Monochromatic Diameter Two Components in Edge Colorings of Complete Graph*

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Abstract

In this paper we review recent findings on large monochromatic components in r-edge-colorings of K_n. Previous research has focused on proving the existence of bounded diameter components, showing that there is always a large monochromatic diameter ≤ 4 component of size $\geq \frac{n}{r-1}$, and it is conjectured that the same holds for diameter 3. We explore the largest monochromatic diameter 2 component, and give constructions for small r that do not have such components of size at least $\frac{n}{r-1}$.

1 Introduction

How many people do we need to ensure that there are 3 who each know each other or 3 who don't know each other? This question was answered long ago [8], and it kicked off the field of Ramsey theory. If we consider the people as vertices of a graph, and their relationship to each other to be an edge, we have a 2-edge-coloring of K_n, the complete graph on n vertices, where the colors represent the relationships 'know each other' and 'don't know each other.'

The field of Ramsey theory aims to find large monochromatic structures in edge colorings of a graph.

Definition 1. *The Ramsey number $R(G_1, G_2, \ldots, G_r)$ denotes the smallest n such that any r-edge-coloring of K_n contains a monochromatic $G_1, G_2, \ldots, G_{r-1}$ or G_r.*

The above problem that we introduced is solved by $R(K_3, K_3) = 6$, which says that 6 is the minimal number such that in every 2-edge-coloring of K_6 there is a monochromatic K_3. In other words, in every set of relationships among any six people, there must either be a group of 3 that know each other or a group of 3 who are strangers to each other.

Rather than looking for values of n containing monochromatic subgraphs, we instead look for subgraphs that exist in all K_n.

Definition 2. *A subgraph H of a graph G is said to be spanning if $V(H) = V(G)$.*

A simple remark by Erdős and Rado [4] states that any 2-coloring of the edges of K_n has a monochromatic spanning component. In other words, either a graph or its complement is spanning. This problem of finding large monochromatic components was then generalized to r colors, for which Gyárfás proved that the largest monochromatic component in an r-edge-coloring of K_n has size $\geq \frac{n}{r-1}$.

Now we define an important application of this area, the affine plane.

Definition 3. *An affine plane of order $r \geq 2$ is an r-uniform, $(r+1)$-regular hypergraph H with r^2 vertices in which any two points are in a unique edge.*

It is known that affine planes exist if n is a prime power [9], but it still remains an important open problem whether they only exist for prime powers.

Theorem 1 (Gyárfás [3]). *The size of the largest monochromatic component in an r-edge-coloring of K_n is $\geq \dfrac{n}{r-1}$ and equality holds if $(r-1)^2 | n$ and there is an affine plane of order $r-1$.*

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He also gives a construction for which equality holds by taking the affine plane and coloring each of the r parallel classes a different color. Then he blows it up by replacing each vertex in the plane with $\frac{n}{(r-1)^2}$ vertices from K_n, and coloring the edges within these clusters arbitrarily. Thus, each monochromatic component has the same size of $(r-1)\frac{n}{(r-1)^2} = \frac{n}{r-1}$.

If there is no affine plane of order $r-1$, which is conjectured [9] to be true when $r-1$ is not a prime power, the following holds.

Theorem 2 (Füredi [2]). *If an affine plane of order $r-1$ does not exist, then the size of the largest monochromatic component in an r-edge-coloring of K_n is $\geq \dfrac{n}{r-1-(r-1)^{-1}}$.*

Note that this is larger than the tight bound in case affine planes exist, showing that there is a key difference in the size of the largest monochromatic component depending on the existence of the affine plane. That is, to find the size of the largest monochromatic component in general is extremely difficult.

We now consider an extension of the original problem, and we first introduce the definition of diameter of a graph.

Definition 4. *The diameter of a graph is the maximum distance between any two vertices, where the distance is the length of the shortest path between the two vertices.*

We will now look for different structures: graphs with bounded diameter. Instead of looking for three people who know each other, we look for a group of people each of whom 'know each other indirectly.' More exactly, each pair of people have one other person in the group that they both know. In graph theoretic terms, this corresponds to a monochromatic diameter 2 component. An important diameter 3 graph is a double star.

Definition 5. *A graph G is a double star if it can be constructed by connecting the centers of two vertex disjoint stars. A triple star is similar, just that a third vertex disjoint star is connected to a double star.*

The search for monochromatic double stars in complete graphs can be best summarized by the following problem.

