The Vulcan Game of Kal-Toh: Finding or Making Triconnected Planar Subgraphs

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Abstract. In the game of *Kal-toh* depicted in the television series *Star Trek: Voyager*, players attempt to create convex polyhedra by adding to a jumbled collection of metal rods. Inspired by this fictional game, we formulate graph-theoretical questions about polyhedral subgraphs, i.e., subgraphs that are triconnected and planar. The problem of determining the existence of a polyhedral subgraph within a graph G is shown to be NP-complete, and we also give some non-trivial upper bounds for the problem of determining the minimum number of edge additions necessary to guarantee the existence of a polyhedral subgraph in G.

1 Introduction

Kal-toh is a fictional game from the television series *Star Trek: Voyager*. It was first introduced by the character Tuvok in the episode entitled "Alter Ego" from the show's third season. Tuvok belongs to an alien race known as the Vulcans, known for their superior intelligence, mastery of logic, and highly analytical minds. The cultural impact of Kal-toh among the Vulcans can be compared to the Human game of chess; achieving *Grandmaster* status in either game requires intellect, dedication, and a lifetime of study. (However, this comparison is viewed as somewhat insulting to Tuvok, who remarked that "Kal-toh is to chess as chess is to tic-tac-toe.")

Unfortunately, very few details of the game are explained to the viewer, so the following description is largely based on personal interpretation. The game of Kal-toh is either played alone or between two players and uses small metal rods, which appear to be connected to one another at their endpoints. Initially the game appears as a seemingly random structure of interconnected rods in three dimensions. One "move" consists of removing a rod and placing it elsewhere in the structure, with the ultimate goal of forming a convex polyhedron. One episode depicts a convex polyhedron being formed by using every rod in the structure, whereas another episode depicts a player forming a convex polyhedron using only a subset of the rods.

This paper considers a collection of interesting problems in graph algorithms which are inspired by the one-player game of Kal-toh. For simplicity, we ignore any physical constraints imposed by the length or weight of the metal rods and

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will assume that any rod added to the configuration must be placed exactly between the endpoints of two existing rods. This allows us to use a graph to model any configuration of the game, using edges to represent rods and vertices to represent the connection of two rods at their endpoints. This model also seems appropriate given Tuvok's comment that Kal-toh "is not about striving for balance, but about finding the seeds of order even in the midst of profound chaos." We will also make the assumption that a *new* edge is added to the graph on each move, instead of first being removed from elsewhere in the graph. This assumption is made in order to make a single move as simple as possible.

We can then view a configuration of the game as a simple, non-geometric graph G, and we wish to find a subgraph that corresponds to the graph of a convex polyhedron. It is well-known (by Steinitz's Theorem) that a graph G is the graph of a convex polyhedron if and only if G is a simple graph which is triconnected and planar. Hence we want to find a subgraph H of G that is planar and triconnected. With this formulation in mind, we can ask the following questions:[1]

- Does there exist a subgraph H of G such that H is both triconnected and planar?
- If at least one such H exists:
  - Which subgraph(s) has/have the largest number of vertices?
  - Which subgraph(s) is/are maximally planar (i.e., the addition of any one edge would violate planarity)?
  - Which subgraph(s) is/are minimally triconnected (i.e., the removal of any one edge would violate triconnectivity)?
- If no such H exists:
  - Can a triconnected planar subgraph H of G be created by adding a single edge to G?
  - Can a triconnected planar subgraph H of G be created by adding $l > 1$ edges to G?
  - What is the minimum number of edges that must be added to G such that there exists a triconnected planar subgraph H of G?
  - What is the maximum number of edges that can be added to G without creating a triconnected planar subgraph H of G? In other words, what number of edges guarantees the existence of such an H?

To study the answers to some of these questions, we formulate Kal-toh as the following problem:

KAL-TOH$_{l,\geq k}(G)$ Let G be a graph on $n \geq 4$ vertices. Does G have a triconnected planar subgraph H on at least $4 \leq k \leq n$ vertices after the addition of at most l edges?

[1] In this paper, we only study the one-player version of KAL-TOH; a preliminary exploration of the (much more challenging) two-player version can be found in [1].

