that $f(\mathcal{R}) = f(\mathcal{P}_a)$. Consequently, $g_{\mathcal{P}_a}(\lambda) = g_{\mathcal{R}}(\lambda)$ for every λ, and in particular $g_{\mathcal{P}_a}(\lambda) = g^{s,t}(\lambda)$ for every $\lambda \in [\lambda_a, \lambda_b]$.

Next, we show that $g_{\mathcal{P}_b}(\lambda) = g^{s,t}(\lambda)$ for every $\lambda \in [\lambda_a, \lambda_b]$. By the definition of $\mathcal{P}_b$, it follows that $|\mathcal{R}| \ge |\mathcal{P}_b|$. Applying Theorem 2.1 with $\mathcal{P} = \mathcal{R}$, $\mathcal{Q} = \mathcal{P}_b$, $\alpha = \lambda_a$, and $\lambda = \lambda_b$, we obtain that $\mathcal{P}_b$ is either a refinement or an $\{s,t\}$-refinement of $\mathcal{R}$, and consequently $|\mathcal{P}_b| \ge |\mathcal{R}|$. Thus $|\mathcal{R}| = |\mathcal{P}_b|$. Since $f(\mathcal{R}) - \lambda_b|\mathcal{R}| = f(\mathcal{P}_b) - \lambda_b|\mathcal{P}_b|$ and $|\mathcal{R}| = |\mathcal{P}_b|$, it follows that $f(\mathcal{R}) = f(\mathcal{P}_b)$. Consequently, $g_{\mathcal{P}_b}(\lambda) = g_{\mathcal{R}}(\lambda)$ for every λ, and in particular $g_{\mathcal{P}_b}(\lambda) = g^{s,t}(\lambda)$ for every $\lambda \in [\lambda_a, \lambda_b]$. ☐

The following lemma characterizes the behavior of the partitions minimizing $g^{s,t}(\lambda)$ at the leftmost and rightmost breakpoints. The proof does not rely on submodularity and holds for arbitrary set functions.

Lemma 2.12. *Let a, b be the leftmost and rightmost breakpoints of $g^{s,t}$ and let*

$$\mathcal{P}' \in \arg\min\{f(\pi)\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } |\pi| = 2\},$$
$$\mathcal{Q}' \coloneqq \{\{v\}\colon v \in V\}.$$

Then, $g^{s,t}(\lambda) = g_{\mathcal{P}'}(\lambda)$ for $\lambda \in (-\infty, a]$ and $g^{s,t}(\lambda) = g_{\mathcal{Q}'}(\lambda)$ for $\lambda \in [b, \infty)$.

Proof. We first show that $g^{s,t}(\lambda) = g_{\mathcal{P}'}(\lambda)$ for all $\lambda \in (-\infty, a]$. Since a is the leftmost breakpoint of $g^{s,t}$, there exists an $\{s,t\}$-separating partition $\mathcal{R}$ such that $g^{s,t}(\lambda) = g_{\mathcal{R}}(\lambda)$ for every $\lambda \in (-\infty, a]$. Assume that $|\mathcal{R}| \ge 3$. Let $\lambda < \min a, (f(\mathcal{R}) - f(\mathcal{P}'))/(|\mathcal{R}| - 2)$. For such λ, we have

$$g^{s,t}(\lambda) \le g_{\mathcal{P}'}(\lambda) = f(\mathcal{P}') - 2\lambda < f(\mathcal{R}) - \lambda|\mathcal{R}| = g_{\mathcal{R}}(\lambda) = g^{s,t}(\lambda),$$

a contradiction. Thus we conclude that $|\mathcal{R}| = 2$, yielding $g_{\mathcal{P}'}(\lambda) = g_{\mathcal{R}}(\lambda)$ for every $\lambda \in (-\infty, a]$.

Next, we show that $g^{s,t}(\lambda) = g_{\mathcal{Q}'}(\lambda)$ for all $\lambda \in [b, \infty)$. Since b is the rightmost breakpoint of $g^{s,t}$, there exists an $\{s,t\}$-separating partition $\mathcal{R}$ such that $g^{s,t}(\lambda) = g_{\mathcal{R}}(\lambda)$ for every $\lambda \in [b, \infty)$. Assume that $|\mathcal{R}| < |V|$. Let $\lambda > \max b, (f(\mathcal{Q}') - f(\mathcal{R}))/(|V| - |\mathcal{R}|)$. For such λ, we have

$$g^{s,t}(\lambda) \le g_{\mathcal{Q}'}(\lambda) = f(\mathcal{Q}') - \lambda|V| < f(\mathcal{R}) - \lambda|\mathcal{R}| = g_{\mathcal{R}}(\lambda) = g^{s,t}(\lambda),$$

a contradiction. Thus we conclude that $|\mathcal{R}| = |V|$, yielding $\mathcal{R} = \mathcal{Q}'$. ☐

Lemma 2.13. *Let λ be a breakpoint of $g^{s,t}$ and let*

$$\mathcal{P} \in \arg\min\{|\pi|\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}.$$

Then, there exists a sequence $\mathcal{P} = \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{r+1}$ of $\{s,t\}$-separating partitions such that

1. *$\mathcal{R}_{r+1} \in \arg\max\{|\pi|\colon \pi$ is an $\{s,t\}$-separating partition with $g_\pi(\lambda) = g^{s,t}(\lambda)\}$,*

2. $\mathcal{R}_1$ *is either a refinement of* $\mathcal{P}$ *up to one set or an* $\{s,t\}$*-refinement of* $\mathcal{P}$ *up to two sets,*
3. $\mathcal{R}_{i+1}$ *is a refinement of* $\mathcal{R}_i$ *up to one set for every* $i \in [r]$*,*
4. $|\mathcal{R}_{i+1}| > |\mathcal{R}_i|$ *for every* $i \in \{0, 1, 2, \dots, r\}$*, and*
5. $g_{\mathcal{R}_i}(\lambda) = g^{s,t}(\lambda)$ *for every* $i \in \{0, 1, \dots, r+1\}$*.*

Proof. Let $\mathcal{Q}' \in \arg\max\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}$. By Theorem 2.1, $\mathcal{Q}'$ is either a refinement or an $\{s,t\}$-refinement of $\mathcal{P}$. Let $U := \emptyset$ if $\mathcal{Q}'$ is a refinement of $\mathcal{P}$ and $U := X \cup Y$ if $\mathcal{Q}'$ is an $\{s,t\}$-refinement of $\mathcal{P}$ along (X, Y). It then follows that $|\{A \in \mathcal{P} \colon A \subseteq V \setminus U\}| \leq |\{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\}|$.

First, consider the case when $|\{A \in \mathcal{P} \colon A \subseteq V \setminus U\}| = |\{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\}|$. Then, we have $\{A \in \mathcal{P} \colon A \subseteq V \setminus U\} = \{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\}$. Consequently, $\mathcal{Q}'$ is a refinement of $\mathcal{P}$ up to one set or an $\{s,t\}$-refinement of $\mathcal{P}$ up to two sets. Since λ is a breakpoint of $g^{s,t}$, it follows that the slope of the function $g_{\mathcal{P}}$ at λ is strictly more than the slope of $g_{\mathcal{Q}'}$ at λ and consequently, $|\mathcal{P}| < |\mathcal{Q}'|$. If $\mathcal{Q}'$ is a refinement of $\mathcal{P}$, then $U = \emptyset$ and consequently, $\{A \in \mathcal{P} \colon A \subseteq V \setminus U\} = \{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\}$ implies that $\mathcal{P} = \mathcal{Q}'$, thus contradicting $|\mathcal{P}| < |\mathcal{Q}'|$. Hence, $\mathcal{Q}'$ is an $\{s,t\}$-refinement of $\mathcal{P}$ up to two sets. Therefore, setting $r = 0$, $\mathcal{R}_0 = \mathcal{P}$, and $\mathcal{R}_1 = \mathcal{Q}'$ satisfies the requirements of the lemma.

Now consider the case when $|\{A \in \mathcal{P} \colon A \subseteq V \setminus U\}| < |\{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\}|$. We know that for every $B \in \mathcal{Q}'$ there exists $A \in \mathcal{P}$ with $B \subseteq A$. By the assumption of the case, there exists $A \in \mathcal{P}$ with $A \subseteq V \setminus U$ that contains at least two parts of $\mathcal{Q}'$. Let $P_1, \dots, P_{r+1}$ be such parts of $\mathcal{P}$, that is, $P_i \subseteq V \setminus U$ and each P_i contains at least two parts of $\mathcal{Q}'$ for every $i \in [r+1]$. For each $i \in [r+1]$, we define

$$\mathcal{R}_i := \{B \in \mathcal{Q}' \colon B \subseteq U\} \cup \Big\{B \in \mathcal{Q}' \colon B \subseteq \bigcup_{j=1}^{i} P_j\Big\} \cup \Big\{A \in \mathcal{P} \colon A \subseteq (V \setminus U) \setminus \bigcup_{j=1}^{i} P_j\Big\},$$

$$\mathcal{S}_i := \{A \in \mathcal{P} \colon A \subseteq U\} \cup \Big\{A \in \mathcal{P} \colon A \subseteq \bigcup_{j=1}^{i} P_j\Big\} \cup \Big\{B \in \mathcal{Q}' \colon B \subseteq (V \setminus U) \setminus \bigcup_{j=1}^{i} P_j\Big\}.$$

Moreover, define

$$\begin{aligned}
\mathcal{R}_0 &:= \{B \in \mathcal{Q}' \colon B \subseteq U\} \cup \{A \in \mathcal{P} \colon A \subseteq V \setminus U\} \\
\mathcal{S}_0 &:= \{A \in \mathcal{P} \colon A \subseteq U\} \cup \Big\{B \in \mathcal{Q}' \colon B \subseteq V \setminus U\Big\}.
\end{aligned}$$

Observe that $\mathcal{R}_{r+1} = \mathcal{Q}'$, $\mathcal{S}_{r+1} = \mathcal{P}$, $\mathcal{R}_i$ is a refinement of $\mathcal{R}_{i-1}$ up to one set for every $i \in [r+1]$, and if $U = \emptyset$, then $\mathcal{R}_0 = \mathcal{P}$, whereas if $U \neq \emptyset$, then $\mathcal{R}_0$ is an $\{s,t\}$-refinement of $\mathcal{P}$ up to two sets. Since $\mathcal{R}_i$ is a strict refinement of $\mathcal{R}_{i-1}$ up to one set for every $i \in [r+1]$, we immediately have that $|\mathcal{R}_i| > |\mathcal{R}_{i-1}|$ for every $i \in [r+1]$. Similarly, $|\mathcal{S}_{i-1}| > |\mathcal{S}_i|$ for every $i \in [r+1]$.

We claim that $g_{\mathcal{R}_i}(\lambda) = g^{s,t}(\lambda) = g_{\mathcal{S}_i}(\lambda)$ for every $i \in \{0, 1, \dots, r\}$. To see this, note that $\mathcal{R}_i$ and $\mathcal{S}_i$ are $\{s,t\}$-separating partitions of V with $\mathcal{R}_i \cup \mathcal{S}_i = \mathcal{P} \cup \mathcal{Q}'$, yielding $g^{s,t}(\lambda) \leq g_{\mathcal{R}_i}(\lambda)$ and $g^{s,t}(\lambda) \leq g_{\mathcal{S}_i}(\lambda)$ for $i \in \{0, 1, \dots, r\}$. Hence

$$\begin{aligned}
2g^{s,t}(\lambda) &\leq g_{\mathcal{R}_i}(\lambda) + g_{\mathcal{S}_i}(\lambda) \\
&= f(\mathcal{R}_i) + f(\mathcal{S}_i) - \lambda(|\mathcal{R}_i| + |\mathcal{S}_i|) \\
&= f(\mathcal{P}) + f(\mathcal{Q}') - \lambda(|\mathcal{P}| + |\mathcal{Q}'|) \\
&= g_{\mathcal{P}}(\lambda) + g_{\mathcal{Q}'}(\lambda) \\
&= 2g^{s,t}(\lambda),
\end{aligned}$$

implying $g_{\mathcal{R}_i}(\lambda) = g^{s,t}(\lambda) = g_{\mathcal{S}_i}(\lambda)$.