Problem 1 (Gyárfás, Problem 4.2 in [4]). *For $r \geq 3$, is there a monochromatic double star of size asymptotic to $n/(r-1)$ in every r-coloring of K_n?*

This problem is still unsolved, and Gyárfás posed the following weaker version.

Problem 2 (Gyárfás, Problem 4.3 in [4]). *Given positive numbers n, r, is there a constant d (perhaps $d = 3$) such that in every r-coloring of K_n there is a monochromatic subgraph of diameter at most d with at least $n/(r-1)$ vertices?*

This was proved in affirmative for three colors by Mubayi [5].

Theorem 3 (Mubayi [5]). *Every 3-edge-coloring of K_n contains a monochromatic component of diameter ≤ 4 on at least $\lceil n/2 \rceil$ ($n/2+1$ if $n \equiv 2 \pmod 4$) vertices.*

Problem 2 in general with r colors has been solved by Ruszinkó with $d = 5$.

Theorem 4 (Ruszinkó [6]). *In every r-edge-coloring of K_n there is a monochromatic connected subgraph of diameter at most 5 on at least $n/(r-1)$ vertices.*

The proof relies on a theorem of Mubayi, which states that a complete bipartite graph on n vertices colored with r colors has a monochromatic double star of size n/r.

Theorem 4 has been improved by Letzter to diameter 4.

Theorem 5 (Letzter [7]). *Let $G = K_n$ be r-edge-colored with $r \geq 3$. Then G contains a monochromatic triple star with at least $\dfrac{n}{r-1}$ vertices.*

She uses an approach similar to Ruszinkó's.

Summarizing, it is not yet known if a diameter at most three monochromatic subgraph on at least $n/(r-1)$ vertices does exist.

The purpose of this paper is to explore, in problem 2, the statement "*perhaps $d = 3$.*" How do we know that this is the case? Can we find colorings with no large diameter 2 components? A theorem of Erdős and Fowler answers this question for $r = 2$.

Theorem 6 (Erdős, Fowler [1]). *Every 2-edge-coloring of K_n contains a monochromatic component of diameter ≤ 2 on at least $3n/4$ vertices.*

They give the following example to show that the bound given in Theorem 6 is sharp. Partition the set of vertices evenly into parts A_1, A_2, A_3, A_4 of size $\leq \lceil n/4 \rceil$. For $j > i$ color all edges red between A_i and A_j if $j - i = 1$, else color them blue. Color the edges inside each A_i arbitrarily, see Figure 1.

The above example is sharp because the largest monochromatic diameter 2 component of this graph must be a subset of some $A_i \cup A_j \cup A_k$ with $i \neq j \neq k$, which means it has size $\leq \frac{3n}{4}$. That is, a spanning monochromatic subgraph of diameter two does not necessarily exist in any 2-coloring of K_n.

This coloring is essentially a partitioning of K_4 into two Hamiltonian paths — that is, paths that span the vertex set, and then "blowing up" the graph for general n. We extend this result of Erdős and Fowler for $r = 3, 4, 5$, and 6 showing that a monochromatic subgraph of diameter two on $n/(r-1)$ vertices does not necessarily exist.

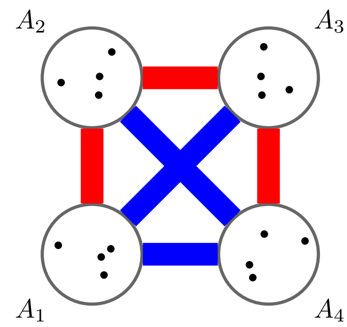

Figure 1: A coloring of K_4 with small diameter 2 components

2 Colorings With No Large Diameter Two Components

Definition 6. *A k-factorization of a graph G is a partition on the edges of G into disjoint k-factors, where k-factors are spanning subgraphs with each vertex having degree k.*

Our approach to r-color complete graphs such that they have no large monochromatic diameter 2 components revolves around taking suitable chosen 'small' complete graphs K_{n_r} and k-factoring them in a way that avoids large ($\geq n/(r-1)$) diameter two components. Coloring each factor a different color, there will be obviously no monochromatic connected diameter 2 subgraph in K_{n_r}. Then we blow up the vertices into clusters of suitable size $\lfloor n/n_r \rfloor$, and distribute the remaining $n \mod n_r$ vertices as evenly as possible. All the edges between two clusters inherit the color of the edge associated to K_{n_r}. The edges inside the clusters are colored arbitrarily. This way we obtain colorings of K_n where every monochromatic connected diameter 2 subgraph is of size $< n/(r-1)$. First we present an application of this method for $r = 3$.

