Available online at www.sciencedirect.com
ScienceDirect
Int. J. Production Economics 112 (2008) 782–795
international journal of production economics
www.elsevier.com/locate/ijpe

Optimal ordering quantities for multi-products with stochastic demand: Return-CVaR model

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Received 26 January 2006; accepted 11 April 2007
Available online 28 July 2007

Abstract

Multi-product-ordering problem with stochastic demand is often optimized by the expected cost reduction or the expected profit improvement, in most of previous works. Moreover, risk preference of decision maker is often ignored. Evidently, this approach is impracticable when the corresponding volatility is larger. Based on the outstanding characteristics of conditional value at risk (CVaR) as a risk measure, the paper proposes an optimal-order model for multi-product with CVaR constraints, also the model is finally formulated as a linear programming problem. The model is simulated for the case of a newsvendor to analyze to what degree it succeeds. The solution, in fact, is bound consistent fully with the decision maker's intuition on return-risk decision making. Finally, results of return-CVaR model and the classical model are compared and shows that return-CVaR model is more flexible than the classical model.

Keywords: Stochastic demand; Multi-product order; Conditional value at risk (CVaR); Risk control; Linear programming model

1. Introduction

The business environment today can be characterized by fast product development, short selling season and highly volatile demand. These futures account for an increase of demand uncertainty in many types of products in the market (e.g. fashion apparels, toys, newspapers, Newsweek and electronics). The risk associated the sales of these products is evidently getting larger and larger. Consequently, in recent years, companies are increasingly focusing on incorporating risk control in decision making for achieving efficiencies in supply chains. However, how to decide product-ordering quantities often baffles the decision maker.

The optimal-order problem in this paper is also closely related to the multi-product newsvendor problem (MPNP) of the inventory theory. In this problem the product demands are considered independent and several units of the products are to be sold within a specified time and after that time the product must be discarded (see Khouja, 1999). In the MPNP, the seller is to determine the optimal supply levels under the assumptions of stochastic demand and fixed product prices. The classical MPNP focuses on a sole objective, of either minimizing the expected cost or maximizing the

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0925-5273/$ - see front matter
doi:10.1016/j.ijpe.2007.04.014

expected profit. However, the performance measure which considers only to expected value lacks precision when the associated volatility is large. Further, the classical MPNP ignores an important aspect that different decision makers have different risk preferences, although the risk preference problem has been extensively studied in the single-product newsvendor problem in the literature. It has been shown that: (a) a risk-averse retailer's optimal order quantity (i.e., the one that maximizes the retailer's expected utility) will be less than the order quantity that maximizes expected profit (see Horowitz, 1970; Baron, 1973; Eeckhoudt et al., 1995), (b) this optimal order quantity decreases with increase in risk aversion (see Bouakiz and Sobel, 1992; Eeckhoudt et al., 1995; Agrawal and Seshadri, 2000 for structural properties). Furthermore, Keren and Pliskin (2006) proposed a benchmark solution for the risk-averse single-product newsvendor problem. They derived the first-order conditions for optimality for the problem of a risk-averse expected-utility maximizing newsvendor.

Obviously, decision maker, in the same way, has risk preference in the MPNP setting. In general the multi-product ordering problem is more common in real business environment, than the single-product ordering problem. Moreover, procuring various products is also a way of decreasing risk for a stock maker. Therefore, although non-empirical study on the MPNP could prove that risk preference will impact the optimal ordering quantity, putting risk control and decision maker's risk preference into the decision-making process in the MPNP becomes crucial.

Generally, utility function or Mean–Variance (MV) model is used as a tool reflecting risk preference in the literature on newsvendor problem (e.g., see Agrawal and Seshadri, 2000; Bouakiz and Sobel, 1992; Chen and Federguen, 2000; Choi, 2002; Choi et al., 2004, 2008). However, utility function is difficult to determine in true life MV has an ever larger limitation, e.g., since MV treats positive variance and negativity variance similarly, it is not a downside-risk measure. However, most people are downside-risk averse, which results in the solution obtained by MV being inconsistent with individual's normal psychological reaction to risk some time. Motivated by the suggestion of Geunes and Pardalos (2003) that conditional value at risk (CVaR), widely used in portfolio investment recently, might be exploited in solving supply chain planning problems, we attempt to apply CVaR to investigate the ordering problem.

CVaR develops from value at risk (VaR). By definition, with respect to a specified probability level β, the β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α, and β-CVaR is the conditional expectation of losses above that amount α. Financial analysts and government agencies have long relied on VaR in measuring the risk associated with an investment portfolio. However, in a scenario-analysis framework, Rockafellar and Uryasev (2002) showed that the problem of minimizing the CVaR of a portfolio can be formulated as a linear program and that CVaR provides a more consistent and mathematically well-behaved risk measure than does the more commonly used VaR. Moreover, constraints on CVaR can also be stated as linear constraints, leading to methods for minimizing or constraining an effective risk measure without substantially increasing modeling complexity. CVaR method requires random variable in a problem satisfying time serial character such as of the stock price in portfolio investment. By the square, demand, random variable in the MPNP, possesses time series character. Therefore we can, in a sense, view each product as an investment instrument and a multi-product-ordering quantity problem as a "product-portfolio" problem. The number of products that a retailer orders provides a measure of the level of diversification of the product portfolio, and greater diversification implies less exposure to risk. Hence, CVaR can be used as a risk measure tool of the expected inventory shortages in excess of some fixed stock value in our optimal-order quantity model.

The purpose of this paper is to develop a risk decision model of the MPNP to provide a tail-fit optimal decision-making policy for the decision maker. Additionally, our paper offers a promising direction for future research at the intersection of financial engineering and supply chain management by incorporating CVaR measures in supply chain inventory optimization.

This paper is organized as follows. After this introduction, in Section 2, pertinent literature is reviewed. Section 3 describes CVaR simply. Section 4 formulates the return-CVaR model of the multi-product-ordering problem. Section 5 shows a numerical example to illustrate the application of this model and makes a comparison analysis between the classical model and our model. Section 6 presents the conclusions of this paper.

2. Brief literature review

Since the introduction of the multi-product newsvendor model by Hadley and Whitin (1963), many researchers have extended the model to two or more ex ante linear constraints, such as budget or volume constraints. Most of them researched on problem-solving approach to this problem with special demand distribution or general demand distribution, not considering risk preference.

For example, Nahmias and Schmidt (1984) provide four heuristics for solving the single-constraint MPNP under normally distributed demand. Ben-Daya and Raouf (1993) solve the MPNP for the uniform demand case under two linear constraints using Lagrange multipliers. Lau and Lau (1995, 1996) develop algorithms to solve the problem for general demand distributions. They provide heuristics for an efficient search across Lagrange multipliers and discuss managerial insights into the problem based on these multipliers. Because the solution relies on Lagrange multipliers, the method becomes cumbersome with more than one constraint.

