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## * Knapsack problem using dynamic programming.

$$table[i,j] = \text{maximum}\{table[i-1,j],\ v_i + table[i-1, j-w_i]\} \quad \text{if } j \geq w_i$$

or

$$table[i-1,j] \quad \text{if } j < w_i$$

Ex. For the given instance of problem obtain the optimal solution for the knapsack.

| item | weight | value |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 4 |
| 3 | 4 | 5 |
| 4 | 5 | 6 |

The capacity of knapsack is $W = 5$.

Sol$^n$ :→ Initially, $table[i,0] = 0$ & $table[0,j] = 0$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | | | | | |
| 2 | 0 | | | | | |
| 3 | 0 | | | | | |
| 4 | 0 | | | | | |

$$table[i,j] = \begin{cases} \max\{table[i-1,j],\ v_i + table[i-1, j-w_i]\} & \text{if } j \geq w_i \\ \text{or} \\ table[i-1,j] & \text{if } j < w_i \end{cases}$$

table [1, 1] with $i=1$, $j=1$, $w_i = 2$ and $v_i = 3$

As $j < w_i$, we will obtain table [1, 1] as

$$table[1,1] = table[i-1, j]$$
$$= table[0, 1]$$

$\therefore \boxed{table[1,1] = 0}$

table [1, 2] with $i=1$, $j=2$, $w_i = 2$ and $v_i = 3$

As $j \geq w_i$, we will obtain table [1,2] as

$$table[1,2] = \cancel{table} \max\{table[i-1,j],\ v_i + table[i-1, j-w_i]\}$$
$$= \max\{table[0,2],\ 3 + table(0,0]\}$$
$$= \max\{0,\ 3+0\}$$
$$= \max\{0, 3\}$$

$\therefore \boxed{table[1,2] = 3}$

table [1, 3] with $i=1$, $j=3$, $w_i = 2$ and $v_i = 3$

As $j \geq w_i$; we will obtain table [1, 3] as

$$table[1,3] = \max\{table[i-1,j],\ v_i + table[i-1, j-w_i]\}$$
$$= \max\{table[0,3],\ 3 + table[0, 1]\}$$
$$= \max\{0; 3+0\}$$

$\therefore \boxed{table[1,3] = 3}$

table [1,4] with $i=1$, $j=4$, $w_i=2$ and $v_i=3$.

As $j \geq w_i$, we will obtain table [1,4] as

$$\text{table}[1,4] = \max\{\text{table}[i-1,j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[0,4],\ 3 + \text{table}[0,2]\}$$
$$= \max\{0,\ 3+0\}$$

$\therefore \boxed{\text{table}[1,4] = 3}$

table [1,5] with $i=1$, $j=5$, $w_i=2$ and $v_i=3$.

As $j \geq w_i$, we will obtain table [1,5] as

$$\text{table}[1,5] = \max\{\text{table}[i-1,j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[0,5],\ 3 + \text{table}[0,3]\}$$
$$= \max\{0,\ 3+0\}$$

$\therefore \boxed{\text{table}[1,5] = 3}$

The table with these values can be

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | | | | | |
| 3 | 0 | | | | | |
| 4 | 0 | | | | | |

Now, let's fill up next row of the table.

table [2,1] with $i=2$, $j=1$, $w_i=3$ and $v_i=4$

As $j < w_i$, we will obtain table [2,1] as

$$\text{table}[2,1] = \text{table}[i-1,j]$$
$$= \text{table}[1,1]$$
$$= 0$$

$\therefore \boxed{\text{table}[2,1] = 0}$

table $[2,2]$ with $i=2$, $j=2$, $w_i=3$ and $v_i=4$.

As $j < w_i$, we will obtain table $[2,2]$ as

$$\text{table}[2,2] = \text{table}[i-1, j]$$
$$= \text{table}[1,2]$$
$$= 3$$

$\therefore \boxed{\text{table}[2,2] = 3}$

table $[2,3]$ with $i=2$, $j=3$, $w_i=3$ and $v_i=4$.