Theorem 7. *There exists a 3-edge-coloring of K_n with the largest monochromatic diameter ≤ 2 subgraph of size $\leq \lceil \frac{3n}{7} \rceil$.*

Proof. We color K_n as follows: partition the vertex set V into $A_1, A_2, \ldots, A_7$ with $\lfloor \frac{n}{7} \rfloor \leq |A_i| \leq \lceil \frac{n}{7} \rceil$ and $\sum_{i=1}^{7} |A_i| = n$. Color the edges between A_i and A_j $c = min\{|i-j|, 7-|i-j|\}$. This allows for 3 colors, and color the edges within A_i arbitrarily for $i = 1 \ldots 7$, see Figure 2. The largest monochromatic diameter 2 subgraph may clearly contain vertices from at most three clusters, i.e., its size is $\leq \lceil \frac{3n}{7} \rceil$. This is smaller than $\frac{n}{r-1} = \frac{n}{2}$ for $r = 3$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5. $\square$

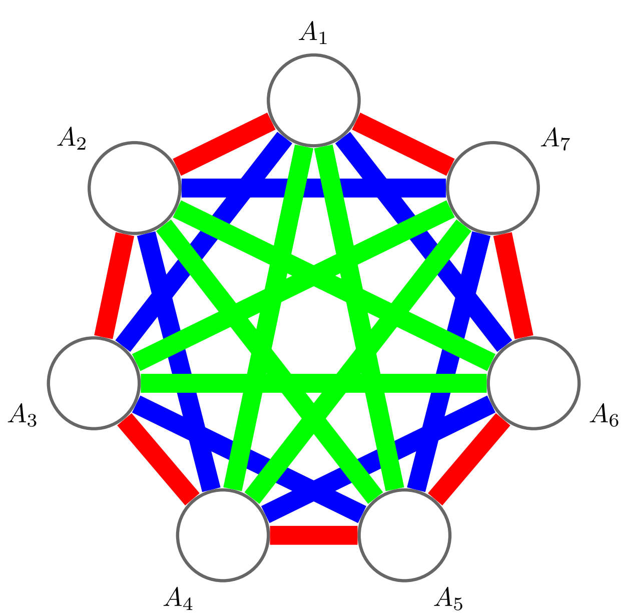

Figure 2: A 3-edge-coloring of K_7 used to color K_n

Here we performed a 2-factorization of K_7, which is a decomposition into Hamiltonian cycles. It is a classical result in graph theory that for any odd ℓ, K_ℓ can be decomposed into Hamiltonian cycles. In order to obtain suitable colorings for larger r, first we choose a suitable n_r, decompose the edges of K_{n_r} into Hamiltonian cycles and define the factors to be the unions of particular cycles. All edges in a given factor will be colored the same. Also, we will be choosing n_r to be prime, which allows us to define our cycles as in the proof of Theorem 7, i.e., $C_i = \{(j,k) : i = min\{|k-j|, n_r - |k-j|\}\}$.

Theorem 8. *There exists a* 4*-edge-coloring of* K_n *with the largest monochromatic diameter* ≤ 2 *subgraph of size* $\leq \lceil \frac{5n}{17} \rceil$.

Proof. Let $n_4 = 17$, and decompose K_{17} into 8 Hamiltonian cycles as above, i.e., $C_i = \{(j,k) : i = min\{|k-j|, 17 - |k-j|\}\}$. The unions of pairs of Hamiltonian cycles $G_1 = (C_1 \cup C_2)$, $G_2(C_3 \cup C_6)$, $G_3(C_4 \cup C_8)$, $G_4(C_5 \cup C_7)$, form a 4 factorization of K_{17}. Notice that G_i-s are isomorphic where the isomorphism is simply renumbering the vertex v as $v \cdot i^{-1}$ mod 17. Color all the edges i in G_i and blow-up K_{17} to K_n as we did before (see Figure 3). All that's left to show that G_1 has no diameter 2 subgraphs with at least 6 vertices.

