Appl Math Optim 42:19–33 (2000)
DOI: 10.1007/s002450010003

Applied Mathematics & Optimization

Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework*

X. Y. Zhou and D. Li

Department of Systems Engineering and Engineering Management,
The Chinese University of Hong Kong,
Shatin, Hong Kong
{xyzhou,dli}@se.cuhk.edu.hk

Communicated by M. Nisio

Abstract. This paper is concerned with a continuous-time mean-variance portfolio selection model that is formulated as a bicriteria optimization problem. The objective is to maximize the expected terminal return and minimize the variance of the terminal wealth. By putting weights on the two criteria one obtains a single objective stochastic control problem which is however not in the standard form due to the variance term involved. It is shown that this nonstandard problem can be "embedded" into a class of auxiliary stochastic linear-quadratic (LQ) problems. The stochastic LQ control model proves to be an appropriate and effective framework to study the mean-variance problem in light of the recent development on general stochastic LQ problems with indefinite control weighting matrices. This gives rise to the efficient frontier in a closed form for the original portfolio selection problem.

Key Words. Continuous time, Mean-variance, Portfolio, Efficient frontier, Linear-quadratic control.

AMS Classification. Primary 90A09, Secondary 93E20.

* This research was supported by the RGC Earmarked Grants CUHK 4125/97E, CUHK 4054/98E, and CUHK 4130/97E.

1. Introduction

Portfolio selection is to seek a best allocation of wealth among a basket of securities. The mean-variance approach by Markowitz [18], [19] provides a fundamental basis for portfolio construction in a single period. The most important contribution of this model is that it quantifies the risk by using the variance, which enables investors to seek the highest return after specifying their acceptable risk level. This approach became the foundation of modern finance theory and inspired literally hundreds of extensions and applications. In particular, in the case where the covariance matrix is positive definite and short-selling is allowed, an analytic solution was obtained by Merton [20]. Moreover, Perold [23] developed a general technique to locate the efficient frontier when the covariance matrix is nonnegative definite.

After Markowitz's pioneering work, the mean-variance model was soon extended to multiperiod portfolio selection; see, for example, [21], [25], [12], [7], [10], [6], [11], and [24]. However, to our best knowledge no analytical result, comparable to those in the single-period model, has been reported in the literature on mean-variance efficient frontiers for multiperiod portfolio selections. Researches on multiperiod portfolio selections have been dominated by those of maximizing expected utility functions of the terminal wealth, namely maximizing $E[U(x(T))]$ where U is a utility function. Specifically, the investment situations where U is of a power form, log form, exponential form, or quadratic form have been extensively investigated. The resulting portfolio policies, in these situations, are often shown to be myopic optimal policies. However, when using utility functions of the terminal wealth in multiperiod portfolio selection, besides the difficulty in eliciting utility functions from the investors, tradeoff information between the risk and the expected return is implicit which makes an investment decision much less intuitive. In this sense, Markowitz's mean-variance approach has not been fully utilized in the dynamic, multiperiod setting.

If in a multiperiod model one is to mimic Markowitz's formulation completely, then one should minimize an objective function involving a term $[Ex(T)]^2$ (due to the variance term) or, more generally, a term of the form $U(Ex(T))$ where U is a *nonlinear* utility function. The seemingly harmless difference between $EU[x(T)]$ and $U[Ex(T)]$ actually causes a major difficulty for the latter in view of applying the dynamic programming method. More precisely, for the objective function of the form $EU[x(T)]$, dynamic programming is applicable due to the so-called "smoothing property" $E(E(U[x(T)]|\mathcal{F}_m)|\mathcal{F}_n) = E(U[x(T)]|\mathcal{F}_n)$, where $\{\mathcal{F}_k, k = 1, 2, \ldots\}$ is the underlying filtration and $n \leq m$. However, no analogous relation, such as $E(U[Ex(T)|\mathcal{F}_m]$ $|\mathcal{F}_n) = U[Ex(T)|\mathcal{F}_n]$, is available for the case when the objective function involves a nonlinear $U[Ex(T)]$.

The mean-variance hedging problem has been studied in [9], [5], and [26] where an optimal dynamic strategy is sought to hedge contingent claims in an imperfect market with two assets. Optimal hedging policies [9], [5], [26] were obtained primarily based on the so-called projection theorem. In particular, in [5], the result was derived under the assumption that all the coefficients (interest rate, volatility rate, etc.) are deterministic, time-invariant constants. On the other hand, in [27] the variance minimization problem was solved by introducing an equality constraint on the expected return and then incorporating this constraint into the objective function using the Lagrangian approach. In general, the success of the dual search associated with the Lagrangian method depends on some concavity of the underlying problem, which is sometimes difficult to verify in the context of portfolio selection.

