Risk Parity, Maximum Diversification, and Minimum Variance: An Analytic Perspective

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Abstract

Analytic solutions to Risk Parity, Maximum Diversification, and Minimum Variance portfolios provide useful perspectives about their construction and composition. Individual asset weights depend on both systematic and idiosyncratic risk in all three risk-based portfolios, but systematic risk eliminates many investable assets in long-only constrained Maximum Diversification and Minimum Variance portfolios. On the other hand, all investable assets are included in Risk Parity portfolios, and idiosyncratic risk has little impact on the magnitude of the weights. The algebraic forms for optimal asset weights derived in this paper yield generalizable properties of risk-based portfolios, in contrast to empirical simulations that employ a specific set of historical returns, proprietary risk models, and multiple constraints. The analytic solutions reveal precisely how various kinds of predicted risk impact the relative magnitude of security weights under each type of risk-based portfolio construction.

Risk Parity, Maximum Diversification, and Minimum Variance: An Analytic Perspective

Portfolio construction techniques based on predicted risk, without expected returns, have become popular in the last decade. In terms of individual asset selection, Minimum Variance and more recently Maximum Diversification objective functions have been explored, motivated in part by the cross-sectional equity risk anomaly first documented in Ang, Hodrick, Xing, and Zhang [2006]. Application of these objective functions to large (e.g., 1000 stock) investable sets requires sophisticated estimation techniques for the risk model. On the other end of the spectrum, the principal of Risk Parity, traditionally applied to small-set (e.g., 2 to 10) asset allocation decisions, has been proposed for large-set security selection applications. Unfortunately, most of the published research on these low-risk structures is based on standard unconstrained portfolio theory, matched with long-only simulations. The empirical results in such studies are specific to the investable set, time period, maximum weight constraints and other portfolio limitations, as well as the risk model. The purpose of this paper is to compare and contrast risk-based portfolio construction techniques using long-only analytic solutions. We also provide a simulation of risk-based portfolios for large-cap U.S. stocks using the CRSP database from 1968 to 2011. The back-test is performed using a single-index model, standard OLS risk estimates, and without maximum position or other portfolio constraints, leading to easily replicable results.

The concept of Risk Parity has evolved over time from the original concept embedded in research by Bridgewater in the 1990's. Initially, an asset allocation portfolio was said to be in "parity" when weights are proportional to asset-class inverse volatility. For example, if the equity sub-portfolio has a forecasted volatility of 15% and the fixed-income sub-portfolio has a volatility of just 5%, then a combined portfolio of 75 percent fixed-income and 25 percent equity (i.e., three times as much fixed-income) is said to be in parity. This early definition of Risk Parity ignored correlations, even as the concept was applied to more than two asset classes. A more complete definition, which considers correlations, was formalized by Qian [2006] who couched the property in terms of a risk budget where weights are adjusted so that each asset has the same contribution to portfolio risk. Maillard, Roncalli, and Teiletche [2010] call this an "equal risk contribution" portfolio, and analyzed properties of an unconstrained analytic solution. An equivalent "portfolio beta" interpretation by Lee [2011] is that Risk Parity is achieved when weights are proportional to the inverse of their beta with respect to the final portfolio. Risk Parity is not a traditional mean-variance objective function and has been numerically difficult to implement on large-scale investable sets. Our new analytic solution allows for quick calculation of Risk Parity weights on any sized investable set for a general linear risk model.

Chow, Hsu, Kalesnik, and Little [2011], provide a review of Minimum Variance portfolios, which have been defined and analyzed from the start of modern portfolio theory in the 1960s. The objective function is minimization of ex-ante portfolio risk, irrespective of forecasted returns, so that the Minimum Variance portfolio lies on the left-most tip of the ex-ante efficient frontier. Minimum Variance portfolios equalize the *marginal* contributions of each asset to portfolio risk, in contrast to the Risk Parity portfolio which equalizes each asset's *total* risk contribution. Thus, Risk Parity portfolios generally lie within, rather than on, the efficient frontier. As a special case of the popular mean-variance objective function, Minimum Variance portfolios can be constructed using standard optimization software, given a specified asset covariance matrix. Although substantial analytic work exists on unconstrained portfolios, an analytic solution for long-only constrained Minimum Variance portfolios was first derived by Clarke, de Silva, and Thorley [2011].

Maximum Diversification portfolios use a recently introduced objective function by Choueifaty and Coignard [2008] that maximizes the ratio of weighted-average asset volatilities to portfolio volatility. Like Minimum Variance, Maximum Diversification portfolios equalize the marginal contributions of each asset, given a small change in the asset's weight. However, the objective function is motivated by maximization of the portfolio Sharpe ratio, where expected asset returns are assumed to be proportional to asset risk. Thus, the Maximum Diversification portfolio is the tangent (highest Sharpe ratio) portfolio on the efficient frontier, if average asset returns increase proportionally with risk. On the other hand, if asset returns decrease with risk, the Maximum Diversification portfolio will be on the lower half of the traditional efficient frontier and clearly sub-optimal. In this paper, we provide an analytic solution for long-only constrained Maximum Diversification portfolios, similar to Clarke, de Silva, and Thorley [2011].