We use $\textsc{Kal-Toh}_{l,=k}(G)$ to denote the problem where we wish to specify the *exact* number of vertices required for the triconnected planar subgraph. We use the notation $\textsc{Kal-Toh}_{l,\geq k}$ when graph G is clear from the context.

Our interpretation of the game appears to be a mixture of problems related to (incremental) planarity testing (see e.g. [6,4,2,8] and the references therein) and triconnectivity testing and augmentation (see e.g. [5,9,7,3] and the references therein).

Some papers combine both topics, i.e., how to make a planar graph triconnected while maintaining planarity; see e.g. [10]. But to the best of our knowledge, the $\textsc{Kal-Toh}_{l,\geq k}(G)$ problem, i.e., how to add edges such that a *subgraph* is planar and triconnected, has not previously been studied.

Our paper examines various instances of this problem in further detail. Section 2 considers $\textsc{Kal-Toh}_{0,\geq 4}$, which asks if G contains a triconnected planar subgraph H. We show that it is NP-complete to answer this question. Section 3 considers the case $l > 0$ in which the addition of edges is allowed. In addition to some results on simple instances of the game, in Section 4 we show that the addition of at most $k-1$ edges to a connected graph G always suffices to form a triconnected subgraph H with k vertices, and this is best-possible.

2 Recognizing a Winning Graph

We begin by showing the NP-completeness of $\textsc{Kal-Toh}_{0,\geq 4}$: without requiring the addition of edges, does a graph G contain *any* triconnected planar subgraph?

Our reduction used in proving the NP-completeness of $\textsc{Kal-Toh}_{0,\geq 4}$ will be from the NoncrossingCycle problem in an orthogonal geometric graph G, defined below.

Definition 1. *An* orthogonal geometric graph *is a graph drawn in the plane such that each edge is represented by a path of contiguous axis-parallel line segments such that two line segments intersect only if they cross, and no line segment intersects a non-incident vertex. A* crossing *is a point that belongs to the interior of two segments of edges.*

For an orthogonal geometric graph G, the NoncrossingCycle *problem asks if there exists a cycle in G that contains no crossing.*

Figure 1 shows an example of an orthogonal geometric graph containing a non-crossing cycle. The NoncrossingCycle problem was shown to be NP-complete in [11]. We now show how to reduce this problem to $\textsc{Kal-Toh}_{0,\geq 4}$.

Let G be an orthogonal geometric graph. Preprocess the graph G by placing a dummy vertex at each *bend*, which is the common point of any two consecutive segments belonging to the same edge. Furthermore, if two line segments l_1 and l_2 both cross a line segment l_3, place a dummy vertex along l_3 between these crossings. This preprocessing of G guarantees that each crossing is surrounded by exactly four vertices, and every edge consists of a single line segment with

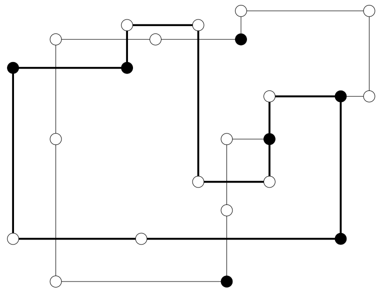

Fig. 1. An orthogonal geometric graph (black vertices) with a non-crossing cycle (in bold). White vertices are added in the pre-processing step.

at most one crossing on it. Note that the preprocessing does not change our problem; adding the dummy vertices will neither add nor remove a non-crossing cycle from G.

Now we construct the graph H that is the instance for $\text{KAL-TOH}_{0,\geq 4}$ by replacing each vertex and crossing in G by a *spine* in H, where a spine is a path with two edges. Figure 2 shows an example.

We replace edges in G as follows. By our preprocessing, an edge $e = (v, w) \in G$ consists of a horizontal or vertical segment with at most one crossing and with a vertex at both ends. If there is no crossing along e, it is represented in H by a *vertex-segment gadget*, otherwise it is represented by two *crossing-segment gadgets*. Here a vertex-segment gadget consists of a 6-cycle, with each cycle-vertex connected to one of the vertices at the spines of v and w. A crossing-segment gadget consists of a 7-cycle, with 6 of the cycle-vertices connected to the vertices at the spines of the crossing and one of the endpoints of e, and the 7th cycle-vertex connected to the 7th cycle-vertex of the other crossing-gadget of e. All connections are done in such a way that the gadgets replacing e form a planar graph. See Figure 2 for an example.