The purpose of this paper is to introduce the stochastic linear-quadratic (LQ) control as a general framework to study the mean-variance optimization/hedging problem, taking advantage of the general stochastic LQ theory developed recently [3], [4]. Within this framework we seek an analytical optimal portfolio policy and an explicit expression of the efficient frontier for a continuous-time mean-variance portfolio selection problem. To achieve this, we have first to handle the difficulty (which was discussed earlier) arising from the term $[Ex(T)]^2$ for the problem with time-varying coefficients. The idea is to embed the original (not readily solvable) problem into a tractable auxiliary problem, following a similar embedding technique introduced by Li and Ng [15] for the multiperiod model, then to show that this auxiliary problem actually is a stochastic optimal LQ problem and can be solved explicitly by LQ theory. The optimal solution to the original problem can then be located via the solution to the auxiliary problem. One of the promising features of this approach is that it bridges portfolio selection problems and standard stochastic control models (the theory of which is now very rich; see [8] and [29]) and provides a uniform and general framework to deal with more complicated situations. For example, a portfolio selection problem with *random* coefficients is studied in [16] based on LQ theory and backward stochastic differential equations. A Black–Scholes model with mean-variance hedging is investigated in [14] within the LQ framework. Moreover, we expect to use the embedding technique to handle more general problems such as those with *nonconcave* functions/*nonconvex* constraints.

The organization of the rest of the paper is as follows. In Section 2 we formulate a continuous-time mean-variance portfolio selection model and give some preliminaries. Section 3 is devoted to the construction of an auxiliary stochastic LQ problem. In Section 4 the solution to a general stochastic LQ problem is presented. Section 5 then applies the general results obtained in Section 4 to solve the auxiliary problem. In Section 6 an efficient frontier in a closed form is obtained for the original mean-variance model. Finally, Section 7 concludes the paper.

2. Problem Formulation

Throughout this paper $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \ldots, W^m(t))'$ is defined. We denote by $L^2_{\mathcal{F}}(0, T; R^m)$ the set of all R^m-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |f(t)|^2\, dt < +\infty$.

Notation. We make the following additional notation:

M' is the transpose of any vector or matrix M;
M^j is the jth entry of any vector M;
$|M| = \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$;
S^n is the space of all $n \times n$ symmetric matrices;
S^n_+ is the subspace of all nonnegative definite matrices of S^n;

$\hat{S}^n_+$ is the subspace of all positive definite matrices of S^n;
$C([0,T];X)$ is the Banach space of X-valued continuous functions on $[0,T]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space X;
$L^2(0,T;X)$ is the Hilbert space of X-valued integrable functions on $[0,T]$ endowed with the norm $(\int_0^T \|f(t)\|^2_X dt)^{1/2}$ for a given Hilbert space X.

Suppose there is a market in which $m+1$ assets (or securities) are traded continuously. One of the assets is the *bond* whose price process $P_0(t)$ is subject to the following (deterministic) ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t)\,dt, & t \in [0,T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{2.1}$$

where $r(t) > 0$ is the *interest rate* (of the bond). The other m assets are *stocks* whose price processes $P_1(t), \ldots, P_m(t)$ satisfy the following stochastic differential equation:

$$\begin{cases} dP_i(t) = P_i(t)\left\{b_i(t)\,dt + \sum_{j=1}^m \sigma_{ij}(t)\,dW^j(t)\right\}, & t \in [0,T], \\ P_i(0) = p_i > 0, \end{cases} \tag{2.2}$$

where $b_i(t) > 0$ is the *appreciation rate*, and $\sigma_i(t) \equiv (\sigma_{i1}(t), \ldots, \sigma_{im}(t))\colon [0,T] \to R^m$ is the *volatility* or the *dispersion* of the stocks. Define the *covariance matrix*

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} \equiv (\sigma_{ij}(t))_{m\times m}. \tag{2.3}$$

The basic assumption throughout this paper is

$$\sigma(t)\sigma(t)' \geq \delta I, \qquad \forall t \in [0,T], \tag{2.4}$$

for some $\delta > 0$. This is the so-called *nondegeneracy* condition. We also assume that all the functions are measurable and uniformly bounded in t.

Consider an investor whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Suppose he/she decides to hold $N_i(t)$ shares of ith asset ($i = 0, 1, \ldots, m$) at time t. Then

$$x(t) = \sum_{i=0}^m N_i(t)P_i(t), \qquad t \geq 0. \tag{2.5}$$

Assume that the trading of shares takes place continuously and transaction cost and consumptions are not considered. Then one has

$$\begin{cases} dx(t) = \sum_{i=0}^m N_i(t)\,dP_i(t) \\ \qquad\quad = \left\{r(t)N_0(t)P_0(t) + \sum_{i=1}^m b_i(t)N_i(t)P_i(t)\right\}\,dt \\ \qquad\qquad + \sum_{i=1}^m N_i(t)P_i(t)\sum_{j=1}^m \sigma_{ij}(t)\,dW_j(t) \\ \qquad\quad = \left\{r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]\,u_i(t)\right\}\,dt \\ \qquad\qquad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t)\,dW^j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{2.6}$$

where

$$u_i(t) \equiv N_i(t)P_i(t), \qquad i = 0, 1, 2, \ldots, m, \tag{2.7}$$

denotes the total market value of the investor's wealth in the ith bond/stock. We call $u(t) = (u_1(t), \ldots, u_m(t))'$ a *portfolio* of the investor. The objective of the investor is to maximize the mean terminal wealth, $Ex(T)$, and at the same time to minimize the variance of the terminal wealth

$$\mathrm{Var}\, x(T) \equiv E\,[x(T) - Ex(T)]^2 = Ex(T)^2 - [Ex(T)]^2\,. \tag{2.8}$$

This is a *multi-objective optimization* problem with two criteria in conflict.

Definition 2.1. A portfolio $u(\cdot)$ is said to be *admissible* if $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$.

