Discrete Mathematics (II) Spring 2020

Lecture 1: Lattice(I)

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Lattice is a special algebra structure. It is also a part of theoretic foundation of model theory, which formalizes the semantics of logic. It is also applied in computer system area, compiler. Lattice is the theoretic foundation of the framework to analyze program.

1 Introduction

In the first half of the nineteenth century, George Boole's attempt to formalize *propositional logic* led to the concept of *boolean algebras*. While investigating the axiomatics of boolean algebras at the end of the nineteenth century, Charles S. Pierce and Ernst Schröder, both provided many contributions to mathematical logic, found it useful to introduce the *lattice* concept. Independently, Richard Dedekind's research on ideals of algebraic numbers also led to the same discovery.

Although some of the early results of these mathematicians are very elegant and far from trivial, they did not attract the attention of the mathematical community. It was the work of Garrett Birkhoff in the mid-1930s that kicked off the general development of lattice theory. In a brilliant series of papers, he demonstrated the importance of lattice theory and showed that it provides a unifying framework for hitherto unrelated developments in many mathematical disciplines. Birkhoff attempted to "sell" it to the general mathematical community, which he did with astonishing success in the first edition of his monograph *Lattice Theory*.

The explosive growth of this field continued. While the 1960s provided under 1,500 papers and books, the seventies up to 2,700, the eighties over 3,200, the nineties almost 3,600, and the first decade of this century about 4,000.

There are numerous topics covered by lattice theory. Several monographs are on lattice theory include Birkhoff's classic. In our course, we only introduce some elementary concepts and results of this theory. Traditionally, lattice theory should be taught under the category of abstract algebra. However, it can be explored from two different points of view. First, a Lattice is just a partial order with two special operations, which will be discussed latter in detail. Second, it can also be defined in algebra approach.

2 Review of Partial Order Set

In order to show a lattice from the order's perspective, we first simply review the partial order and some concepts associated with it, such as bound and extreme elements.

Definition 1. *Given a set A and a relation R on it, $< A, R >$ is called a partially ordered set(***poset** *in brief) if R is* reflexive, antisymmetric *and* transitive*.*

Once a order is defined, elements in the set could be compared between each other to distinguish the bigger or the smaller one. So it is intuitive to define some general terms such as extreme element as the following.

Definition 2. *Given a poset $< A, \leq >$, we have:*

1. *a is maximal if there does not exist $b \in A$ such that $a \leq b$.*
2. *a is minimal if there does not exist $b \in A$ such that $b \leq a$.*
3. *a is greatest if for every $b \in A$, we have $b \leq a$.*
4. *a is least if for every $b \in A$, we have $a \leq b$.*

When we consider a subset of a poset A. A bound of it could be found, which is an element in A.

Definition 3. *Given a poset $< A, \leq >$ and a set $S \subseteq A$.*

1. *$u \in A$ is a* upper bound *of S if $s \leq u$ for every $s \in S$.*
2. *$l \in A$ is a* lower bound *of S if $l \leq s$ for every $s \in S$.*

You can observe that there sometimes exist more than one upper/lower bound of a given subset. We are specially interested in some special one, such as the least or the greatest. So the following concepts are derived.

Definition 4. *Given a poset $< A, \leq >$ and a set $S \subseteq A$.*

1. *u is a* least upper bound *of S, ($LUB(S)$), if u is the upper bound of S and $u \leq u'$ for any other upper bound u' of S.*
2. *l is a* greatest lower bound *of S, ($GLB(S)$), if l is the upper bound of S and $l' \leq l$ for any other lower bound l' of S.*

It is obvious that $LUB(S)$ is the least element of the set of upper bounds of S.

Example 1. *Given two poset described in Figure 1.*

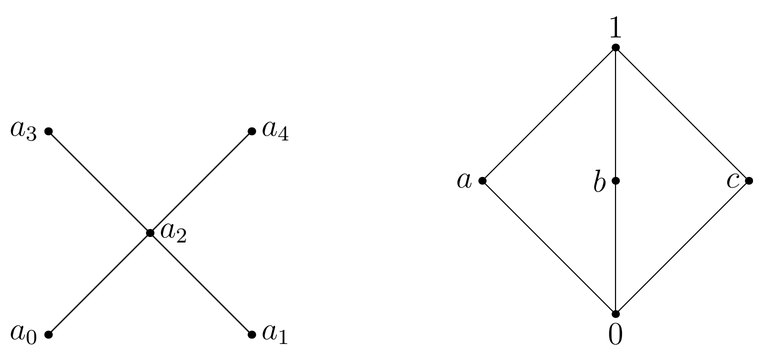


Figure 1: Extreme element of poset

They can demonstrate concepts mentioned before.

Theorem 5. *A subset of a poset has at most one LUB or GLB.*

The proof is left as an exercise.

3 Lattice

In this section, we will illustrate *lattice* in two approaches. One is to define it on the base of a partial order. And another is to define it in traditionally algebraic style.