In the first section, we review our analytic solutions for the three risk-based portfolios under the simplifying assumption of a single-factor risk model, with derivations for a more general multi-factor model provided in the technical appendix. We explain how the analytic solutions provide intuition for the comparative weight structure of Risk Parity, Maximum Diversification, and Minimum Variance portfolios. The equations show that asset weights decrease with both systematic and idiosyncratic risk for all three portfolios, although the form and impact of these two sources of risk vary. For example, Minimum Variance weights are generally proportional to inverse variance (standard deviation squared), while Maximum Diversification and Risk Parity weights are generally proportional to inverse volatility (standard deviation). The analytic solutions reveal why long-only Maximum Diversification and Minimum Variance portfolios employ a relatively small portion (e.g., 100 out of 1000) of the investable set, in contrast to Risk Parity portfolios which include all of the assets the specified set.

In addition to intuition about portfolio composition, the long-only analytic solutions also provide simple numerical recipes for actual portfolio construction using large investable sets. In the second section, we report the performance results for all three risk-based portfolios using the largest 1000 U.S. stocks from 1968 to 2011. While numerous back-tests of Minimum Variance and Maximum Diversification portfolios have been published in recent years, this study includes the first simulation of a large (i.e., 1000 asset) Risk Parity portfolio. The third section of the paper compares and contrasts the asset weight distribution of the three risk-based portfolios using the analytic solutions on a specific date, January 2012. The paper concludes with a summary of various perspectives about the three risk-based portfolio construction techniques that are gleaned from the analytic solutions.

Analytic Solutions to Risk-Based Portfolios

We first review properties of the single-factor risk model and acquaint the reader with our mathematical notation. Under a single-factor asset covariance matrix, individual assets have only one source of common risk, resulting in the familiar decomposition of the i^{th} asset's total risk into systematic and idiosyncratic components

$$\sigma_i^2 = \beta_i^2 \ \sigma_F^2 + \sigma_{\varepsilon,i}^2 \ . \tag{1}$$

In Equation 1, σ_F is the risk of the common factor, for example the capitalization-weighted market portfolio, and $\sigma_{\varepsilon,i}$ is the i^{th} asset's idiosyncratic risk. The asset's exposure to the systematic risk factor, β_i, is by definition equal to the ratio of asset risk to factor risk, multiplied by the correlation coefficient between the asset and risk factor returns,

$$\beta_i = \frac{\sigma_i}{\sigma_F} \rho_i . \tag{2}$$

The single-factor model also yields simple relationships for the pair-wise association between any two assets. Specifically, the covariance between two assets, *i* and *j*, is $\sigma_{i,j} = \beta_i \beta_j \sigma_F^2$, and the correlation coefficient is the product of their correlations to the common factor, $\rho_{i,j} = \rho_i \, \rho_j$.

As shown in Clarke, de Silva, and Thorley [2011], and reviewed in the technical appendix, individual asset weights in the long-only Minimum Variance portfolio under a single factor risk model can be written as

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2} \left(1 - \frac{\beta_i}{\beta_L} \right) \text{ for } \beta_i < \beta_L \text{ else} = 0 \tag{3}$$

where β_L is a "long-only threshold" beta and σ_{MV} is the risk of the minimum variance portfolio. According to Equation 3, individual assets are only included in the long-only portfolio if their factor beta is lower than the threshold beta, β_L, which for large investment sets can exclude a majority of the assets. High idiosyncratic risk in the denominator of the first term of Equation 3 lowers the asset weight, but cannot by itself drive an asset out of solution. The second term, in the parentheses of Equation 3, indicates that asset weights increase from zero with the only other asset-specific (i.e., subscripted by *i*) parameter, β_i, and that the highest weight is given to the lowest beta asset. In fact, in the absence of the cross-sectional variations due to idiosyncratic risk, Minimum Variance portfolio weights fall on a kinked line when plotted against factor beta, visually like the expiration-date payoff profile to a put option. Specifically, weights are zero above the long-only threshold beta, β_L, and lie on a negatively sloped line for betas below the long-only threshold.

The technical appendix shows that weights in the Maximum Diversification portfolio can be written in a similar form to Minimum Variance weights, but are driven by the *correlation* of the asset to the common risk factor,

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \frac{\rho_i}{\rho_L} \right) \text{ for } \rho_i < \rho_L \text{ else } = 0 \tag{4}$$

where ρ_L is a "long-only threshold" correlation and σ_{MD} is the Maximum Diversification portfolio risk. According to Equation 4, individual assets are only included in the long-only Maximum Diversification portfolio if their correlation to the common risk factor is lower than the threshold correlation. As in Equation 3 for Minimum Variance portfolio weights, high idiosyncratic risk in the denominator of the first term of Equation 4 lowers the asset weight, but does not by itself drive an asset out of the Maximum Diversification portfolio.