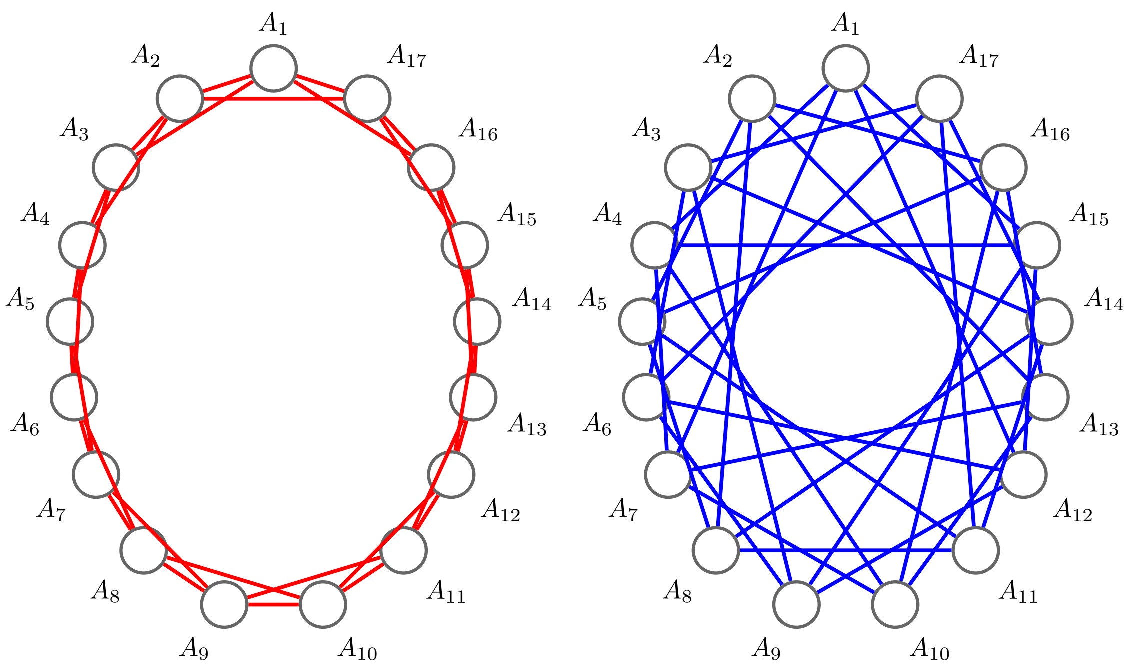

Figure 3: $C_1 \cup C_2$ is Isomorphic to $C_3 \cup C_6$

To prove that any subgraph induced by 6 vertices in G_1 has two vertices of distance at least 3, we assume there is a collection of 6 vertices in G_1 that don't satisfy this. Call one of the vertices v. Clearly, there are four vertices in each direction reachable in two steps in G_1 from v, for a total of 8 possibilities; the remaining 5 vertices must be in these spots. Name the vertex farthest to the left v_L and the one farthest to the right v_R. As there are only 4 spots on each side, both sides must have at least one vertex, meaning v is between v_L and v_R. So there are 4 vertices between v_L and v_R, i.e., their distance is at least 3.

This implies that no six vertices in a color class can each be at distance two from each other, so the largest monochromatic diameter 2 subgraph has 5 vertices of K_{17} to be "blown up", having size $\leq 5\lceil \frac{n}{17} \rceil \leq \lceil \frac{5n}{17} \rceil$. This is smaller than $\frac{n}{r-1} = \frac{n}{3}$ for $r = 4$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5.

□

Theorem 9. *There exists a* 5*-edge-coloring of* K_n *with the largest monochromatic diameter* ≤ 2 *subgraph of size* $\leq \lceil \frac{7n}{31} \rceil$.

Proof. Decompose K_{31} into 15 Hamiltonian cycles $C_i = \{(j,k) : i = min\{|k-j|, 31 - |k-j|\}\}$, $i = 1 \ldots, 15$ and then color as follows (see Figure 4):

- Color $C_1 \cup C_2 \cup C_3$ red.
- Color $C_4 \cup C_5 \cup C_6$ blue.
- Color $C_7 \cup C_8 \cup C_9$ green.
- Color $C_{10} \cup C_{11} \cup C_{13}$ purple.
- Color $C_{12} \cup C_{14} \cup C_{15}$ orange.

Notice that these color classes are not isomorphic, but there are only two different isomorphism classes. Therefore, one can relatively easily check, that no 8 vertices in any of the color classes induce a diameter 2 subgraph. Theorem 9 follows by our standard 'blow up' technique. $\lceil \frac{7n}{31} \rceil$ is less than $\frac{n}{r-1} = \frac{n}{4}$ for $r = 5$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5.