Definition 2.2. The mean-variance portfolio optimization problem is denoted as

$$\begin{array}{ll} \text{Minimize} & (J_1(u(\cdot)), J_2(u(\cdot))) \equiv (-Ex(T), \mathrm{Var}\, x(T)) \\ \quad \text{subject to} & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6).} \end{cases} \end{array} \tag{2.9}$$

Moreover, an admissible portfolio $\bar{u}(\cdot)$ is called an *efficient portfolio* of the problem if there exists no admissible portfolio $u(\cdot)$ such that

$$J_1(u(\cdot)) \le J_1(\bar{u}(\cdot)), \qquad J_2(u(\cdot)) \le J_2(\bar{u}(\cdot)), \tag{2.10}$$

and at least one of the inequalities holds *strictly*. In this case, we call $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in R^2$ an *efficient point*. The set of all efficient points is called the *efficient frontier*.

In other words, an efficient portfolio is one where there exists no other portfolio better than it with respect to both the mean and variance criteria. The problem then is to identify the efficient portfolios along with the efficient frontier. By standard multi-objective optimization theory, an efficient portfolio can be found by solving a single-objective optimization problem where the objective is a weighted average of the two original criteria under certain convexity conditions (see, e.g., [30]), which are satisfied in the present case. The efficient frontier can then be generated by varying the weights. Therefore, the original problem can be solved via the following optimal control problem:

$$\begin{array}{ll} \text{Minimize} & J_1(u(\cdot)) + \mu J_2(u(\cdot)) \equiv -Ex(T) + \mu\, \mathrm{Var}\, x(T) \\ \quad \text{subject to} & \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \\ (x(\cdot), u(\cdot)) \text{ satisfy (2.6),} \end{cases} \end{array} \tag{2.11}$$

where the parameter (representing the weight) $\mu > 0$. Denote the above problem by $P(\mu)$. Define

$$\Pi_{P(\mu)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } P(\mu)\}\,. \tag{2.12}$$

3. Construction of an Auxiliary Problem

Note that problem $P(\mu)$ is *not* a standard stochastic optimal control problem and is hard to solve directly due to the term $[EX(T)]^2$ in its cost function, which is nonseparable in the sense of dynamic programming (see the discussion in the Introduction).

We now propose to embed problem $P(\mu)$ into a tractable auxiliary problem that turns out to be a stochastic LQ problem. To do this, set the following problem:

$$\begin{aligned} &\text{Minimize} \quad J(u(\cdot);\mu,\lambda) \equiv E\{\mu x(T)^2 - \lambda x(T)\} \\ &\quad \text{subject to} \quad \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0,T;R^m), \\ (x(\cdot),u(\cdot)) \text{ satisfy (2.6)}, \end{cases} \end{aligned} \tag{3.1}$$

where the parameters $\mu > 0$ and $-\infty < \lambda < +\infty$. We call the above, problem $A(\mu,\lambda)$. Define

$$\Pi_{A(\mu,\lambda)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control of } A(\mu,\lambda)\}. \tag{3.2}$$

The following result shows the relationship between problems $P(\mu)$ and $A(\mu,\lambda)$.

Theorem 3.1. *For any $\mu > 0$, one has*

$$\Pi_{P(\mu)} \subseteq \bigcup_{-\infty<\lambda<+\infty} \Pi_{A(\mu,\lambda)}. \tag{3.3}$$

Moreover, if $\bar{u}(\cdot) \in \Pi_{P(\mu)}$, then $\bar{u}(\cdot) \in \Pi_{A(\mu,\bar{\lambda})}$ with $\bar{\lambda} = 1 + 2\mu E\bar{x}(T)$, where $\bar{x}(\cdot)$ is the corresponding wealth trajectory.

Proof. We only need to prove the second assertion as the first one is a direct consequence of the second. Let $\bar{u}(\cdot) \in \Pi_{P(\mu)}$. If $\bar{u}(\cdot) \notin \Pi_{A(\mu,\bar{\lambda})}$, then there exist $u(\cdot)$ and the corresponding $x(\cdot)$ such that

$$\mu(Ex(T)^2 - E\bar{x}(T)^2) - \bar{\lambda}(Ex(T) - E\bar{x}(T)) < 0. \tag{3.4}$$

Set a function

$$\pi(x,y) = \mu x - \mu y^2 - y. \tag{3.5}$$

It is a concave function in (x,y) and

$$\pi(Ex(T)^2, Ex(T)) = -Ex(T) + \mu \operatorname{Var} x(T), \tag{3.6}$$

which is exactly the objective function of problem $P(\mu)$. The concavity of π implies (noting $\pi_x(x,y) = \mu$ and $\pi_y(x,y) = -(1+2\mu y)$)

$$\begin{aligned} \pi(Ex(T)^2, Ex(T)) &\le \pi(E\bar{x}(T)^2, E\bar{x}(T)) + \mu(Ex(T)^2 - E\bar{x}(T)^2) \\ &\qquad - (1+2\mu E\bar{x}(T))(Ex(T) - E\bar{x}(T)) \\ &< \pi(E\bar{x}(T)^2, E\bar{x}(T)), \end{aligned} \tag{3.7}$$

where the last inequality is due to (3.4). By (3.7), $\bar{u}(\cdot)$ is not optimal for problem $P(\mu)$, leading to a contradiction. $\square$

The implication of Theorem 3.1 is that any optimal solution of problem $P(\mu)$ (as long as it exists) can be found via solving problem $A(\mu,\lambda)$. Notice that the auxiliary problem $A(\mu,\lambda)$ is a standard stochastic optimal control problem parameterized by (μ,λ), which has an objective function of the form $EU(x(T))$ as well as an LQ structure.