We will now show that G has a non-crossing cycle if and only if H has a triconnected planar subgraph.

First assume we are given a non-crossing cycle C of G. We can construct a corresponding triconnected planar subgraph H' of H as follows. Consider any edge (v_1, v_2) in C, and let S_1 and S_2 be the vertex-spines of v_1 and v_2, respectively, in H. Then in H' we include S_1 and S_2. If (v_1, v_2) has no crossing, then we include its vertex-segment gadget. If it does, we include its crossing-spine and both crossing-segment gadgets. It is easy to see that the subgraph H' is triconnected and planar.

Now we show the other direction, i.e., we show that if H contains a triconnected planar subgraph, then there exists a non-crossing cycle C. Presume a triconnected planar subgraph H' of H exists. We think of H' as being obtained from H by removing enough edges to achieve planarity while at the same time maintaining triconnectivity. The following are easy to observe:

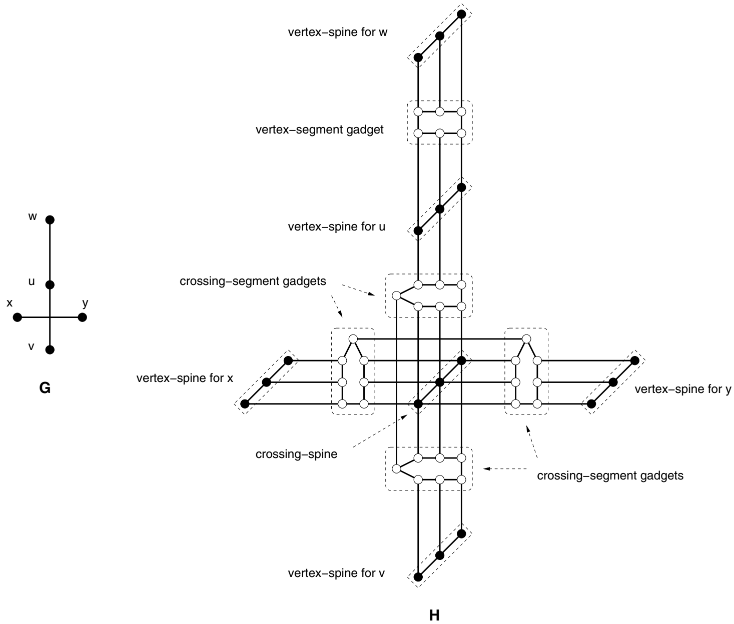

Fig. 2. The graph H is constructed from the graph G using spines and segment gadgets

Lemma 1. *1. For any segment gadget in H, either all or none of the edges incident to its vertices are in H'.*

2. Any spine in H is connected to either 0 or 2 non-empty segment gadgets in H'.

3. If a crossing-spine in H is connected to two crossing-segment gadgets R_1 and R_2 that are both non-empty in H', then R_1 and R_2 belong to the same edge.

Proof. (1) All vertices not on a spine have degree 3 in H, so if any segment gadget has some, but not all, edges in H', then H' contains contains a vertex of degree 1 or 2, contradicting triconnectivity. (2) If 3 or 4 gadgets at a spine are non-empty, then (since the gadgets contain all their edges by (1)) this gives a $K_{3,3}$ and violates planarity. If exactly one gadget is non-empty, then the middle vertex of the spine has degree 2 and the graph is not triconnected. (3) The 7th cycle-vertex on the crossing-gadget R_1 has degree 3 in H and must retain all three edges in H' by triconnectivity, hence the other crossing-gadget of the same edge must exist in H' and be R_2. □

So if there is a triconnected planar subgraph H' of H, we define C to be a subset of the edges of G as follows. Consider any edge (v_1, v_2) in the graph G. Then the edge (v_1, v_2) is in the set C if and only if the gadgets replacing (v_1, v_2) in H belong to H' (they must be in H' entirely or not at all by Lemma 1(1) and (3)).

We first note that the set C is non-empty. For otherwise, H' would be a subforest of paths on three vertices by Lemma 1(1) because no two spines would be connected by a common segment gadget. But this is a contradiction because H' must be triconnected. Therefore by Lemma 1(2), the vertices in C all have degree two. They must form a cycle in G because H' is triconnected. Furthermore, Lemma 1(3) guarantees that C is a non-crossing cycle.