□

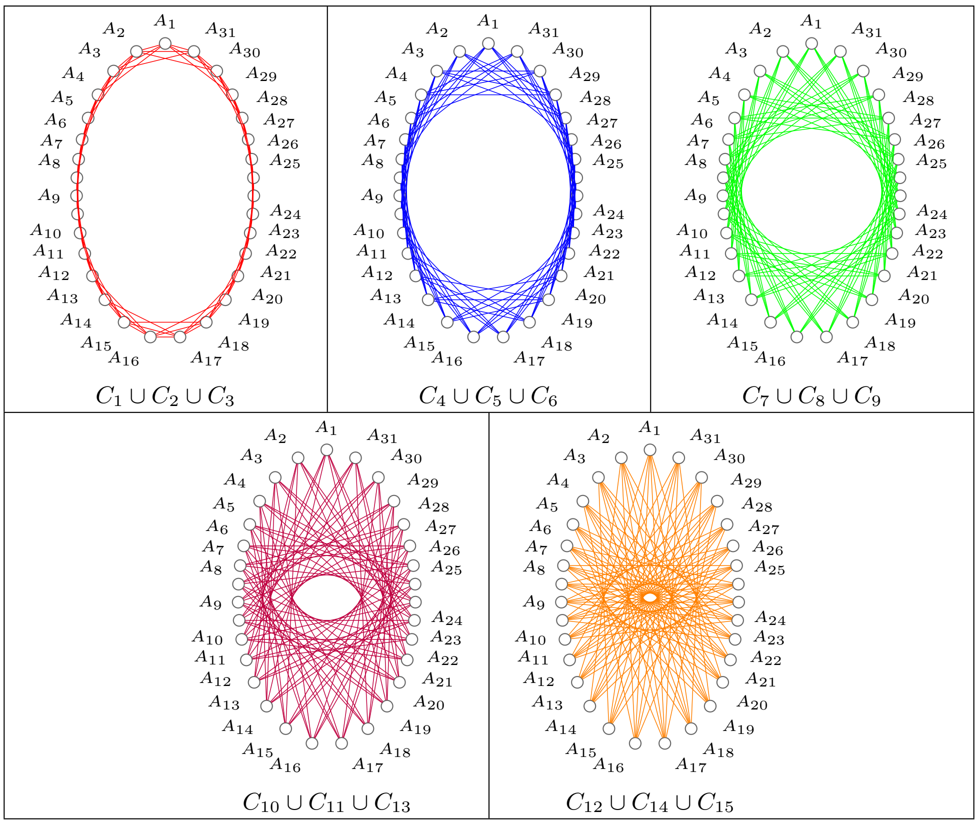

Figure 4: The 5-edge coloring for K_n made by blowing up K_{31}

Theorem 10. *There exists a* 6*-edge-coloring of* K_n *with the largest monochromatic diameter* ≤ 2 *subgraph of size* $\leq \lceil \frac{9n}{47} \rceil$.

Proof. Decompose K_{47} into 23 Hamiltonian cycles $C_i = \{(j,k) : i = min\{|k-j|, 31-|k-j|\}\}$, $i = 1, 2, \ldots, 23$ and then color as follows (see Figure 5):

- Color $C_1 \cup C_2 \cup C_3 \cup C_4$ red.
- Color $C_5 \cup C_{10} \cup C_{15} \cup C_{20}$ blue.
- Color $C_6 \cup C_{12} \cup C_{18} \cup C_{23}$ green.
- Color $C_7 \cup C_{14} \cup C_{21} \cup C_{19}$ purple.
- Color $C_9 \cup C_{11} \cup C_{13} \cup C_{17}$ orange.
- Color $C_8 \cup C_{16} \cup C_{12}$ yellow.

That no 10 vertices in any of the color classes induce a diameter 2 subgraph is verified computationally. Theorem 10 follows by our standard 'blow up' technique. $\lceil \frac{9n}{47} \rceil$ is less than $\frac{n}{r-1} = \frac{n}{5}$ for $r = 5$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5. □

3 Final Remarks

By Erdős and Fowler (Theorem 6) and Letzter (Theorem 5) in case of two colors, the size of the largest monochromatic diameter two subgraph existing in every 2 coloring is strictly less then the size of the largest monochromatic diameter four subgraph. We showed that the same phenomena holds if the number of colors is $3, 4, 5$ or 6.

Based on this we conjecture the following.

Conjecture 1. *For arbitrary number of colors r, the size of the largest monochromatic diameter two subgraph existing in every r coloring is strictly less then the size of the largest monochromatic diameter four subgraph.*

So Gyárfás's suggestion of *probably* $d = 3$ in Problem 2 seems to be accurate.