3.1 Order's Perspective

Definition 6. *A* lattice *(structure) is a poset $\langle A, \leq \rangle$ in which any two elements a, b have a $LUB(a, b)$ and a $GLB(a, b)$.*

From now on, we define $a \cup b = LUB(a, b)$ and $a \cap b = GLB(a, b)$ in brief. We also call them *join* and *meet* respectively. With the following example, we show you why the notations are taken.

Example 2. *$\langle \mathcal{P}(A), \subseteq \rangle$, with two operations $\cup$(union) and $\cap$(intersection), is a lattice.*

The verification is simple. You just follow the definition of lattice.

There are several means to represent lattice. For lattice is generally a partial order, it can be described by Hasse diagram. Another way is to use table, for a lattice is also determined by two operations.

Example 3. *$\langle \mathcal{P}(a, b), \subseteq \rangle$, with two operations $\cup$(defined as union) and $\cap$(defined as intersection).*

It is a lattice according to previous Example 2. It can be described by a Hasse diagram as shown in Figure 2.

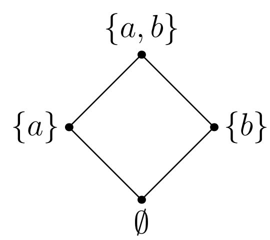


Figure 2: Hasse diagram of $\langle \mathcal{P}(\{a, b\}), \subseteq \rangle$

It can aslo be described by joint and meet table as shown in the following Table 1. Obviously,

$\cup$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

(a) Joint table

$\cap$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a, b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$

(b) Meet table

Table 1: $\langle \mathcal{P}(\{a, b\}), \subseteq \rangle$ in table

joint/meet table is symmetric. Sometimes, they could be merged into a table by taken upper/lower triangle matrix respectively. For operation $\cup$(union) and $\cap$(intersection), there are many properties, such as commutative law, associative law,idempotent law, and eve distributive law etc..

In previous example, we have found many properties. However, a lattice generally have the following common properties.

Proposition 7. *The Lattice with operation $\cap$ and $\cup$ satisfies:*

1. *Commutative:* $a \cap b = b \cap a, a \cup b = b \cup a.$
2. *Associative:* $(a \cap b) \cap c = a \cap (b \cap c), (a \cup b) \cup c = a \cup (b \cup c).$
3. *Idempotent:* $a \cap a = a, a \cup a = a.$
4. *Absorption:* $(a \cup b) \cap a = a, (a \cap b) \cup a = a.$

Proof. The proof is directly deduced from definition, which is left as an exercise. $\square$

3.2 Algebraic Perspective

In abstract algebra, we first learn semigroup. Then more and more complicated structure are introduced by adding more constraints. Similarly, We also introduce semilattice here.

Definition 8. *A semilattice is an algebra $\mathcal{S} = (S, *)$ satisfying, for all $x, y, z \in S$,*

1. $x * x = x,$
2. $x * y = y * x,$
3. $x * (y * z) = (x * y) * z.$

More specifically, the operator $*$ can be substituted as $\cup, \cap$ and forms a join-semilattice and meet-semilattice correspondingly.

Example 4. *Given a set A. Consider the partially ordered set $\langle \mathcal{P}(A), \subseteq \rangle$. Then $\langle \mathcal{P}(A), \cup \rangle$ is a semilattice.*

For this case, it is easy to verify that all three properties are satisfied.

Based on semilattice, we can furthermore define lattice by defining two operations $\cap$ and $\cup$ on some set as following:

Definition 9. *Given a structure $\mathcal{L} = (L, \cap, \cup)$, it is a lattice if it subjects to:*

1. $(L, \cap)$ *and* $(L, \cap)$ *are two semilattices.*
2. $(a \cup b) \cap a = a, (a \cap b) \cup a = a.$

This is a typical style of defining a structure in algebra.

In another word, Lattice is just a structure with two operations $\cap$ and $\cup$ on some set which meets 4 properties mentioned previously in Proposition 7. Therefore, we have the following theorem to guarantee the equivalence of two different definitions.

Theorem 10. *If L is any set in which there are two operation defined as $\cup$ and $\cap$ satisfying the last four properties, then L is a lattice.*

Proof. We first show that L has the following two properties:

1. $a \cup b = a$ and $a \cap b = b$ are equivalent.

   if $a \cup b = a$, we get $a \cap b = (a \cup b) \cap b = b$ by absorption.

2. $< S, \leq >$ is a poset if $b \leq a$ is defined by $a \cap b = b$ or $a \cup b = a$.

   (i) $a \cup a = a$ imply $a \leq a$.

   (ii) Suppose $a \leq b$ and $b \leq a$, we have $a = a \cup b = b \cup a = b$.

   (iii) If $a \leq b, b \leq c$, then $a \cup b = b, b \cup c = c$. Hence,

$$a \cup c = a \cup (b \cup c) = (a \cup b) \cup c = b \cup c = c$$

We now prove that $a \cup b$ is the LUB. Since $(a \cup b) \cap a = a, a \leq a \cup b$. Similarly $b \leq a \cup b$. Now given any c such that $a, b \leq c$, we have $a \cup c = c$ and $b \cup c = c$. Hence

$$(a \cup b) \cup c = a \cup (b \cup c) = a \cup c = c$$

and $a \cup b \leq c$.