The proposed embedding scheme does not depend on the problem constraint represented by the system dynamics (2.6), thus making an analytical solution possible. The condition that the objective function is concave with respect to $Ex(T)$ and $Ex(T)^2$ is a more relaxed assumption than the condition that the continuous-time portfolio problem is concave itself.

4. Solutions to General LQ Problems

In this section we solve a general stochastic LQ problem that includes problem $A(\mu, \lambda)$ introduced in the previous section as a special case. It should be noted that problem $A(\mu, \lambda)$, as we will see from the next section, is a singular LQ problem that cannot be solved by the conventional approach as developed by Wonham [28] and Bensoussan [2] among others. Indeed, study on the general (possibly singular) stochastic LQ problem is interesting in its own right, and will in turn elicit deeper insights into the mean-variance problem.

The system under consideration in this section is governed by the following linear Ito's stochastic differential equation (SDE):

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]\,dt + \sum_{j=1}^{m} D_j(t)u(t)\,dW^j(t), \\ x(0) = x_0 \in R^n, \end{cases} \tag{4.1}$$

where x_0 is the initial state, $W(t) \equiv (W^1(t), \ldots, W^m(t))'$ is a given m-dimensional Brownian motion over $[0, T]$ on a given filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$, and $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$ is a control.

For each $u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m)$, the associated cost is

$$J(u(\cdot)) = E\left\{\int_0^T \tfrac{1}{2}[x(t)'Q(t)x(t) + u(t)'R(t)u(t)]\,dt + \tfrac{1}{2}x(T)'Hx(T)\right\}. \tag{4.2}$$

The solution $x(\cdot)$ of the SDE (4.1) is called the *response* of the control $u(\cdot)$, and $(x(\cdot), u(\cdot))$ is called an *admissible pair*. The objective of the optimal control problem is to minimize the cost function $J(u(\cdot))$ over $L^2_{\mathcal{F}}(0, T; R^m)$.

In the rest of this paper we may write A for a (deterministic or stochastic) process $A(t)$, omitting the variable t, whenever no confusion arises. Under this convention, when $A \in C([0, T]; S^n)$, $A \geq (>) 0$ means $A(t) \geq (>) 0,\ \forall t \in [0, T]$.

We introduce the following assumption for the coefficients of the above problem.

(A) The data appearing in the LQ problem satisfy
$A \in C([0, T]; R^n)$,
$B, D_j \in C([0, T]; R^{n\times m})$,
$f \in L^2(0, T; R^n)$,
$Q \in C([0, T]; S^n_+)$,
$R \in C([0, T]; S^m)$,
$H \in S^n_+$.

Note that here we do *not* assume that R is positive definite (therefore, R could be zero, indefinite, or even negative definite) as opposed to almost all the relevant existing research. Namely, the problem under consideration may include singular situations.

We introduce the following *stochastic Riccati equation*:

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)'P(t) - Q(t) \\ \qquad\qquad + P(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'P(t), \\ P(T) = H, \\ K(t) \equiv R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t) > 0, \qquad \forall t \in [0, T], \end{cases} \tag{4.3}$$

along with an equation

$$\begin{cases} \dot{g}(t) = -A(t)'g(t) + P(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'g(t) \\ \qquad\qquad -P(t)f(t), \\ g(T) = 0. \end{cases} \tag{4.4}$$

Theorem 4.1. *If* (4.3) *and* (4.4) *admit solutions* $P \in C([0, T]; S_+^n)$ *and* $g \in C([0, T]; R^n)$, *respectively, then the stochastic LQ problem* (4.1)–(4.2) *has an optimal feedback control*

$$u^*(t, x) = -\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'(P(t)x + g(t)). \tag{4.5}$$

Moreover, the optimal cost value is

$$J^* = \frac{1}{2}\int_0^T \left(2f(t)'g(t) - g(t)B(t)\left(R(t) + \sum_{j=1}^{m} D_j(t)'P(t)D_j(t)\right)^{-1} B(t)'g(t)\right) dt \\ + \frac{1}{2}x_0'P(0)x_0 + x_0 g(0). \tag{4.6}$$

Proof. Applying Ito's formula, we get

$$\frac{1}{2}d(x'Px) = \frac{1}{2}\left\{\sum_{j=1}^{m} u'D_j'PD_ju + x'(-Q + PBK^{-1}B'P)x \right. \\ \left. + 2u'B'Px + 2x'Pf\right\} dt + \frac{1}{2}\{\cdots\}\,dW(t) \tag{4.7}$$

and

$$d(x'g) = \{u'B'g + x'PBK^{-1}B'g + f'g - x'Pf\} + \{\cdots\}\,dW(t). \tag{4.8}$$