Erlebacher (2000) develops both optimal and heuristic solutions for the MPNP with one capacity constraint. He begins by proving the optimality of the order quantities for two special cases: the first case is when the cost structure is the same for all the considered items, and the second case is concerned with uniform probability density function of demand distribution for all items. Then, he proceeds by developing heuristics for a few general probability distribution functions. These heuristics are based on the results of the aforementioned cases.

In a sequel of articles, Abdel-Malek et al. (2004), Abdel-Maleka and Montanari (2005a, b), and Abdel-Malek and Areeratchakul (2007) introduce several approaches to solve the constrained MPNP. In their publication, they present exact, approximate and generic solution methods for the model with budget constraint. In both the approximate and the generic (iterative), they showed how to determine the resulting error level at each step allowing the user to proceed to the desired level of accuracy (see Abdel-Malek et al., 2004; Abdel-Maleka and Montanari, 2005a). Then they developed a methodology to examine the dual of the solution space of the MPNP with two constraints and proposed an approach to obtain the optimum batch size of each product. The approach is based on utilizing the Lagrangian Multipliers, Leibniz Rule, Kuhn–Tucker conditions, and when necessary it engages iterative techniques to obtain the optimum or near-optimum solution values. Among the important features of the developed approach is its applicability to general probability distribution functions of products' demands (see Abdel-Maleka and Montanari, 2005b). In addition, they propose a quadratic programming approach for solving the MPNP with side constraints. Among its salient features are the facts that it utilizes familiar packages to solve the problem such as Excel Solver and Lingo; it can accommodate lower bounds of products' demands that are larger than zero, and it facilitates the performance of sensitivity analysis tasks (see Abdel-Malek and Areeratchakul, 2007).

Niederhoff (2007) provides an approximating programming technique to solve the constrained MPNP for any demand distribution by taking advantage of the separable nature of the problem. Sensitivity analysis of the linear program provides managerial insight into the effects of parameters of the problem on the optimal solution and future decisions.

The studies mentioned have made significant contribution to the MPNP. However, it as noted in Section 1 studies incorporating risk control or decision maker's preference in the MPNP do not exist. To complement the existing literature, we, from risk perspective, analyze the constrained MPNP and provide a return-CVaR model for stock makers, using CVaR techniques.

3. Simple description of conditional value at risk

Let $L(\mathbf{x}, \mathbf{y})$ be the loss associated with the decision vector $\mathbf{x}$ (to be chosen from a certain subset $\mathbf{X}$ of $\Re^n$) and the random vector $\mathbf{y}$ (in $\Re^n$). (We use boldface type for vectors to distinguish them from scalars.) Here the vectors $\mathbf{x}$ can be interpreted as representing a product portfolio, and $\mathbf{X}$ can be interpreted as the set of available portfolios (subject to various constraints). The vector $\mathbf{y}$ stands for uncertainties, (e.g. those in market demand of the paper), which can affect the loss. Of course, the loss might be negative and thus, in effect, constitute a gain.

For each $\mathbf{x}$, loss $L(\mathbf{x}, \mathbf{y})$ is a random variable having a distribution in $\Re$ induced by that of $\mathbf{y}$. For convenience, the underlying probability distribution of $\mathbf{y}$ in $\Re$ will be assumed to have density denoted by $p(\mathbf{y})$.

The probability of $L(\mathbf{x}, \mathbf{y})$ not exceeding a threshold α is the given by

$$\Psi(\mathbf{x},\alpha) = \int_{L(\mathbf{x},\mathbf{y})\leqslant\alpha} p(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

As a function of α for fixed $\mathbf{x}$, $\Psi(\mathbf{x},\alpha)$ is the cumulative distribution function for the loss associated with $\mathbf{x}$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

The β-VaR and β-CVaR values for the loss random variable associated with $\mathbf{x}$ and any specified probability level β in $(0, 1)$ will be denoted by $\alpha_\beta(\mathbf{x})$ and $\phi_\beta(\mathbf{x})$, respectively. They are given by

$$\alpha_\beta(\mathbf{x}) = \min\{\alpha \in \Re : \Psi(\mathbf{x},\alpha) \geqslant \beta\} \tag{1}$$

and

$$\begin{aligned}\phi_\beta(\mathbf{x}) &= E[L(\mathbf{x},\mathbf{y})|L(\mathbf{x},\mathbf{y}) \geqslant \alpha_\beta(\mathbf{x})]\\ &= \alpha_\beta(\mathbf{x}) + E[L(\mathbf{x},\mathbf{y}) - a_\beta(\mathbf{x})|L(\mathbf{x},\mathbf{y}) \geqslant \alpha_\beta(\mathbf{x})]\\ &= \frac{1}{1-\beta}\int_{L(\mathbf{x},\mathbf{y})\geqslant a_\beta(\mathbf{x})} L(\mathbf{x},\mathbf{y})p(\mathbf{y})\,\mathrm{d}\mathbf{y}.\end{aligned} \tag{2}$$

It is difficult to handle CVaR directly because of the VaR function $\alpha_\beta(\mathbf{x})$ involved in its definition, unless we have an analytical representation for VaR. Therefore, Rockafellar and Uryasev (2000) defined a much simpler function

$$F_\beta(\mathbf{x},\alpha) = \alpha + \frac{1}{1-\beta}\int_{L(\mathbf{x},\mathbf{y})\geqslant\alpha} [L(\mathbf{x},\mathbf{y}) - \alpha]^+ p(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{3}$$

where $[t]^+ = t$ when $t > 0$ but $[t]^+ = 0$ when $t \leqslant 0$. Then we have

$$\phi_\beta(\mathbf{x}) = \min_\alpha F_\beta(\mathbf{x},\alpha),$$

which shows that we can use $\min_\alpha F_\beta(\mathbf{x},\alpha)$ for representing CVaR. See details in Rockafellar and Uryasev (2000).

A typical approach in risk management is to estimate and control VaR with a specified confidence level. The problem of controlling VaR can be formalized as a mathematical programming problem with VaR constraints. However, such a problem is very difficult to solve using formal optimization methods because VaR is non-convex w.r.t $\mathbf{x}$ and has many local minima. Uryasev (2000) showed that in contrast to VaR constraints, CVaR constraints can be easily handled using formal optimization approaches and furthermore constraining CVaR also restricts VaR because CVaR $\geqslant$ VaR. Therefore Uryasev (2000) proposed using more conservative CVaR constraints replace VaR constraints in the optimal problems. CVaR constraints can be stated as $F_\beta(\mathbf{x},\alpha) \leqslant C_\beta$, where C_β is some constant constraining CVaR at some confidential levels. Decision makers can also set multi-CVaR constraints at different confidential levels according to real requirements. If a risk constraints is active, i.e., $F_\beta(\mathbf{x}^*,\alpha^*) = C_\beta$, then the optimal value α^* equals to β-VaR. A detail description of CVaR can be found in Rockafellar and Uryasev (2000).