Concluding the above, if $U = \emptyset$, then the sequence $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{r+1}$ satisfies the properties of the lemma. We now consider the case where $U \neq \emptyset$. If $|\{A \in \mathcal{P} : A \subseteq U\}| < |\{B \in \mathcal{Q}' : B \subseteq U\}|$, then the sequence $\mathcal{P}, \mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{r+1}$ satisfies the properties of the lemma. Suppose $|\{A \in \mathcal{P} : A \subseteq U\}| = |\{B \in \mathcal{Q}' : B \subseteq U\}|$. Then, we observe that $|\mathcal{S}_0| = |\mathcal{Q}'|$ and moreover, $g^{s,t}(\lambda) = g_{\mathcal{S}_0}(\lambda)$. Consequently, $\mathcal{S}_0 \in \arg\max\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}$. Hence, the sequence $\mathcal{S}_{r+1}, \mathcal{S}_r, \dots, \mathcal{S}_0$ satisfies the properties of the lemma. ∎

We now prove Theorem 2.10.

Proof of Theorem 2.10. Let $\lambda_1 < \cdots < \lambda_{\ell-1}$ be the breakpoints of $g^{s,t}$. We will use Lemmas 2.13 and 2.11 alternatively to construct the claimed sequence of partitions along with the critical value sequence. Let

$$\mathcal{P}_1 \in \arg\min\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_1) = g^{s,t}(\lambda_1)\}.$$

By Lemma 2.12, we have $\mathcal{P}_1 \in \arg\min\{f(\mathcal{P}) \colon \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition with } |\mathcal{P}| = 2\}$ and $g^{s,t}(\lambda) = g_{\mathcal{P}'_1}(\lambda)$ if $\lambda \in (-\infty, \lambda_1]$.

We proceed as follows for $i = 1, 2, 3, \ldots, \ell - 1$. We will inductively ensure that the $\{s,t\}$-separating partition $\mathcal{P}_i$ is such that $\mathcal{P}_i \in \arg\min\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_i) = g^{s,t}(\lambda_i)\}$. Applying Lemma 2.13 at the breakpoint λ_i and the $\{s,t\}$-separating partition $\mathcal{P}_i$ gives a sequence of $\{s,t\}$-separating partitions $\mathcal{P}_i = \mathcal{R}^i_0, \mathcal{R}^i_1, \ldots, \mathcal{R}^i_{r_i+1}$ satisfying the properties guaranteed by the lemma. We set $\mathcal{P}_{i+1} := \mathcal{R}^i_{r_i+1}$. By the first property of Lemma 2.13, we know that

$$\mathcal{P}_{i+1} = \mathcal{R}^i_{r_i+1} \in \arg\max\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_i) = g^{s,t}(\lambda_i)\}.$$

Applying Lemma 2.11 to adjacent breakpoints $\lambda_i < \lambda_{i+1}$ of $g^{s,t}$, we conclude that

$$\mathcal{P}_{i+1} \in \arg\min\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_{i+1}) = g^{s,t}(\lambda_{i+1})\}.$$

This implies that $\mathcal{P}_{i+1}$ satisfies the induction hypothesis for every $i \in [\ell-2]$ and moreover, $\mathcal{P}_\ell = \{\{v\} : v \in V\}$ by Lemma 2.12.

With this construction, we now consider the sequence

$$\mathcal{P}_1, \mathcal{R}^1_1, \mathcal{R}^1_2, \ldots, \mathcal{R}^1_{r_1+1} = \mathcal{P}_2, \ldots, \mathcal{P}_i, \mathcal{R}^i_1, \mathcal{R}^i_2, \ldots, \mathcal{R}^i_{r_i+1} = \mathcal{P}_{i+1}, \ldots, \mathcal{P}_{\ell-1}, \mathcal{R}^\ell_1, \mathcal{R}^\ell_2, \ldots, \mathcal{R}^\ell_{r_\ell+1} = \mathcal{P}_\ell$$

together with the critical value sequence

$$\lambda_1, \underbrace{\lambda_1, \ldots, \lambda_1}_{r_1 \text{ copies}}, \lambda_2, \ldots, \lambda_i, \underbrace{\lambda_i, \ldots, \lambda_i}_{r_i \text{ copies}}, \lambda_{i+1}, \ldots, \lambda_{\ell-1}, \underbrace{\lambda_{\ell-1}, \ldots, \lambda_{\ell-1}}_{r_{\ell-1} \text{ copies}}.$$

We claim that this sequence of partitions satisfies properties (1)-(4) of Definition 1.7. The critical value sequence is non-decreasing and hence, we have (1). We have already seen that $\mathcal{P}_1 \in \arg\min\{f(\mathcal{P}) \colon \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition with } |\mathcal{P}| = 2\}$ and $\mathcal{P}_\ell = \{\{v\} : v \in V\}$ and hence, we have (3). We now show (2). By Lemma 2.12, we have that $g^{s,t}(\lambda) = g_{\mathcal{P}_1}(\lambda)$ for $\lambda \in (-\infty, \lambda_1]$ and $g^{s,t}(\lambda) = g_{\mathcal{P}_\ell}(\lambda)$ for $\lambda \in [\lambda_{\ell-1}, \infty)$. Let $i \in [\ell-1]$. We observe that $g^{s,t}(\lambda_i) = g_{\mathcal{R}^i_j}(\lambda_i)$ by construction of $\mathcal{R}^i_j$ via Lemma 2.13. It suffices to show that $g^{s,t}(\lambda) = g_{\mathcal{P}_{i+1}}(\lambda)$ for every $\lambda \in [\lambda_i, \lambda_{i+1}]$. This follows by applying Lemma 2.11 by considering adjacent breakpoints $\lambda_i < \lambda_{i+1}$ of $g^{s,t}$ and the partition $\mathcal{P}_{i+1} \in \arg\max\{|\pi| \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda_i) = g^{s,t}(\lambda_i)\}$. Finally, we show (4). Let $i \in [\ell-1]$. We observe that $\mathcal{R}^i_1$ is either a refinement of $\mathcal{P}_i$ or an $\{s,t\}$-refinement of $\mathcal{P}_i$ up to two sets and $|\mathcal{R}^i_1| > |\mathcal{P}_i|$ by Lemma 2.13. Moreover, for $j > 1$, we also have that $\mathcal{R}^i_j$ is either a refinement of $\mathcal{R}^i_{j-1}$ or an $\{s,t\}$-refinement of $\mathcal{R}^i_{j-1}$ up to two sets and $|\mathcal{R}^i_j| > |\mathcal{R}^i_{j-1}|$ by Lemma 2.13.

□

2.3 Algorithm

In this section, we present a polynomial-time algorithm to compute an $\{s,t\}$-separating principal partition sequence. Since $g^{s,t}$ is a piecewise-linear concave function with at most $|V| - 2$ breakpoints, all of its breakpoints can be found in polynomial time using the Newton–Dinkelbach method, provided that the value of $g^{s,t}(\lambda)$ can be computed for any given λ; see e.g. [9, 30]. We now show that $g^{s,t}(\lambda)$ is indeed computable in polynomial time. In particular, we can obtain a partition $\mathcal{P} \in \arg\min\{g_\pi(\lambda) \colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}$.

Theorem 2.14. *Given a submodular function $f\colon 2^V \to \mathbb{R}$ via its valuation oracle and $\lambda \in \mathbb{R}$, there exist polynomial-time algorithms to find a partition in the following collections:*

1. $\arg\min\{f(\mathcal{P}) - \lambda|\mathcal{P}| \colon \mathcal{P}$ *is a partition of* $V\}$, *and*
2. $\arg\min\{f(\mathcal{P}) - \lambda|\mathcal{P}| \colon \mathcal{P}$ *is an* $\{s,t\}$*-separating partition of* $V\}$.

Proof. The ideas for both problems are similar. The problem in (1) was already known to be solvable in polynomial time – see e.g. [27, 30]. We show that the same approach also extends to solve the problem in (2). Consider the function $f_\lambda\colon 2^V \to \mathbb{R}$ defined as $f_\lambda(U) \coloneqq f(U) - \lambda$ if $\emptyset \neq U \subsetneq V$ and $f_\lambda(U) = 0$ otherwise. Then, the function f_λ is intersecting submodular – see e.g. [27].

For a function $p\colon 2^V \to \mathbb{R}$, consider the function $p^\vee\colon 2^V \to \mathbb{R}$ defined as

$$p^\vee(U) \coloneqq \min\{p(\mathcal{P})\colon \mathcal{P} \text{ is a partition of } U\}.$$

The function $p^\vee$ is known as the Dilworth truncation of p. Suppose p is intersecting submodular. Then $p^\vee$ is submodular. Moreover, given evaluation oracle access to p, there exists a polynomial-time algorithm to find a partition $\mathcal{P}$ of a given subset U such that $p^\vee(U) = p(\mathcal{P})$ [15]. For any submodular function $q\colon 2^V \to \mathbb{R}$, the complement function $\overline{q}\colon 2^V \to \mathbb{R}$ defined by $\overline{q}(U) \coloneqq q(V \setminus U)$ for every $U \subseteq V$ is also submodular. Consequently, the function $m \coloneqq \overline{f_\lambda^\vee}$ is submodular and admits a polynomial-time evaluation oracle via the oracle for f. We note that the function $m\colon 2^V \to \mathbb{R}$ is given by

$$m(U) \coloneqq \min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P} \text{ is a partition of } V \setminus U\}.$$

and there exists a polynomial time algorithm that takes a subset U as input and finds a partition $\mathcal{P}$ of $V \setminus U$ such that $m(U) = f(\mathcal{P}) - \lambda|\mathcal{P}|$. Next, consider the function $b\colon 2^V \to \mathbb{R}$ defined as

$$b(U) \coloneqq f(U) - \lambda + m(U).$$

Claim 2.15. *We have the following:*

1. $\min\{b(U)\colon s \in U \subseteq V - t\} = \min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P}$ *is an* $\{s,t\}$*-separating partition of* $V\}$.
2. $\min\{b(U)\colon U \subseteq V\} = \min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P}$ *is a partition of* $V\}$.

Proof. We prove (1); the proof of (2) is analogous. Let $W \in \arg\min\{b(U)\colon s \in U \subseteq V - t\}$ and $\mathcal{P}_1 \in \arg\min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P}$ is a partition of $V \setminus W\}$. Consider $\mathcal{P}_2 \coloneqq \{W\} \cup \mathcal{P}_1$. Then, $\mathcal{P}_2$ is an $\{s,t\}$-separating partition of V such that $b(W) = f(W) - \lambda + f(\mathcal{P}_1) - \lambda|\mathcal{P}_1| = f(\mathcal{P}_2) - \lambda|\mathcal{P}_2|$, showing that the minimum on the left hand side is at least the minimum on the right hand side.