Integrating both (4.7) and (4.8) from 0 to T, taking expectations, adding them together, and noting (4.2), one obtains

$$\begin{aligned} J(u(\cdot)) &= \frac{1}{2}E\int_0^T \{u'Ku + 2u'B'(Px + g) + x'PBK^{-1}B'Px \\ &\qquad + 2x'PBK^{-1}B'g + 2f'g\}\,dt + \frac{1}{2}x_0'P(0)x_0 + x_0g(0) \\ &= \frac{1}{2}E\int_0^T \{[u + K^{-1}B'(Px + g)]'K[u + K^{-1}B'(Px + g)] + 2f'g \\ &\qquad - gBK^{-1}B'g\}\,dt + \frac{1}{2}x_0'P(0)x_0 + x_0g(0). \end{aligned} \tag{4.9}$$

It follows immediately that the optimal feedback control is given by (4.5) and the optimal value is given by (4.6) provided that the corresponding equation (4.1) under (4.5) has a solution. However, under (4.5), the system (4.1) reduces to

$$\begin{cases} dx(t) = [A(t)x(t) - B(t)K(t)^{-1}B(t)'(P(t)x(t) + g(t))]\,dt \\ \qquad\qquad - \sum_{j=1}^{m} D_j(t)K(t)^{-1}B(t)'(P(t)x(t) + g(t))\,dW^j(t), \\ x(0) = x_0. \end{cases} \tag{4.10}$$

This is a nonhomogeneous linear stochastic differential equation. Since $P \in C([0,T]; S_+^n)$, $g \in C([0,T]; R^n)$, and $K^{-1} \in C([0,T]; S_+^m)$, (4.10) admits one and only one solution. This completes the proof. □

LQ models constitute an extremely important class of optimal control problems and their optimal solutions can be obtained explicitly via the Riccati equations, due to the nice underlying structures (see [1], [2], [13], and [28]). Mean-variance portfolio selection problems have inherent LQ structures and therefore it is very natural to solve them through solving the stochastic LQ problems. The general stochastic Riccati equation is introduced in [3] as a backward SDE of the Pardoux–Peng type [22] for the case where all the coefficients are random. It reduces to (4.3) for the present case. A rather surprising discovery in [3] is that a stochastic LQ problem may still be meaningful and solvable even when the running control weighting cost R is *indefinite*, as opposed to the conventional belief (due to the deterministic case) that the positive definiteness of R is absolutely *necessary* for the LQ problem to be sensible. This phenomenon has to do with the deep nature of uncertainty as well as the way of controlling the uncertainty. While a detailed discussion on this point can be found in [3], one may also look at the stochastic Riccati equation (4.3) to get some rough idea as to why R may be allowed to be indefinite (or zero in particular). Indeed, even when $R \le 0$, the presence of the term $\sum_{j=1}^{m} D_j' P D_j$ may offer compensation if it is positive enough so that $R + \sum_{j=1}^{m} D_j' P D_j > 0$. The mean-variance portfolio model exemplifies such situations as we will see in the subsequent sections.

5. Solution to the Auxiliary Problem

Now we return to solving problem $A(\mu, \lambda)$ introduced in Section 3, which specializes the general model discussed in Section 4. Set

$$\gamma = \frac{\lambda}{2\mu} \quad \text{and} \quad y(t) = x(t) - \gamma. \tag{5.1}$$

Then problem $A(\mu, \lambda)$ is equivalent to minimizing

$$E[\tfrac{1}{2}\mu y(T)^2] \tag{5.2}$$

subject to

$$\begin{cases} dy(t) = \{A(t)y(t) + B(t)u(t) + f(t)\}\,dt \\ \qquad\qquad + \sum_{j=1}^{m} D_j(t)u(t)\,dW^j(t), \\ y(0) = x_0 - \gamma, \end{cases} \tag{5.3}$$

where

$$\begin{cases} A(t) = r(t), & B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t)), \\ f(t) = \gamma r(t), & D_j(t) = (\sigma_{1j}(t), \ldots, \sigma_{mj}(t)). \end{cases} \tag{5.4}$$

Thus, problem (5.2)–(5.3) is a special case of problem (4.1)–(4.2) with

$$(Q(t), R(t)) = (0, 0), \qquad H = \mu, \tag{5.5}$$

and $A(t)$, $B(t)$, $f(t)$, and $D_j(t)$ given by (5.4).

Note that $R(t) = 0$ in this problem. This is why the mean-variance model gives rise to an inherently singular stochastic LQ problem. Also, a special feature of the problem is that the state $x(t)$ is one-dimensional, so is the unknown $P(t)$ of the corresponding stochastic Riccati equation (4.3). This makes it easier to solve (4.3) explicitly. Denote

$$\rho(t) = B(t)\left[\sum_{j=1}^{m} D_j(t)' D_j(t)\right]^{-1} B(t)' = B(t)[\sigma(t)\sigma(t)']^{-1} B(t)'. \tag{5.6}$$

Then (4.3) reduces to

$$\begin{cases} \dot{P}(t) = (\rho(t) - 2r(t))\, P(t), \\ P(T) = \mu, \\ P(t)[\sigma(t)\sigma(t)'] > 0, \qquad t \in [0, T]. \end{cases} \tag{5.7}$$

