

**The Classical Risk Model with a Constant Dividend Barrier:  
Analysis of the Gerber-Shiu Discounted Penalty Function**

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## **Abstract**

The classical compound Poisson risk model is considered in the presence of a constant dividend barrier. An integro-differential equation for the Gerber-Shiu discounted penalty function is derived and solved. The solution is a linear combination of the Gerber-Shiu function with no barrier and the solution of the associated homogeneous integro-differential equation. This latter function is proportional to the product of an exponential function and a compound geometric distribution function. The results are then used to find the Laplace transform of the time to ruin, the distribution of the surplus before ruin, and moments of the deficit at ruin. The special cases where the claim size distribution is exponential and a mixture of two exponentials are considered in some detail. The integro-differential equation is then extended to the stationary renewal risk model.

**Keywords:** ruin, integro-differential equation, Lundberg equation, renewal equation, compound geometric, time of ruin, surplus before ruin, deficit at ruin, exponential distribution, mixture of exponentials, stationary renewal risk process, Sparre Andersen process.

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## 1. Introduction

In the classical continuous time compound Poisson risk model, the number of claims from an insurance portfolio is assumed to follow a Poisson process  $\{N_t\}$  with Poisson parameter  $\lambda$ . The individual claim sizes  $Y_1, Y_2, \dots$ , independent of  $\{N_t\}$ , are positive, independent and identically distributed random variables with common distribution function (df)  $P(y) = 1 - \bar{P}(y) = Pr(Y \leq y)$ , moments  $p_j = \int_0^\infty y^j dP(y)$  for  $j = 0, 1, 2, \dots$ , and Laplace-Stieltjes transform  $\tilde{p}(z) = \int_0^\infty e^{-zy} dP(y)$ . The aggregate claims process is  $\{S(t); t \geq 0\}$  where  $S(t) = Y_1 + Y_2 + \dots + Y_{N_t}$  (with  $S(t) = 0$  if  $N_t = 0$ ). The insurer's surplus process without a barrier is  $\{U(t); t \geq 0\}$  with  $U(t) = u + ct - S(t)$  or  $dU(t) = cdt - dS(t)$ . In the above,  $u = U(0) \geq 0$  is the initial surplus,  $c = \lambda p_1(1 + \theta)$  is the premium rate per unit time, and  $\theta > 0$  is the relative security loading. A barrier strategy considered in this paper assumes that there is a horizontal barrier of level  $b \geq u$  such that when the surplus reaches level  $b$ , dividends are paid continuously at rate  $c$  until a new claim occurs. Let  $U_b(t)$  be the surplus process with initial surplus  $U_b(0) = u$  under the barrier strategy above. Thus,  $U_b(t)$  is capped at level  $b$  and it can be expressed as

$$dU_b(t) = \begin{cases} cdt - dS(t), & U_b(t) < b, \\ -dS(t), & U_b(t) = b. \end{cases}$$

Define  $T_b = \inf\{t : U_b(t) < 0\}$  to be the first time that the surplus becomes negative. The stopping time  $T_b$  is referred to as the time of ruin. The special case  $b = \infty$  is considered in classical ruin theory. In particular, there is an extensive literature on the probability of ruin  $\psi(u) = Pr\{T_\infty < \infty\}$  and the joint distribution of the time of ruin  $T_\infty$ , the surplus before ruin  $U(T_\infty -)$ , and the deficit at ruin  $|U(T_\infty)|$ . The interested reader is directed to Gerber, Goovaerts and Kaas (1987), Dufresne and Gerber (1988), Dickson (1992), Dickson and Waters (1992), Dickson, Dos Reis and Waters (1995), Dickson and Dos Reis (1996), Gerber and Shiu (1997, 1998), Schmidli (1999), Lin and Willmot (1999, 2000), and references therein.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. They include Bühlmann (1970), Segerdahl (1970), Gerber (1972), Gerber (1973), Gerber (1979), Gerber (1981), Paulsen and Gjessing (1997), Gerber and Shiu (1998), Albrecher and Kainhofer (2002), and Højgaard (2002). The main focus of these papers/books is on optimal dividend policies and the time of ruin under various barrier strategies and other economic conditions. Although some elegant results have been obtained for optimal dividend policies, few results on analytic and probabilistic properties of the time of ruin, apart from its Laplace transform (see Gerber, 1979), have been obtained. Moreover, there has not been much research on the surplus before ruin and the deficit at ruin under a barrier strategy, as opposed to that for the compound Poisson risk model without a barrier.

In this paper, we will study the time of ruin  $T_b$  and its related functions such as the surplus before ruin  $U_b(T_b-)$  and the deficit at ruin  $|U_b(T_b)|$ . By using a renewal equation approach, we will be able to obtain a number of analytic and probabilistic properties of these quantities. Our analysis will involve the Gerber-Shiu discounted penalty function (Gerber and Shiu, 1998) that is defined below.

Let  $w(x_1, x_2)$ ,  $0 \leq x_1, x_2 < \infty$ , be a nonnegative function. For  $\delta \geq 0$ , define

$$m_b(u) = E\{e^{-\delta T_b} w(U_b(T_b-), |U_b(T_b)|) I(T_b < \infty)\}, \quad (1.1)$$

where  $I(T_b < \infty) = 1$  if  $T_b < \infty$  and  $I(T_b < \infty) = 0$  otherwise. We remark that as will be seen later, the use of the indicator function  $I(\cdot)$  is unnecessary in some cases, in particular when  $b$  is finite since  $T_b$  is a finite-valued random variable. As a result, there is no need to consider the ruin probability, as opposed to the classical ruin case. The function  $m_b(u)$  in (1.1) is useful for deriving results in connection with joint and marginal distributions of  $T_b$ ,  $U_b(T_b-)$ , and  $|U_b(T_b)|$ . While  $\delta$  may be interpreted as a force of interest, the function (1.1) may also be viewed in terms of a Laplace transform with  $\delta$  serving as the argument. In particular, if we let

$w(x_1, x_2) = 1$ , (1.1) is the Laplace transform of the time of ruin  $T_b$ . If we let  $\delta = 0$  and  $w(x_1, x_2) = I(x_1 \leq x)I(x_2 \leq y)$ , (1.1) becomes the joint df of the surplus before ruin and the deficit at ruin. Furthermore, if  $\delta = 0$  and  $w(x_1, x_2) = x_1^n$ , we obtain the  $n$ -th moment of the surplus before ruin. Likewise, if  $\delta = 0$  and  $w(x_1, x_2) = x_2^n$ , we obtain the  $n$ -th moment of the deficit at ruin. For other functions of interest, see Gerber and Shiu (1997, 1998), Lin and Willmot (1999, 2000), and Willmot and Lin (2001).