A solution to our problem can be verified in polynomial time by performing planarity and triconnectivity tests on the graph H', so it is in NP. Furthermore, our reduction requires polynomial time.

Theorem 1. *It is* NP*-complete to determine if a graph G contains a triconnected planar subgraph.*

Note that Theorem 1 also holds if we were to replace "subgraph" by "induced subgraph" because in no part of our proof do we ever specifically require that H' be a non-induced subgraph. Theorem 1 also holds if we were to replace "triconnected" by "having minimum vertex-degree of 3," or if we add the restriction that G is triconnected.

Recall that $\textsc{Kal-Toh}_{0,\geq k}$ asks whether or not a given graph G has a triconnected planar subgraph on at least k vertices. By Theorem 1, $\textsc{Kal-Toh}_{0,\geq k}$ is NP-complete, because for $k = 4$ this is the same as asking whether G has a planar triconnected subgraph.

Corollary 1. $\textsc{Kal-Toh}_{0,\geq 4}$ *is* NP*-complete.*

One might wonder whether $\textsc{Kal-Toh}_{0,\geq k}$ becomes easier as the value of k gets larger. For example, if this question were instead asked for *induced* subgraphs, then $\textsc{Kal-Toh}_{0,\geq k}$ for the case $k = n$ is easily answered by running planarity and triconnectivity tests on the input graph G. With a very similar proof, reducing from $\textsc{HamiltonianCycle}$ in 3-regular graphs, we can show that $\textsc{Kal-Toh}_{0,\geq k}$ remains NP-complete for $k = \frac{3n}{4}$ in Theorem 2. Details are left to the reader.

Theorem 2. $\textsc{Kal-Toh}_{0,\geq \frac{3n}{4}}$ *is* NP*-complete.*

3 $\textsc{Kal-Toh}_{l,\geq 4}$ for $l > 0$

The previous section showed that $\textsc{Kal-Toh}_{0,\geq 4}$ is NP-complete. On the other hand, $\textsc{Kal-Toh}_{6,\geq 4}$ is polynomial: We may add 6 edges, and we can then always create a K_4, which is planar and triconnected, so the answer is always Yes. (We assume throughout that $n \geq 4$.) We now show that even $\textsc{Kal-Toh}_{3,\geq 4}$ is polynomial, while the complexity status of $\textsc{Kal-Toh}_{2,\geq 4}$ and $\textsc{Kal-Toh}_{1,\geq 4}$ remains open.

We first show that both $\textsc{Kal-Toh}_{5,\geq 4}$ and $\textsc{Kal-Toh}_{4,\geq 4}$ can be answered in linear time. For $l = 5$, a subgraph that is K_4 can be constructed by adding up to five edges between the endpoints of any preexisting edge of G and any two other vertices. For $l = 4$, a subgraph that is K_4 can be constructed by adding four edges between the endpoints of any two preexisting edges of G (plus another vertex if the two edges are incident to a common vertex). Testing whether such vertices/edges exist and finding them can easily be accomplished in linear time.

KAL-TOH$_{3,\geq 4}$ can also be answered in linear time, by testing for slightly more complicated subgraphs.

Lemma 2. *If we can create some triconnected planar subgraph H of G by adding at most three edges, then G must have one of the graphs seen in Figure 3(i-iii) as a (not necessarily induced) subgraph.*

Proof. Let H be a triconnected planar graph formed by adding at most three edges to G, and let $G|_{V(H)}$ denote the graph G restricted to the vertices of H (excluding the added edges). For a contradiction, assume that G did not originally contain any of the graphs seen in Figure 3(i-iii). Then the components of G (and hence also of $G|_{V(H)}$) are single vertices, single edges, and 2-paths. For any such component, the ratio of edges to vertices is at most $\frac{2}{3}$, therefore $G|_{V(H)}$ has at most $\frac{2}{3}|V(H)|$ edges.