Clearly, the solution of (5.7) is given by

$$P(t) = \mu e^{-\int_t^T (\rho(s) - 2r(s))ds}. \tag{5.8}$$

Note that the third constraint in (5.7) is satisfied automatically due to the assumption (2.4). Moreover, (4.4) becomes

$$\begin{cases} \dot{g}(t) = (\rho(t) - r(t))\, g(t) - \gamma r(t) P(t), \\ g(t) = 0, \end{cases} \tag{5.9}$$

which evidently admits a unique solution $g \in C([0, T]; R^1)$. The optimal feedback control (4.5) then gives

$$\bar{u}(t, y) \equiv (\bar{u}_1(t, y), \ldots, \bar{u}_m(t, y)) = -[\sigma(t)\sigma(t)']^{-1} B(t)' \left(y + \frac{g(t)}{P(t)}\right). \tag{5.10}$$

Let $h(t) = g(t)/P(t)$. Then noting (5.7) and (5.9), one has

$$\begin{aligned} \dot{h}(t) &= \frac{P(t)\dot{g}(t) - \dot{P}(t)g(t)}{P(t)^2} \\ &= \frac{r(t)P(t)g(t) - \gamma r(t)P(t)^2}{P(t)^2} \\ &= r(t)h(t) - \gamma r(t). \end{aligned}$$

Since $h(T) = 0$, we can solve $h(\cdot)$ to get

$$\frac{g(t)}{P(t)} = h(t) = \gamma(1 - e^{-\int_t^T r(s)ds}). \tag{5.11}$$

Substituting (5.11) into (5.10), and noting (5.1), we arrive at

$$\begin{aligned}\bar{u}(t,x) &\equiv (\bar{u}_1(t,x),\ldots,\bar{u}_m(t,x))\\ &= -[\sigma(t)\sigma(t)']^{-1}B(t)'[x-\gamma+\gamma(1-e^{-\int_t^T r(s)ds})]\\ &= [\sigma(t)\sigma(t)']^{-1}B(t)'(\gamma e^{-\int_t^T r(s)ds}-x).\end{aligned}\tag{5.12}$$

6. Efficient Frontier

In this section we proceed to derive the efficient frontier for the original mean-variance problem (2.9). Under the optimal feedback control (5.12) (for problem $A(\mu,\lambda)$), the wealth equation (2.6) evolves as

$$\begin{cases} dx(t) = \{(r(t)-\rho(t))x(t)+\gamma e^{-\int_t^T r(s)ds}\rho(t)\}\,dt\\ \qquad\qquad + B(t)(\sigma(t)\sigma(t)')^{-1}\sigma(t)(\gamma e^{-\int_t^T r(s)ds}-x(t))\,dW(t),\\ x(0)=x_0.\end{cases}\tag{6.1}$$

Moreover, applying Ito's formula to $x(t)^2$, we obtain

$$\begin{cases} dx(t)^2 = \{(2r(t)-\rho(t))x(t)^2+\gamma^2 e^{-2\int_t^T r(s)ds}\rho(t)\}\,dt\\ \qquad\qquad + 2x(t)B(t)(\sigma(t)\sigma(t)')^{-1}\sigma(t)(\gamma e^{-\int_t^T r(s)ds}-x(t))\,dW(t),\\ x(0)^2=x_0^2.\end{cases}\tag{6.2}$$

Taking expectations on both sides of (6.1) and (6.2), we conclude that $Ex(t)$ and $Ex(t)^2$ satisfy the following two nonhomogeneous linear ordinary differential equations:

$$\begin{cases} dEx(t) = \{(r(t)-\rho(t))Ex(t)+\gamma e^{-\int_t^T r(s)ds}\rho(t)\}\,dt,\\ Ex(0)=x_0,\end{cases}\tag{6.3}$$

and

$$\begin{cases} dEx(t)^2 = \{(2r(t)-\rho(t))Ex(t)^2+\gamma^2 e^{-2\int_t^T r(s)ds}\rho(t)\}\,dt,\\ Ex(0)^2=x_0^2.\end{cases}\tag{6.4}$$

Solving (6.3) and (6.4), we can express $Ex(T)$ and $Ex(T)^2$ as explicit functions of γ,

$$Ex(T)=\alpha x_0+\beta\gamma,\qquad Ex(T)^2=\delta x_0^2+\beta\gamma^2,\tag{6.5}$$

where

$$\alpha = e^{\int_0^T (r(t)-\rho(t))dt},\qquad \beta = 1-e^{-\int_0^T \rho(t)dt},\qquad \delta = e^{\int_0^T (2r(t)-\rho(t))dt}.\tag{6.6}$$

By Theorem 3.1, an optimal solution of problem $P(\mu)$, if it exists, can be found by selecting $\bar{\lambda}$ so that (noting (6.5) and (5.1))

$$\bar{\lambda} = 1+2\mu E\bar{x}(T) = 1+2\mu\left(\alpha x_0+\beta\frac{\bar{\lambda}}{2\mu}\right).$$

This yields

$$\bar{\lambda} = \frac{1 + 2\mu\alpha x_0}{1-\beta} = e^{\int_0^T \rho(t)dt} + 2\mu x_0 e^{\int_0^T r(t)dt}. \tag{6.7}$$