In this paper, we obtain the general solution for the Gerber-Shiu function (1.1) and study several of its special cases. In Section 2, we derive an integral equation for (1.1) by conditioning on the time of the first claim as in Willmot and Dickson (2003). Similar ideas can also be found in Gerber (1969), Sundt and Teugels (1995), Dickson (1998), Dickson and Hipp (1998, 2001), and Cai and Dickson (2002). Through differentiation of both sides of the integral equation with respect to the initial surplus  $u$ , we obtain an integro-differential equation that permits a general solution (due to its special structure) to be found. In Sections 3 and 4, we show that the Gerber-Shiu discounted penalty function under the constant barrier strategy is a linear combination of the Gerber-Shiu discounted penalty function without a barrier and a solution to an excessive or proper defective renewal equation. The latter can be expressed as a product of an exponential function and the df of a compound geometric distribution. In Section 5, we apply the results in Sections 3 and 4 to the time of ruin  $T_b$ , the surplus before ruin  $U_b(T_b-)$ , and the deficit at ruin  $|U_b(T_b)|$ . Section 6 considers further simplifications of (1.1) under two claim amount distributions of special interest: exponential and the mixture of two exponentials. Section 7 extends the integro-differential equation and boundary condition for the Gerber-Shiu function derived in Section 2 for the compound Poisson risk model to the stationary renewal risk model.

## 2. An integro-differential equation

In this section, we first derive an integral equation for  $m_b(u)$ . This equation utilizes

the regenerative property of the Poisson process at claim instants and is obtained by conditioning on the time of the first claim. We then convert the integral equation into an integro-differential equation and identify its boundary condition under the barrier strategy. The basic idea is due to Willmot and Dickson (2003).

Let  $t$  be the time of the first claim and  $y$  be the amount of the claim. There are two possibilities. First,  $t < (b - u)/c$  and the surplus has not yet reached the barrier. In this case, the surplus immediately before time  $t$  is  $u + ct$ . Secondly,  $t \geq (b - u)/c$  and the surplus immediately before time  $t$  is  $b$ . Arguing heuristically, since the “probability” that the claim occurs at time  $t$  is  $\lambda e^{-\lambda t} dt$  and the “probability” of the claim amount being  $y$  is  $dP(y)$ , we have, for  $0 \leq u \leq b$ ,

$$m_b(u) = \int_0^{(b-u)/c} \lambda e^{-(\lambda+\delta)t} \gamma_b(u+ct) dt + \int_{(b-u)/c}^{\infty} \lambda e^{-(\lambda+\delta)t} \gamma_b(b) dt, \quad (2.1)$$

where

$$\gamma_b(t) = \int_0^t m_b(t-y) dP(y) + \zeta(t), \quad (2.2)$$

and

$$\zeta(t) = \int_t^{\infty} w(t, y-t) dP(y). \quad (2.3)$$

We remark that  $\zeta(t)$  is a known function which is independent of  $b$  and  $m_b(u)$ , and was introduced by Gerber and Shiu (1998, eq. (2.17)). Note that (2.1) may be rewritten as

$$m_b(u) = \frac{\lambda}{c} \int_u^b e^{-\frac{\lambda+\delta}{c}(t-u)} \gamma_b(t) dt + \frac{\lambda}{\lambda+\delta} \gamma_b(b) e^{-\frac{\lambda+\delta}{c}(b-u)}, \quad 0 \leq u \leq b. \quad (2.4)$$

We now derive an integro-differential equation for  $m_b(u)$  by differentiating (2.4) with respect to  $u$  as follows. For  $0 \leq u \leq b$ ,

$$m'_b(u) = -\frac{\lambda}{c} \gamma_b(u) + \left( \frac{\lambda+\delta}{c} \right) \frac{\lambda}{c} \int_u^b e^{-\frac{\lambda+\delta}{c}(t-u)} \gamma_b(t) dt + \left( \frac{\lambda+\delta}{c} \right) \frac{\lambda}{\lambda+\delta} \gamma_b(b) e^{-\frac{\lambda+\delta}{c}(b-u)}.$$

It follows from (2.4) that for  $0 \leq u \leq b$ ,

$$m'_b(u) = \frac{\lambda+\delta}{c} m_b(u) - \frac{\lambda}{c} \gamma_b(u), \quad (2.5)$$

or equivalently using (2.2),

$$m'_b(u) = -\frac{\lambda}{c} \int_0^u m_b(u-y) dP(y) + \frac{\lambda+\delta}{c} m_b(u) - \frac{\lambda}{c} \zeta(u), \quad 0 \leq u \leq b. \quad (2.6)$$

An interesting and also important fact regarding (2.6) is that the equation itself does not involve the barrier level  $b$ . Hence, the Gerber-Shiu function for all positive levels  $b$ , including  $b = \infty$ , satisfies (2.6). The only difference between these functions is the boundary condition which we now identify. Letting  $u = b$  in (2.4), we obtain

$$m_b(b) = \frac{\lambda}{\lambda+\delta} \gamma_b(b), \quad (2.7)$$

and substitution into (2.5) yields immediately that

$$m'_b(b) = 0. \quad (2.8)$$

Thus, the Gerber-Shiu function with barrier level  $b$  satisfies (2.6) and the boundary condition (2.8).

Finally in this section, we show that the converse implication also holds (i.e., the solution to the integro-differential equation (2.6) with boundary condition (2.8) is indeed the solution to (2.4)). Rewriting (2.5) as

$$\frac{d}{du} \left[ e^{-\frac{\lambda+\delta}{c}u} m_b(u) \right] = -\frac{\lambda}{c} e^{-\frac{\lambda+\delta}{c}u} \gamma_b(u),$$

and integrating both sides from  $u$  to  $b$  (after replacement of  $u$  by  $t$ ), we obtain

$$e^{-\frac{\lambda+\delta}{c}u} m_b(u) - e^{-\frac{\lambda+\delta}{c}b} m_b(b) = \frac{\lambda}{c} \int_u^b e^{-\frac{\lambda+\delta}{c}t} \gamma_b(t) dt. \quad (2.9)$$

Equation (2.5) with  $u = b$  and (2.8) imply that (2.7) holds. Inserting (2.7) into (2.9) therefore yields (2.4). Hence, we can simply consider (2.6) with condition (2.8) instead of (2.4) from now on. The advantage of using this approach will become evident in the next section as the structure of (2.6) allows us to obtain the general

solution.

### 3. A representation of the discounted penalty function

In this section, we show that the Gerber-Shiu discounted penalty function with a barrier can be expressed as the sum of two functions. The first function is the Gerber-Shiu discounted penalty function without a barrier. As a result, it is the solution to a defective renewal equation, as shown in Gerber and Shiu (1998). Furthermore, the Gerber-Shiu discounted penalty function without a barrier has been studied extensively in recent years, and has proven to be very useful in unifying and extending classical ruin theoretic results. As will be seen in later sections, many of these results are transferable to the barrier case due to the special structure of the Gerber-Shiu discounted penalty function with a barrier. The second function is independent of the choice of the penalty function  $w(x_1, x_2)$ . In fact, we will show in Section 4 that this second function is proportional to the product of an exponential function and the df of a compound geometric distribution. Moreover, in some cases, it may also be expressed as a solution to an excessive or proper renewal equation, depending on whether  $\delta > 0$  or not.