4. Model formulations

4.1. Problem description

Consider a scenario of a distribution center (DC) procuring n products from suppliers. In the formulation of the model the following assumptions are made:

- Decision maker is downside-risk averse, not risk neutral.
- All the market demands of these n products are stochastic.
- The products should be sold for some specified fixed time after which they have few residual values.
- Procuring prices and selling prices are decided by suppliers.
- Budget is limited.
- Ordering quantity has upper or lower bounds.

Since the demand is uncertain, the DC that is risk averse has to incorporate the future sale risk into decision-making framework. Accordingly, a problem that considers what the optimal procuring amounts of n products should be, and, which can satisfy the return maximization considering the risk preference and a set of constraints is proposed.

The notation for the mathematical model is as follows:

Index set

i product families procured form suppliers, $i \in \{1, \ldots, n\}$,

Variables

x_i ordering quantity of product i; decision variable,

y_i market demand of product i; random variable.

Input parameters

c_i procuring cost (including procuring price and ordering cost) of product i,
p_i sale price of product i,
q_i the maximal supply quantity of product i,
h_i unit holding cost for product i,
s_i cost of revenue loss per unit of product i,
g_i the minimal ordering quantity of product i,
w budget
$f_i(y_i)$ probability density function of demand for product i

Additional notation is introduced as needed.

4.2. The classical MPNP model

Let us review the classical MPNP before giving our model for comparing the results between the classical MPMP and our model. There are two familiar model formulations for MPNP also they use different object functions:

$$\text{Minimize} \quad \text{EC} = \sum_{i=1}^{n}\left[c_i x_i + h_i \int_0^{x_i} (x_i - y_i) f_i(y_i)\,\mathrm{d}y_i + s_i \int_{x_i}^{\infty} (y_i - x_i) f_i(y_i)\,\mathrm{d}y_i\right], \tag{4}$$

or

$$\text{Maximize} \quad \text{EP} = \sum_{i=1}^{n}\left[p_i x_i - c_i x_i - (h_i + p_i) \int_0^{x_i} (x_i - y_i) f_i(y_i)\,\mathrm{d}y_i - s_i \int_{x_i}^{\infty} (y_i - x_i) f_i(y_i)\,\mathrm{d}y_i\right], \tag{5}$$

subject to

$$\sum_{m=1}^{M} \varpi_{m_i} x_i \leqslant C_m, \tag{6}$$

where EC and EP express the expected cost and the expected profit, respectively, ϖ_{m_i} is the coefficient of product i of resource m, which could be cost, supply capability, space, etc., and C_m is the available resource for constraint m.

In this paper, constraints (6) can be formulated as

$$\sum_{i=1}^{n} c_i x_i \leqslant w, \tag{7}$$

$$x_i \leqslant q_i, \tag{8}$$

$$x_i \geqslant g_i, \tag{9}$$

$$w \in \Re, \quad w \geqslant 0, \tag{10}$$

$$x_i \geqslant 0, \tag{11}$$

$$p_i, c_i, h_i, q_i, g_i \in \Re, \quad p_i, c_i, h_i, q_i, g_i \geqslant 0, \quad i = 1, \ldots, n. \tag{12}$$

Constraint (7) means that the capital for procuring products cannot exceed budget w. Constraints (8) and (9) states that the ordering quantity of a product cannot exceed its upper and lower bounds of supply. Finally, constraints (10), (11) and (12) enforce, respectively, non-negativity, integer and n-dimension restrictions.

Evidently, the classical MPNP considers only linear constraints, and the non-linear constraints, e.g. risk or loss constraint, are excluded. Some researchers, e.g. those mentioned in Section 2, give solving approaches to their two models. One of them, generic iterative method (GIM) proposed by Abdel-Malek et al. (2004) and Abdel-Maleka and Montanari (2005a) is selected here as a method of solving Eqs. (5) and (6). In order to compare the results generated by the classical model and return-CVaR model, let $s_i = 0$. The reason is that the decision maker is assumed downside-risk averse in this paper.

4.3. Return-CVaR MPNP model

4.3.1. Constraints

The model is subject to two groups of constraints, i.e., risk constraints and non-risk constraints.

4.3.1.1. Risk constraints—CVaR constraints. Let $L(\mathbf{x}, \mathbf{y})$ be a loss function of a product portfolio, which is valued linearly, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$. Let

$$L(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n}\left[(c_i + h_i)\max\left(x_i - y_i, 0\right)\right]. \tag{13}$$

$L(\mathbf{x}, \mathbf{y})$ is apparently a piecewise linear function w.r.t. $\mathbf{x}$ and its vector form is

$$L(\mathbf{x}, \mathbf{y}) = \boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x} - \mathbf{y})^{+}, \tag{14}$$

where $\boldsymbol{\lambda} = (\mathbf{c} + \mathbf{h})$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$, and other parameter vectors in this paper have the same expression forms.

According to formula (3) in Section 3, CVaR can be expressed as

$$F_\beta(\mathbf{x},\alpha)=\alpha+\frac{1}{1-\beta}\int_{\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x}-\mathbf{y})^+\geqslant\alpha}\left[\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x}-\mathbf{y})^+-\alpha\right]^+f(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{15}$$

Let us consider the case in which an analytical repression of the density function $f(\mathbf{y})$ is not available, but we have J scenarios, $\mathbf{y}^j$, $\mathbf{y}^j=(y_1^j,\ldots,y_n^j)$, $j=1,\ldots,J$, sampled from density $f(\mathbf{y})$. For example, we may have historical observations of demands for products, or we may use Monte Carlo simulations to obtain demands of products. In this case, the function of $F_\beta(\mathbf{x},\alpha)$ can be approximately calculated as follows:

$$\widetilde{F}_\beta(\mathbf{x},a)=\alpha+v\sum_{j=1}^{J}\left[\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x}-\mathbf{y}^j)^+-\alpha\right]^+,$$

where $v=((1-\beta)J)^{-1}$. Therefore, CVaR constraint can be stated as

$$\widetilde{F}_\beta(\mathbf{x},a)=\alpha+v\sum_{j=1}^{J}\left[\boldsymbol{\lambda}^{T}\left(\mathbf{x}-\mathbf{y}^j\right)^+-\alpha\right]^+\leqslant C_\beta. \tag{16}$$