We conjecture that our method could be used to prove Conjecture 1 for an arbitrary number of colors. We can show that a prime n_r for every $r \geq 3$ that meets our needs, i.e. it can be factored into appropriately sized classes with small stars does exist. One possible approach is to ensure that each color class is isomorphic to one that is easily proven not to have large diameter 2 components, as we did in Theorem 8. However we have yet to find a way to partition the cycles into color classes in a way that ensures the graph will have no large monochromatic diameter two components.

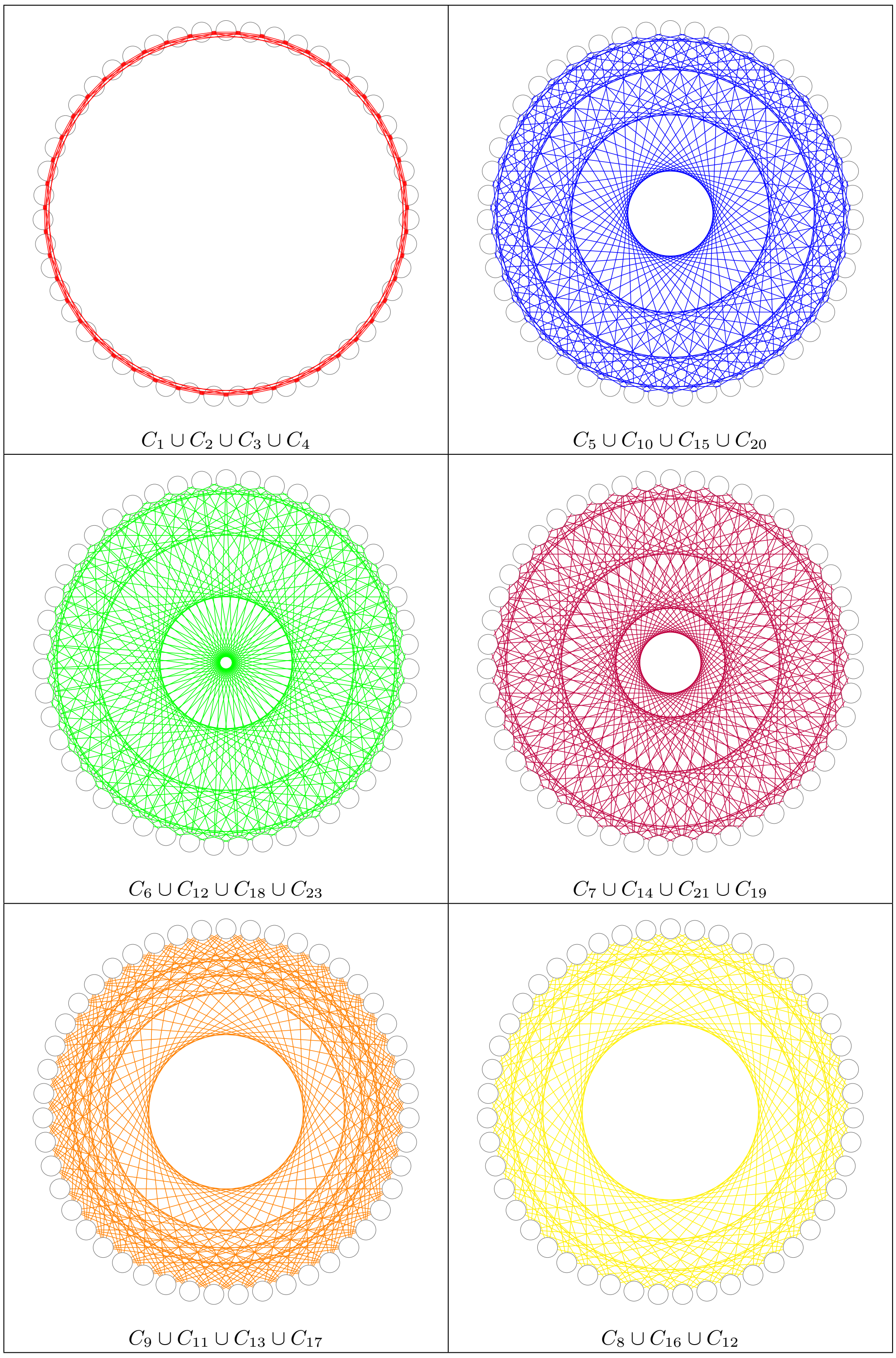

Figure 5: The 6-edge coloring for K_n made by blowing up K_{47}

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