We begin by considering two functions on  $[0, \infty)$ . The first function is

$$m_\infty(u) = E\{e^{-\delta T_\infty} w(U(T_\infty -), |U(T_\infty)|) I(T_\infty < \infty)\}, \quad (3.1)$$

the Gerber-Shiu discounted penalty function with no barrier. As derived in Section 2,  $m_\infty(u)$  is a solution to the integro-differential equation

$$m'_\infty(u) = -\frac{\lambda}{c} \int_0^u m_\infty(u-y) dP(y) + \frac{\lambda + \delta}{c} m_\infty(u) - \frac{\lambda}{c} \zeta(u), \quad (3.2)$$

for  $0 \leq u < \infty$ . We note that this equation is the same as (2.6) except that  $b = \infty$ . The second function  $v(u)$  is a nontrivial solution to the following homogeneous integro-differential equation

$$v'(u) = -\frac{\lambda}{c} \int_0^u v(u-y) dP(y) + \frac{\lambda + \delta}{c} v(u), \quad (3.3)$$



with initial condition defined (without loss of generality) to be  $v(0) = 1$ . In order to proceed, we need to show that  $v'(u) > 0$  for  $0 \leq u < \infty$ . Thus, let  $u_0$  be the largest value such that  $v(u)$  is strictly increasing and  $v'(u) > 0$  for  $0 \leq u < u_0$ . Then,  $u_0 > 0$  since  $v'(0) = \frac{\lambda+\delta}{c} > 0$ . If  $u_0$  is finite, then  $v'(u_0) = 0$ . However, if  $P(u_0) > 0$ ,

$$v'(u_0) = -\frac{\lambda}{c} \int_0^{u_0} v(u_0 - y) dP(y) + \frac{\lambda + \delta}{c} v(u_0) > -\frac{\lambda}{c} v(u_0) P(u_0) + \frac{\lambda + \delta}{c} v(u_0) \geq 0,$$

whereas if  $P(u_0) = 0$ ,

$$v'(u_0) = -\frac{\lambda}{c} \int_0^{u_0} v(u_0 - y) dP(y) + \frac{\lambda + \delta}{c} v(u_0) = \frac{\lambda + \delta}{c} v(u_0) > 0.$$

Hence,  $v'(u_0) > 0$ , a contradiction. Therefore,  $u_0 = \infty$ .

The general solution of (3.2) or (2.6) is of the form

$$m(u) = m_\infty(u) + \eta v(u), \quad (3.4)$$

where  $\eta$  is an arbitrary constant. In this case,  $m_\infty(u)$  may be viewed as a particular solution to (3.2) and  $v(u)$  as a fundamental solution to the homogeneous equation (3.3). It follows from the general theory of differential equations that every solution of the nonhomogeneous equation can be expressed in the form (3.4) (e.g., see Petrovski, 1966, p. 119). Therefore, to obtain  $m_b(u)$ , we only need to identify the boundary condition defined by (2.8). Since  $v'(b) > 0$ , with

$$\eta = -\frac{m'_\infty(b)}{v'(b)},$$

(2.8) is satisfied and the solution for  $m_b(u)$  becomes

$$m_b(u) = m_\infty(u) - \frac{m'_\infty(b)}{v'(b)} v(u), \quad 0 \leq u \leq b. \quad (3.5)$$

Gerber and Shiu (1998) have shown that  $m_\infty(u)$  is the solution of a defective renewal equation, described as follows. Let  $\rho = \rho(\delta)$  and  $-\kappa = -\kappa(\delta)$  be the nonnegative and negative solutions of the equation

$$z + \frac{\lambda}{c} \tilde{p}(z) - \frac{\lambda + \delta}{c} = 0. \quad (3.6)$$

It is straightforward to verify that  $\rho$  always exists and that  $-\kappa$  exists if the claim size random variable  $Y$  is light tailed, i.e.  $\lim_{z \rightarrow z_0+} \tilde{p}(z) = \infty$  where  $(z_0, \infty)$  is the domain of  $\tilde{p}(z)$ . Gerber and Shiu (1998) called equation (3.6) the Lundberg fundamental equation. It reduces to the adjustment coefficient equation when  $\delta = 0$ . The two roots  $\rho$  and  $-\kappa$  of (3.6) have played a key role in analyzing the Gerber-Shiu function  $m_\infty(u)$  (e.g., see Gerber and Shiu, 1998, and Lin and Willmot, 1999). Moreover, if  $\delta = 0$ , then  $\rho = 0$  and  $\kappa$  is the adjustment coefficient. Gerber and Shiu (1998) have shown that  $m_\infty(u)$  satisfies the following defective renewal equation

$$m_\infty(u) = \left(1 - \frac{\delta}{c\rho}\right) \int_0^u m_\infty(u-y) g_1(y) dy + \frac{\lambda}{c} e^{\rho u} \int_u^\infty e^{-\rho y} \zeta(y) dy, \quad (3.7)$$

where  $g_1(x)$  a probability density function (pdf) given by

$$g_1(x) = \frac{e^{\rho x} \int_x^\infty e^{-\rho y} dP(y)}{\int_0^\infty e^{-\rho y} \bar{P}(y) dy}, \quad x \geq 0. \quad (3.8)$$

The asymptotic behaviour of  $m_\infty(u)$  is easily obtained by applying the Key Renewal Theorem (e.g., see Resnick, 1992). In particular, it is easily verified that

$$\left(1 - \frac{\delta}{c\rho}\right) \int_0^\infty e^{\kappa y} g_1(y) dy = 1. \quad (3.9)$$

That is,  $-\kappa$  is the Malthusian parameter for the defective renewal equation (3.7). Thus,  $m_\infty(u) \sim C_1 e^{-\kappa u}$  as  $u \rightarrow \infty$ , where  $C_1$  a constant and  $a(u) \sim b(u)$ ,  $u \rightarrow \infty$ , is interpreted as  $\lim_{u \rightarrow \infty} a(u)/b(u) = 1$ .

The fact that  $m_\infty(u)$  is the solution to a defective renewal equation also allows for it to be explicitly expressed in terms of a compound geometric tail, thereby providing the means to analyze  $m_\infty(u)$  thoroughly (e.g., see Lin and Willmot, 1999, 2000, for details). Properties of  $m_\infty(u)$  and its applications have been studied extensively by Gerber and Shiu (1997, 1998), Lin and Willmot (1999, 2000), Willmot and Lin (2001), and references therein. As a result, we may utilize the properties of  $m_\infty(u)$  to analyze  $m_b(u)$ , the Gerber-Shiu function with a barrier.

#### 4. Analysis of the function $v(u)$

In this section, we consider the function  $v(u)$ . First of all, we show that the general solution for  $v(u)$  is proportional to the product of an exponential function and the df of a compound geometric distribution. The idea and derivation for this decomposition are due to Gerber and may be found in Bühlmann (1970, Section 6.4.9).