Indeed, after replacing the terms $[\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x}-\mathbf{y}^j)^+-\alpha]^+$ in $\tilde{F}_\beta(\mathbf{x},a)$ by auxiliary real variables z_j for $j=1,\ldots,J$, and imposing constraints

$$z_j\geqslant\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{x}-\mathbf{y}^j)^+-\alpha, \tag{17}$$

$$z_j\geqslant 0,\quad z_j\in\Re, \tag{18}$$

we can use constraints (17) and (18) and

$$\alpha+v\sum_{j=1}^{J}z_j\leqslant C_\beta\quad\text{and}\quad j=1,\ldots,J \tag{19}$$

to replace constraint (16). Further, the term $(\mathbf{x}-\mathbf{y}^j)^+$ in constraint (17) can also be replaced by auxiliary real vector variables $\mathbf{e}^j=(e_1^j,\ldots,e_n^j)^{\mathrm{T}}$ for $j=1,\ldots,J$, when imposing constraints

$$\mathbf{x}-\mathbf{y}^j\leqslant\mathbf{e}^j,\quad \mathbf{e}^j\geqslant 0. \tag{20}$$

Constraint (17) can be then expressed as

$$z_j\geqslant\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{e}^j-\alpha,\quad j=1,\ldots,J. \tag{21}$$

Thus, in terms of above analyzed, we show that the CVaR constraint in optimization problems can be approximated by a set of linear constraints. If constraint (16) is active, then the optimal value α^* equals VaR (similar inducing process is seen in Rockafellar and Uryasev, 2000).

4.3.1.2. Non-risk constraints. Non-risk constraints in return-CVaR MPNP model are given by (7)–(12).

4.3.2. Objective function

Object function is given by (5), where $s_i=0$. Its vector expression is

$$EP(\mathbf{x})=[\mathbf{p}-\mathbf{c}]^{\mathrm{T}}\mathbf{x}-(\mathbf{p}+\mathbf{h})^{\mathrm{T}}\int_0^x(\mathbf{x}-\mathbf{y})f(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{22}$$

As we dealt with integral in Eq. (15), Eq. (22) can be computed approximately by

$$\tilde{E}P(\mathbf{x})=[\mathbf{p}-\mathbf{c}]^{\mathrm{T}}\mathbf{x}-\frac{(\mathbf{p}+\mathbf{h})^{\mathrm{T}}}{J}\sum(\mathbf{x}-\mathbf{y}^j)^+. \tag{23}$$

Similar to transforming $(\mathbf{x}-\mathbf{y}^j)^+$ into a series of linear equations in Section 4.3.1.1, Eq. (23) is equivalent to maximizing the linear expression

$$\tilde{E}P(\mathbf{x})=[\mathbf{p}-\mathbf{c}]^{\mathrm{T}}\mathbf{x}-\frac{(\mathbf{p}+\mathbf{h})^{\mathrm{T}}}{J}\sum\mathbf{u}^j,$$

subject to the linear constraints

$$\mathbf{x}-\mathbf{y}^j\leqslant\mathbf{e}^j,\quad \mathbf{e}^j\geqslant\mathbf{0}\quad\text{and}\quad\mathbf{e}^j\in\Re^n.$$

4.3.3. Risk-decision model

As a result, according to the analysis above, we obtain the final risk decision model:

$$\underset{\mathbf{x}\in R^n,\alpha\in R}{\mathrm{Min}}\ -\tilde{E}P(\mathbf{x})=-[\mathbf{p}-\mathbf{c}]^{\mathrm{T}}\mathbf{x}+\frac{(\mathbf{p}+\mathbf{h})^{\mathrm{T}}}{J}\sum\mathbf{u}^j$$

$$\text{s.t.}\begin{cases} z_j\geqslant\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{e}^j-\alpha\\ \alpha+v\sum_{j=1}^{J}z_j\leqslant C_\beta\\ \mathbf{x}-\mathbf{y}^j\leqslant\mathbf{e}^j\\ \mathbf{c}^{\mathrm{T}}\mathbf{x}\leqslant w\\ \mathbf{x}\leqslant\mathbf{q}\\ \mathbf{x}\geqslant\mathbf{g}\\ z_j,w\geqslant 0\\ \mathbf{e}^j,\mathbf{x}\geqslant 0\\ C_\beta,w\in\Re\\ \mathbf{x},\mathbf{e}^j\in\Re^n\\ \mathbf{p},\mathbf{c},\mathbf{h},\mathbf{q},\mathbf{g},\boldsymbol{\lambda}\in\Re^n\\ \mathbf{p},\mathbf{c},\mathbf{h},\mathbf{q},\mathbf{g},\boldsymbol{\lambda}\geqslant 0\\ j=1,\ldots,J \end{cases}.$$

Evidently, this risk-decision model is a linear programming model and we can utilize LP solver to solve it easily, even if it has many variables.

5. Case analysis

5.1. General characters of return-risk decision

In order to understand the model that, we proposed at a deep level, we give a case analysis. From intuition, one expect, risk decision model often has the following characters:

(1) Expected return of the risk-averse decision maker is smaller than that of the risk-preferring decision maker.
(2) The ordering quantity of the product with lower demand volatility and the higher return rate is closer to the expected market demand than the product with the higher demand volatility and the lower return rate.
(3) The degree of preference of a product changes with change in risk tolerance of individual. When the tolerance is low the, individual favors the product with the lower volatility; once it increases, individual favors the product with the higher return rate, bit by bit.
(4) Provided the several products have the same return rates, the product with lower volatility more preferred; inversely, the product with the higher return rate will obtain more preference if the volatilities of the products are the same.
(5) The more the number of products that a retailer orders, the less risk exposure the product portfolio brings.

According to these five characters we analyze the solution obtained from the model in detail and help readers to understand the trade-off between risk and return based on different values of risk constraints and different budgets.

5.2. Case description

A newsvendor in S city distributes 10 types of weeklies. Assume the market demands of weeklies are random variables, inter-independent, and follow normal distributions. Other assumptions are the same as Section 4.1. Let $h_i = 0$, $d_i = 0$, $r_i = 0$, $g_i = 0$, $\beta = 0.05$.

Table 1 gives the procuring prices, the sales prices and the maximal supply quantities of 10 weeklies in specified term, $t_0 \sim t_0 + \tau$, where t_0 is the initial moment. And the means and the standard deviations of product market demands, denoted by mu_i and σ_I, respectively, also are given in Table 1. Scenarios samples $\mathbf{y}^j$ were generated by Mento Carlo simulation approach. The return-CVaR model was computed using Matlab version 7.0 as a solver.