We have $\sum_{v\in H} \deg_H(v) = 2 \cdot |E(H)|$, and also $|E(H)| \geq \frac{3}{2} \cdot |V(H)|$ by the triconnectivity of H. If $a \leq 3$ is the number of edges added to G to obtain H, then $\frac{3}{2}|V(H)| \leq |E(H)| = |E\left(G|_{V(H)}\right)| + a \leq \frac{2}{3}|V(H)| + a$. But $|V(H)| \geq 4$ by triconnectivity, so this implies $a \geq \frac{5}{6}|V(H)| \geq \frac{20}{6} > 3$, a contradiction. □

Theorem 3. KAL-TOH$_{3,\geq 4}$ *can be answered in linear time.*

Proof. If G contains any of the graphs in Figure 3(i-iii) as subgraph, then we add 3 edges to G at these vertices (plus one more vertex in case of graph (i)) and obtain K_4, so the answer to KAL-TOH$_{3,\geq 4}$ is YES. Otherwise, by Lemma 2, the answer to KAL-TOH$_{3,\geq 4}$ is NO. The latter happens if and only if the components of G consist only of isolated vertices, single edges, and 2-paths. This can be tested in linear time. □

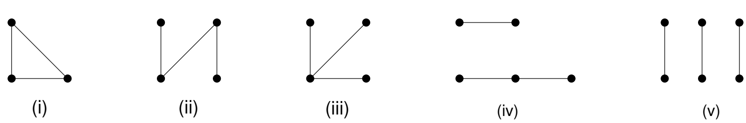

Fig. 3. Graphs on three edges. The answer to KAL-TOH$_{3,\geq 4}$ is YES if and only if G contains any of (i-iii) as a subgraph. The answer to KAL-TOH$_{\lceil \frac{3}{2}k \rceil -3,\geq k}$ is YES if and only if G contains any of (i-v) as a subgraph.

Unfortunately, we are unable to use the same reasoning for the problem KAL-TOH$_{2,\geq 4}$. As seen in Figure 4, A and B are the only two possible subgraphs of K_4 which contain $6-2=4$ edges. If a graph G contains either of these as subgraphs, then the answer to $\textsc{Kal-Toh}_{2,\geq 4}$ is $\textsc{Yes}$. However, the other direction does not hold. For a counterexample, consider the dodecahedron having any two of its edges removed; see Figure 4. The answer to $\textsc{Kal-Toh}_{2,\geq 4}$ will be $\textsc{Yes}$ for this graph, as the dodecahedron (which is triconnected and planar) can be formed by adding the two missing edges, but this graph contains neither A nor B as a subgraph. In fact, there is an infinite set of "desirable" graphs that can be made 3-connected after adding two edges, yet none of the graphs contains a smaller desirable graph as a subgraph. (They can be constructed by adding layers of faces of degree 6 to the dodecahedron and then deleting two edges that are not on a 5-cycle.)

Hence we conjecture that $\textsc{Kal-Toh}_{2,\geq 4}$ (and similarly $\textsc{Kal-Toh}_{1,\geq 4}$) are NP-complete. In fact, we believe that almost the same reduction as for $\textsc{Kal-Toh}_{0,\geq 4}$ would work; the only changes required are: replace the segment-gadgets with larger graphs that have only 5-cycles and 6-cycles as faces and are cyclically 4-edge connected, and use more edges to connect such gadgets to (longer) spines and to the opposite crossing-segment-gadget. However, the details of this remain to be worked out.

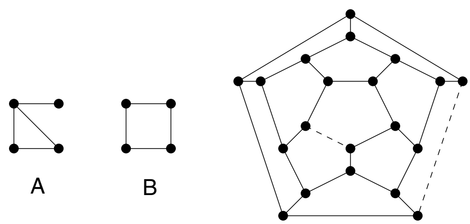

Fig. 4. The answer to $\textsc{Kal-Toh}_{2,\geq 4}$ is $\textsc{Yes}$ if G contains A or B as a subgraph. But the reverse is not true; the dodecahedron having any two of its edges removed serves as a counterexample.

4 Creating Larger Graphs

The results in the previous section were obtained by creating K_4, which can always be done with 6 edges. But the convex polyhedra created in the *Star Trek: Voyager* episodes appear to be much more complex. What can be said about creating triconnected planar graphs that have a given size k? Or in other words, what can we say about $\textsc{Kal-Toh}_{x,\geq k}$ for $k \geq 5$? We do not have a complete characterization here, but we give some bounds on x such that the answer is always $\textsc{Yes}$. We study this first for arbitrary graphs and then for connected graphs.

