HOMEWORK 1

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1. We saw in class that any group G of size p^n for some prime n has a nontrivial center, in fact $\#Z(G) \geq p$. Thus, $\#(G/Z(G)) = p^a$ for some $a < n$ (because $Z(G)$ is a subgroup). Iterating this argument, we see that if $G^1 = G$ and $G^i = G^{i-1}/Z(G^{i-1})$ for $i \geq 2$ there must be some $k \leq n$ such that $G^k = \{id\}$, and therefore G itself is nilpotent.

2. If $H \subset N_G(K)$, then it is also a subgroup, because H is a subgroup of G. Similarly K is normal in its normalizer $N_G(K)$ and therefore a subgroup as well. Using the fact that H and K are conjugate sylow-p subgroups *in* $N_G(K)$, we know there is a $g \in N_G(K)$ such that $gHg^{-1} = K$. This means however that H commutes with G, so this implies $gg^{-1}H = H = K$, which is a contradiction.

3. As recommended, consider the action of H on Λ by conjugation. The orbit stabilizer theorem tells us that

$$\#\Lambda = \sum_{\text{conjugacy class } \mathcal{C}} \frac{\#H}{\#Z_H(\lambda)}$$

where λ some element in $\mathcal{C}$. The centralizer of λ in H is however $Z_H(\lambda) = \{h \in H : h\lambda h^{-1} = \lambda\}$, which we saw in 2. can only be H if $\lambda = H$. This means $\dfrac{\#H}{\#Z_H(\lambda)} = 1 \iff \lambda = H$, and otherwise $\#Z_H(\lambda)$ must strictly divide $\#H$ and therefore be $0 \mod p$. Thus

$$\#\Lambda = \sum_{\text{conjugacy class } \mathcal{C}} \frac{\#H}{\#Z_H(\lambda)} \equiv \frac{\#H}{\#Z_H(H)} \mod p \equiv 1 \mod p.$$

4. Let n_p be the number of p-Sylow subgroups and n_q the number of q-Sylow subgroups. Then problem 3 tells us that $n_p \equiv 1 \mod p$ and $n_q \equiv 1 \mod q$. The rest of the third Sylow theorem also shows $n_q | p$ and $n_p | q$. Assuming WLOG that $p < q$, we either have $n_p = n_q = 1$, in which case the group is abelian by conjugacy, or $q \equiv 1 \mod p$ and $n_q = p$. In this case we still have $n_p = 1$, call this sole Sylow-p subgroup H. Given any $g \in G$ if $gHg^{-1} \neq H$ we would have a different Sylow-p subgroup, so H is normal in G. Also, G/H has size q and is therefore abelian, which means H_p is nilpotent ($\implies$ solvable) so G itself is solvable as well.

5. Applying the orbit-stabilizer theorem we know

$$\#S = \sum_{orbits\ \mathcal{O}} \frac{\#G}{\#Stab(s)},$$

where $s \in \mathcal{O}$ is any element in the orbit. Because $Stab(s)$ is a subgroup of G its cardinality must divide G and therefore be a power if 2, so the quotient $\frac{\#G}{\#Stab(s)}$ must also be a power of 2. If every stabilizer had size strictly less than 32, then the sum would be $0 \mod 2$ which means $121 = \#S \equiv 0 \mod 2$, which is a contradiction. This at least some s has $Stab(s) = G$, which means $g \cdot s = s$ for all $g \in G$.