We begin by introducing a function  $\Psi(u)$  such that

$$v(u) = \frac{1 - \Psi(u)}{1 - \Psi(0)} e^{\rho u}, \quad (4.1)$$

where  $\Psi(0)$  will be determined shortly. Thus, it follows from (3.3) and (3.6) that

$$\begin{aligned} \Psi'(u) &= \frac{\lambda}{c} \int_0^u [1 - \Psi(u - y)] e^{-\rho y} dP(y) + \left( \rho - \frac{\lambda + \delta}{c} \right) [1 - \Psi(u)] \\ &= \frac{\lambda}{c} \int_0^u [1 - \Psi(u - y)] e^{-\rho y} dP(y) - \frac{\lambda}{c} \tilde{p}(\rho) [1 - \Psi(u)]. \end{aligned} \quad (4.2)$$

Let  $\hat{P}(y) = 1 - \bar{\hat{P}}(y)$  be an Esscher-transformed df such that

$$d\hat{P}(y) = \frac{e^{-\rho y} dP(y)}{\tilde{p}(\rho)}, \quad (4.3)$$

with mean  $\hat{p}_1 = \int_0^\infty y d\hat{P}(y) = \int_0^\infty y e^{-\rho y} dP(y) / \tilde{p}(\rho)$ . Also, define

$$\hat{c} = \frac{c}{\tilde{p}(\rho)} \quad (4.4)$$

and

$$\Psi(0) = \hat{\phi} = \frac{\lambda \hat{p}_1}{\hat{c}}. \quad (4.5)$$

With these definitions, it is clear that  $0 < \Psi(0) < 1$ , and it is possible to rewrite (4.2) so that  $\Psi(u)$  is the solution to the equation

$$\Psi'(u) = \frac{\lambda}{\hat{c}} \int_0^u [1 - \Psi(u - y)] d\hat{P}(y) - \frac{\lambda}{\hat{c}} [1 - \Psi(u)],$$

with initial value given by (4.5). Therefore,  $\Psi(u)$  is the probability of ruin for the compound Poisson risk model without a barrier, where  $\lambda$  is the Poisson parameter,  $\hat{P}(y)$  is the df of the individual claim sizes, and  $\hat{c}$  is the annual premium rate. As a

result,  $\Psi(u)$  is the tail of a compound geometric distribution with geometric parameter  $\hat{\phi}$  and secondary distribution having pdf  $\hat{g}(y) = \bar{\hat{P}}(y)/\hat{p}_1$ . Consequently, if  $\delta = 0$ , then  $\rho = 0$  which implies that (4.3) and (4.4) simplify to give  $\hat{P}(y) = P(y)$  and  $\hat{c} = c$ , respectively. In this case,  $\Psi(u) = \psi(u)$ , the probability of ruin in the usual sense, and

$$v_0(u) = v(u)|_{\delta=0} = \frac{1 - \psi(u)}{1 - \psi(0)} . \quad (4.6)$$

The analysis of  $v(u)$  is, by the discussion above and (4.1), therefore essentially equivalent to the analysis of the compound geometric tail  $\Psi(u)$ . Much is known about analytic properties of compound geometric tails (e.g., see Willmot and Lin, 2001, and references therein), and we will not discuss this issue in full detail here. One has immediately from (4.1) that

$$v(u) \sim \frac{1}{1 - \Psi(0)} e^{\rho u}, \quad u \rightarrow \infty. \quad (4.7)$$

It is instructive to consider the adjustment coefficient  $\hat{\kappa}$  for the related ruin problem discussed above. Let  $\hat{\theta}$  be the relative security loading satisfying  $\hat{c} = \lambda \hat{p}_1 (1 + \hat{\theta})$ , or equivalently,  $c = \lambda \hat{p}_1 (1 + \hat{\theta}) \tilde{p}(\rho)$ . Then,  $\hat{\kappa}$  (if it exists) satisfies

$$1 + (1 + \hat{\theta}) \hat{p}_1 \hat{\kappa} = \int_0^\infty e^{\hat{\kappa} y} d\hat{P}(y) = \frac{\tilde{p}(\rho - \hat{\kappa})}{\tilde{p}(\rho)}.$$

Multiplication by  $\tilde{p}(\rho)$  implies that

$$\tilde{p}(\rho) + \frac{c}{\lambda} \hat{\kappa} = \tilde{p}(\rho - \hat{\kappa}).$$

Since  $\rho$  satisfies (3.6), we immediately have that

$$(\rho - \hat{\kappa}) + \frac{\lambda}{c} \tilde{p}(\rho - \hat{\kappa}) - \frac{\lambda + \delta}{c} = 0.$$

Moreover, if  $-\kappa$  satisfies (3.6), i.e.,

$$-\kappa + \frac{\lambda}{c} \tilde{p}(-\kappa) - \frac{\lambda + \delta}{c} = 0,$$

then it is clear by comparison of the above two equations that

$$\hat{\kappa} = \rho + \kappa. \quad (4.8)$$

Cramer's asymptotic ruin formula then yields  $\Psi(u) \sim \hat{C}e^{-(\rho+\kappa)u}$  as  $u \rightarrow \infty$  where  $\hat{C}$  is a constant. Since (4.1) may be expressed as

$$v(u) - \frac{1}{1 - \Psi(0)}e^{\rho u} = -\frac{\Psi(u)}{1 - \Psi(0)}e^{\rho u},$$

it follows that

$$v(u) - \frac{1}{1 - \Psi(0)}e^{\rho u} \sim -\frac{\hat{C}}{1 - \Psi(0)}e^{-\kappa u}, \quad u \rightarrow \infty, \quad (4.9)$$

a refinement of (4.7). Also, one has (e.g., see Willmot and Lin, 2001, p. 109) that  $\hat{C}_Le^{-(\rho+\kappa)u} \leq \Psi(u) \leq \hat{C}_Ue^{-(\rho+\kappa)u}$ ,  $u \geq 0$ , where

$$\hat{C}_L = \inf_{x \geq 0} \frac{e^{\hat{\kappa}x} \int_x^\infty \hat{g}(y)dy}{\int_x^\infty e^{\hat{\kappa}y} \hat{g}(y)dy} \text{ and } \hat{C}_U = \sup_{x \geq 0} \frac{e^{\hat{\kappa}x} \int_x^\infty \hat{g}(y)dy}{\int_x^\infty e^{\hat{\kappa}y} \hat{g}(y)dy},$$

which implies from (4.1) that

$$\frac{e^{\rho u} - \hat{C}_Ue^{-\kappa u}}{1 - \Psi(0)} \leq v(u) \leq \frac{e^{\rho u} - \hat{C}_Le^{-\kappa u}}{1 - \Psi(0)}, \quad u \geq 0. \quad (4.10)$$

It is interesting to note that if  $\kappa > 0$  satisfying (3.6) exists, then  $v(u)$  can be expressed as a solution to either a proper (if  $\delta = 0$ ) or excessive (if  $\delta > 0$ ) renewal equation. To identify the solution of the integro-differential equation (3.3), we proceed as follows. Taking Laplace transforms of both sides of (3.3) for sufficiently large  $z$  (i.e.,  $z > \rho$ , as is evident from (4.7)) yields

$$z\tilde{v}(z) - v(0) = -\frac{\lambda}{c}\tilde{p}(z)\tilde{v}(z) + \frac{\lambda + \delta}{c}\tilde{v}(z).$$

Since  $v(0) = 1$ , we obtain

$$\left(z + \frac{\lambda}{c}\tilde{p}(z) - \frac{\lambda + \delta}{c}\right)\tilde{v}(z) = 1. \quad (4.11)$$

Recalling that  $-\kappa$  is the negative root of (3.6), we may rewrite (4.11) as

$$\left[ z + \kappa + \frac{\lambda}{c} (\tilde{p}(z) - \tilde{p}(-\kappa)) \right] \tilde{v}(z) = 1. \quad (4.12)$$

Further algebraic manipulation of (4.12) then yields

$$\tilde{v}(z) = \frac{\lambda}{c} \frac{\tilde{p}(-\kappa) - \tilde{p}(z)}{z + \kappa} \tilde{v}(z) + \frac{1}{z + \kappa}. \quad (4.13)$$

Inverting the Laplace transforms in (4.13) leads to (e.g., using Proposition 2.1 of Willmot and Dickson, 2003)

$$v(u) = \left( 1 + \frac{\delta}{c\kappa} \right) \int_0^u v(u-y) g_2(y) dy + e^{-\kappa u}, \quad (4.14)$$

where  $g_2(x)$  is a pdf defined by

$$g_2(x) = \frac{e^{-\kappa x} \int_x^\infty e^{\kappa y} dP(y)}{\int_0^\infty e^{\kappa y} \bar{P}(y) dy}, \quad x \geq 0. \quad (4.15)$$

The reader may verify that the solution to (4.14) does in fact satisfy (3.3). It is easy to see that when  $\delta > 0$ , the renewal equation (4.14) is excessive and when  $\delta = 0$ , the renewal equation (4.14) is proper. Similar to (3.9), we have that

$$\left( 1 + \frac{\delta}{c\kappa} \right) \int_0^\infty e^{-\rho y} g_2(y) dy = 1.$$

Thus,  $\rho$  is the Malthusian parameter for the excessive renewal equation (4.14), which is expected because of (4.7).