Equation 4 also has a middle term; the ratio of the asset risk, σ_i, to the weighted average risk of the assets in the long-only portfolio, σ_A. This middle term, like the first term, sizes the weight rather than dictates the asset's inclusion in the Maximum Diversification portfolio. However, total asset risk in the numerator of the middle term does tend to offset the idiosyncratic risk squared, so that the Maximum Diversification portfolio weights are approximately proportional to the inverse of asset return standard deviation, as opposed to the inverse of variance in Minimum Variance portfolio weights.[1] The result of this structure is that weights in the Maximum Diversification portfolio tend to be less concentrated than weights in the Minimum Variance portfolio, because inverse standard deviation has less extreme values than inverse variance. Finally, the last term in the parentheses of Equation 4 indicates that asset weights increase from zero with lower correlation, and that the highest weight is given to the lowest correlation asset. In the absence of the variations due to idiosyncratic risk, Maximum Diversification portfolio weights lie a kinked line when plotted against correlation rather than beta.

The technical appendix provides a new and conceptually important solution for individual assets weights in a Risk Parity portfolio. The equation for Risk Parity weights is somewhat different in algebraic form than the two objective-function based portfolios;

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2}\left[\left(\frac{\beta_i^{\,2}}{\gamma^2} + \frac{1}{N}\frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2}\right)^{1/2} - \frac{\beta_i}{\gamma}\right] \tag{5}$$

where γ is a constant across all assets, σ_{RP} is the Risk Parity portfolio risk, and *N* is the number of assets. High idiosyncratic risk in the denominator of the first term of Equation 5 lowers the asset weight, but the similarity to Equations 3 and 4 ends there. As shown using partial derivatives in the technical appendix, asset weights decline monotonically with asset beta, but *all* assets have some positive weight so that the Risk Parity portfolio is long-only by definition. For positive beta assets, the square-brackets term in Equation 5 must be positive because the square-root of squared beta over gamma plus some small (divided by *N*) positive value is greater than the beta over gamma subtracted at the end. On the other hand, if the asset beta is negative, then the weight is clearly positive and large. In the absence of variations due to idiosyncratic risk in Equation 5, Risk Parity weights plotted against factor beta lie along a hyperbolic curve, somewhat like the pre-expiration payoff profile for a put option. In other words, Risk Parity portfolio weights tend asymptotically towards zero with higher beta, and tend asymptotically towards a line with a negative slope for lower beta.

Equations 3, 4, and 5 for individual asset weights provide insights into risk-based portfolio construction, as well as formulas for determining weights without the need for an optimizer or complex numerical search routines. In Equations 3 and 4, assets need only be sorted by increasing beta and correlation, respectively, and then compared to the threshold parameters to determine which assets are in solution. The sum of raw weights can then be scaled to one to accommodate the portfolio constants outside of the parentheses. Equation 5 for the Risk Parity portfolio is inherently different in that portfolio constants, which depend on the final assets weights, are embedding *within* the parentheses, making the weights endogenously determined. However, a fast convergence routine can be designed using Equation 5 by initially assuming equal weights, and then iteratively calculating the portfolio parameters and assets weights until they converge. Unlike other numerical routines that have been proposed for Risk Parity weights, the calculation of asset weights using Equation 5 is almost instantaneous and can be applied to large (i.e., 1000 asset) investment sets in a simple Excel spreadsheet. While we focus on the intuition of a single-factor risk model in the body of this paper, the technical appendix provides a general *K*-factor solution to the Risk Parity portfolio, as well as the other two risk-based structures.

One Thousand Stock Empirical Example

In this section, we use Equations 3, 4, and 5 to construct risk-based and benchmark portfolios from the largest 1000 common stocks in the CRSP database at the end of each month from 1968 to 2011. To keep the risk model estimation process as simple as possible, beta and idiosyncratic risk are based on standard OLS regressions for the prior 60 months, using the value-weighted CRSP index as the common factor. To translate historical betas into predicted betas, the entire set of 1000 is adjusted ½ towards one each month. Similarly, the set of 1000 log idiosyncratic risks is adjusted ⅓ towards the average log idiosyncratic risk for that month.[2] The results in this paper are generally robust to the specific shrinkage process, although some sort of Bayesian adjustment is needed in order to avoid extreme predicted risk parameter values and thus extreme weights. A full 60 months of prior return data is required for a stock to be considered for the portfolios, but no other data scrubbing, stock selection, or maximum weight constraints are employed.

While we report full-period performance statistics below, we note that the risk-structure of the equity market and consequently the composition of the various risk-based portfolios has changed dramatically over the last 44 years. For example, the predicted risk of the market portfolio, based on our simple 60-month rolling window, varied between a low of about 10 percent in the early 1960s, to highs of about 20 percent in the late 1970's, shortly after the 1987 market crash, and again around the turn of the century. Idiosyncratic risk, on the other hand, has been fairly stable at about 25 to 30 percent for the average large-cap stock, except for several years around the turn of the century when the average idiosyncratic risk reached a high of about 40 percent. While the average predicted market beta naturally remained close to one over time, the spread of predicted betas varied substantially. For example, the lowest systematic risk stocks out of all 1000 in any given month typically had betas of 0.3 to 0.5, but the minimum predicted beta went into negative territory for several years in the mid 1990's. For the Minimum Variance portfolio, the long-only threshold beta varied between 0.6 and 0.8 over time, meaning that only stocks with lower predicted beta values were admitted into solution each month. The result is that the long-only Minimum Variance portfolio averaged only about 62 stocks in solution, with a low of 30 and a high of 121. Similarly, the long-only Maximum Diversification portfolio averaged about 83 stocks in solution, with a low of 50 and a high of 134. While these counts may indicate higher portfolio concentration than some managers are comfortable with, they are largely a function of the aggressiveness of the shrinkage process used to translate historical security risk values into predicted values.