Hence the optimal control for problem $P(\mu)$ is given by (5.12) with $\gamma = \bar{\gamma} = \bar{\lambda}/2\mu$ and $\bar{\lambda}$ given by (6.7). In this case the corresponding variance of the terminal wealth is

$$\begin{aligned} \operatorname{Var} \bar{x}(T) &= E\bar{x}(T)^2 - [E\bar{x}(T)]^2 \\ &= \beta(1-\beta)\bar{\gamma}^2 - 2\alpha\beta x_0\bar{\gamma} + (\delta - \alpha^2)x_0^2 \\ &= \frac{1-\beta}{\beta}\left[\beta^2\bar{\gamma}^2 - 2\frac{\alpha\beta^2 x_0\bar{\gamma}}{1-\beta} + \frac{\beta(\delta-\alpha^2)}{1-\beta}x_0^2\right] \\ &= \frac{1-\beta}{\beta}\left[(\beta\bar{\gamma} + \alpha x_0)^2 - 2\frac{\alpha\beta^2 x_0\bar{\gamma}}{1-\beta} + \frac{\beta(\delta-\alpha^2)}{1-\beta}x_0^2\right]. \end{aligned} \tag{6.8}$$

Substituting $\beta\bar{\gamma} = E\bar{x}(T) - \alpha x_0$ (due to (6.5)) in the above and noting (6.6), we obtain

$$\begin{aligned} \operatorname{Var} \bar{x}(T) &= \frac{1-\beta}{\beta}\left[(E\bar{x}(T))^2 - 2\frac{\alpha}{1-\beta}x_0 E\bar{x}(T) + \frac{\beta\delta + \alpha^2}{1-\beta}x_0^2\right] \\ &= \frac{1-\beta}{\beta}(E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt})^2 \\ &= \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}}(E\bar{x}(T) - x_0 e^{\int_0^T r(t)dt})^2. \end{aligned} \tag{6.9}$$

To summarize the above discussion, we have the following result.

Theorem 6.1. *The efficient frontier of the bicriteria optimal portfolio selection problem* (2.9), *if it ever exists, must be given by* (6.9).

The relation (6.9) reveals explicitly the tradeoff between the mean (return) and variance (risk). For example, if one has set an expected return level, then (6.9) tells the risk he/she has to take; and vice versa. In particular, if one cannot take any risk, namely, $\operatorname{Var}(\bar{x}(T)) = 0$, then $E\bar{x}(T)$ has to be $x_0 e^{\int_0^T r(t)dt}$ meaning that he/she can only put his/her money in the bond. Another interesting phenomenon is that the efficient frontier (6.9) involves a perfect square. This is due to the possible inclusion of the bond in a portfolio. In the case when the riskless bond is excluded from consideration, the efficient frontier may no longer be a perfect square, which means one cannot have a risk-free portfolio.

If we denote by $\sigma_{\bar{x}(T)}$ the standard deviation of the terminal wealth, then (6.9) gives

$$E\bar{x}(T) = x_0 e^{\int_0^T r(t)dt} + \sqrt{\frac{1 - e^{-\int_0^T \rho(t)dt}}{e^{-\int_0^T \rho(t)dt}}}\sigma_{\bar{x}(T)}. \tag{6.10}$$

Hence the efficient frontier in the mean–standard-deviation diagram is a straight line, which is also termed the *capital market line* (see, e.g., [17]). The slope of this line, $k = \sqrt{(1 - e^{-\int_0^T \rho(t)dt})/e^{-\int_0^T \rho(t)dt}}$, is called the *price of risk*.

Now we discuss an example. Suppose a market has a bond with a nominal annual interest rate $r = 6\%$, and a stock with a nominal annual appreciation rate $b = 12\%$ and a standard deviation $\sigma = 15\%$. Using $T = 1$ (year) and $\rho = ((b - r)/\sigma)^2 = 0.16$, we can plot the capital market line (6.10) as

$$E\bar{x}(1) = x_0 e^{0.06} + 0.4165\sigma_{\bar{x}(1)}. \tag{6.11}$$

Now, consider an (aggressive) inestor who has an initial fund $x_0 = \$1$ million and wishes to obtain an expected return of 20% in one year. We calculate how much risk he has to bear in order to achieve that level of expected return. Taking $x_0 = \$1$ million and $E\bar{x}(1) = \$1.2$ million in (6.11), we obtain $\sigma_{\bar{x}(1)} = \$0.3317$ million, implying that the standard deviation of his goal is as high as 33.17%! Next, we calculate his portfolio. By virtue of the first equation of (6.5), we obtain

$$\gamma = \frac{1.2 - e^{-0.1}}{1 - e^{-0.16}} = 1.9963.$$

Thus, by (5.12), the amount of money he should invest in the stock as a function of time and wealth is

$$\bar{u}(t, x) = 2.6667(1.9963e^{0.06(t-1)} - x).$$

In particular, at the initial time $t = 0$, $\bar{u}(0, x_0) = \$2.3468$ million, meaning that he needs to short the bond (to borrow money) for an amount \$1.3468 million and invest it in the stock together with his initial endowment \$1 million. This is a very aggressive policy indeed.