## 5. On the time of ruin $T_b$ , the surplus before ruin $U_b(T_b-)$ , and the deficit at ruin $|U_b(T_b)|$

In this section, we discuss probabilistic properties of the time of ruin  $T_b$ , the surplus before ruin  $U_b(T_b-)$ , and the deficit at ruin  $|U_b(T_b)|$  for  $b < \infty$ . The case with  $b = \infty$  is discussed extensively in Lin and Willmot (2000).

First of all,  $T_b$  is a finite-valued random variable, i.e.,  $Pr(T_b < \infty) = 1$ . This can be easily verified by setting  $\delta = 0$  and  $w(x_1, x_2) = 1$  into (1.1), and using the facts

that  $m_\infty(u) = \psi(u)$  and (4.6) holds. Hence, for  $b < \infty$ ,  $m_b(u)$  may be rewritten as

$$m_b(u) = E\{e^{-\delta T_b} w(U_b(T_b-), |U_b(T_b)|)\}.$$

It follows from (3.5) and (4.1) that for  $0 \leq u \leq b$ ,

$$m_b(u) = m_\infty(u) + \frac{m'_\infty(b)e^{-\rho b}}{\Psi'(b) - \rho[1 - \Psi(b)]}[1 - \Psi(u)]e^{\rho u}. \quad (5.1)$$

We now consider  $E(e^{-\delta T_b})$ , the Laplace transform of  $T_b$ , which corresponds to the case  $w(x_1, x_2) = 1$ . Let

$$\overline{K}(u) = m_\infty(u)|_{w(x_1, x_2)=1} = E\{e^{-\delta T_\infty} I(T_\infty < \infty)\}.$$

Lin and Willmot (1999) have shown that  $\overline{K}(u)$  is the solution of

$$\overline{K}(u) = \phi_1 \int_0^u \overline{K}(u-y)g_1(y)dy + \phi_1 \int_u^\infty g_1(y)dy,$$

with  $\phi_1 = 1 - \delta/(c\rho)$ . Thus,  $\overline{K}(u)$  is the tail of a compound geometric distribution with geometric parameter  $\phi_1$  and secondary distribution having claim size pdf  $g_1(x)$  given by (3.8). Hence,  $\overline{K}(u)$  may be expressed as

$$\overline{K}(u) = \sum_{n=1}^{\infty} (1 - \phi_1) \phi_1^n \overline{G_1^{*n}}(u), \quad (5.2)$$

where  $\overline{G_1^{*n}}(x)$  is the tail of the  $n$ -fold convolution of  $g_1(x)$ . Similarly, since  $\Psi(u)$  is the tail of a compound geometric distribution with geometric parameter  $\hat{\phi}$  and secondary distribution having pdf  $\hat{g}(y)$ , we have

$$\Psi(u) = \sum_{n=1}^{\infty} (1 - \hat{\phi}) \hat{\phi}^n \overline{\hat{G}^{*n}}(u), \quad (5.3)$$

where  $\overline{\hat{G}^{*n}}(x)$  is the tail of the  $n$ -fold convolution of  $\hat{g}(x)$ . Therefore, we may express the Laplace transform of  $T_b$  as

$$E(e^{-\delta T_b}) = \overline{K}(u) + \frac{\overline{K}'(b)e^{-\rho b}}{\Psi'(b) - \rho[1 - \Psi(b)]}[1 - \Psi(u)]e^{\rho u}, \quad (5.4)$$

where  $\overline{K}(u)$  and  $\Psi(u)$  are given by (5.2) and (5.3), respectively.

As a special case, the Laplace transform of the time of the first drop below the initial surplus  $u$  may be obtained via

$$E(e^{-\delta T_b})\big|_{u=0} = \phi_1 + \frac{\overline{K}'(b)e^{-\rho b}}{\Psi'(b) - \rho[1 - \Psi(b)]}(1 - \hat{\phi}). \quad (5.5)$$

Further simplification of (5.4) is possible for claim size distributions such as the exponential, the mixture of two exponentials, and the mixture of Erlangs. This is due to the fact that analytic expressions  $\overline{K}(u)$  and  $\Psi(u)$  exist for these three distribution classes (see Lin and Willmot, 1999, for details). Such simplifications will enable us to derive the moments of the time of ruin  $T_b$  for the exponential and the mixture of two exponentials claim sizes in the next section.

We next turn our attention to the distribution of the surplus before ruin  $U_b(T_b-)$ . Letting  $\delta = 0$  and  $w(x_1, x_2) = w_1(x_1)$ , the Gerber-Shiu function becomes

$$m_{b,1}(u) = m_b(u)|_{\delta=0, w(x_1, x_2)=w_1(x_1)} = E\{w_1(U_b(T_b-))\}.$$

To derive a general expression of  $m_{b,1}(u)$ , note that  $\delta = 0$  implies that (4.6) is the solution to the homogeneous renewal equation (4.14). Hence,

$$m_{b,1}(u) = m_{\infty,1}(u) + \frac{1 - \psi(u)}{\psi'(b)} m'_{\infty,1}(b). \quad (5.6)$$

It follows from (2.3) that  $\zeta(y) = w_1(y)\overline{P}(y)$  and from (3.7) and (3.8) that

$$m_{\infty,1}(u) = \frac{1}{1 + \theta} \int_0^u m_{\infty,1}(u - y) dP_1(y) + \frac{1}{1 + \theta} \int_u^\infty w_1(y) dP_1(y), \quad (5.7)$$

since  $\rho = 0$  and  $G_1(y) = P_1(y)$ , the equilibrium df of  $P(y)$ . Lin and Willmot (2000) have shown that (5.7) can be solved by

$$m_{\infty,1}(u) = \frac{1}{\theta} \left\{ \int_0^u \psi(u - y) w_1(y) dP_1(y) + \int_u^\infty w_1(y) dP_1(y) - \psi(u) \int_0^\infty w_1(y) dP_1(y) \right\}. \quad (5.8)$$



Differentiating (5.8) and evaluating at  $u = b$  yields

$$m'_{\infty,1}(b) = \frac{1}{\theta} \left\{ \int_0^b \psi'(b-y)w_1(y)dP_1(y) - \psi'(b) \int_0^\infty w_1(y)dP_1(y) \right\} - \frac{1}{1+\theta} \frac{\overline{P}(b)}{p_1} w_1(b). \quad (5.9)$$

Substituting (5.8) and (5.9) into (5.6), we obtain

$$\begin{aligned} m_{b,1}(u) &= \frac{1}{\theta} \left\{ - \int_0^u [1 - \psi(u-y)]w_1(y)dP_1(y) + \frac{1 - \psi(u)}{\psi'(b)} \int_0^b \psi'(b-y)w_1(y)dP_1(y) \right\} \\ &+ \frac{1 - \psi(u)}{(1+\theta)[- \psi'(b)]} \frac{\overline{P}(b)}{p_1} w_1(b). \end{aligned} \quad (5.10)$$

We remark that the upper limit in each of the two integrals in (5.10) is  $b$  or less, reflecting the fact that the surplus under the barrier strategy is capped at  $b$ .