The average performance from 1968 to 2011 of the three risk-based portfolios, as well as the market (value-weighted) portfolio are tabulated in Exhibit 1 and plotted in Exhibit 2. Based on DeMiguel, Garlappi, and Uppal [2009] we also include an equal-weighted 1000 stock portfolio as a form of "naive diversification". The returns throughout this study are reported in excess of the contemporaneous risk-free rate, measured by one-month Treasury bill returns from Ibbotson Associates. While high average returns are not the explicit goal of risk-based portfolio construction, all three risk-based portfolios outperform the excess market return of 5.1 percent. The Risk Parity portfolio had an excess return of 7.2 percent, closely matching the return on the market-wide equal-weighted portfolio. The Maximum Diversification portfolio had an excess return of 5.5 percent, and the Minimum Variance portfolio had an excess return of 5.7 percent, similar to the capitalization-weighted market portfolio. As is common practice, the realized returns of each asset class are shown as *average* annual (multiplication by 12) monthly excess returns, although investors with longer holding periods may be more interested in *compound* annual returns. Thus, Exhibit 2 also provides an "x" for the position of the compound annual return to each portfolio, calculated over the entire 1968 to 2011 period.[3] The realized average return performance of the risk-based portfolios is interesting, but not part of the portfolio construction process. Conclusions regarding the relative performance of the various portfolio construction techniques will depend on the cross sectional relationship between risk and return among the investable securities during the simulation period.

In order to make this point about the historical performance in empirical simulations clear, Exhibit 3 shows the performance of risk-sorted quintile portfolios for the same 1000 U.S.

stocks over the same 1968 to 2012 period. Each quintile portfolio contains 200 equally-weighted stocks, assigned to portfolios each month based on their prior 60-month standard deviation of excess returns. Exhibit 3 is consistent with the "low risk anomaly" first documented by Ang, Hodrick, Xing, and Zhang [2006] within the cross-section of U.S. stocks.[4] The lowest and second-to-the-lowest risk portfolios (i.e., Quint 1 and Quint 2) have the traditional risk-return pattern, but the other three portfolios have an *inverse* relationship between risk and return. This potentially perverse relationship between risk and reward is even worse when measured by compound returns, as shown with an "x" for each portfolio in Exhibit 3. The high ex-ante risk quintile does in fact have the highest realized risk at about 27.3 percent, but with a compound annual excess return of only 2.4 percent over the last 44 years. While inconsistent with long-standing academic views of risk and return, Baker and Wurgler [2011] provide a reasonable explanation for the low risk anomaly based on individual investor preferences for high-risk stocks, along with constraints associated with benchmarking that prohibit larger institutions from fully exploiting the anomaly. Frazzini and Pedersen [2011] also provide an explanation based on the lack of access to leverage for individual investors, who bid up the price (i.e., lower the subsequent return) of high risk stocks.

For the three risk-based portfolios in Exhibit 1, the explicit objective of low risk is best achieved by the Minimum Variance portfolio, with a realized risk of 12.5 percent compared to the market portfolio risk of 15.6 percent. As a result, the Sharpe ratio for 1968 to 2011 is highest for the Minimum Variance portfolio, followed by the Risk Parity and then the Maximum Diversification portfolio.[5] Exhibit 1 also reports on each portfolio's market beta, the average number of positions over time, and the average Effective N, as defined by Strongin, Petsch, and Sharenow [2000]. The market betas (single time-series regression of realized portfolio returns) for the various portfolios are about one, but with a notably lower beta of 0.52 for the Minimum Variance portfolio, a key source of its low realized risk. While the market portfolio includes all 1000 securities each month, market-capitalization weighting leads to an average Effective N of only 138.5. Alternatively, the average Effective N of the Risk Parity portfolio is 933.5, close to an equally-weighted portfolio. The Effective N of the Maximum Diversification and Minimum Variance portfolios is even lower than the market portfolio, with average values of 46.7 and 36.0, respectively. Some investors traditionally view more securities in solution as equivalent to better diversification and lower risk. However, the relatively low number of securities in the Maximum Diversification and Minimum Variance portfolios illustrate that risk reduction is best achieved by selecting fewer less correlated and less risky securities rather than just adding more securities. Of course, *realized* risk minimization depends on the accuracy of the security risk forecasts, specifically the spread in systematic and idiosyncratic risk. Alternatively, if a manager does not want to infer much from historical differences in security risks, the predicted security risk parameters would be "shrunk" towards a common value, and the risk-based portfolios described in Equations 3, 4, and 5 would converge to equal-weighted portfolios.[6]