5.3. Computational results and discussion

Calculate ordering quantities of various weeklies given the different risk tolerances (C_β); 100, 300, 500, 1000, 1500 or 2000 and the different budgets (w); 4000, 6000, 8000, 12,000, 14,000, 16,000, 18,000, 20,000, 22,000, 24,000 or 26,000. The results are illustrated in Tables 2–8.

To analyze the results conveniently, we use the matrix diagram of return rate b_i and volatility δ_i (represented by the value of coefficient variance, C.V.) as a benchmark. Return rates and volatilities of ten weeklies are calculated by the formula $b_i = (p_i - c_i)/c_i$ and $\delta_i = \sigma_i/mu_i$, respectively. Return rates and volatilities matrix diagram (see Fig. 1) give an intuitive idea of return and risk of weeklies.

5.3.1. Results analysis on character 1

Fig. 2 shows the expected returns based on different risk tolerances and different ordering budgets. The outline CVaR-Cnst 100 expresses $C_\beta = 100$ and the others have the similar meaning. We find that the expected return of the risk-averse

Table 1
Basic information about weeklies during $t_0 \sim t_0 + \tau$

Weekly, i	c_i	p_i	q_i	mu_i	σ_i	Weekly, i	c_i	p_i	q_i	mu_i	σ_i
1	1.4	2	2200	2500	1200	6	2.8	4	1300	1000	100
2	1.1	1.5	2000	1800	250	7	4	7	580	600	500
3	1.5	2.5	2800	3000	2200	8	6	8	600	650	150
4	1.4	2	2500	2000	400	9	4.5	8	480	500	115
5	1.1	1.8	1900	1800	620	10	4	5.8	650	600	50

Table 2
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 100$

Budget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	137	0	943	0	126	0	359	161	2717
6000	0	0	137	270	934	144	126	0	359	466	3600
8000	0	0	137	903	934	541	126	0	359	466	4457
10,000	0	678	137	1162	952	794	112	0	359	521	5257
12,000	109	12,881	160	1236	952	810	80	184	359	527	5933
14,000	109	14,02	137	1348	952	845	13	352	349	533	6177
16,000	109	1402	137	1334	952	852	13	352	349	533	6177

Table 3
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 300$

Budget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	112	0	400	0	346	0	446	0	2991
6000	0	0	851	0	1008	0	265	0	387	203	4072
8000	0	0	851	315	1015	131	263	0	386	502	4952
10,000	0	0	851	946	1015	530	263	0	386	502	5809
12,000	109	605	849	1207	1008	807	251	0	386	527	6607
14,000	109	1281	840	1276	1035	817	235	194	385	532	7278
16,000	109	1422	790	1436	1035	863	189	358	368	534	7580
18,000	109	1422	790	1436	1035	863	189	358	368	534	7580

Table 4
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 500$

Budget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	0	0	0	0	460	0	480	0	3060
6000	0	0	901	0	945	0	394	0	451	0	4325
8000	0	0	1072	0	1201	0	346	0	437	428	5256
10,000	0	0	1072	568	1201	326	345	0	438	502	6118
12,000	0	0	1072	1112	1201	769	345	0	438	502	6975
14,000	109	1241	1072	1277	1202	835	333	0	435	534	7726
16,000	109	1365	1069	1436	1174	863	312	282	432	534	8374
18,000	109	1449	1068	1491	1167	892	281	408	401	542	8535
20,000	109	1449	1068	1491	1167	892	281	408	401	542	8535

decision maker (with lower C_β) is always less than that of the risk preferred (with higher C_β). Further, the expected return on a product-portfolio increases with ordering capital improvement, but it cannot increase endlessly, i.e., it has a threshold. For example, when $C_\beta = 100$, $w \geqslant 14{,}000$, the expected return is 6177 and the ordering quantities of different products are nearly fixed (see Table 2). The result is coherent with the practical return-risk decision-making process of individuals. Because once one has a certain risk tolerance, his expected return must have a threshold.

5.3.2. Results analysis on character 2

Let $\xi_i = x_i^*/mu_i$, $i = 1, \ldots, 10$, where x_i^* is the optimal ordering quantity of weekly i. The aim of introducing $\xi = (\xi_1, \ldots, \xi_{10})^{\mathrm{T}}$ is to evaluate accurately the favorite degrees of decision maker upon the different weeklies, since the absolute value of the

Table 5
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 1000$

Buget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	0	0	0	0	460	0	480	0	3060
6000	0	0	1013	0	0	0	580	0	480	0	4433
8000	0	0	1558	0	1317	0	513	0	480	0	5700
10,000	0	0	1685	0	1353	0	485	0	471	481	6601
12,000	0	0	1687	622	1353	349	483	0	471	520	7460
14,000	0	0	1629	1615	1353	482	482	0	470	521	8316
16,000	109	1189	1724	1423	1363	457	457	0	466	534	9065
18,000	109	1402	1736	1466	1363	448	448	281	456	542	9723
20,000	109	1505	1587	1631	1363	922	416	449	441	562	9994
22,000	109	1505	1587	1631	1363	922	416	449	441	562	9994

Table 6
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 1500$

Buget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	0	0	0	0	460	0	480	0	3060
6000	0	0	1031	0	0	0	580	0	480	0	4433
8000	0	0	2000	0	472	0	580	0	480	0	5750
10,000	0	0	2000	0	1600	0	580	0	480	190	6882
12,000	0	0	2000	175	1600	54	580	0	480	590	7773
14,000	0	0	2000	813	1600	448	580	0	480	591	8630
16,000	109	0	2000	1382	1600	847	580	0	480	574	9586
18,000	109	1365	2000	1571	1548	903	580	37	480	552	10,214
20,000	109	1441	2000	1598	1471	916	576	358	480	559	10,858
22,000	114	1536	2000	1736	1448	943	492	494	471	572	11,012
24,000	118	1536	2000	1731	1448	943	492	494	471	572	11,012

Table 7
Expected returns and ordering quantities of the optimal weekly portfolio for various scenarios with different ordering budgets at $C_\beta = 2000$

Budget	Optimal ordering quantity										Return
	1	2	3	4	5	6	7	8	9	10	
4000	0	0	0	0	0	0	460	0	480	0	3060
6000	0	0	1013	0	0	0	580	0	480	0	4433
8000	0	0	2000	0	472	0	580	0	480	0	5750
10,000	0	0	2000	0	1600	0	580	0	480	190	6882
12,000	25	0	2000	110	1600	57	580	0	480	602	7774
14,000	64	0	2000	752	1600	430	580	0	480	602	8631
16,000	106	0	2000	1374	1600	812	580	0	480	602	9488
18,000	810	0	2000	1803	1600	961	580	0	480	603	10,346
20,000	936	1451	2000	1805	1600	963	580	44	480	592	11,703
22,000	871	1491	2000	1802	1600	961	580	390	480	586	11,729
24,000	642	1590	2000	1782	1572	956	580	517	480	579	11,834
26,000	640	1590	2000	1782	1572	957	580	517	480	579	11,834