We are now able to identify the distribution of the surplus before ruin. By setting  $w_1(x_1) = I(x_1 \leq y)$  for  $0 \leq y \leq b$  into (5.10) and differentiating, we can obtain the following results:

(i) for  $0 < y < b$ ,  $U_b(T_b-)$  has pdf

$$f_s(y) = \begin{cases} \frac{1}{\theta} \frac{\overline{P}(y)}{p_1} \left\{ \frac{[1-\psi(u)]\psi'(b-y)}{\psi'(b)} - [1 - \psi(u-y)] \right\}, & 0 < y \leq u, \\ \frac{1}{\theta} \frac{\overline{P}(y)}{p_1} \frac{[1-\psi(u)]\psi'(b-y)}{\psi'(b)}, & u < y < b. \end{cases} \quad (5.11)$$

(ii)  $U_b(T_b-)$  has a probability mass at  $b$  given by

$$Pr\{U_b(T_b-) = b\} = \frac{1}{1+\theta} \frac{\overline{P}(b)}{p_1} \frac{1 - \psi(u)}{[-\psi'(b)]}. \quad (5.12)$$

We also remark that by defining  $w_1(x_1) = x_1^k$ , the Gerber-Shiu function in this case becomes the  $k$ -th moment of  $U_b(T_b-)$  and the moments of  $U_b(T_b-)$  can be obtained from (5.10). Alternatively, we may use (5.11) and (5.12) to calculate the moments directly.

Finally in this section, we derive an analytic expression for the  $k$ -th moment of the deficit at ruin  $|U(T_b)|$ . Letting  $\delta = 0$  and  $w(x_1, x_2) = x_2^k$  for  $k = 1, 2, \dots$ , the Gerber-Shiu function becomes

$$m_{b,2}(u) = m_b(u)|_{\delta=0, w(x_1, x_2)=x_2^k} = E\{|U_b(T_b)|^k\}.$$

Similar to the surplus case, the expression for  $m_{b,2}(u)$  is given by

$$m_{b,2}(u) = m_{\infty,2}(u) + \frac{1 - \psi(u)}{\psi'(b)} m'_{\infty,2}(b), \quad (5.13)$$

where

$$m_{\infty,2}(u) = \frac{1}{1 + \theta} \int_0^u m_{\infty,2}(u - y) dP_1(y) + \frac{k}{1 + \theta} \int_u^\infty \int_y^\infty (t - y)^{k-1} dP_1(t) dy.$$

Lin and Willmot (2000, Corollaries 3.1 and 4.1) have shown that for  $k = 1, 2, \dots$ ,

$$\begin{aligned} m_{\infty,2}(u) &= E\{|U(T_\infty)|^k I(T_\infty < \infty)\} \\ &= -\frac{p_{k+1}}{(k+1)p_1\theta} \psi(u) - \frac{1}{p_1\theta} \sum_{j=0}^{k-2} \binom{k}{j} p_{k-j} \int_u^\infty (x - u)^j \psi(x) dx \\ &\quad + k \int_u^\infty (x - u)^{k-1} \psi(x) dx. \end{aligned} \quad (5.14)$$

Of course, (5.14) holds true provided that all claim size moments and integrals in (5.14) exist. The derivative of  $m_{\infty,2}(u)$  at  $u = b$  can also be obtained via (5.14). For  $k = 1$ , it is given by

$$m'_{\infty,2}(b) = -\frac{p_2}{2p_1\theta} \psi'(b) - \psi(b), \quad (5.15)$$

and for  $k > 1$ ,

$$\begin{aligned} m'_{\infty,2}(b) &= -\frac{p_{k+1}}{(k+1)p_1\theta} \psi'(b) + \frac{p_k}{p_1\theta} \psi(b) + \frac{k}{p_1\theta} \sum_{j=0}^{k-3} \binom{k-1}{j} p_{k-j-1} \int_b^\infty (x - b)^j \psi(x) dx \\ &\quad - k(k-1) \int_b^\infty (x - b)^{k-2} \psi(x) dx. \end{aligned} \quad (5.16)$$

Substituting (5.14), (5.15), and (5.16) into (5.13), it is possible to obtain an analytic expression for  $E\{|U_b(T_b)|^k\}$ . Furthermore, if a closed-form expression for  $\psi(u)$  is available (e.g. claim sizes are exponential, a combination of exponentials, or a mixture of Erlangs), simpler closed-form expressions for the moments of  $|U_b(T_b)|$  can be obtained.

## 6. The exponential and mixture of two exponentials claim sizes

In this section, we consider the cases when the individual claim size is an exponential and a mixture of two exponentials. We begin with the exponential case where the claim size df is of the form

$$P(y) = 1 - e^{-\mu y}, \quad y \geq 0, \quad (6.1)$$

with Laplace-Stieltjes transform  $\tilde{p}(z) = \mu/(\mu + z)$ . The Lundberg equation (3.6) simplifies to give

$$cz^2 + (c\mu - \lambda - \delta)z - \delta\mu = 0,$$

which yields two useful identities:

$$\kappa - \rho = \mu - \frac{\lambda + \delta}{c} \quad \text{and} \quad \kappa\rho = \frac{\delta\mu}{c}. \quad (6.2)$$

We now derive the Laplace transform of the time of ruin  $T_b$ . It follows from (6.2) that

$$\phi_1 = 1 - \frac{\delta}{c\rho} = \frac{\mu - \kappa}{\mu}.$$

With  $P(y)$  given by (6.1), it also follows from (3.8) and (5.2) that

$$\overline{K}(u) = \phi_1 e^{-\kappa u} = \frac{\mu - \kappa}{\mu} e^{-\kappa u}.$$

Moreover, since  $P_1(y) = P(y)$ , it is easy to verify that

$$\begin{aligned} \hat{P}(y) &= 1 - e^{-(\mu+\rho)y}, \\ \hat{c} &= \frac{\lambda}{\mu - \kappa} \quad \text{and} \quad \hat{\phi} = \frac{\lambda\hat{p}_1}{\hat{c}} = \frac{\mu - \kappa}{\mu + \rho}. \end{aligned}$$

Thus, (5.3) yields

$$\Psi(u) = \hat{\phi} e^{-(1-\hat{\phi})(\mu+\rho)u} = \frac{\mu - \kappa}{\mu + \rho} e^{-(\kappa+\rho)u}.$$