As $j \geq w_i$, we will obtain table $[2,3]$ as

$$\text{table}[2,3] = \max\{\text{table}[i-1,j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[1,3],\ 4 + \text{table}[1,0]\}$$
$$= \max\{3,\ 4+0\}$$
$$= 4$$

$\therefore \boxed{\text{table}[2,3] = 4}$

table $[2,4]$ with $i=2$, $j=4$, $w_i=3$ and $v_i=4$.

As $j \geq w_i$, we will obtain table $[2,4]$ as

$$\text{table}[2,4] = \max\{\text{table}[i-1,j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[1,4],\ 4 + \text{table}[1,1]\}$$
$$= \max\{3,\ 4+0\}$$
$$= 4$$

$\therefore \boxed{\text{table}[2,4] = 4}$

table $[2,5]$ with $i=2$, $j=5$, $w_i=3$ and $v_i=4$

As $j \geq w_i$, we will obtain table $[2,5]$ as

$$\text{table}[2,5] = \max\{\text{table}[i-1,j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[1,5],\ 4 + \text{table}[1,2]\}$$
$$= \max\{3,\ 4+3\}$$
$$= 7$$

$\therefore \boxed{\text{table}[2,5] = 7}$

The table with these computed values will be

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | | | | | |
| 4 | 0 | | | | | |

Now, let's fill up next row of the table.

table [3, 1] with $i = 3$, $j = 1$, $w_i = 4$ and $v_i = 5$

As $j < w_i$, we will obtain table [3, 1] as

table [3, 1] = table [i-1, j]
= table [2, 1]
= 0

$\therefore$ $\boxed{\text{table } [3, 1] = 0}$

table [3, 2] with $i = 3$, $j = 2$, $w_i = 4$ and $v_i = 5$

As $j < w_i$, we will obtain table [3, 2] as

table [3, 2] = table [i-1, j]
= table [2, 2]
= 3

$\therefore$ $\boxed{\text{table } [3, 2] = 3}$

table [3, 3] with $i = 3$, $j = 3$, $w_i = 4$ and $v_i = 5$.

As $j < w_i$, we will obtain table [3, 3] as

table [3, 3] = table [i-1, j]
= table [2, 3]
= 4

$\therefore$ $\boxed{\text{table } [3, 3] = 4}$

table [3, 4] with $i = 3$, $j = 4$, $w_i = 4$ and $v_i = 5$

As $j \geq w_i$; we will obtain table [3, 4] as

table [3, 4] = max { table [i-1, j], $v_i$ + table [i-1, j-$w_i$] }
= max { table [2, 4], 5 + table [2, 0] }

$= \max\{4, 5+0\}$

$= 5$

$\therefore \boxed{table[3,4] = 5}$

table [3,5] with $i=3$, $j=5$, $w_i=4$ and $v_i=5$

As $j \geq w_i$, we will obtain table [3,5] as

$$\begin{aligned} table[3,5] &= \max\{table[i-1,j],\ v_i + table[i-1, j-w_i]\} \\ &= \max\{table[2,5],\ 5 + table[2,1]\} \\ &= \max\{7, 5+0\} \\ &= 7 \end{aligned}$$

$\therefore \boxed{table[3,5] = 7}$

The table with these computed values will be

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | | | | | |

Now, let's fill up next row of the table.

table [4,1] with $i=4$, $j=1$, $w_i=5$ and $v_i=6$.

As $j < w_i$, we will obtain table [4,1] as

$$\begin{aligned} table[4,1] &= table[i-1,j] \\ &= table[3,1] \\ &= 0 \end{aligned}$$

$\therefore \boxed{table[4,1] = 0}$

table [4,2] with $i=4$, $j=2$, $w_i=5$ and $v_i=6$

As $j < w_i$, we will obtain table [4,2] as

$$\begin{aligned} table[4,2] &= table[i-1,j] \\ &= table[3,2] \\ &= 3 \end{aligned}$$