For the reverse direction, let $\mathcal{P}_2 \in \arg\min\{f(\pi) - \lambda|\pi|\colon \pi$ is an $\{s,t\}$-separating partition of $V\}$. Let W be the part in $\mathcal{P}_2$ containing s, implying $t \notin W$. Consider $\mathcal{P}_1 \coloneqq \mathcal{P}_2 \setminus \{W\}$. Then, W is a subset of V such that $b(W) = f(W) - \lambda + m(W) \leq f(W) - \lambda + f(\mathcal{P}_1) - \lambda|\mathcal{P}_1| = f(\mathcal{P}_2) - \lambda|\mathcal{P}_2|$, finishing the proof of the claim. □

Since f and m are submodular, it follows that the function b is also submodular. Moreover, we have a polynomial-time evaluation oracle for b using the evaluation oracle for f. Thus, we can compute $U \in \arg\min\{b(U)\colon s \in U \subseteq V - t\}$ (and similarly, $U \in \arg\min\{b(U)\colon U \subseteq V\}$) via submodular minimization in polynomial time using the evaluation oracle for b. Now, we recall that there exists a polynomial-time algorithm to compute a partition $\mathcal{P}'$ of $V \setminus U$ such that $m(U) = f(\mathcal{P}') - \lambda|\mathcal{P}'|$. Now, we return the partition $\mathcal{P} \coloneqq \mathcal{P}' \cup \{U\}$. This partition is indeed in $\arg\min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P}$ is an $\{s,t\}$-separating partition of $V\}$ (and in $\arg\min\{f(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P}$ is a partition of $V\}$ respectively) by Claim 2.15. □

Using Theorem 2.14 and the Newton-Dinkelbach method (e.g., see [9, 30]), we can compute the breakpoints of $g^{s,t}$ in polynomial time. In order to compute an $\{s,t\}$-separating principal partition sequence from the breakpoints, we follow the proof of Theorem 2.10 on the existence. In particular, at each breakpoint λ of $g^{s,t}$, it suffices to find a partition in each of the following collections:

$$\arg\min\{|\pi|\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}, \tag{1}$$

$$\arg\max\{|\pi|\colon \pi \text{ is an } \{s,t\}\text{-separating partition with } g_\pi(\lambda) = g^{s,t}(\lambda)\}. \tag{2}$$

This can be done as follows: for a sufficiently small constant ε[1], consider the function $c\colon 2^V \to \mathbb{R}$ defined as $c(A) := \varepsilon$ for every non-empty subset $A \subseteq V$ and $c(\emptyset) := 0$. Now, consider the functions $f_1 := f + c$, and $f_2 := f - c$. Then, using Theorem 2.14, we can find partitions in

$$\arg\min\{f_i(\mathcal{P}) - \lambda|\mathcal{P}|\colon \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition}\}$$

for $i \in \{1,2\}$, which must then be partitions in the collection 1 and 2, respectively.

3 Approximation Algorithm for $\{s,t\}$-Sep-Submod-k-Part

In this section, we design an algorithm for $\{s,t\}$-Sep-Submod-k-Part via $\{s,t\}$-separating principal partition sequences and show that it achieves an approximation factor of $4/3$ for monotone submodular functions and 2 for posimodular submodular functions. We recall $\{s,t\}$-Sep-Submod-k-Part below:

$\{s,t\}$-Sep-Submod-k-Part
Input: A submodular function $f\colon 2^V \to \mathbb{R}_{\geq 0}$ given by a value oracle, distinct elements $s,t \in V$, and $k \in \mathbb{Z}_{\geq 0}$.
Goal:

$$\min\left\{\sum_{i=1}^{k} f(V_i)\colon \{V_i\}_{i=1}^{k} \text{ is a } \{s,t\}\text{-separating partition of } V \text{ into } k \text{ non-empty parts}\right\}.$$

Our algorithm computes an $\{s,t\}$-separating principal partition sequence as mentioned in Theorem 1.8. If the sequence contains a partition with exactly k parts, the algorithm returns this partition. Otherwise, it considers the two partitions $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ in the sequence with $|\mathcal{P}_{i-1}| < k < |\mathcal{P}_i|$. By the properties of the $\{s,t\}$-separating principal partition sequence, we know that $\mathcal{P}_i$ is either a refinement of $\mathcal{P}_{i-1}$ up to one set or an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ up to two sets. Suppose first that $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i-1}$ up to one set $X \in \mathcal{P}_{i-1}$. Then the algorithm proceeds similarly to the k-partitioning algorithm of [6, 32]: it obtains an $\{s,t\}$-separating k-partition $\mathcal{P}$ from $\mathcal{P}_{i-1}$ by replacing X with the $k - |\mathcal{P}_{i-1}|$ cheapest parts of $\mathcal{P}_i$ contained within X and an additional part containing the remainder. Now suppose that $\mathcal{P}_i$ is an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ up to two sets, namely $X \in \mathcal{P}_{i-1} \setminus \mathcal{P}_i$ and $Y \in \mathcal{P}_i \setminus \mathcal{P}_{i-1}$. In this case, the algorithm constructs three $\{s,t\}$-separating k-partitions σ_1, σ_2, and π, and returns the one with the smallest objective value, as follows: σ_1 is obtained from $\mathcal{P}_{i-1}$ by replacing X with the $k - |\mathcal{P}_{i-1}|$ cheapest parts of $\mathcal{P}_i$ contained within X and an additional part containing the remainder; σ_2 is obtained from $\mathcal{P}_{i-1}$ by replacing X with the $k - |\mathcal{P}_{i-1}| - 1$ cheapest parts of $\mathcal{P}_i$ contained within X, a part $X \cap Y$, and an additional part containing the remainder; π is obtained from $\mathcal{P}_i$ by replacing the $|\mathcal{P}_i| - k + 1$ most expensive parts of $\mathcal{P}_i$ contained within X by their union. The algorithm is presented as Algorithm 1. We discuss two examples based on Figures 2a and 2b to illustrate the partitions created by the algorithm.

Example 3.1. First, consider the two partitions in Figure 2a. Suppose $\mathcal{P}_{i-1} = \{X, C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4\}$ (solid lines), $\mathcal{P}_i = \{B_1, B_2, B_3, C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4\}$ (dashed lines), and $k = 12$. Suppose $f(B_1) \leq f(B_2) \leq f(B_3)$. Then, the partition returned by the algorithm is $\mathcal{P} = \{C_1, C_2, C_3, C_4, C_5, C_6, D_1, D_2, D_3, D_4, B_1, B_2 \cup B_3\}$.

Next, consider the two partitions in Figure 2b. Suppose $\mathcal{P}_{i-1} = \{X, C_1, C_2, D_1, D_2, D_3, D_4\}$ (solid lines), $\mathcal{P}_i = \{Y, B_1, B_2, B_3, B_4, B_5, B_6, D_1, D_2, D_3, D_4\}$ (dashed lines), and $k = 9$. Suppose $f(B_1) \leq f(B_2) \leq \cdots \leq f(B_6)$. Then, the partitions created by the algorithm are $\sigma_1 = \{C_1, C_2, D_1, D_2, D_3, D_4, B_1, B_2, (X \cap Y) \cup \bigcup_{j=3}^{6} B_j\}$, $\sigma_2 = \{C_1, C_2, D_1, D_2, D_3, D_4, B_1, X \cap Y, \bigcup_{j=2}^{6} B_j\}$, and $\pi = \{Y, D_1, D_2, D_3, D_4, B_1, B_2, B_3, \bigcup_{j=4}^{6} B_j\}$.

[1] This ε can be found in polynomial time, provided that a bound on the bit complexity of the query responses is given in advance. This assumption on the bit complexity of query responses is natural, since without such a bound there is no guarantee that any operation can be done in polynomial time.

Algorithm 1 Approximation Algorithm for $\{s,t\}$-Sep-Submod-k-Part.

Input: A submodular function $f\colon 2^V \to \mathbb{R}$ given by value oracle and an integer $k \geq 2$.
Output: An $\{s,t\}$-separating k-partition $\mathcal{P}$ of V.

Use Theorem 1.8 to compute an $\{s,t\}$-separating principal partition sequence $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of the submodular function f.
if $\exists\, j \in [\ell]\colon |\mathcal{P}_j| = k$ **then**
    Return $\mathcal{P} \coloneqq \mathcal{P}_j$.
end if
Let $i \in \{2, \ldots, \ell\}$ such that $|\mathcal{P}_{i-1}| < k < |\mathcal{P}_i|$.
if $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i-1}$ up to one set **then**
    Let $X \in \mathcal{P}_{i-1}$ be the part refined by $\mathcal{P}_i$.
    Let $\mathcal{P}'$ be the parts of $\mathcal{P}_i$ that are contained in X.
    Let $\mathcal{P}' \coloneqq \{B_1, \ldots, B_{|\mathcal{P}'|}\}$ such that $f(B_1) \leq \ldots \leq f(B_{|\mathcal{P}'|})$.
    return $\mathcal{P} \coloneqq \big(\mathcal{P}_{i-1} \setminus \{X\}\big) \cup \Big\{B_i \colon i \in \big[k - |\mathcal{P}_{i-1}|\big]\Big\} \cup \Big\{\bigcup_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big\}$.
else if $\mathcal{P}_i$ is an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ up to two sets **then**
    Let $X \in \mathcal{P}_i \setminus \mathcal{P}_{i-1}$ and $Y \in \mathcal{P}_{i-1} \setminus \mathcal{P}_i$ such that $\mathcal{P}_i$ is an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ along (X, Y).
    Let $\mathcal{P}'$ be the parts of $\mathcal{P}_i$ that are contained in X.
    Let $\mathcal{P}' \coloneqq \{B_1, \ldots, B_{|\mathcal{P}'|}\}$ such that $f(B_1) \leq \ldots \leq f(B_{|\mathcal{P}'|})$.
    Compute the following three partitions:

$$\sigma_1 \coloneqq \big(\mathcal{P}_{i-1} \setminus \{X\}\big) \cup \Big\{B_i \colon i \in \big[k - |\mathcal{P}_{i-1}|\big]\Big\} \cup \Big\{(X \cap Y) \cup \bigcup_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big\}.$$

$$\sigma_2 \coloneqq \big(\mathcal{P}_{i-1} \setminus \{X\}\big) \cup \Big\{B_i \colon i \in \big[k - |\mathcal{P}_{i-1}| - 1\big]\Big\} \cup \{X \cap Y\} \cup \Big\{\bigcup_{j=k-|\mathcal{P}_{i-1}|}^{|\mathcal{P}'|} B_j\Big\}.$$

$$\pi \coloneqq \bigg(\mathcal{P}_i \setminus \Big\{B_i \colon |\mathcal{P}_i| - k + 1 \leq i \leq |\mathcal{P}'|\Big\}\bigg) \cup \Big\{\bigcup_{j=|\mathcal{P}_i|-k+1}^{|\mathcal{P}'|} B_j\Big\}.$$

    return $\mathcal{P} \coloneqq \arg\min\{f(\sigma_1), f(\sigma_2), f(\pi)\}$.
end if

Since an $\{s,t\}$-separating principal partition sequence can be computed in polynomial time by Theorem 1.8, Algorithm 1 can indeed be implemented to run in polynomial time. Moreover, the algorithm returns an $\{s,t\}$-separating k-partition. The rest of the section is devoted to bounding the approximation factor.

Our first lemma shows an easy case under which the algorithm identifies the optimum. This lemma has appeared in the literature before in the context of submodular k-partitioning (e.g., see [32]). We include a proof since our problem of interest is the $\{s,t\}$-separating submodular k-partitioning problem.

Lemma 3.2. *Let $\mathcal{P}^*$ be an optimal $\{s,t\}$-separating k-partition and let $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ be an $\{s,t\}$-separating principal partition sequence of f. If there exists $j \in [\ell]$ such that $|\mathcal{P}_j| = k$, then $f(\mathcal{P}_j) \leq f(\mathcal{P}^*)$.*

Proof. Let λ_j be a value such that $g^{s,t}(\lambda_j) = g_{\mathcal{P}_j}(\lambda_j)$ - such a value exists since $\mathcal{P}_j$ is a partition in an $\{s,t\}$-separating principal partition sequence. Then, we have the following:

$$f(\mathcal{P}^*) - \lambda_j k = g_{\mathcal{P}^*}(\lambda_j) \geq g_{\mathcal{P}_j}(\lambda_j) = f(\mathcal{P}_j) - \lambda_j |\mathcal{P}_j| = f(\mathcal{P}_j) - \lambda_j k,$$

yielding $f(\mathcal{P}^*) \geq f(\mathcal{P}_j)$. □

Let $\mathcal{P}^*$ be an optimal $\{s,t\}$-separating k-partition. Consider an $\{s,t\}$-separating principal partition sequence $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of the submodular function f with critical values $\lambda_1, \ldots, \lambda_{\ell-1}$, such that there exists no $j \in [\ell]$ with $|\mathcal{P}_j| = \ell$. Choose $i \in \{2, \ldots, \ell\}$ satisfying $|\mathcal{P}_{i-1}| < k < |\mathcal{P}_i|$. Let $\mathcal{P}', B_1, \ldots, B_{|\mathcal{P}'|}$ be defined as in Algorithm 1, and let $\mathcal{P}$ be the partition returned by the algorithm.