It then follows from (5.4) that

$$\begin{aligned} E(e^{-\delta T_b}) &= \frac{\mu - \kappa}{\mu} e^{-\kappa u} + \frac{\mu - \kappa}{\mu} \frac{\kappa e^{-\kappa b}}{(\rho + \mu)\rho e^{\rho b} + (\mu - \kappa)\kappa e^{-\kappa b}} \left[ (\rho + \mu)e^{\rho u} - (\mu - \kappa)e^{-\kappa u} \right] \\ &= \frac{\lambda(\kappa e^{-\kappa b} e^{\rho u} + \rho e^{\rho b} e^{-\kappa u})}{c[(\rho + \mu)\rho e^{\rho b} + (\mu - \kappa)\kappa e^{-\kappa b}]}. \end{aligned} \quad (6.3)$$

We note that (6.3) is also obtained in Segerdahl (1970, eq. (18.5)) and in Gerber (1979, Chapter 10, eq. (2.4)), in slightly different but equivalent forms. Furthermore, moments of  $T_b$  can be acquired via differentiation of (6.3) and evaluation of the resulting expressions at  $\delta = 0$ . For instance, the mean of  $T_b$  is given by

$$E(T_b) = \frac{c\mu e^{R(b-u)}(c\mu e^{Ru} - \lambda)}{\lambda(c\mu - \lambda)^2} - \frac{1 + \mu u}{c\mu - \lambda},$$

where  $R = \kappa(0) = \theta\mu/(1 + \theta)$ . We remark that the result for  $E(T_b)$  is in agreement with the result obtained by Segerdahl (1970, eq. (18.11)). Higher moments of  $T_b$  can be obtained similarly although tediously. However, in the case of zero initial surplus (i.e.,  $u = 0$ ), the mean and variance of  $T_b$  are easily obtained as

$$E(T_b)|_{u=0} = \frac{c\mu e^{Rb} - \lambda}{\lambda(c\mu - \lambda)}$$

and

$$Var(T_b)|_{u=0} = \frac{(c\mu + \lambda)(c^2\mu^2 e^{2Rb} - \lambda^2) - 2\lambda\mu e^{Rb}(c\mu - \lambda)[\lambda b + c(2 + \mu b)]}{\lambda^2(c\mu - \lambda)^3}.$$

The distribution of the surplus before ruin can be computed easily with the aid of (5.11) and (5.12). Since  $\psi(u) = e^{-Ru}/(1 + \theta)$ , we have

$$f_s(y) = \begin{cases} \frac{\mu}{\theta} e^{-\frac{\mu}{1+\theta}y} (1 - e^{-Ry}), & 0 < y \leq u, \\ \frac{\mu}{\theta(1+\theta)} e^{-\frac{\mu}{1+\theta}y} (1 + \theta - e^{-Ru}), & u < y < b, \end{cases}$$

and

$$Pr\{U_b(T_b-) = b\} = \frac{1}{\theta} e^{-\frac{\mu}{1+\theta}b} (1 + \theta - e^{-Ru}).$$

All the moments of the surplus can be computed easily using incomplete gamma functions, but we omit the computation here.

We now consider a mixture of two exponentials claim size distributions

$$P(y) = 1 - \omega e^{-\alpha y} - (1 - \omega) e^{-\beta y}, \quad y \geq 0.$$

In this situation, it is known that (3.6) yields two negative roots and one nonnegative root, namely  $-r$ ,  $-\kappa$ , and  $\rho$  in ascending order, when the Laplace transform  $\tilde{p}(z)$  is treated as a rational function. Furthermore, both pdfs  $g_1(y)$  and  $g_2(y)$  are a combination of two exponentials. More precisely, for  $i = 1, 2$ ,

$$g_i(y) = p_i \alpha e^{-\alpha y} + (1 - p_i) \beta e^{-\beta y},$$

where

$$p_1 = \frac{\omega(\beta + \rho)}{\omega\beta + (1 - \omega)\alpha + \rho} \text{ and } p_2 = \frac{\omega(\beta - \kappa)}{\omega\beta + (1 - \omega)\alpha - \kappa}.$$

In order to derive the Laplace transform of  $T_b$ , we first remark that the form of (5.2) implies that a closed-form expression for  $\overline{K}(u) = E\{e^{-\delta T_\infty} I(T_\infty < \infty)\}$  exists and is given by

$$\overline{K}(u) = \frac{\phi_1}{r - \kappa} [(\varphi - \kappa)e^{-\kappa u} + (r - \varphi)e^{-ru}],$$

where  $\varphi = \alpha(1 - p) + \beta p$  (e.g., see Willmot and Lin, 2001, eq. (4.1.20)). While a similar approach could be employed to obtain a closed-form expression for  $\Psi(u)$  from (5.3), let us instead use the renewal equation (4.14) to obtain  $v(u)$  directly. Formally taking Laplace transforms of (4.14), we obtain

$$\tilde{v}(z) = \phi_2 \tilde{v}(z) \tilde{g}_2(z) + \frac{1}{z + \kappa},$$

where  $\phi_2 = 1 + \delta/(c\kappa)$  and  $\tilde{g}_2(z) = \int_0^\infty e^{-zx} g_2(x) dx$ . Thus,

$$\begin{aligned} \tilde{v}(z) &= \frac{1}{[1 - \phi_2 \tilde{g}_2(z)](z + \kappa)} \\ &= \frac{1}{1 - \phi_2 \left[ \frac{q\alpha}{z + \alpha} + \frac{(1-q)\beta}{z + \beta} \right] (z + \kappa)} \\ &= \frac{(z + \alpha)(z + \beta)}{[(z + \alpha)(z + \beta) - \phi_2 q \alpha (z + \beta) - \phi_2 (1 - q) \beta (z + \alpha)](z + \kappa)}. \end{aligned} \quad (6.4)$$

The reader may verify that the denominator of (6.4) is proportional to the left hand side of (3.6), and therefore we may rewrite  $\tilde{v}(z)$  as

$$\tilde{v}(z) = \frac{(z + \alpha)(z + \beta)}{(z - \rho)(z + \kappa)(z + r)}$$

$$= \frac{C_\rho}{z - \rho} + \frac{C_\kappa}{z + \kappa} + \frac{C_r}{z + r},$$

where

$$C_\rho = \frac{(\alpha + \rho)(\beta + \rho)}{(\kappa + \rho)(r + \rho)}, \quad C_\kappa = -\frac{(\alpha - \kappa)(\beta - \kappa)}{(r - \kappa)(\rho + \kappa)} \text{ and } C_r = -\frac{(\alpha - r)(\beta - r)}{(\kappa - r)(\rho + r)}.$$

Therefore,  $v(u) = C_\rho e^{\rho u} + C_\kappa e^{-\kappa u} + C_r e^{-r u}$ . As a result, the Laplace transform of the time of ruin  $T_b$  may be obtained via (5.1), and it is a linear combination of  $e^{\rho u}$ ,  $e^{-\kappa u}$ , and  $e^{-r u}$ .

The distribution of the surplus before ruin may be computed using (5.11) and (5.12) while the moments of the deficit at ruin may be computed using (5.13), (5.14), and (5.16), but we omit both of them here due to their tedious computation.