Table 8
Expected returns, Expected loss(E-loss) and ordering quantities of optimal weekly portfolios for various scenarios with different ordering budgets solved by the classical MPNP model

Budget	Optimal ordering quantity										Return	E-Loss
	1	2	3	4	5	6	7	8	9	10		
6000	0	0	528	0	1054	0	127	0	397	439	3622	118.2
8000	0	0	651	302	1093	367	149	0	402	491	4531	143.4
10000	0	0	682	928	1102	732	155	0	403	496	5398	151.3
12,000	340	0	933	1280	1179	820	202	0	413	518	6304	234.4
14,000	833	0	1253	1445	1276	861	264	0	427	535	7135	399.1
16,000	708	1302	1155	1403	1247	851	245	188	423	530	7806	344.9
18,000	902	1384	1311	1467	1293	867	276	360	430	537	8510	450.2
20,000	1192	1475	1591	1564	1377	891	332	435	442	548	9160	684.5
22,000	1480	1551	1924	1660	1474	915	401	488	457	560	9679	1037.9
24,000	1757	1618	2285	1753	1579	938	477	532	474	570	10022	1521.4
26,000	2018	1718	2655	1839	1686	9599	555	5704	492	581	10156	2146.3

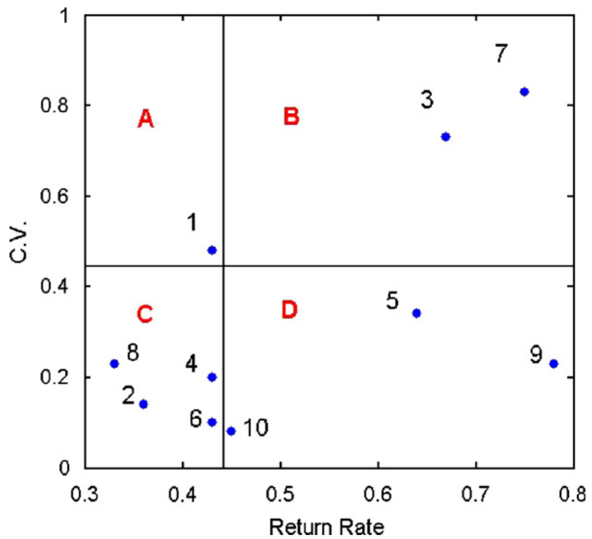

Fig. 1. Return rates and volatilities matrix diagram.

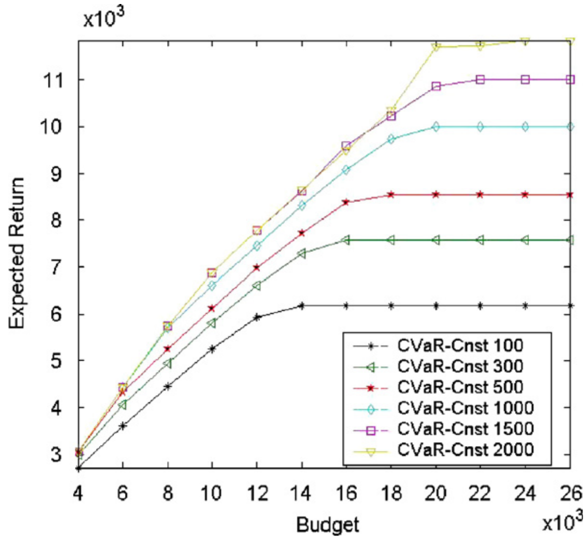

Fig. 2. Expected returns of optimal weekly portfolio for various scenarios with different CVaR constraints and ordering budgets.

ordering quantity as a measure of favorite degree is inaccurate given the upper bound of the ordering quantity and a certain demand capacity. According to the above assumption (Section 4.3), there is $x_i^* \leqslant mu_i$, so $0 \leqslant \xi_i \leqslant 1$. If $\xi_i \to 1$, then the optimal quantity of weekly i is close to its mean of market demand, which also reflects the favorite degree of decision maker.

Fig. 3 illustrates ξ of different weeklies when $w = 12{,}000$. We find weekly 5, 9 and 10 belonging to D in Fig. 1. have higher ξ and the fluctuation range of ξ is lower (see F–H in Fig. 3); weekly 1 locating A in Fig. 1. has lower ξ and at the same time the fluctuation range of ξ is also lower (see E in Fig. 3). All of these show the results accord with character 2. Moreover, in spite of the change of C_β, change rates of the ordering quantities of weeklies 1, 5, 9 and 10 are lower, which accounts for decision maker always favoring the weekly with higher return rate and lower volatility and disliking the weeklies with lower return rate and higher volatility under any risk tolerance.

5.3.3. Results analysis on character 3

Weeklies 3 and 7 locate in B and weeklies 2, 4, 6 and 8 locate in C (see Fig. 1). Therefore, ordering quantities of weeklies 3 and 7 should be lower when

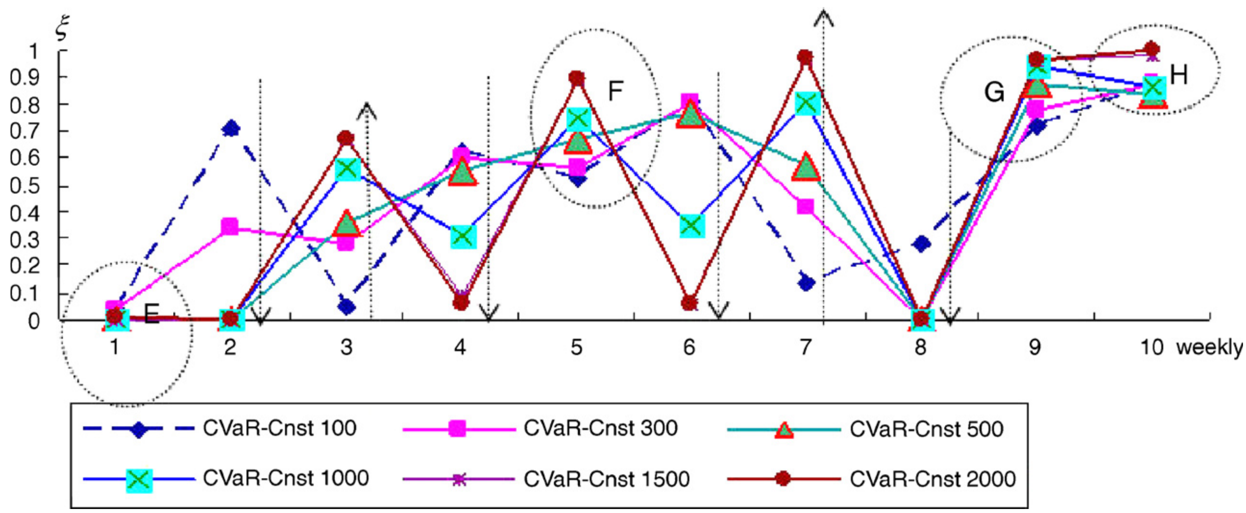

Fig. 3. Rate of ordering quantity and market demand mean for various CVaR constraints for weeklies $i = 1, \ldots, 10$.