We begin with a lower bound for the optimum objective value. This lemma has also appeared in the literature before in the context of submodular k-partitioning (e.g., see [6]). Our proof in the context of $\{s,t\}$-separating k-partitioning problem is similar.

Lemma 3.3. *We have the following:*

1. $f(\mathcal{P}^*) \geq \left(\frac{|\mathcal{P}_i|-k}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}_{i-1}) + \left(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}_i)$.
2. $f(\mathcal{P}^*) \geq f(\mathcal{P}_{i-1})$.

Proof. For (1), let λ_{i-1} be the value such that $g_{\mathcal{P}_{i-1}}(\lambda_{i-1}) = g_{\mathcal{P}_i}(\lambda_{i-1}) = g^{s,t}(\lambda_{i-1})$. Then, we have

$$f(\mathcal{P}_{i-1}) - \lambda_{i-1}|\mathcal{P}_{i-1}| = f(\mathcal{P}_i) - \lambda_{i-1}|\mathcal{P}_i|,$$

yielding $\lambda_{i-1} = \frac{f(\mathcal{P}_i)-f(\mathcal{P}_{i-1})}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}$. By the definition of $\{s,t\}$-separating principal partition sequences, we also have

$$f(\mathcal{P}^*) - \lambda_{i-1} \cdot k = g_{\mathcal{P}^*}(\lambda_{i-1}) \geq g^{s,t}(\lambda_{i-1}) = g_{\mathcal{P}_i}(\lambda_{i-1}) = f(\mathcal{P}_i) - \lambda_{i-1}|\mathcal{P}_i|,$$

which implies $f(\mathcal{P}^*) \geq f(\mathcal{P}_i) + \lambda_{i-1}(k - |\mathcal{P}_i|)$. Combining these observations, we get

$$f(\mathcal{P}^*) \geq f(\mathcal{P}_i) + \frac{f(\mathcal{P}_i) - f(\mathcal{P}_{i-1})}{|P_i| - |P_{i-1}|}(k - |\mathcal{P}_i|) = \frac{|\mathcal{P}_i| - k}{|\mathcal{P}_i| - |\mathcal{P}_{i-1}|} f(\mathcal{P}_{i-1}) + \frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}_i| - |\mathcal{P}_{i-1}|} f(\mathcal{P}_i).$$

Now we prove (2). Let $P_1^*, \ldots, P_k^*$ denote the parts of $\mathcal{P}^*$ and set $k' \coloneqq |\mathcal{P}_{i-1}|$. Note that $k' < k$. Consider the k'-partition $\mathcal{Q}$ obtained as $Q_1 \coloneqq P_1^*, \ldots, Q_{k'-1} \coloneqq P_{k'-1}^*, Q_{k'} \coloneqq \cup_{j=k'}^k P_j^*$. Then, using submodularity and $f(\emptyset) \geq 0$, we get

$$f(\mathcal{P}^*) = \sum_{i=1}^{k} f(P_i^*) \geq \left(\sum_{i=1}^{k'-1} f(P_i^*)\right) + f\left(\bigcup_{j=k'}^{k} P_j^*\right) = \sum_{i=1}^{k'} f(Q_i) = f(\mathcal{Q}).$$

That is, $\mathcal{Q}$ is a k'-partition, while $\mathcal{P}_{i-1}$ is an optimal k'-partition by Lemma 3.2, meaning that $f(\mathcal{Q}) \geq f(\mathcal{P}_{i-1})$. These observations together imply $f(\mathcal{P}^*) \geq f(\mathcal{P}_{i-1})$. ☐

We need the following proposition about posimodular submodular functions from [6].

Proposition 3.4 (Chandrasekaran, Wang). *Let $f\colon 2^V \to \mathbb{R}_{\geq 0}$ be a submodular function on a ground set V. If f is posimodular, then*

$$f(T) \leq f(S) + f(S \setminus T)$$

for all $T \subseteq S \subseteq V$.

Next, we show two upper bounds on the objective value of the solution returned by the algorithm. Variants of this lemma have appeared in the literature before in the context of Submod-k-Part–see [6]. Our main contribution is proving similar upper bounds in the context of $\{s,t\}$-Sep-Submod-k-Part–recall that the problem has an additional constraint and consequently, the algorithm is also different. We note that the structure of $\{s,t\}$-separating principal partition sequence is more complicated than that of a principal partition sequence.

Lemma 3.5. *We have the following:*

1. $f(\mathcal{P}) \leq f(\mathcal{P}_i)$.
2. *If f is posimodular, then* $f(\mathcal{P}) \leq f(\mathcal{P}_{i-1}) + 2\left(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\right) f(\mathcal{P}_i)$.
3. *If f is monotone, then* $f(\mathcal{P}) \leq f(\mathcal{P}_{i-1}) + \left(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\right) f(\mathcal{P}_i)$.

Proof. We consider two cases.

Case 1. $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i-1}$ up to one set. In this case, the argument proceeds analogously to the proof of the algorithm for submodular k-partition via principal partition sequences in [6]. We include the proof here for completeness. Recall that $\mathcal{P} := (\mathcal{P}_{i-1}\backslash\{S\}) \cup \left\{B_1, \dots, B_{k-|\mathcal{P}_{i-1}|}\right\} \cup \left\{\bigcup_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\right\}$.

We have

$$\begin{aligned}
f(\mathcal{P}) &= f(\mathcal{P}_{i-1}) - f(S) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) + f\Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big) \\
&\leq f(\mathcal{P}_{i-1}) - f(S) + \sum_{j=1}^{|\mathcal{P}'|} f(B_j) && \text{(by submodularity)} \\
&= f(\mathcal{P}_i). && \text{(since } \mathcal{P}_i = (\mathcal{P}_{i-1} \setminus \{S\}) \cup \{B_1, \dots, B_{|\mathcal{P}'|}\},
\end{aligned}$$

showing (1). If f is posimodular, we get

$$\begin{aligned}
f(\mathcal{P}) &= f(\mathcal{P}_{i-1}) - f(S) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) + f\Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big) \\
&\leq f(\mathcal{P}_{i-1}) + 2 \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) && \text{(by Proposition 3.4 and submodularity)} \\
&\leq f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|}\Big) f(\mathcal{P}') && \text{(by the choice of } B_1, \dots, B_{k-|\mathcal{P}_{i-1}|}) \\
&\leq f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|}\Big) f(\mathcal{P}_i) && \text{(since } B_1, \dots, B_{|\mathcal{P}'|} \text{ are parts in } \mathcal{P}_i) \\
&= f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}_i| - |\mathcal{P}_{i-1}| + 1}\Big) f(\mathcal{P}_i),
\end{aligned}$$

proving (2). Finally, if f is monotone, then

$$\begin{aligned}
f(\mathcal{P}) &= f(\mathcal{P}_{i-1}) - f(S) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) + f\Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big) \\
&\leq f(\mathcal{P}_{i-1}) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) && \text{(by monotonicity)} \\
&\leq f(\mathcal{P}_{i-1}) + \frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|} f(\mathcal{P}') && \text{(by the choice of } B_1, \dots, B_{k-|\mathcal{P}_{i-1}|}) \\
&\leq f(\mathcal{P}_{i-1}) + \frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|} f(\mathcal{P}_i) && \text{(since } B_1, \dots, B_{|\mathcal{P}'|} \text{ are parts in } \mathcal{P}_i) \\
&= f(\mathcal{P}_{i-1}) + \Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}_i| - |\mathcal{P}_{i-1}| + 1}\Big) f(\mathcal{P}_i),
\end{aligned}$$

yielding (3).

Case 2. $\mathcal{P}_i$ is an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ up to two sets. This case is non-trivial and our proof is different from that of the previous one. Let $X \in \mathcal{P}_i \setminus \mathcal{P}_{i-1}$ and $Y \in \mathcal{P}_{i-1} \setminus \mathcal{P}_i$ such that $\mathcal{P}_i$ is an $\{s,t\}$-refinement of $\mathcal{P}_{i-1}$ along (X, Y).

We have

$$f(\mathcal{P}) \leq f(\pi)$$

$$
\begin{aligned}
&= f(\mathcal{P}_i) - \sum_{j=|\mathcal{P}_i|-k+1}^{|\mathcal{P}'|} f(B_j) + f\Big(\bigcup\nolimits_{j=|\mathcal{P}_i|-k+1}^{|\mathcal{P}'|} B_j\Big) \\
&\le f(\mathcal{P}_i), && \text{(by submodularity)}
\end{aligned}
$$

showing (1). If f is posimodular then, by submodularity and Proposition 3.4 applied to $T \coloneqq (X \cap Y) \cup \Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big)$ and $S \coloneqq X$, we get

$$
\begin{aligned}
f\Big((X \cap Y) \cup \Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big)\Big) &\le f(X) + f\Big(\bigcup\nolimits_{j=1}^{k-|\mathcal{P}_{i-1}|} B_j\Big) \\
&\le f(X) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j).
\end{aligned} \tag{3}
$$

Observe that

$$
\begin{aligned}
|\mathcal{P}'| &= |\{B \in \mathcal{P}_i \colon B \subseteq X\}| \\
&\ge |\{B \in \mathcal{P}_i \colon B \subseteq X\}| - |\{A \in \mathcal{P}_{i-1} \colon A \subseteq Y\}| + 1 \\
&= |\mathcal{P}_i| - |\mathcal{P}_{i-1}| + 1.
\end{aligned} \tag{4}
$$

Hence, we have that

$$
\begin{aligned}
f(\mathcal{P}) &\le f(\sigma_1) \\
&= f(\mathcal{P}_{i-1}) - f(X) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) + f\Big((X \cap Y) \cup \Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|} B_j\Big)\Big) \\
&\le f(\mathcal{P}_{i-1}) + 2 \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(B_j) && \text{(by inequality (3))} \\
&\le f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|}\Big) \sum_{j=1}^{|\mathcal{P}'|} f(B_j) && \text{(by definition of } B_1, \dots, B_{k-|\mathcal{P}_{i-1}|}) \\
&= f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|}\Big) f(\mathcal{P}') \\
&\le f(\mathcal{P}_{i-1}) + 2\Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}'|}\Big) f(\mathcal{P}_i) && \text{(since } \mathcal{P}' \subseteq \mathcal{P}_i) \\
&\le 2f(\mathcal{P}_{i-1}) + \Big(\frac{k - |\mathcal{P}_{i-1}|}{|\mathcal{P}_i| - |\mathcal{P}_{i-1}| + 1}\Big) f(\mathcal{P}_i), && \text{(by inequality (4))}
\end{aligned}
$$

proving (2). Finally, assume that f is monotone. Let σ_1 and σ_2 be as defined in Algorithm 1. Denote by $A_1, \dots, A_{|\mathcal{P}'|+1}$ an ordering of $\mathcal{P}' \cup \{X \cap Y\}$ such that $f(A_1) \le \ldots \le f(A_{|\mathcal{P}'|+1})$. Let

$$
\sigma \coloneqq \Big(\mathcal{P}_{i-1} \setminus \{X\}\Big) \cup \Big\{A_i \colon i \in \big[k - |\mathcal{P}_{i-1}|\big]\Big\} \cup \Big\{\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|+1} A_j\Big\}.
$$