## 7. Extensions to the stationary renewal risk model

In this section, we show that the simple integro-differential equation of Section 2 extends in a relatively straightforward manner to the equilibrium (or stationary) renewal risk process (e.g., see Willmot and Dickson, 2003, and references therein). For the (ordinary) renewal risk process, the distributions of the time until the first claim occurs and the subsequent interclaim times are generalized from a common exponential distribution (with mean  $1/\lambda$ ) to a common arbitrary distribution with df  $A(t) = 1 - \overline{A}(t)$  and pdf  $a(t) = A'(t)$ . Let the mean be  $E_A = \int_0^\infty \overline{A}(t) dt < \infty$ . The Gerber-Shiu discounted penalty function in the renewal risk model with a constant barrier  $b$  is, by analogy with (1.1)

$$m_{b,r}(u) = E\{e^{-\delta T_{b,r}} w(U_b(T_{b,r}-), |U_b(T_{b,r})|) I(T_{b,r} < \infty)\},$$

where  $T_{b,r}$  is the time to ruin. For the equilibrium renewal risk model, the time until the first claim occurs has pdf  $\overline{A}(t)/E_A$ , and all other interclaim times have df  $A(t)$ . Adopting the same notational convention, let the discounted penalty function for the equilibrium renewal risk model with a constant barrier  $b$  be given by

$$m_{b,e}(u) = E\{e^{-\delta T_{b,e}} w(U_b(T_{b,e}-), |U_b(T_{b,e})|) I(T_{b,e} < \infty)\}.$$

For more details on these models, we refer the reader to Willmot and Dickson (2003), and references therein.

By conditioning on the time and amount of the first claim, as in the derivation of (2.1), one finds for the (ordinary) renewal risk model that for  $0 \leq u \leq b$ ,

$$m_{b,r}(u) = \int_0^{(b-u)/c} e^{-\delta t} \gamma_{b,r}(u+ct) a(t) dt + \int_{(b-u)/c}^{\infty} e^{-\delta t} \gamma_{b,r}(b) a(t) dt, \quad (7.1)$$

where

$$\gamma_{b,r}(t) = \int_0^t m_{b,r}(t-y) dP(y) + \zeta(t), \quad (7.2)$$

and  $\zeta(t)$  is given by (2.3). A change of variable in the first integral in (7.1) implies that

$$m_{b,r}(u) = \frac{1}{c} \int_u^b e^{-\delta(\frac{t-u}{c})} \gamma_{b,r}(t) a\left(\frac{t-u}{c}\right) dt + \gamma_{b,r}(b) \int_{(b-u)/c}^{\infty} e^{-\delta t} a(t) dt. \quad (7.3)$$

For the equilibrium renewal process, the time of the first claim has pdf  $\bar{A}(t)/E_A$ , after which the process behaves as an (ordinary) renewal risk process. Thus, (7.3) is replaced by

$$m_{b,e}(u) = \frac{1}{cE_A} \int_u^b e^{-\delta(\frac{t-u}{c})} \gamma_{b,r}(t) \bar{A}\left(\frac{t-u}{c}\right) dt + \frac{1}{E_A} \gamma_{b,r}(b) \int_{(b-u)/c}^{\infty} e^{-\delta t} \bar{A}(t) dt, \quad (7.4)$$

where  $0 \leq u \leq b$ . Note that

$$\frac{d}{du} \left[ e^{-\delta(\frac{t-u}{c})} \bar{A}\left(\frac{t-u}{c}\right) \right] = \frac{1}{c} e^{-\delta(\frac{t-u}{c})} \left[ \delta \bar{A}\left(\frac{t-u}{c}\right) + a\left(\frac{t-u}{c}\right) \right],$$

and differentiation of (7.4) yields, with the help of (7.3),

$$\begin{aligned} m'_{b,e}(u) &= \frac{\delta}{c} m_{b,e}(u) + \frac{1}{cE_A} m_{b,r}(u) - \frac{1}{cE_A} \gamma_{b,r}(u) \\ &+ \frac{1}{cE_A} \gamma_{b,r}(b) \left[ e^{-\delta(\frac{b-u}{c})} \bar{A}\left(\frac{b-u}{c}\right) - \int_{(b-u)/c}^{\infty} e^{-\delta t} a(t) dt - \delta \int_{(b-u)/c}^{\infty} e^{-\delta t} \bar{A}(t) dt \right]. \end{aligned}$$

Integration by parts yields

$$\delta \int_x^{\infty} e^{-\delta t} \bar{A}(t) dt = e^{-\delta x} \bar{A}(x) - \int_x^{\infty} e^{-\delta t} a(t) dt, \quad x \geq 0. \quad (7.5)$$

Thus, with  $x = (b - u)/c$  in (7.5), the final term in the expression for  $m'_{b,e}(u)$  above vanishes, thereby leaving

$$m'_{b,e}(u) = \frac{\delta}{c}m_{b,e}(u) + \frac{1}{cE_A}m_{b,r}(u) - \frac{1}{cE_A}\gamma_{b,r}(u), \quad 0 \leq u \leq b, \quad (7.6)$$

or equivalently using (7.2),

$$m'_{b,e}(u) = \frac{\delta}{c}m_{b,e}(u) + \frac{1}{cE_A}m_{b,r}(u) - \frac{1}{cE_A} \int_0^u m_{b,r}(u - y)dP(y) - \frac{1}{cE_A}\zeta(u). \quad (7.7)$$

When  $A(t) = 1 - e^{-\lambda t}$ , it follows that  $E_A = 1/\lambda$ ,  $m_{b,r}(u) = m_{b,e}(u) = m_b(u)$ , implying that (7.7) is a generalization of (2.6).

It follows from (7.3) and (7.4) that

$$m_{b,r}(b) = \gamma_{b,r}(b) \int_0^\infty e^{-\delta t} a(t) dt, \quad (7.8)$$

and

$$m_{b,e}(b) = \frac{1}{E_A} \gamma_{b,r}(b) \int_0^\infty e^{-\delta t} \overline{A}(t) dt, \quad (7.9)$$

both of which generalize (2.7). Thus, from (7.6),

$$m'_{b,e}(b) = \frac{1}{cE_A} \gamma_{b,r}(b) \left[ \delta \int_0^\infty e^{-\delta t} \overline{A}(t) dt + \int_0^\infty e^{-\delta t} a(t) dt - 1 \right],$$

and since (7.5) holds with  $x = 0$ , it follows that

$$m'_{b,e}(b) = 0, \quad (7.10)$$

generalizing (2.8).

The solution to (7.7) for  $m_{b,e}(u)$  in terms of  $m_{b,r}(u)$  is straightforward. Replace  $u$  by  $t$  and rewrite (7.6) as

$$\frac{d}{dt} \left[ e^{-\frac{\delta}{c}t} m_{b,e}(t) \right] = \frac{1}{cE_A} e^{-\frac{\delta}{c}t} [m_{b,r}(t) - \gamma_{b,r}(t)].$$

Integrate from  $u$  to  $b$  to obtain

$$e^{-\frac{\delta}{c}b} m_{b,e}(b) - e^{-\frac{\delta}{c}u} m_{b,e}(u) = \frac{1}{cE_A} \int_u^b e^{-\frac{\delta}{c}t} [m_{b,r}(t) - \gamma_{b,r}(t)] dt,$$



i.e., for  $0 \leq u \leq b$ ,

$$m_{b,e}(u) = e^{-\frac{\delta}{c}(b-u)} m_{b,e}(b) + \frac{1}{cE_A} \int_u^b e^{-\frac{\delta}{c}(t-u)} [\gamma_{b,r}(t) - m_{b,r}(t)] dt, \quad (7.11)$$

where  $m_{b,e}(b)$  is given by (7.9).

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