7. Concluding Remarks

This paper investigates a dynamic continuous-time mean-variance portfolio selection problem in the spirit of Markowitz's original work. An efficient frontier is traced out in a closed form for the model where the interest/appreciation rates and volatility rate are nonconstants. The basic idea of solving the problem is to embed the original problem into a stochastic LQ control problem. This represents a new approach different from the existing literature, which is however completely natural in view of the inherent LQ structure of the mean-variance model. More importantly, this approach opens up possible ways of solving more general portfolio selection models. One such example is the problem with random interest/appreciation rates and volatility rate, which can be handled by the general stochastic LQ technique; see [3] and [4] (in this case the corresponding Riccati equation is a backward SDE, the theory of which has been developed extensively in recent years; see [22] and [29]). Another interesting observation is that the embedding scheme does not depend on the feasible set of the state, as opposed to the

Lagrangian approach where a constraint $Ex(T) = \varepsilon$ is added (see [5], [26], and [27]). The latter requires convexity of the constraints in order to guarantee its applicability, which in turn requires linearity of the state equation. Hence, our approach allows for the possibility of tackling portfolio selection problems with nonlinear state equations and objective functions of nonlinear form $U(Ex(T), Ex(T)^2)$ via stochastic control theory. These are the problems currently under investigation.

The mean-variance portfolio problem studied in this paper also nicely exemplifies the general stochastic LQ models where the control running cost R is *indefinite*. To be specific, observe that there is no *direct* running cost (over the time period $[0, T]$) associated with a control (portfolio) in the cost function (3.1) or (5.2), namely, $R \equiv 0$ in this case. However, the portfolio does influence the diffusion term of the system dynamics (see (2.6)) which gives rise to a "hidden" running cost $\sum_{j=1}^{m} D_j(t)'P(t)D_j(t) \equiv P(t)\sigma(t)\sigma(t)'$ (with $P(\cdot)$ given by (5.7)) that must be taken into consideration and may be regarded as a "cost equivalence" of the risk. To balance between this uncertainty/risk cost and the potential return is exactly the goal of a mean-variance portfolio selection problem.

Acknowledgments

The authors are grateful to an anonymous referee for his/her comments which led to an improved version of the paper.

References

1. Anderson BDO, Moore JB (1989) Optimal Control—Linear Quadratic Methods. Prentice-Hall, Englewood Cliffs, NJ
2. Bensoussan A (1982) Lecture on stochastic control, part I. In: Nonlinear Filtering and Stochastic Control, Mitter SK, Moro A (eds). Lecture Notes in Mathematics, vol 972. Springer-Verlag, Berlin, pp 1–39
3. Chen S, Li X, Zhou XY (1998) Stochastic linear quadratic regulators with indefinite control weight costs. SIAM J Control Optim 36: 1685–1702
4. Chen S, Zhou XY (to appear) Stochastic linear quadratic regulators with indefinite control weight costs, II. SIAM J Control Optim
5. Duffie D, Richardson H (1991) Mean-variance hedging in continuous time. Ann Appl Probab 1: 1–15
6. Dumas B, Liucinao E (1991) An exact solution to a dynamic portfolio choice problem under transaction costs. J Finance 46: 577–595
7. Elton EJ, Gruber MJ (1975) Finance as a Dynamic Process. Prentice-Hall, Englewood Cliffs, NJ
8. Fleming WH, Soner HM (1993) Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, New York
9. Föllmer H, Sondermann D (1986) Hedging of non-redundant contingent claims. In: Contributions to Mathematical Economics, Mas-Colell A, Hildenbrand W (eds). North-Holland, Amsterdam, pp 205–223
10. Francis JC (1976) Investments: Analysis and Management. McGraw-Hill, New York
11. Grauer RR, Hakansson NH (1993) On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: a comparison of returns and investment policies. Management Sci 39: 856–871
12. Hakansson NH (1971) Multi-period mean-variance analysis: toward a general theory of portfolio choice. J Finance 26: 857–884
13. Kalman RE (1960) Contributions to the theory of optimal control. Bol Soc Mat Mexicana 5: 102–119

14. Kohlmann M, Zhou XY (to appear) Relationship between backward stochastic differential equations and stochastic controls: an LQ approach. SIAM J Control Optim
15. Li D, Ng WL (to appear) Optimal dynamic portfolio selection: multi-period mean-variance formulation. Math Finance
16. Lim AEB, Zhou XY (1999) Mean-variance portfolio selection with random parameters. Preprint
17. Luenberger DG (1998) Investment Science. Oxford University Press, New York
18. Markowitz H (1952) Portfolio selection. J Finance 7: 77–91
19. Markowitz H (1959) Portfolio Selection: Efficient Diversification of Investment. Wiley, New York
20. Merton RC (1972) An analytic derivation of the efficient portfolio frontier. J Finance Quant Anal 7: 1851–1872
21. Mossin J (1968) Optimal multiperiod portfolio policies. J Business 41: 215–229
22. Pardoux E, Peng S (1990) Adapted solution of backward stochastic equation. System Control Lett 14: 55–61
23. Perold A (1984) Large-scale portfolio optimization. Management Sci 30: 1143–1160
24. Pliska SR (1997) Introduction to Mathematical Finance. Blackwell, Malden
25. Samuelson PA (1969) Lifetime portfolio selection by dynamic stochastic programming. Rev Econom Statist 51: 239–246
26. Schweizer M (1995) Variance-optimal hedging in discrete time. Math Oper Res 20: 1–32
27. White DJ (1974) Dynamic programming and probabilistic constraints. Oper Res 22: 654–664
28. Wonham WM (1968) On a matrix Riccati equation of stochastic control. SIAM J Control 6: 312–326
29. Yong J, Zhou XY (1999) Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, New York
30. Yu PL (1971) Cone convexity, cone extreme points, and nondominated solutions in decision problem with multiobjectives. J Optim Theory Appl 7: 11–28

Accepted 24 *November* 1999