Then, $f(\sigma) = \min\{f(\sigma_1), f(\sigma_2)\}$. Observe that

$$
\begin{aligned}
|\mathcal{P}'| &= |\{B \in \mathcal{P}_i \colon B \subseteq X\}| \\
&\ge |\{B \in \mathcal{P}_i \colon B \subseteq X\}| - |\{A \in \mathcal{P}_{i-1} \colon A \subseteq Y\}| \\
&= |\mathcal{P}_i| - |\mathcal{P}_{i-1}|.
\end{aligned} \tag{5}
$$

Thus we have

$$
\begin{aligned}
f(\mathcal{P}) &\le \min\{f(\sigma_1), f(\sigma_2)\} \\
&= f(\sigma) \\
&= f(\mathcal{P}_{i-1}) - f(X) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(A_j) + f\Big(\bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|+1} A_j\Big) \\
&\le f(\mathcal{P}_{i-1}) + \sum_{j=1}^{k-|\mathcal{P}_{i-1}|} f(A_j) && \text{(by monotonicity and } \bigcup\nolimits_{j=k-|\mathcal{P}_{i-1}|+1}^{|\mathcal{P}'|+1} A_j \subseteq X\text{)} \\
&\le f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}'|+1}\Big) \sum_{j=1}^{|\mathcal{P}'|+1} f(A_j) && \text{(by definition of } A_1, \ldots, A_{k-|\mathcal{P}_{i-1}|}\text{)} \\
&= f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}'|+1}\Big)\Big(f(\mathcal{P}') + f(X \cap Y)\Big) \\
&\le f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}'|+1}\Big)\Big(f(\mathcal{P}') + f(Y)\Big) && \text{(by monotonicity)} \\
&\le f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}'|+1}\Big) f(\mathcal{P}_i) && \text{(since } Y \in \mathcal{P}_i \text{ and } \mathcal{P}' \subseteq \mathcal{P}_i\text{)} \\
&\le f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big) f(\mathcal{P}_i), && \text{(by inequality (5))}
\end{aligned}
$$

yielding (3). □

We next refine the upper bounds established in Lemma 3.5 using the lower bounds of Lemma 3.3.

Lemma 3.6. *We have the following:*

1. $f(\mathcal{P}) \le \left(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{k-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}^*) - \left(\frac{|\mathcal{P}_i|-k}{k-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}_{i-1})$.
2. *If f is posimodular, then* $f(\mathcal{P}) \le \left(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\right)\left(2f(\mathcal{P}^*) + \left(\frac{2k-|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}_{i-1})\right)$.
3. *If f is monotone, then* $f(\mathcal{P}) \le \left(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\right)\left(f(\mathcal{P}^*) + \left(\frac{k-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\right) f(\mathcal{P}_{i-1})\right)$.

Proof. We have

$$
\begin{aligned}
f(\mathcal{P}) &\le f(\mathcal{P}_i) && \text{(by Lemma 3.5(1))} \\
&= \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{k-|\mathcal{P}_{i-1}|}\Big)\Big(\Big(\frac{|\mathcal{P}_i|-k}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_i)\Big) - \Big(\frac{|\mathcal{P}_i|-k}{k-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1}) \\
&\le \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{k-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}^*) - \Big(\frac{|\mathcal{P}_i|-k}{k-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1}), && \text{(by Lemma 3.3(1))}
\end{aligned}
$$

showing (1). If f is posimodular, we get

$$
\begin{aligned}
f(\mathcal{P}) &\le f(\mathcal{P}_{i-1}) + 2\Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big) f(\mathcal{P}_i) && \text{(by Lemma 3.5(2))} \\
&= \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big)\Big(\Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1}) + 2\Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_i)\Big) \\
&\le \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big)\Big(2f(\mathcal{P}^*) + \Big(\frac{2k-|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1})\Big), && \text{(by Lemma 3.3(1))}
\end{aligned}
$$

proving (2). Finally, if f is monotone, then

$$\begin{aligned}
f(\mathcal{P}) &\leq f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big) f(\mathcal{P}_i) && \text{(by Lemma 3.5(3))} \\
&= \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big)\Big(\Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1}) + \Big(\frac{k-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_i)\Big) \\
&\leq \Big(\frac{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|+1}\Big)\Big(f(\mathcal{P}^*) + \Big(\frac{k-|\mathcal{P}_{i-1}|+1}{|\mathcal{P}_i|-|\mathcal{P}_{i-1}|}\Big) f(\mathcal{P}_{i-1})\Big), && \text{(by Lemma 3.3(1))}
\end{aligned}$$

yielding (3). ◻

We now bound the approximation factor of Algorithm 1. Using the bounds in Lemma 3.6, the rest of the argument is identical to the argument in [6] and we repeat it here for the sake of completeness. We need the following proposition that is implicit in [6].

Proposition 3.7 (Chandrasekaran, Wang)**.** *Suppose $A, B \geq 1$ are integers. Then, for every $c \in \mathbb{R}_{\geq 0}$, the following two inequalities hold:*

$$\min\Big\{\frac{A+B}{A} - \frac{B}{A}\cdot c,\ \Big(\frac{A+B}{A+B+1}\Big)\Big(2 + \frac{A-B+1}{A+B}\cdot c\Big)\Big\} \leq 2\Big(1 - \frac{1}{A+B+2}\Big),\ \textit{and} \tag{6}$$

$$\min\Big\{\frac{A+B}{A} - \frac{B}{A}\cdot c,\ \Big(\frac{A+B}{A+B+1}\Big)\Big(1 + \frac{1+A}{A+B}\cdot c\Big)\Big\} \leq \frac{4}{3}\Big(1 - \frac{1}{3(A+B)+4}\Big). \tag{7}$$

We are now ready to bound the approximation factor of Algorithm 1 for posimodular submodular functions and monotone submodular functions.

Theorem 3.8. *Let f be a submodular function, $k \geq 2$ be an integer, and let $\mathcal{P}^*$ denote an optimal $\{s,t\}$-separating k-partition. Let $\mathcal{P}$ denote the partition returned by Algorithm 1.*

1. *If f is posimodular, then $f(\mathcal{P}) \leq 2f(\mathcal{P}^*)$.*
2. *If f is monotone, then $f(\mathcal{P}) \leq \frac{4}{3}f(\mathcal{P}^*)$.*

Proof. First, if $f(\mathcal{P}^*) = 0$, then by Lemma 3.3(2) we also have $f(\mathcal{P}_{i-1}) \leq f(\mathcal{P}^*) = 0$, and hence $f(\mathcal{P}) = 0$ by Lemma 3.6(1). Consequently, the returned partition is optimal. We may therefore assume that $f(\mathcal{P}^*) > 0$. Define $c \coloneqq f(\mathcal{P}_{i-1})/f(\mathcal{P}^*)$, $A \coloneqq k - |\mathcal{P}_{i-1}|$, and $B \coloneqq |\mathcal{P}_i| - k$. Note that $A, B \geq 1$.

Assume first that f is posimodular. Then, the upper bounds of Lemma 3.6 can be rewritten as

$$\begin{aligned}
\frac{f(\mathcal{P})}{f(\mathcal{P}^*)} &\leq \min\Big\{\frac{A+B}{A} - \frac{B}{A}\cdot c,\ \Big(\frac{A+B}{A+B+1}\Big)\Big(2 + \frac{A-B+1}{A+B}\cdot c\Big)\Big\} \\
&\leq 2\Big(1 - \frac{1}{A+B+2}\Big) && \text{(by Proposition 3.7)} \\
&\leq 2\Big(1 - \frac{1}{n}\Big), && \text{(since } A+B = |\mathcal{P}_i| - |\mathcal{P}_{i-1}| \leq n-2)
\end{aligned}$$

proving (1).

Now suppose that f is monotone. Then, the upper bounds of Lemma 3.6 can be rewritten as

$$\begin{aligned}
\frac{f(\mathcal{P})}{f(\mathcal{P}^*)} &\leq \min\Big\{\frac{A+B}{A} - \frac{B}{A}\cdot c,\ \Big(\frac{A+B}{A+B+1}\Big)\Big(1 + \frac{1+A}{A+B}\cdot c\Big)\Big\} \\
&\leq \frac{4}{3}\Big(1 - \frac{1}{3(A+B)+4}\Big) && \text{(by Proposition 3.7)} \\
&\leq \frac{4}{3}\Big(1 - \frac{1}{3n-2}\Big), && \text{(since } A+B = |\mathcal{P}_i| - |\mathcal{P}_{i-1}| \leq n-2)
\end{aligned}$$

proving (2). ◻

4 Hypergraph Orientations

In this section, we present an application of the polynomial-time computability of $\{s,t\}$-separating principal partition sequence to hypergraph orientations. We recall that a *directed hypergraph* $\overrightarrow{G} = (V, E, \text{head}\colon E \to V)$ is specified by a vertex set V, hyperedge set E where each $e \in E$ is a subset of V, and a function head: $E \to V$ with the property that head$(e) \in e$ for each $e \in E$. For a subset $U \subseteq V$, we define $\delta^{in}_{\overrightarrow{G}}(U) := \{e \in E\colon \text{head}(e) \in U, e \setminus U \neq \emptyset\}$ and the function $d^{in}_{\overrightarrow{G}}\colon 2^V \to \mathbb{Z}$ defined by $d^{in}_{\overrightarrow{G}}(U) := |\delta^{in}_{\overrightarrow{G}}(U)|$. It is well-known that the function $d^{in}_{\overrightarrow{G}}$ is submodular. By Menger's theorem, $\overrightarrow{G}$ is k-hyperarc-connected if and only if $d^{in}_{\overrightarrow{G}}(U) \geq k \ \forall\, \emptyset \neq U \subsetneq V$. Moreover, $\overrightarrow{G}$ is $(k,(s,t),\ell)$*-hyperarc-connected* if and only if the following two conditions hold:

$$
\begin{aligned}
&d^{in}_{\overrightarrow{G}}(U) \geq k \ \forall\, \emptyset \neq U \subsetneq V,\\
&d^{in}_{\overrightarrow{G}}(U) \geq \ell \ \forall\, t \in U \subseteq V - s.
\end{aligned}
$$

Frank, Király, and Király gave a complete characterization for the existence of an orientation $\overrightarrow{G}$ of a given hypergraph G with specified vertices s and t such that (1) $\overrightarrow{G}$ is k-hyperarc-connected, (2) $\overrightarrow{G}$ has k_1 hyperedge-disjoint paths from s to t and (3) $\overrightarrow{G}$ has k_2 hyperedge-disjoint paths from t to s, where $k_1, k_2 \geq k$. Lemma A.5 in the appendix shows that this problem is equivalent to $(k,(s,t),\ell)$-Conn-Orient. Frank, Király, and Király [12, Theorem 5.1] showed the following characterization for the existence of a $(k,(s,t),\ell)$-hyperarc-connected orientation.