Similarly, we can prove that $a \cap b$ is the GLB. Totally, L is a lattice with $\cup$ and $\cap$. $\square$

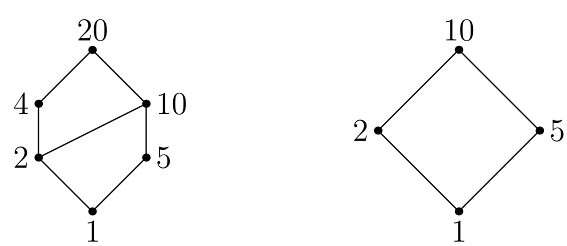
4 Sublattice

Similar to group, given some subset, we have the following concept.

Definition 11. *A subset S of a lattice L is called sublattice if it is* closed *under the operation $\cup$ and $\cap$.*

It should be mentioned that operations in S are constrained by L. But a common mistake often takes place, which operations in S are redefined.

Example 5. *Given two posets described in Figure 3*



(a) Lattice (b) Sublattice

Figure 3: Sublattice of $\langle \{20, 10, 5, 4, 2, 1\}, | \rangle$

It obvious that lattice in Figure 3b is a sublattice of lattice in Figure 3a

For sublattice is define relatively. Dually, we can also define a concept *extension* as following:

Definition 12. *If S is a sublattice of L, L is an extension of S.*

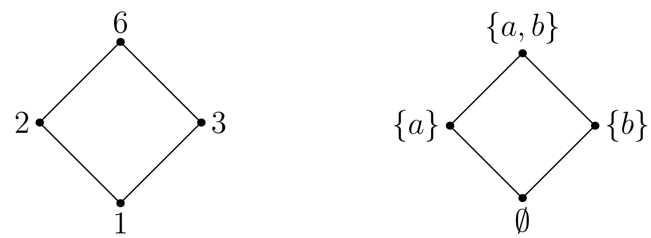
In some case, we have the following special sublattice.

Definition 13. *The subset S of the lattice L is called* convex *if $a, b \in S, c \in L$, and $a \leq c \leq b$ imply that $c \in S$.*

Given two lattices, they sometimes have the same topology structure by means of Hasse diagram. Consider the following example.

Example 6. *Given a poset $\langle \{1, 2, 3, 6\}, | \rangle$, it is isomorphic to $\langle \mathcal{P}(\{a, b\}), \subseteq \rangle$.*

They have the Hasse diagrams as shown in Figure 4. It is easy to construct a bijection between



(a) $< \{1, 2, 3, 6\}, | >$ (b) $< \mathcal{P}(\{a, b\}), \subseteq >$

Figure 4: Finite Boolean algebra example

$\{1, 2, 3, 6\}$ and $\mathcal{P}(\{a, b\})$. And the mapping is not unique.

Lattice is a specail type of a partial order. Generally, we have the assertion.

Theorem 14. *Given two lattice L and L', a bijection $f : L \to L'$ from L to L' is an isomorphism if and only if $a \leq b$ in L implies $f(a) \leq f(b)$ in L'.*

Proof. If f is isomorphic, f is order preserving. Suppose $a \leq b$, we have $a \cup b = b$. Then we have $f(a \cup b) = f(a) \cup f(b) = f(b)$, which means $f(a) \leq f(b)$. Conversely, $f(a) \leq f(b)$ also implies $a \leq b$ because f^{-1} is also a bijection.

Suppose f is an order preserving bijection, f is isomorphic. Given $a, b \in L$, let $d = a \cup b$, we have $f(a) \cup f(b) \leq f(d)$ because $a, b \leq d$ leads $f(a), f(b) \leq f(d)$. For any $f(e) = e' \in L'$ such that $f(a), f(b) \leq e'$, we have $a, b \leq e$. So $a \cup b = d \leq e$. Then we have $f(d) \leq f(e) = e'$, which means $f(a \cup b) = f(a) \cup f(b)$. Similarly, we can prove $f(a \cap b) = f(a) \cap f(b)$. $\square$

Exercies

1. Define a partial order $|$ on the set N of natural numbers by stipulating that $m|n$ iff m is a divisor of n.

(a) Verify that $\langle N, | \rangle$ is a lattice.

(b) Find the solution of $m \cup n$ and $m \cap n$.

2. Show that every chain is a lattice.

3. Prove that the absorption identities imply the idempotency of $\cup$ and $\cap$.

4. In a lattice, if we have $a \leq b$ and $c \leq d$, prove or disprove that:

   (a) $a \cup c \leq b \cup d$,

   (b) $a \cap c \leq b \cap d$.

5. Show that a lattice L is a chain iff every nonempty subset of L is a sublattice.

6. Let $L = P(S)$ be the lattice of all subsets of a set S under the relation of containment. Let T be a subset of S. Show that $P(T)$ is a sublattice of L.

7. Let L be a lattice and let a and b be element of L such that $a \leq b$. The **interval** $[a, b]$ is defined as the set of all $x \in L$ such that $a \leq x \leq b$. Prove that $[a, b]$ is a sublattice of L.

8. Describe a practical method of checking associativity in a join-table and in a meet-table.