<u>Weight Distributions as a Function of Risk</u>

We now turn to Equations 3, 4, and 5 for a closer examination of weight structure of the three risk-based portfolios at a specific point in time, January 2012. The January 2012 date is chosen because the weights employ the most recent five years of historical data, January 2007 to December 2011. To support the subsequent weight charts, Exhibit 4 plots the predicted beta and idiosyncratic risk of all 1000 stocks as of January 2012. Beta ranges from about 0.4 to 2.7, and idiosyncratic risk ranges from about 15 percent to over 105 percent. The ranges of security risk for 2012 are high by historical standards, although not as high as the around the turn of the century. As is typical over time, the scatter-plot in Exhibit 4 shows that the two measures of asset risk are somewhat correlated and that common factor beta, as well as idiosyncratic risk, is positively skewed. The predicted common factor risk (i.e., market portfolio) in January 2012 is 19.3 percent, relatively high by historical standards. As we shall see, the impact of higher systematic risk leads to even more concentration in the Minimum Variance and Maximum Diversification portfolios than is typical over time.

Exhibit 5 shows asset weights plotted against beta for the three risk-based portfolios; Minimum Variance (● markers), Maximum Diversification (◊ markers), and Risk-Parity (+ markers). Only 45 of the 1000 investable stocks are in solution for the long-only Minimum Variance portfolio in January 2012, ranging from a weight of about 8.0 percent to zero. The stocks with positive Minimum Variance weights all have betas below the long-only threshold beta of 0.7 for this period, in accordance with Equation 3. The Minimum Variance weights tend to fall on a negatively sloped line, also in accordance with Equation 3, although the correspondence is not perfect because of the impact of idiosyncratic risk. For example, the stock with the lowest beta of 0.4 in Exhibit 4 has a Minimum Variance weight in Exhibit 5 of about 3.6 percent, much lower than the largest weight of 8.0 percent. Notice that the impact of idiosyncratic risk on this Minimum Variance portfolio weight is greater than for the same 0.4 beta stock in the Maximum Diversification portfolio, because stock-specific idiosyncratic risk squared in the denominator of the first term in Equation 3 is not offset by total stock risk, as in the numerator of the second term in Equation 4.

The long-only Maximum Diversification portfolio has 63 of the 1000 investable stocks in January 2012, also with weights ranging from about 8.0 percent to zero. The stocks with positive weights tend to have low betas, but several have high betas, including one Maximum Diversification weight of about 0.3% for a stock with a beta of 2.4. When the Maximum Diversification weights are plotted against factor correlation (not shown) instead of factor beta, the positive weights are all associated with stocks that have correlations below the long-only threshold correlation of about 0.4 for this period, in accordance with Equation 4. For example, the 2.4 beta stock mentioned above has the highest idiosyncratic risk in this set (see highest dot in Exhibit 4) leading to relatively low correlation to the common risk factor and thus a positive Maximum Diversification weight in Exhibit 5.

As shown using the right-hand scale in Exhibit 5, the Risk Parity portfolio weights are positive for all 1000 securities, and the largest weight of about 0.25 percent is given to the stock with the lowest beta. The Risk Parity weights in Exhibit 5 are remarkably aligned with factor beta along a curve that has the hyperbolic functional form indicated by Equation 6. Because the Risk Parity portfolio includes all 1000 stocks, no single stock has an exceptionally large weight, and the impact of the idiosyncratic risk on the weights is negligible. The lower concentration of the Risk Parity portfolio does *not* mean, however, that it has less risk ex-ante than the other two portfolios. By design, the Minimum Variance portfolio has the lowest ex-ante risk of all three (or any other long-only portfolio) and will generally have the lowest risk ex-post depending on the accuracy of the risk model. Finally, we note as discussed in Lee [2011] that the Risk Parity portfolio weights are perfectly aligned with the inverse of another asset beta: the beta of each stock to the final Risk Parity portfolio (not shown).

Charts of risk-based portfolio weights similar to Exhibit 5 for dates other than January 2012 illustrate the dynamic nature of market risk and thus risk-based portfolio construction over time. In the mid-1990s, some of the largest 1000 U.S. common stocks exhibited negative market factor betas, and are consequently given relatively large weights in each of the three risk-based portfolios, but particularly the Risk Parity portfolio. For example, the Best Buy Corporation, a potentially countercyclical stock, had a predicted beta of about -0.2 in 1995, and a weight in the Risk Parity portfolio of over one percent. As explained in the technical appendix, Risk Parity portfolios can only accommodate assets with limited magnitude of negative common factor beta before the goal of parity across all assets becomes untenable. On the other hand, a key determinant of the concentration in the Maximum Diversification and Minimum Variance portfolios is the value of the long-only threshold parameters, which are in turn a function of the level of systematic risk, in addition to the range of betas.