Theorem 4.1 (Frank, Király, Király)**.** *Let $G = (V,E)$ be a hypergraph, $s,t \in V$, and $k, \ell \in \mathbb{Z}_{\geq 0}$. The hypergraph G has a $(k,(s,t),\ell)$-hyperarc-connected orientation if and only if*

$$
|\delta_G(\mathcal{P})| \geq \sum_{X \in \mathcal{P}} p^{s,t}_{k,\ell}(X) \quad \textit{for all partition } \mathcal{P} \textit{ of } V, \tag{8}
$$

where $\delta_G(\mathcal{P})$ is the set of hyperedges in G that intersect at least two parts of $\mathcal{P}$ and $p^{s,t}_{k,\ell}\colon 2^V \to \mathbb{R}$ is defined as

$$
p^{s,t}_{k,\ell}(X) = \begin{cases} 0 & \textit{if } X = V \textit{ or } X = \emptyset,\\ \max\{k,\ell\} & \textit{if } t \in X \subseteq V - s,\\ k & \textit{otherwise.} \end{cases} \tag{9}
$$

The proof of Theorem 4.1 given by Frank, Király, and Király [12] is not constructive. That is, the proof does not lead to a polynomial-time algorithm to verify whether a given hypergraph G with specified vertices s,t satisfies (8) and if so, then find a $(k,(s,t),\ell)$-hyperarc-connected orientation of G. We resolve this issue by giving a polynomial-time algorithm in Section 4.1. In Section 4.2, we address the optimization problems of maximizing k for a given ℓ (or maximizing ℓ for a given k) so that G has a $(k,(s,t),\ell)$-hyperarc-connected orientation, give a min-max relation, and show that the corresponding minimization problem is solvable in polynomial time via our results on $\{s,t\}$-separating principal partition sequence. We use the following result due to Frank, Király, and Király [12]. We note that this result, in particular, gives a polynomial-time algorithm for $(k,(s,t),\ell)$-Conn-Orient for the case $\ell = k$.

Theorem 4.2 (Frank, Király, Király)**.** *Let $G = (V,E)$ be a hypergraph and $k \in \mathbb{Z}_{\geq 0}$. Then, G has a k-hyperarc-connected orientation if and only i*

$$
|\delta_G(\mathcal{P})| \geq k|\mathcal{P}| \textit{ for all partition } \mathcal{P} \textit{ of } V, \tag{10}
$$

where $\delta_G(\mathcal{P})$ is the set of hyperedges in G that intersect at least two parts of $\mathcal{P}$. Moreover, there exists a polynomial-time algorithm to verify whether a given hypergraph satisfies (10), and if so, then find a k-hyperarc-connected orientation of the given hypergraph.

4.1 Algorithm for $(k,(s,t),\ell)$-Hyperarc-Connected Orientation Problems

In this section, we give a polynomial-time algorithm to verify whether a given hypergraph G with specified vertices s,t satisfies (8) and if so, then find a $(k,(s,t),\ell)$-hyperarc-connected orientation of G.

Theorem 4.3. *Let $G=(V,E)$ be a hypergraph with specified vertices $s,t\in V$ and $k,\ell\in\mathbb{Z}_{\geq 0}$.*

1. *There exists a polynomial-time algorithm to verify if (8) holds for every partition $\mathcal{P}$ of V and if not, then return a partition $\mathcal{P}$ for which (8) is violated.*
2. *If (8) holds for every partition $\mathcal{P}$ of V, then there exists a polynomial-time algorithm to find a $(k,(s,t),\ell)$-hyperarc-connected orientation.*

We first address the problem of verifying whether a given hypergraph G with specified vertices s,t satisfies (8). The following lemma proves the first part of Theorem 4.3.

Lemma 4.4. *Given a hypergraph $G=(V,E)$ with specified vertices $s,t\in V$ and $k,\ell\in\mathbb{Z}_{\geq 0}$, and $p^{s,t}_{k,\ell}\colon 2^V\to\mathbb{R}$ defined as in (9), there exists a polynomial-time algorithm to solve the following optimization problem:*

$$\min\left\{|\delta_G(\mathcal{P})|-p^{s,t}_{k,\ell}(\mathcal{P})\colon \mathcal{P}\textit{ is a partition of }V\right\}.$$

Proof. Let $\overrightarrow{G}$ be an arbitrary orientation of G. We observe that $|\delta_G(\mathcal{P})|=\sum_{P\in\mathcal{P}} d^{in}_{\overrightarrow{G}}(P)$ for every partition $\mathcal{P}$ of V. Therefore, the optimization problem is equivalent to the following:

$$\min\left\{d^{in}_{\overrightarrow{G}}(\mathcal{P})-p^{s,t}_{k,\ell}(\mathcal{P})\colon \mathcal{P}\text{ is a partition of }V\right\}.$$

Based on the definition of the function $p^{s,t}_{k,\ell}$, the problem reduces to solving the following two problems.

$$\min\left\{d^{in}_{\overrightarrow{G}}(\mathcal{P})-k|\mathcal{P}|\colon \mathcal{P}\text{ is a partition of }V\right\},\text{ and}$$
$$\min\left\{d^{in}_{\overrightarrow{G}}(\mathcal{P})-k(|\mathcal{P}|-1)-\ell\colon \mathcal{P}\text{ is an }\{s,t\}\text{-separating partition of }V\right\}.$$

Since, $d^{in}_{\overrightarrow{G}}$ is submodular and we have a polynomial-time evaluation oracle for it, the latter two problems can be solved in polynomial time by Theorem 2.14. □

We now prove the second part of Theorem 4.3. We need the following lemma that is implicit in [12].

Proposition 4.5 (Frank, Király, Király)**.** *Let $G=(V,E)$ be a hypergraph and $x\in\mathbb{Z}^V$ a vector such that $x(V)=|E|$ and $x(Y)\geq i_G(Y)$ for every $Y\subseteq V$, where $i_G(Y)$ is the number of hyperedges in G that are contained in Y. Then, there exists a polynomial-time algorithm to find an orientation $\overrightarrow{G}$ of G with indegree vector x.*

The following lemma concludes the proof of Theorem 4.3.

Lemma 4.6. *Let $G=(V,E)$ be a hypergraph with specified vertices $s,t\in V$ and $k,\ell\in\mathbb{Z}_{\geq 0}$ such that (8) holds for every partition $\mathcal{P}$ of V. Then, there exists a polynomial-time algorithm to find a $(k,(s,t),\ell)$-hyperarc-connected orientation of G.*

Proof. We state the algorithm that is implicit in the proof of Frank, Király, and Király [12] in Algorithm 2. The correctness of the algorithm follows from the proof in [12]. We will show here that the algorithm can be implemented to run in polynomial time.

We now show that Step 2 can be implemented in polynomial-time. Let $v\in V$. We observe that the problem of computing

$$x_v:=\min\left\{|\delta_G(\mathcal{P})|-p^{s,t}_{k,\ell}(\mathcal{P})\colon \mathcal{P}\text{ is a partition of }V\text{ with }\{v\}\text{ as a singleton part of }\mathcal{P}\right\}$$

Algorithm 2 An algorithm for finding a $(k,(s,t),\ell)$-hyperarc-connected orientation

Input: A connected hypergraph $G=(V,E)$, two vertices $s,t\in V$, and $k,\ell\in\mathbb{Z}_{\geq 0}$ such that (8) holds.
Output: A $(k,(s,t),\ell)$-hyperarc-connected orientation of G.

1: **for** each $v\in V$ **do**
2: Compute $x_v \coloneqq \min\left\{|\delta_G(\mathcal{P})| - p^{s,t}_{k,\ell}(\mathcal{P})\colon \mathcal{P} \text{ is a partition of } V \text{ with } \{v\} \text{ as a singleton part of } \mathcal{P}\right\}$.
3: **end for**
4: **return** an orientation $\overrightarrow{G}$ of G with in-degree vector x.

reduces to the problem of computing y_v defined as follows, where the hypergraph $G-v$ is obtained from G by deleting the vertex v and all hyperedges incident to v:

$$y_v \coloneqq \min\left\{|\delta_{G-v}(\mathcal{P})| - p^{s,t}_{k,\ell}(\mathcal{P})\colon \mathcal{P} \text{ is a partition of } V\setminus\{v\}\right\}.$$

Indeed, this is because $x_v = y_v + |\delta_G(v)| - p^{s,t}_{k,\ell}(\{v\})$. Observe that the problem of computing y_v is solvable in polynomial time by Lemma 4.4.

Next, we show that Step 4 can be implemented to run in polynomial time. Frank, Király, and Király [12] showed that, assuming (8), the vector $x\in\mathbb{Z}^V$ computed in the first three steps satisfies $x\geq 0$, $x(V)=|E|$, and $x(Y)\geq i_G(Y)+p^{s,t}_{k,\ell}(Y)$ for every $Y\subseteq V$, where $i_G(Y)$ denotes the number of hyperedges of G contained in Y. Hence, by Proposition 4.5, there exists a polynomial-time algorithm that finds an orientation $\overrightarrow{G}$ with in-degree vector x. Moreover, for every $Y\subseteq V$, we have $d^{in}_{\overrightarrow{G}}(Y) = x(Y) - i_G(Y) \geq p^{s,t}_{k,\ell}(Y)$. □

4.2 Min-Max Relations for $(k,(s,t),\ell)$-Hyperarc-Connected Orientation Problems

In this section, we consider two optimization variants of $(k,(s,t),\ell)$-hyperarc-connected orientation problems, present min-max relations for both, and outline a polynomial-time algorithm to solve the corresponding minimization problem using our results on $\{s,t\}$-separating principal partition sequence. Our min-max relations follow immediately from the known characterization for the existence of $(k,(s,t),\ell)$-hyperarc-connected orientation [12]. Our main contribution is a polynomial-time algorithm to solve the minimization problem that arises in the min-max relation.

Firstly, we consider the problem of maximizing ℓ for a given k such that a given hypergraph G with vertices $s,t\in V$ has a $(k,(s,t),\ell)$-connected orientation and give a min-max relation. For this maximization problem, we work under the assumption that G has a k-hyperarc-connected orientation – otherwise, G has no $(k,(s,t),\ell)$-hyperarc-connected orientation for every ℓ.

Theorem 4.7. *Let $G=(V,E)$ be a hypergraph, $s,t\in V$, and $k\in\mathbb{Z}_{\geq 0}$ such that G has a k-hyperarc-connected-orientation. Then,*

$$\begin{aligned}&\max\left\{\ell\colon G \textit{ has a } (k,(s,t),\ell)\textit{-hyperarc-connected orientation}\right\}\\ &\quad= \min\left\{|\delta_G(\mathcal{P})| - k(|\mathcal{P}|-1)\colon \mathcal{P} \textit{ is an } \{s,t\}\textit{-separating partition of } V\right\}.\end{aligned}$$

Proof. We know that both the max and the min are at least k by Theorem 4.2. For $\ell\geq k$, by Theorem 4.1, G has a $(k,(s,t),\ell)$-hyperarc-connected orientation if and only if $|\delta_G(\mathcal{P})|\geq k(|\mathcal{P}|-1)+\ell$ for every $\{s,t\}$-separating partition $\mathcal{P}$ of V. Hence, the theorem follows. □

Next, we consider the problem of maximizing k for a given ℓ such that G has a $(k,(s,t),\ell)$-connected orientation and give a min-max relation. For this maximization problem, we work under the assumption that G has ℓ hyperedge-disjoint paths between s and t – otherwise, G has no $(k,(s,t),\ell)$-hyperarc-connected orientation for every k.

Theorem 4.8. *Let $G = (V, E)$ be a hypergraph, $s, t \in V$, and $\ell \in \mathbb{Z}_{\geq 0}$ such that G has ℓ hyperedge-disjoint paths between s and t. Then,*

$$\max\left\{k\colon G \text{ has a } (k,(s,t),\ell)\text{-hyperarc-connected orientation}\right\} = \min\{\alpha, \beta\},$$

where

$$\alpha := \min\left\{\left\lfloor\frac{|\delta_G(\mathcal{P})|}{|\mathcal{P}|}\right\rfloor : \mathcal{P} \text{ is a partition of } V\right\}, \text{ and}$$
$$\beta := \min\left\{\left\lfloor\frac{|\delta_G(\mathcal{P})| - \ell}{|\mathcal{P}| - 1}\right\rfloor : \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition of } V\right\}.$$

Proof. We first show that max $\geq$ min. Suppose that G has a $(k,(s,t),\ell)$-hyperarc-connected orientation. Let $\mathcal{P}$ be an arbitrary partition of V. By Theorem 4.2, we have $|\delta_G(\mathcal{P})| \geq k|\mathcal{P}|$, and hence $\lfloor |\delta_G(\mathcal{P})|/|\mathcal{P}| \rfloor \geq k$. Next, suppose that $\mathcal{P}$ is an $\{s,t\}$-separating partition of V. By Theorem 4.1, $|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}|-1) + \max\{k,\ell\} \geq k(|\mathcal{P}|-1) + \ell$, and therefore $\lfloor (|\delta_G(\mathcal{P})| - \ell)/(|\mathcal{P}|-1) \rfloor \geq k$.