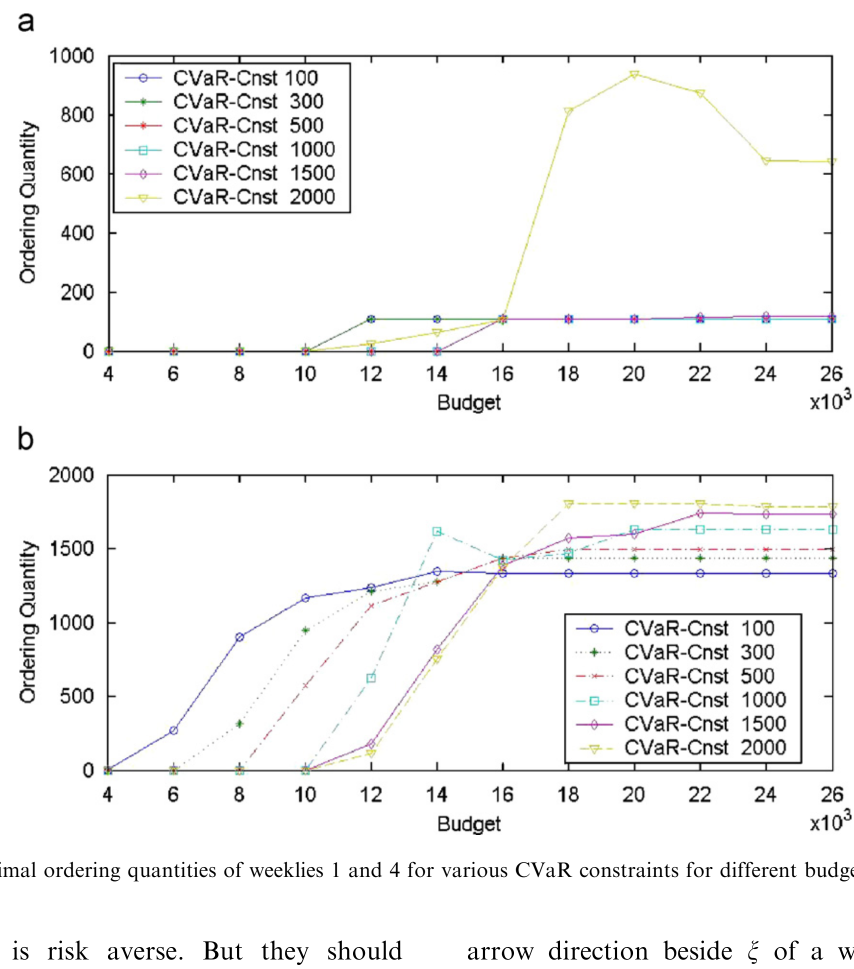

Fig. 4. Optimal ordering quantities of weeklies 1 and 4 for various CVaR constraints for different budget constraints.

decision maker is risk averse. But they should increase following the improvement of the risk tolerance according to character 2, while weeklies 2, 4, 6 and 8 are of opposite type. Let us see Fig. 3. An arrow direction beside ξ of a weekly depicts the change direction of the value of ξ following an increment of risk tolerance, namely, from $C_\beta = 100$ to 2000 in this case. Evidently, the ξ values of weeklies 3 and 7 are increasing and the ξ values of weeklies 2, 4, 6 and 8 are decreasing. For instance, the ξ of weekly 7 (with the second largest return rate and the highest volatility) changes most. In addition, although weekly 8 (with the smallest return rate and the lower volatility) orderes less at $C_\beta = 100$, the ordering quantity decreases to zero along with increase in C_β. The results are coherent with character 3.

5.3.4. *Results analysis on character 4*

Test character 4 exemplified by weeklies 1 and 4 for they have the same return rates, approximately the same mean market demands and different volatilities, i.e., $r_1 = 0.43$, $\delta_1 = 0.48$, $mu_1 = 2500$, $r_4 = 0.43$, $\delta_4 = 0.2$, $mu_4 = 2000$. Fig. 4a illustrates weekly 1 has only less quantity orders on all, but for $C_\beta = 2000$ and $w \geqslant 1600$ scenarios. Weekly 4, according to Fig. 4b, always has some certain ordering quantities at various C_β and w. As a result, a weekly with the lower volatility is preferred on the condition that the return rates are nearly same. Similarly, a weekly with a higher return rate is preferred when volatilities are approximately the same. Therefore, the results satisfy character 4 also.

5.3.5. *Results analysis on character 5*

Denotes by $Z^*_{l,k}$ the optimal product portfolios where $l{:} = C_\beta$, $k{:} = w$. Let budget be 4000 and 8000. Fig. 5 illustrates the relations between the value of CVaR constraint and ordering quantities of different weeklies at $w = 4000$ and 8000. According to Fig. 5, we are informed that $Z^*_{100,4000} = \{3,5,7,9,10\}$, $Z^*_{300,4000} = \{3,5,7,9\}$ and $Z^*_{l\geqslant 500,4000} = \{7,9\}$ at $w = 4000$; $Z^*_{100,8000} = \{3,4,5,6,7,9,10\}$, $Z^*_{500,8000} = \{3,5,7,9,10\}$ and $Z^*_{l\geqslant 1000,8000} = \{3,5,7,9\}$ at $w = 8000$. The number of weeklies in the case of $w = 8000$ is more than for the case $w = 4000$ when C_β is the same, e.g., $C_\beta = 100$ or 500. It shows that

- decision maker prefers more weeklies to decrease risk when C_β is small;
- decision maker favors weekly with higher return rate and number of weeklies is reduced with increase in C_β;
- decision maker with a lager budget possesses the smaller relative risk enduring capability at the same C_β, which leads him to select more products.