4.1 Creating Prisms

A *k-prism* is defined as the graph composed of two disjoint cycles $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_k$ where the edge (v_i, w_i) is added for all i.

Lemma 3. *For any $k \geq 4$, there is a graph G on k vertices that is triconnected, planar, Hamiltonian, contains a triangle, and has $\lceil \frac{3}{2}k \rceil$ edges.*

Proof. If k is even, take the $(\frac{k}{2}+1)$-prism H with two cycles $v_1, v_2, \dots, v_{\frac{k}{2}+1}$ and $w_1, w_2, \dots, w_{\frac{k}{2}+1}$ and edges (v_i, w_i) for all i. This is planar and triconnected and remains so after contracting (v_1, w_1) and (v_2, w_2). The resulting graph satisfies all conditions. For odd k, start with a $(\frac{k+1}{2})$-prism H and contract (v_1, w_1). See Figure 5. The edge-bound holds because the resulting graph has degree 3 at all vertices except at most one vertex of degree 4. □

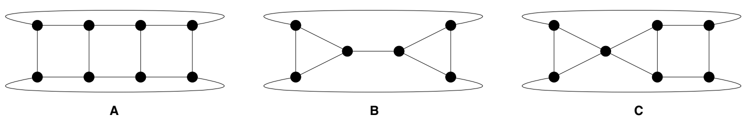

Fig. 5. Graph A is a 4-prism. Graphs B and C are triconnected, planar, Hamiltonian, and contain a triangle.

Note that any triconnected graph on k vertices must have at least $\lceil \frac{3}{2}k \rceil$ edges, so these graphs are smallest possible triconnected planar graphs on k vertices. Obviously they can always be build if we are allowed to add $\lceil \frac{3}{2}k \rceil$ edges to G. We now show, similar to Lemma 2 in the previous section, that we can exactly characterize when these graphs after adding only $\lceil \frac{3}{2}k \rceil - 3$ edges to G, i.e., the resulting subgraph includes at least 3 edges of the input graph.

Lemma 4. KAL-TOH$_{\lceil \frac{3}{2}k \rceil -3,=k}$ *has an answer of* YES *if and only if G contains at least three edges with at most k distinct endpoints.*

Figure 3 shows the possible ways in which three edges on at most k vertices (i.e., on at most k distinct endpoints) can be present in G.

Proof. Assume that the answer to KAL-TOH$_{\lceil \frac{3}{2}k \rceil -3,=k}$ is YES for some graph G. Let H be the triconnected planar subgraph created by the addition of edges to G. G must contain at least three edges among the vertices that define H. Since these edges must be part of the graph H on k vertices, then G must initially contain three edges with at most k distinct endpoints.

To show the other direction, assume that G has at least three edges e_1, e_2, and e_3 having at most k distinct endpoints. Then it contains one of the configurations in Figure 3. (Graphs (iv) and (v) appear only if $k \geq 5$ and graph (v) appears only if $k \geq 6$.) All of these graphs are subgraphs of the graph of Lemma 3. So we can build this graph by adding $\lceil \frac{3}{2}k \rceil - 3$ edges to the three edges of G. □

So the answer to KAL-TOH$_{\lceil \frac{3}{2}k \rceil -3,=k}$ is NO if and only if G contains none of the configurations from Figure 3. This holds if and only if G has maximum degree 2 (by graph (iii)), no component is a 3-cycle (by graph (i)), and no component is path or cycle with 4 or more vertices (by graph (ii)). So all components must be singletons, single edges or two-paths. This can be tested (and the existence of graph (iv) or (v) then checked) in linear time. Therefore, $\textsc{Kal-Toh}_{\lceil\frac{3}{2}k\rceil-3,=k}$ can be answered in linear time.

4.2 Creating Triconnected Planar Subgraphs of Connected Graphs

This section shows a second upper bound for l if the host graph G is connected.

Theorem 4. *Every connected graph G on $n \geq k \geq 4$ vertices has a triconnected planar subgraph H on k vertices after adding at most $k-1$ edges.*

Proof. Compute a spanning tree of G, and delete leaves from it until we are left with a tree T with exactly k vertices. T is connected and planar; we now add $k-1$ (or fewer) edges to make it triconnected without destroying planarity. We have 3 cases.