Next, we show that max $\leq$ min. Suppose that min $\geq k$. Then, we have

$$\begin{aligned} &|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}|-1) + k \quad \text{for all partition } \mathcal{P} \text{ of } V, \text{ and} \\ &|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}|-1) + \ell \quad \text{for all } \{s,t\}\text{-separating partition } \mathcal{P} \text{ of } V. \end{aligned}$$

Consequently,

$$\begin{aligned} &|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}|-1) + k \quad \text{for all partition } \mathcal{P} \text{ of } V, \text{ and} \\ &|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}|-1) + \max\{k,\ell\} \quad \text{for all } \{s,t\}\text{-separating partition } \mathcal{P} \text{ of } V. \end{aligned}$$

Equivalently, $|\delta_G(\mathcal{P})| \geq p_{k,\ell}^{s,t}(\mathcal{P})$ for every partition $\mathcal{P}$ of V. Hence, by Theorem 4.1, the hypergraph G has a $(k,(s,t),\ell)$-hyperarc-connected orientation. ☐

We now outline algorithms to solve the minimization problems in Theorems 4.7 and 4.8 in polynomial time. We need some background. We recall that a *directed hypergraph* $\vec{G} = (V, E, \text{head}\colon E \to V)$ is specified by a vertex set V, hyperedge set E where each $e \in E$ is a subset of V, and a function head: $E \to V$ with the property that head$(e) \in e$ for each $e \in E$. For a subset $U \subseteq V$, we define $\delta_{\vec{G}}^{in}(U) := \{e \in E\colon \text{head}(e) \in U, e \setminus U \neq \emptyset\}$ and the function $d_{\vec{G}}^{in}\colon 2^V \to \mathbb{Z}$ defined by $d_{\vec{G}}^{in}(U) := |\delta_{\vec{G}}^{in}(U)|$. It is well-known that the function $d_{\vec{G}}^{in}$ is submodular.

Let $G = (V, E)$ be a hypergraph and $s, t \in V$. Let $\vec{G} = (V, E, head: E \to V)$ be an arbitrary orientation of G. We observe that for each partition $\mathcal{P}$ of V, we have that $|\delta_G(\mathcal{P})| = \sum_{U \in \mathcal{P}} d_{\vec{G}}^{in}(A) = d_{\vec{G}}^{in}(\mathcal{P})$. We define $g_{\mathcal{P}}(\lambda) := d_{\vec{G}}^{in}(\mathcal{P}) - \lambda|\mathcal{P}|$ for each partition $\mathcal{P}$ of V, $g(\lambda) := \min\{g_{\mathcal{P}}(\lambda)\colon \mathcal{P} \text{ is a partition of } V\}$, and $g^{s,t}(\lambda) := \min\{g_{\mathcal{P}}(\lambda)\colon \mathcal{P} \text{ is a } \{s,t\}\text{-separating partition of } V\}$.

Now, for a given $k \in \mathbb{Z}_{\geq 0}$, the minimization problem in Theorem 4.7 is equivalent to $g^{s,t}(k) + k$ and we note that $g^{s,t}(k)$ can be computed in polynomial time using Theorem 1.8. Next, for a given $\ell \in \mathbb{Z}_{\geq 0}$, we address the minimization problem in Theorem 4.8. Here, we note that α is the floor of the smallest number λ such that $g(\lambda) \leq 0$ and β is the floor of the smallest number λ such that $g^{s,t}(\lambda) \leq \ell - \lambda$. We can compute $g(\lambda)$ and $g^{s,t}(\lambda)$ for all λ in polynomial time using Theorems 1.5 and 1.8 respectively. Consequently, we can compute α and β in polynomial time.

Remark 4.9. If $G = (V, E)$ is a hyperedge-weighted hypergraph with weights $w\colon E \to \mathbb{R}_+$ and $|\delta_G(\mathcal{P})|$ is replaced by $\sum_{e \in \delta_G(\mathcal{P})} w_e$, then the corresponding weighted minimization problems can also be solved in strongly polynomial time using the same approach. For hyperedge-weighted hypergraphs, the maximization orientation problems on the left hand sides can be defined naturally by interpreting the weight of a hyperedge as the number of its parallel copies; we omit the formal definition in the interests of brevity.

5 Conclusion

Motivated by the breadth of applications of the principal partition sequence of submodular functions, we investigated the notion of $\{s,t\}$-separating principal partition sequence. We illustrated two applications of this sequence: we designed approximation algorithms for $\{s,t\}$-Sep-Submod-k-Part for monotone and posimodular submodular functions and polynomial-time algorithms for $(k,(s,t),\ell)$-Conn-Orient in hypergraphs. A natural direction for research is whether variants of principal partitioning sequence provide insights into the approximability of multiway cut. As we have seen, it does provide the current-best approximation factor for Submod-k-Part and $\{s,t\}$-Sep-Submod-k-Part for monotone and posimodular submodular functions. Another interesting direction is whether it is possible to compute principal partition sequence and more generally, $\{s,t\}$-separating principal partition sequence of the cut function/graphic matroid rank function of a given graph in near-linear time. It would also be interesting to understand other applications of the $\{s,t\}$-separating principal partition sequence. E.g., the principal partition sequence of the rank function of the graphic matroid is related to recursive ideal tree packing [8]; do we have a similar connection for $\{s,t\}$-separating principal partition sequence?

We mention another interesting application of $\{s,t\}$-separating principal partition sequence that seems related to the arboricity of a graph. Nash-Williams [34] showed that for a graph $G=(V,E)$, the minimum number of forests whose union is E is equal to $\max_{S\subseteq V}\lceil |E[S]|/(|S|-1)\rceil$. Frank, Király, and Király [11] generalized this result to hypergraphs: for a hypergraph $G=(V,E)$, the minimum number of hyperforests whose union is E is equal to $\max_{S\subseteq V}\lceil |E[S]|/(|S|-1)\rceil$, where $E[S]$ is the set of hyperedges in E that are completely contained in S (see [11] for the definition of hyperforests). We observe that for graphs and hypergraphs, $\max_{S\subseteq V}\lceil |E[S]|/(|S|-1)\rceil = \max\{\lceil |E[\mathcal{P}]|/(|V|-|\mathcal{P}|)\rceil : \mathcal{P}$ is a partition of $V\}$, where $E[\mathcal{P}]$ is the set of hyperedges that do not intersect multiple parts of $\mathcal{P}$. One may generalize the latter maximization problem to consider

$$\max\left\{\left\lceil \frac{|E[\mathcal{P}]|}{|V|-|\mathcal{P}|}\right\rceil : \mathcal{P} \text{ is an } \{s,t\}\text{-separating partition of } V\right\}.$$

This maximum can be computed in polynomial time via our results for $\{s,t\}$-separating principal partition sequence, in particular Theorem 1.8. It would be interesting to understand whether this maximum has a min-max relation similar to arboricity.

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References

[1] F. Barahona. On the k-cut problem. *Operations Research Letters*, 26(3):99–105, 2000.

[2] K. Bérczi, K. Chandrasekaran, T. Király, E. Lee, and C. Xu. Beating the 2-approximation factor for global bicut. *Mathematical Programming*, 177(1):291–320, 2019.

[3] K. Bérczi, K. Chandrasekaran, T. Király, and D. P. Szabo. Approximating submodular matroid-constrained partitioning. *arXiv preprint arXiv:2506.19507*, 2025.

[4] J. Bruno and L. Weinberg. The principal minors of a matroid. *Linear Algebra and Its Applications*, 4(1):17–54, 1971.

[5] K. Chandrasekaran, C. Chekuri, and S. Kulkarni. On Deleting Vertices to Reduce Density in Graphs and Supermodular Functions. In *52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025)*, pages 43:1–43:20, 2025.

[6] K. Chandrasekaran and W. Wang. Approximating submodular k-partition via principal partition sequence. *SIAM Journal on Discrete Mathematics*, 38:3198–3219, 2024.

[7] C. Chekuri and S. Li. On the hardness of approximating the k-way hypergraph cut problem. *Theory of Computing*, 16(1), 2020.

[8] C. Chekuri, K. Quanrud, and C. Xu. LP Relaxation and Tree Packing for Minimum k-Cut. *SIAM Journal on Discrete Mathematics*, 34(2):1334–1353, 2020.

[9] W. H. Cunningham. Optimal attack and reinforcement of a network. *Journal of the ACM (JACM)*, 32(3):549–561, 1985.

[10] M. P. Desai, H. Narayanan, and S. B. Patkar. The realization of finite state machines by decomposition and the principal lattice of partitions of a submodular function. *Discrete Applied Mathematics*, 131(2):299–310, 2003.

[11] A. Frank, T. Király, and M. Kriesell. On decomposing a hypergraph into k connected sub-hypergraphs. *Discrete Applied Mathematics*, 131(2):373–383, 2003.

[12] A. Frank, T. Király, and Z. Király. On the orientation of graphs and hypergraphs. *Discrete Applied Mathematics*, 131(2):385–400, 2003.

[13] S. Fujishige. Lexicographically optimal base of a polymatroid with respect to a weight vector. *Mathematics of Operations Research*, 5(2):186–196, 1980.

[14] S. Fujishige. Principal structures of submodular systems. *Discrete Applied Mathematics*, 2:77–79, 1980.

[15] S. Fujishige. *Submodular functions and optimization*. Elsevier, 2005.

[16] S. Fujishige. Theory of principal partitions revisited. In W. Cook, L. Lovász, and J. Vygen, editors, *Research Trends in Combinatorial Optimization*, pages 127–162. Springer, Berlin, Heidelberg, 2009.

[17] O. Goldschmidt and D. Hochbaum. A Polynomial Algorithm for the k-cut Problem for Fixed k. *Mathematics of Operations Research*, 19(1):24–37, Feb 1994.

[18] E. Harb, K. Quanrud, and C. Chekuri. Faster and scalable algorithms for densest subgraph and decomposition. In *Advances in Neural Information Processing Systems*, 2022.

[19] M. Iri. A min-max theorem for the ranks and term-ranks of a class of matrices: An algebraic approach to the problem of the topological degrees of freedom of a network. *Transactions of the Institute of Electronics and Communication Engineers of Japan*, 51:180–187, 1968.

[20] M. Iri. The maximum-rank minimum-term-rank theorem for the pivotal transforms of a matrix. *Linear Algebra and Its Applications*, 2(4):427–446, 1969.

[21] M. Iri. A review of recent work in Japan on principal partitions of matroids and their applications. *Annals of the New York Academy of Sciences*, 319(1):306–319, 1979.

[22] M. Iri. Structural theory for the combinatorial systems characterized by submodular functions. In *Progress in Combinatorial Optimization*, pages 197–219. Academic Press, New York, 1984.

[23] G. Kishi and Y. Kajitani. Maximally distant trees and principal partition of a linear graph. *IEEE Transactions on Circuit Theory*, 16(3):323–330, 1969.

[24] V. Kolmogorov. A faster algorithm for computing the principal sequence of partitions of a graph. *Algorithmica*, 56(4):394–412, 2010.

[25] P. Manurangsi. Inapproximability of Maximum Biclique Problems, Minimum k-Cut and Densest At-Least-k-Subgraph from the Small Set Expansion Hypothesis. *Algorithms*, 11(1):10, 2018.

[26] H. Nagamochi and Y. Kamidoi. Minimum cost subpartitions in graphs. *Information Processing Letters*, 102(2):79–84, 2007.