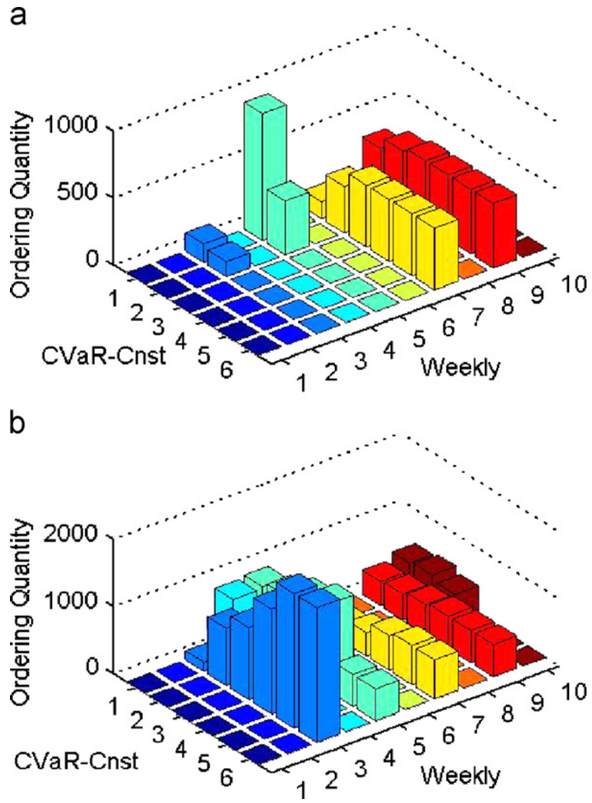

Fig. 5. Relations between values of CVaR constraints and optimal ordering quantities of different weeklies, $w = 4000$ (a) and $w = 8000$ (b).

This is consistent with the principle that the product portfolio decentralizes risk.

5.3.6. *Results comparison between return-CVaR MPNP model and the classical MPNP model*

The method of solving the Classical MPNP Model (formula (5) and (6)) in Section 4.2 is GIM proposed by Abdel-Malek et al. (2004). Abdel-Maleka and Montanari (2005a). The results under different budget constraints are illustrated in Table 8.

Here, we exemplify the case $w = 6000$. Fig. 6 represents 5 weekly portfolios. One is the result of the classical model and the others are the results of return-CVaR model at $C_\beta = 100$, 300, 500 and 1000 (The label 'C-100' means $C_\beta = 100$). We find that the optimal weekly portfolio of the classical model is $\{3,5,7,9,10\}$, which is the same as the one of return-CVaR model at $C_\beta = 300$. However, their ordering quantities are different, which result in different returns. According to Fig. 7, the return of the classical model (i.e., 3622) is closer to the one

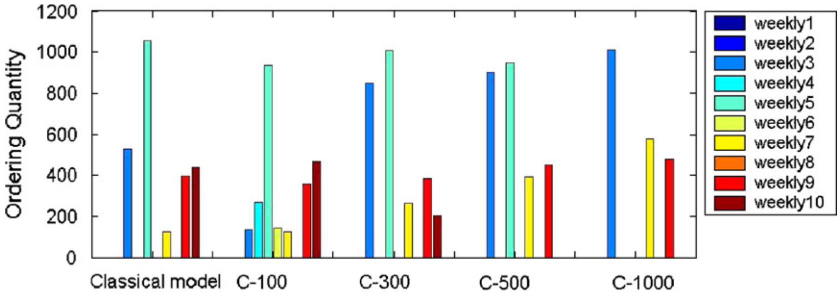

Fig. 6. Results of the classical model and return-CVaR model under various CVaR constraints, $w = 6000$.

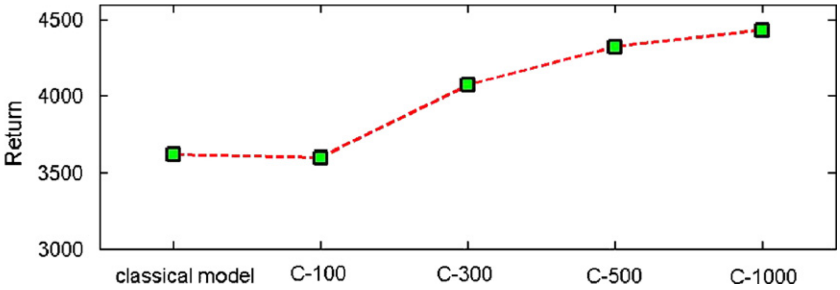

Fig. 7. Return of the classical model and return-CVaR model under various CVaR constraints, $w = 6000$.

(i.e., 3600) at $C_\beta = 100$, whose weekly portfolio is $\{3, 4, 5, 6, 7, 9, 10\}$.

Therefore, by comparing with the classical model, our model's characteristics are summarized as follows:

- our model provides more alternatives than the classical model. The classical model has only one result, while our model has more results based on different risk tolerances, i.e., the value of C_β, at some budget;
- the weekly number of our model is greater than or equal to the one of the classical model when budget and expected return are the same, which is also consistent with human intuition;
- decision maker will select a tailor-fit expected return by adjusting C_β.

Of course, using Monte Carlo technique in the process of solving return-CVaR model will induce that the result is always not the same is each Monte Carlo simulation and the computing time of solving return-CVaR model is longer than Abdel-Malek et al.'s approach to the classical model. However, these limitations cannot slim its practical value.

6. Conclusion

The return-CVaR model presented here provides a tailor-fit optimal decision-making approach for decision maker when facing multi-product ordering quantity problem with stochastic market demand. Since most of previous published papers ignored risk preference of decision maker, the methods and models proposed by those studies cannot aid decision maker to make decision effectively once the corresponding variance is high. The objective of this paper was, therefore, to apply a modeling approach that considers risk preference of decision maker by means of introducing CVaR, a coherent risk measure, to improve the performance of the distribution centers or dealers.

Finally, we examined a simulation case to analyze the solution obtained from our model from the point of view of the five characters of return-risk decision. This helps to understand the trade-off between risk and return based on different values of risk constraints and different budgets. Moreover, by comparing our model with the classical model, we find that our model is flexible and it can reflect decision maker's risk preference. Additionally, our model can be optimized using a linear programming, which allows handling a product portfolio with very large number of products and scenarios. We believe that the optimal ordering policy developed in this paper is applicable in real life.

Future work related to our model will include extending it to consider the ordering policy in the context of multi-stages, which exists widely across enterprises. Another extension of the model relates to the consideration of output arrangement with multi-products. Of course, these problems should be characterized by the stochastic demand and time series.

Acknowledgements

The authors appreciate the efforts and the constructive comments of anonymous reviewers that have helped in the revision process.

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