For example, under the simplifying assumption of homogenous idiosyncratic risk (i.e., $\sigma_{\varepsilon,i}$ is the same for all stocks), Equation A11 in the technical appendix for the Minimum Variance portfolio long-only threshold beta is

$$\beta_L = \frac{\frac{\sigma_\varepsilon^2}{\sigma_F^2} + \sum_{\beta_i < \beta_L} \beta_i^2}{\sum_{\beta_i < \beta_L} \beta_i} . \tag{6}$$

Holding the cross-sectional distribution of betas fixed, the long-only threshold beta increases with the ratio of idiosyncratic to systematic risk in Equation 6, increasing the number of securities in solution. Intuitively, higher idiosyncratic relative to systematic risk in the marketplace allows for more diversification through a larger number of positions in the Minimum Variance portfolio. In addition, because the summations in Equation 6 increase with the number of securities, larger investable sets have comparatively lower threshold betas. As a result, the proportions of securities that are included in the long-only Maximum Diversification and Minimum Variance solutions decrease with larger investable sets, for example 100 out of 1000 investable assets, compared to say 30 out of 100 investable assets.

Summary and Conclusions

High market volatility, increased investor risk-aversion, and the provocative findings of risk anomalies within the equity market have prompted a surge of empirical research on risk-based portfolio strategies. Simulations of Minimum Variance portfolios over different markets, time periods, and constraint sets have proliferated, with additional interest in Maximum Diversification portfolios and the application of Risk Parity to security selection. Most of these empirical studies confirm the finding that risk-based portfolio structures have historically done quite well, due to one or the other linear versions of the risk anomaly or other dynamic aspects of equity market history. Such studies, however, depend on the specification of the risk model, often proprietary or commercially-based, and individual position limits, in addition to the usual vagaries of empirical work. While some analytic perspectives on the properties of objective-function (i.e., Maximum Diversification and Minimum Variance) portfolios has been published using standard unconstrained portfolio theory, implementation is inevitably in a long-only format leading to portfolios that are materially different from their long/short counterparts. In addition, little analytic work on large-set Risk Parity portfolio construction has been developed, beyond the definitional property that total versus marginal risk contributions define the weight structure.

This paper provides analytic solutions to Risk Parity and long-only constrained Maximum Diversification and Minimum Variance portfolios. The optimal weight equations are not strictly closed form, but provide helpful intuition on the construction of risk-based portfolios. Rather than simply supplying historical data to a constrained optimization routine, the analytic solutions allow portfolio managers to understand why any given investable asset is included in the risk-based portfolio, as well as the reasons for the magnitude of its weight. In addition, the single-factor solution allows for simple numerical calculations of long-only weights for large investable sets, yielding an empirical simulation that is easily replicable. The application of these equations to the largest 1000 U.S. stocks over several decades confirms the ex-post superiority of Minimum Variance portfolios in terms of minimizing risk, as well as the less direct benefit of a higher Sharpe Ratio. In addition, the first large-sample empirical results on Risk Parity equity portfolios reported in this study shows promise in terms of a high ex-post Sharpe ratio.

The analytic solutions reveal several important aspects of risk-based portfolio construction. Some are intuitive properties elucidated by the algebraic forms of the optimal risk-based weights, while others are more subtle. First, long-only objective-function-based portfolios exclude a large portion of the investable set, with the proportional exclusion becoming greater with the size of the investable set. Second, all three risk-based portfolios have asset weights that decrease with both systematic and idiosyncratic risk, but systematic (i.e., non-diversifiable) risk is the dominant factor, especially for Risk Parity portfolios. Third, long-only thresholds on asset risk parameters vary with the ratio of systematic risk to average idiosyncratic risk over time. In particular, risk-based portfolios become more concentrated with higher systematic risk and lower levels of idiosyncratic risk. Higher portfolio concentration does not, however, equate to higher risk ex-ante, or even ex-post in terms of the empirical track record of the single-factor model. Fourth, negative beta assets have particularly large weights (intuitively due to hedging properties) in all three risk-based portfolios, although extreme negative beta assets cannot be accommodated under the standard definition of Risk Parity.

As noted in Scherer [2011], the Ang, Hodrick, Xing, and Zhang [2006] historical risk anomaly in the cross-section of stocks can be characterized as asset exposure to systematic risk, idiosyncratic risk, or both. However, the inter-temporal dynamics of the market over time, as well as the "second moment" nature of risk (i.e., return variance) makes it unlikely that the anomaly can be completely reduced to a simple linear factor like value or momentum. In any event, the primary purpose of this study has been to provide general analytic solutions to risk-based portfolios, not just another empirical back-test. Further examination of the historical data in regards to transaction costs and turnover, for example in Li, Sullivan, and Garcia-Feijoo [2012], may shed more light on the exploitability of the low risk anomaly. However, the emergence of new objective functions, specifically designed around the anomaly, may constitute a subtle form of data mining. In that regard, Minimum Variance portfolios, specified from the beginning of modern portfolio theory (i.e., 1960s), may be more robust. On the other hand, Risk Parity, which was conceptualized in the 1990's with broad asset allocation in mind, but now applied to large-set security portfolios, may also be less susceptible to an ex-post bias.