1. T is a path. In this case, we can add $\lceil\frac{3}{2}k\rceil-(k-1) \leq k-1$ edges to T to turn it into a Hamiltonian 3-connected planar graph on k vertices (i.e., the graph of Lemma 3).
2. T has no vertex of degree 2. Then we add edges to connect the leaves of T in a cycle to form the graph H. Such a graph is called a *Halin graph*, which is known to be planar and triconnected [12]. An example of a Halin graph is shown in Figure 6. The tree T has at most $k-1$ leaves (which is tight if T is a star), and so it requires the addition of at most $k-1$ edges.
3. T has k_2 vertices of degree 2 and at least one vertex of degree ≥ 3. Let k_1 be the number of leaves of T. We prove by induction on k_2 that we can add k_1+k_2 edges to T to make it triconnected and planar; this proves the claim since $k_1+k_2<k$. The case $k_2=0$ has been dealt with in the previous case. So presume T has a vertex v of degree 2. Remove v and add an edge between its neighbors u and w. This results in a tree T' with k_1 vertices of degree 1 and $k_2'=k_2-1$ vertices of degree 2. By induction we can add k_1+k_2' edges to T' to create a planar triconnected graph H'. Now remove the added edge (u,w) from H', re-insert v, and connect it to u,w and one arbitrary other vertex z that shared a face with (u,w) in H'. Call the result H. One can easily see that graph H is triconnected since H' was.[2] By choice of z, H is planar since H' was. To obtain H, we added $k_1+k_2'=k_1+k_2-1$ edges that were added to get H', and one edge (v,z), hence k_1+k_2 edges total.

□

In consequence, the answer to $\textsc{Kal-Toh}_{k-1,\geq k}(G)$ is always Yes if G is connected.

The reader may notice that in the above proof we used k_1+k_2 edges in cases (2) and (3), but only $\approx \frac{k}{2}+1=k_1+\frac{k_2}{2}$ edges in case (1). Obviously $k_1+\frac{k_2}{2}$

[2] For example, we can think of H as obtained by adding edges (u,z) and (w,z) to H', which preserves triconnectivity, and then replacing the triangle $\{u,w,z\}$ with a $K_{1,3}$ from a new vertex v to these three vertices. The latter is known as a *ΔY-transformation* and preserves triconnectivity.

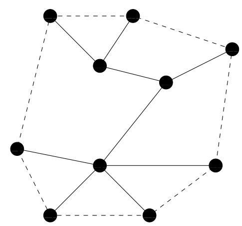

Fig. 6. A Halin graph

edges are always required to turn T into a graph with minimum degree 3. With a significantly more complicated method, we can show that $k_1 + \frac{k_2}{2} + O(1)$ edges are also sufficient; see [1] for details.

5 Conclusions

The topics selected for this paper were originally motivated by the fictional game of *Kal-toh* seen on the television series *Star Trek: Voyager*. Kal-toh is played with a jumbled collection of small metal rods, with the ultimate goal of forming a convex polyhedron by using a subset of these rods as its edges. We studied the one-player version of the game, where a single turn consists of adding a rod somewhere within the existing structure.

Because the exact rules of the game were never formally explained to the viewer, it was necessary to first give a more precise description based on our own interpretation of the game. By having an edge represent a single rod and having a vertex represent the point of contact between two rods, we formulated Kal-toh as a graph-theoretic problem: is it possible to create a triconnected planar subgraph on at least k vertices with the addition of at most l edges?

We first proved NP-completeness of the variant where no edges were to be added to the graph (i.e. when $l = 0$). We then considered cases where edge additions are permitted. We started by showing a few results for small values of l. For larger k, by specifically creating a prism as our triconnected planar subgraph, we demonstrated one upper bound for l. A second approach gives an upper bound of $l \leq k - 1$ for connected graphs, which is tight if G is a star.

As for future work, many alternate interpretations of Kal-toh could be studied. In particular, a more realistic interpretation of Kal-th would be to take the geometry into account: What if the rods have a specific lengths and edges can be added only if edge lengths respect rod lengths, thereby turning the problem into one of rigidity theory?

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