[27] K. Nagano, Y. Kawahara, and S. Iwata. Minimum average cost clustering. *Advances in Neural Information Processing Systems*, 23, 2010.

[28] M. Nakamura. Structural theorems for submodular functions, polymatroids and polymatroid intersections. *Graphs and Combinatorics*, 4(1):257–284, 1988.

[29] H. Narayanan. Theory of matroids and network analysis. *Ph.D. Thesis, Department of Electrical Engineering, Indian Institute of Technology*, 1974.

[30] H. Narayanan. The principal lattice of partitions of a submodular function. *Linear Algebra and its Applications*, 144:179–216, 1991.

[31] H. Narayanan. *Submodular functions and electrical networks*, volume 54 of *Annals of Discrete Mathematics*. North-Holland Publishing Co., Amsterdam, 1997.

[32] H. Narayanan, S. Roy, and S. Patkar. Approximation Algorithms for Min-k-Overlap Problems Using the Principal Lattice of Partitions Approach. *Journal of Algorithms*, 21(2):306–330, 1996.

[33] C. S. J. Nash-Williams. On orientations, connectivity and odd-vertex-pairings in finite graphs. *Canadian Journal of Mathematics*, 12:555–567, 1960.

[34] C. S. J. Nash-Williams. Decomposition of Finite graphs into forests. *Journal of the London Mathematical Society*, 39:12, 1964.

[35] K. Okumoto, T. Fukunaga, and H. Nagamochi. Divide-and-conquer algorithms for partitioning hypergraphs and submodular systems. *Algorithmica*, 62(3):787–806, 2012.

[36] T. Ozawa. Common trees and partition of two-graphs (in Japanese). *Transactions of the Institute of Electronics and Communication Engineers of Japan*, 57(5):383–390, 1974.

[37] S. B. Patkar and H. Narayanan. Improving graph partitions using submodular functions. *Discrete Applied Mathematics*, 131:535–553, 2003.

[38] K. Quanrud. Quotient sparsification for submodular functions. In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 5209–5248, 2024.

[39] R. Ravi and A. Sinha. Approximating k-cuts using network strength as a lagrangean relaxation. *European Journal of Operational Research*, 186(1):77–90, 2008.

[40] R. Santiago. New approximations and hardness results for submodular partitioning problems. In *Proceedings of International Workshop on Combinatorial Algorithms*, IWOCA, pages 516–530, 2021.

[41] H. Saran and V. Vazirani. Finding k Cuts within Twice the Optimal. *SIAM Journal on Computing*, 24(1):101–108, 1995.

[42] N. Tomizawa. Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors (in Japanese). *Transactions of the Institute of Electronics and Communication Engineers of Japan*, 59:83–91, 1976.

[43] L. Zhao, H. Nagamochi, and T. Ibaraki. Greedy splitting algorithms for approximating multiway partition problems. *Mathematical Programming*, 102:167–183, 2005.

A Auxiliary Lemmas

Lemma A.1. *The function $g^{s,t}$ is piecewise linear with at most $|V|-2$ breakpoints.*

Proof. Each $\{s,t\}$-separating partition $\mathcal{P}$ must have at least 2 parts, and no more than $|V|$. Thus there are $|V|-1$ possible values of $|\mathcal{P}|$, and $g^{s,t}$ is the minimum of linear functions with $|V|-1$ different slopes, which is piecewise linear with at most $|V|-1$ pieces. Therefore, $g^{s,t}$ has at most $|V|-2$ breakpoints. $\square$

The following lemma justifies that this assumption can be made without disrupting min-cost $\{s,t\}$-separating partitions with a fixed number of parts.

Lemma A.2. *For every submodular function $f\colon 2^V \to \mathbb{R}_{\geq 0}$, there is a submodular function $h\colon 2^V \to \mathbb{R}_{\geq 0}$ such that*

1. *$h(A)+h(B) > h(A\cap B)+h(A\cup B)$ for every pair $A, B \subseteq V$ of intersecting sets, and*
2. *if $\mathcal{P}$ is an $\{s,t\}$-separating partition such that $h(\mathcal{P}) \leq h(\mathcal{Q})$ for every $\{s,t\}$-separating partition $\mathcal{Q}$ with $|\mathcal{Q}| = |\mathcal{P}|$, then $f(\mathcal{P}) \leq f(\mathcal{Q})$ for every $\{s,t\}$-separating partition $\mathcal{Q}$ with $|\mathcal{Q}| = |\mathcal{P}|$.*

Furthermore, if f is symmetric (or monotone or posimodular), then h can be chosen to be symmetric (monotone or posimodular, respectively).

Proof. Let $G = (V, E)$ be the complete graph on V, let $m := \binom{|V|}{2}$, and fix an $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{4m} \cdot \min\{|f(\mathcal{P}) - f(\mathcal{Q})| \colon \mathcal{P}, \mathcal{Q} \text{ are partitions of } V \text{ with } f(\mathcal{P}) \neq f(\mathcal{Q})\}.$$

Define $w\colon E \to \mathbb{R}$ by $w(e) = \varepsilon$ for all $e \in E$. For any $X \subseteq V$, let $d_w(X) = \varepsilon|X||V \setminus X|$ and $e_w(X) = d_w(X) + \varepsilon\binom{|X|}{2}$; that is, $d_w(X)$ is the total weight of edges leaving X, and $e_w(X)$ is the total weight of edges having at least one end in X. It is easy to check that both d_w and e_w are posimodular, strictly submodular functions. Furthermore, d_w is symmetric, while e_w is monotone. We propose two modifications of f, depending on whether we wish to preserve symmetry or monotonicity; if posimodularity or neither property is needed, either modification works.

If f is symmetric, set $h = f + d_w$; if f is monotone, set $h = f + e_w$. Then $h(\mathcal{P}) \leq f(\mathcal{P}) + 2w(E) = f(\mathcal{P}) + 2m\varepsilon$ for any partition $\mathcal{P}$ of V in either case. Moreover, h is strictly submodular by the submodularity of f, and by strict submodularity of d_w and e_w. The defining property is preserved as well: in the first case, h is symmetric because d_w is symmetric, , and in the second, it is monotone because e_w is monotone. These show that h satisfies property (1).

To show property (2), let $\mathcal{P}$ be an $\{s,t\}$-separating partition such that $h(\mathcal{P}) \leq h(\mathcal{Q})$ for every $\{s,t\}$-separating partition $\mathcal{Q}$ of size $|\mathcal{P}|$, and let $\mathcal{Q}$ be any such partition. We prove two claims that hold no matter we are in the symmetric or the monotone case.

Claim A.3. *If $h(\mathcal{P}) = h(\mathcal{Q})$, then $f(\mathcal{P}) = f(\mathcal{Q})$.*

Proof. Sippose indirectly that $f(\mathcal{P}) \neq f(\mathcal{Q})$. Then,

$$\begin{aligned}
|f(\mathcal{P}) - f(\mathcal{Q})| &= |(f(\mathcal{P}) - f(\mathcal{Q}) - (h(\mathcal{P}) - h(\mathcal{Q})| \\
&\leq |f(\mathcal{P}) - h(\mathcal{P})| + |f(\mathcal{Q}) - h(\mathcal{Q})| \\
&\leq 2m\varepsilon + 2m\varepsilon \\
&< |f(\mathcal{P}) - f(\mathcal{Q})|,
\end{aligned}$$

a contradiction. □

Claim A.4. *If $h(\mathcal{P}) < h(\mathcal{Q})$, then $f(\mathcal{P}) \leq f(\mathcal{Q})$.*

Proof. Suppose indirectly that $f(\mathcal{P}) > f(\mathcal{Q})$. Then,

$$\begin{aligned}
f(\mathcal{P}) &\leq h(\mathcal{P}) \\
&< h(\mathcal{Q}) \\
&\leq f(\mathcal{Q}) + 2m\varepsilon \\
&< f(\mathcal{Q}) + (f(\mathcal{P}) - f(\mathcal{Q}))/2 \\
&< f(\mathcal{P}),
\end{aligned}$$

a contradiction. □

By Claims A.3 and A.4, property (2) is satisfied as well, concluding the proof of the lemma. □

Lemma A.5. *Let $G = (V, E)$ be a hypergraph, $s, t \in V$, and $k, \ell \in \mathbb{Z}_{\geq 0}$ with $\ell \geq k$. Then, G has a $(k, (s,t), \ell)$-hyperarc-connected orientation if and only if for every $k_1, k_2 \in \mathbb{Z}_{\geq k}$ with $k_1 + k_2 = \ell + k$, there exists an orientation $\vec{G'}$ of G such that*

1. *$\vec{G'}$ is k-hyperarc-connected,*
2. *$\vec{G'}$ has k_1 hyperedge-disjoint paths from s to t, or equivalently, $d^{in}_{\vec{G}}(U) \geq k_1$ for all $t \in U \subseteq V - s$, and*
3. *$\vec{G'}$ has k_2 hyperedge-disjoint paths from t to s, or equivalently, $d^{in}_{\vec{G}}(U) \geq k_2$ for all $s \in U \subseteq V - t$.*

Proof. The backwards direction follows by taking $k_1 := \ell$ and $k_2 := k$. For the other direction, let $\overrightarrow{G}$ be a $(k,(s,t),\ell)$-hyperarc-connected orientation of G, and let $k_1, k_2 \in \mathbb{Z}_{\geq k}$ such that $k_1 + k_2 = \ell + k$. By Menger's theorem, there exist ℓ hyperedge-disjoint paths from s to t in $\overrightarrow{G}$ and k hyperedge-disjoint paths from t to s in $\overrightarrow{G}$. If $k_2 = k$, then $\overrightarrow{G'} := \overrightarrow{G}$ satisfies the required properties. Suppose $k_2 > k$. Let $P_1, \ldots, P_\ell$ be ℓ hyperedge-disjoint paths from s to t in $\overrightarrow{G}$. Consider the orientation $\overrightarrow{G'} = (V, E, \text{head}' \colon E \to V)$ obtained from $\overrightarrow{G}$ by reversing the orientations of the edges in $P_1, \ldots, P_{k_2-k}$. That is, for each $j \in [q]$, if P_j consists of edges $e_1, \ldots, e_q$ where $s \in e_1$, $\text{head}(e_i) \in e_{i+1}$ for $i \in [q-1]$ and $\text{head}(e_q) = t$, then in $\overrightarrow{G'}$ we define $\text{head}'(e_1) = s$ and $\text{head}'(e_{i+1}) = \text{head}(e_i)$ for $i \in [q-1]$. For every edge e not contained in the paths, we set $\text{head}'(e) = \text{head}(e)$.

We claim that $\overrightarrow{G'}$ satisfies properties (1)-(3). To see this, consider an arbitrary $U \subseteq V$. If $\{s,t\} \nsubseteq U$ or $\{s,t\} \subseteq U$, then every path $P_1, \ldots P_{k_2-k}$ enters and leaves the set U the same number of times and consequently, $d^{in}_{\overrightarrow{G'}}(U) = d^{in}_{\overrightarrow{G}}(U) \geq k$. If $s \in U$ but $t \notin U$, then $d^{in}_{\overrightarrow{G'}}(U) \geq d^{in}_{\overrightarrow{G}}(U) + k_2 - k \geq k + (k_2 - k) = k_2$. Finally, if $t \in U$ but $s \notin U$, then $d^{in}_{\overrightarrow{G'}}(U) \geq d^{in}_{\overrightarrow{G}}(U) - (k_2 - k) \geq \ell - (k_2 - k) = k + \ell - k_2 = k_1$.

We recall that $k_1, k_2 \geq k$. Hence, the above observations together imply that $\overrightarrow{G'}$ is k-hyperarc-connected. Furthermore, by Menger's theorem, $\overrightarrow{G'}$ has k_1 hyperedge-disjoint paths from s to t and k_2 hyperedge-disjoint paths from t to s. This concludes the proof of the lemma. $\square$