Technical Appendix

The Minimum Variance objective function is minimization of ex-ante (i.e., estimated) portfolio risk,

$$\sigma_P^2 = \mathbf{w}' \, \mathbf{\Omega} \, \mathbf{w} \qquad \text{(A1)}$$

where $\mathbf{w}$ is an N-by-1 vector of asset weights, and $\mathbf{\Omega}$ is an N-by-N asset covariance matrix. One form of the well-known solution to this optimization problem is

$$\mathbf{w}_{MV} = \sigma_{MV}^2 \, \mathbf{\Omega}^{-1} \, \boldsymbol{\iota} \qquad \text{(A2)}$$

where $\boldsymbol{\iota}$ is N-by-1 vector of ones. As a practical matter, $\sigma_{MV}^2 = 1 / \boldsymbol{\iota}' \mathbf{\Omega}^{-1} \, \boldsymbol{\iota}$ in Equation A2 is simply a scaling parameter which enforces the budget constraint that asset weights sum to one.

The Maximum Diversification objective is to maximize the "Diversification Ratio"

$$D_P = \frac{\mathbf{w}' \boldsymbol{\sigma}}{\sqrt{\mathbf{w}' \, \mathbf{\Omega} \, \mathbf{w}}} \qquad \text{(A3)}$$

where $\boldsymbol{\sigma}$ is an N-by-1 vector of asset volatilities; the square-root of the diagonal terms of $\mathbf{\Omega}$. Equation A3 has the form of a Sharpe ratio, where the asset volatility vector, $\boldsymbol{\sigma}$, replaces the expected excess returns vector. Using the well-known solution to the tangent (i.e., maximum Sharpe ratio) portfolio with this substitution gives the optimal Maximum Diversification weight vector as

$$\mathbf{w}_{MD} = \left(\frac{\sigma_{MD}^2}{\sigma_A} \right) \mathbf{\Omega}^{-1} \, \boldsymbol{\sigma} \, . \qquad \text{(A4)}$$

where σ_A is the weighted average asset risk. The key conceptual difference between the Minimum Variance solution in Equation A2 and the Maximum Diversification solution in equation A4 is not the scaling parameters, but the post multiplication of the inverse covariance matrix by the asset risk vector.

The risk-budgeting interpretation of Risk Parity is based on the restatement of Equation A1 as a double sum,

$$\sigma_P^2 = \sum_{i=1}^{N} w_i \sum_{j=1}^{N} w_j \, \mathbf{\Omega}_{i,j} \, . \tag{A5}$$

where $\mathbf{\Omega}_{i,j}$ are the elements in the asset covariance matrix $\mathbf{\Omega}$. A portfolio is said to be in "parity" if the total (rather than marginal) risk contribution is the same for all assets

$$\frac{w_i \sum_{j=1}^{N} w_j \, \mathbf{\Omega}_{i,j}}{\sigma_P^2} = \frac{1}{N} \, . \tag{A6}$$

An equivalent "portfolio beta" interpretation is that Risk Parity is achieved when weights are equal to the inverse of their beta with respect to the final portfolio,

$$w_i = \frac{1}{N \beta_{i,P}} \, . \tag{A7}$$

Note that the asset beta with respect the Risk Parity portfolio in Equation A7 is not the same as β_i, the notation we use for the beta of the asset with respect to the common risk factor.

In the body of this paper, we focus on a single-factor risk model for the asset covariance matrix, a common simplifying assumption in portfolio theory. Using matrix notation, the *N*-by-*N* asset covariance matrix in a single-factor model is

$$\mathbf{\Omega} = \boldsymbol{\beta}\boldsymbol{\beta}' \sigma_F^2 + \mathrm{Diag}(\boldsymbol{\sigma}_\varepsilon^2) \tag{A8}$$

where $\boldsymbol{\beta}$ is an *N*-by-1 vector of risk-factor loadings, σ_F^2 is the risk factor variance, and $\boldsymbol{\sigma}_\varepsilon^2$ is an *N*-by-1 vector of idiosyncratic risks. Using the Matrix Inversion Lemma, the inverse covariance matrix in the single-factor model is analytically solvable

$$\mathbf{\Omega}^{-1} = \mathrm{Diag}(1/\boldsymbol{\sigma}^2) - \frac{\left(\boldsymbol{\beta}/\boldsymbol{\sigma}^2\right)\left(\boldsymbol{\beta}/\boldsymbol{\sigma}^2\right)'}{\frac{1}{\sigma_F^2} + \left(\boldsymbol{\beta}/\boldsymbol{\sigma}^2\right)'\boldsymbol{\beta}} \tag{A9}$$

and can be substituted into the optimization solutions in Equations A2 and A4. Specifically, the individual optimal weights in the long-only Minimum Variance solution are given in Equation 3 of the paper, where β_L is a "long-only threshold" beta that cannot be exceeded in order for an asset to be in solution. The long-only threshold beta is a function of final portfolio risk estimates

$$\beta_L = \frac{\sigma_{MV}^2}{\beta_{MV}\ \sigma_F^2} \tag{A10}$$

where β_{MV} is the risk-factor beta of the long-only Minimum Variance portfolio. As a more practical matter, the threshold beta can be calculated from summations of individual asset risk statistics that come into solution,

$$\beta_L = \frac{\dfrac{1}{\sigma_F^2} + \displaystyle\sum_{\beta_i < \beta_L} \frac{\beta_i^2}{\sigma_{\varepsilon,i}^2}}{\displaystyle\sum_{\beta_i < \beta_L} \frac{\beta_i}{\sigma_{\varepsilon,i}^2}}. \tag{A11}$$