$\therefore \boxed{table[4,2] = 3}$

table [4,3] with $i=4, j=3, w_i=5$ and $v_i=6$

As $j < w_i$, we will obtain table [4,3] as

$$\text{table}[4,3] = \text{table}[i-1, j]$$
$$= \text{table}[3,3]$$
$$= 4$$

$$\therefore \boxed{\text{table}[4,3] = 4}$$

table [4,4] with $i=4, j=4, w_i=5$ and $v_i=6$

As $j < w_i$, we will obtain table [4,4] as

$$\text{table}[4,4] = \text{table}[i-1, j]$$
$$= \text{table}[3,4]$$
$$= 5$$

$$\therefore \boxed{\text{table}[4,4] = 5}$$

table [4,5] with $i=4, j=5, w_i=5$ and $v_i=6$

As $j \geq w_i$, we will obtain table [4,5] as

$$\text{table}[4,5] = \max\{\text{table}[i-1, j],\ v_i + \text{table}[i-1, j-w_i]\}$$
$$= \max\{\text{table}[3,5],\ 6 + \text{table}[3,0]\}$$
$$= \max\{7, 6+0\}$$
$$= 7$$

$$\therefore \boxed{\text{table}[4,5] = 7}$$

Thus the table can be finally as given below.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 |

This is the total value of selected items.

## How to find actual Knapsack items?

→ Following steps are used repeatedly to select actual Knapsack item.

```
Let, i = n and k = W then
while (i > 0 and k > 0)
{
        if (table [i, k] ≠ table [i-1, k]) then
            mark i^th item as in knapsack
            i = i-1 and k = k - wi   // selection of i^th item
    else
            i = i-1          // do not select i^th item
}
```

Let us apply these steps to the above given problem. As we have obtained the final table as.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 → start from here |

Here, $i = 4$ and $k = 5$

i.e, table [4,5] = table [3,5]

∴ Do not select $i^{th}$ item. i.e. $4^{th}$ item

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 |

Now, set $i = i-1$

$i = 3$

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 |

As table [i, K] = table [i-1, K]

i.e. table [3,5] = table [2, 5]

∴ Do not select ith item. i.e. 3rd item.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 |

Now set $i = i-1 = 2$

As table [i, K] ≠ table [i-1, K]

i.e. table [2, 5] ≠ table [1, 5]

∴ select $i^{th}$ item. That is select $2^{nd}$ item.
Now, set $i = i-1$ and $K = K - W_i$ (∵ Here $W_i = 3$)
i.e. $i = 1$ and $K = 5 - 3 = 2$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ✓ 1 | 0 | 0 | 3 | 3 | 3 | 3 |
| ✓ 2 | 0 | 0 | 3 | 4 | 4 | 7 |
| 3 | 0 | 0 | 3 | 4 | 5 | 7 |
| 4 | 0 | 0 | 3 | 4 | 5 | 7 |

As table $[i, K] \neq$ table $[i-1, K]$
i.e. table $[1, 2] \neq$ table $[0, 2]$

∴ select $i^{th}$ item, that is select $1^{st}$ item.
Now, set $i = i-1$ and $K = K - W_i$. (∵ Here $W_i = 2$)
i.e. $i = 0$ and $K = 2 - 2 = 0$.

Thus we have selected item 1 and item 2 for the Knapsack. This solution can also be represented by solution vector $(1, 1, 0, 0)$.

Here, Total Profit = value of item 1 + value of item 2
$= 3 + 4$
$= 7$ //

* Algorithm of Dynamic Knapsack Problem

```
Algorithm Dynamic_Knapsack (n, W, w[], v[])
for (i ← 0 to n) do
{
    for (j ← 0 to W) do
    {
        table [i,0] = 0
        table [0,j] = 0
    }
}
for (i ← 0 to n) do
{
    for (j ← 0 to W) do
    {
        if (j < w[i]) then
            table [i,j] ← table [i-1,j]
        else if (j >= w[i]) then
            table [i,j] ← max(table [i-1,j], (v[i] + table[i-1, j-w[i]])
    }
}
return table [n,W].
```