Equation A11 is a more practical than Equation A10 in the sense that assets can be sorted in order of ascending factor beta, and then compared to the summations until an individual asset beta exceeds the long-only threshold. Together with the fact that portfolio risk is simply a scaling factor in Equation 3 of the paper, numerical search routines are not required to find the set of assets and their optimal weights for long-only Minimum Variance portfolios. Scherer [2011] provides analytic work that is similar in form to Equation 3, but for unconstrained long/short portfolios.

For a *K*-factor risk model, the general unconstrained solution to the Minimum Variance portfolio is

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2}\left(1 - \sum_{k=1}^{K} \frac{\beta_{k,i}}{\beta_k}\right) \tag{A12}$$

where β_k is a portfolio (i.e., not asset specific) parameter, calculated as

$$\beta_k = \frac{\sigma_{MV}^2}{\sum_{l=1}^{K} \beta_{MV,l} \, V_{k,l}} . \tag{A13}$$

In Equation A13, $\beta_{MV,l}$ is the portfolio beta with respect to the l^{th} risk-factor and $V_{k,l}$ is an element of the *K*-by-*K* factor covariance matrix. The *K*-factor solution in Equation A12 also has a long-only *constrained* version, although the criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Substitution of the inverse covariance matrix from Equation A9 into A4, gives a relatively simple solution for individual weights in the Maximum Diversification portfolio as shown in Equation 4 of the paper, which includes the term ρ_L as a "long-only threshold" correlation that cannot be exceeded for an asset to be in solution. The long-only threshold correlation can be described as a function of final portfolio risk estimates

$$\rho_L = \frac{\sigma_{MD}^2}{\sigma_A \, \beta_{MD} \, \sigma_F} \tag{A14}$$

where β_{MD} is the factor beta and σ_A is the average asset risk, respectively. As a more practical matter, the long-only threshold correlation can be calculated from summations of individual asset correlations that come into solution,

$$\rho_L = \frac{1 + \sum_{\rho_i < \rho_L} \frac{\rho_i^2}{1 - \rho_i^2}}{\sum_{\rho_i < \rho_L} \frac{\rho_i}{1 - \rho_i^2}} . \tag{A15}$$

Equation A15 is a more practical than A14 in the sense that assets can be sorted in order of ascending factor correlation, and then compared to the summations until the individual asset correlation exceeds the long-only threshold. Note that assets sorted by factor correlation will be in a somewhat different order than sorting by factor beta due to differences in idiosyncratic risk. Together with the fact that final portfolio risk divided by average asset risk is simply a scaling factor in Equation 4, numerical search routines are not required to find the set of assets and their optimal weights for long-only Maximum Diversification portfolios. Carvalho, Lu, and Moulin [2012] provide analytic work that produces a form similar to Equation 5, but for unconstrained long/short portfolios.

For a K-factor risk model, the general unconstrained solution to the Maximum Diversification portfolio is

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \sum_{k=1}^{K} \frac{\rho_i}{\rho_k} \right) \tag{A16}$$

where ρ_k is a portfolio (i.e., not asset specific) parameter

$$\rho_k = \frac{\sigma_{MD}^2}{\sigma_A \sum_{l=1}^{K} \beta_{MD,l} V_{k,l}} \ . \tag{A17}$$

In Equation A17, $\beta_{MD,l}$ is the portfolio beta with respect to the l^{th} risk-factor and $V_{k,l}$ is an element of the K-by-K factor covariance matrix. The K-factor solution in Equation A16 has a long-only *constrained* version, although criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Risk Parity portfolio weights under a single-factor risk-model can be derived by substituting the individual covariance matrix terms of Equation A8 into A6, and applying the quadratic formula. Using the "positive root" in that formula gives Equation 5 in the body of the paper, where

$$\gamma = \frac{2\,\sigma_{RP}^2}{\beta_{RP}\,\sigma_F^2} \tag{A18}$$

is a constant term across asset weights that includes β_{RP}, the beta of the Risk Parity portfolio with respect to the risk factor. Unlike the equations for individual weights in the Minimum Variance and Maximum Diversification portfolios, Equation 5 cannot be used to perform a simple asset sort to calculate optimal weights because final portfolio terms are embedded in the equation, not simply part of the scaling factor. However, Equation 5 does suggest a quick numerical routine for even large investable sets. The numerical process starts with an equally-weighted portfolio, and then iteratively calculates the portfolio parameters (β_{RP} and σ_{RP}) and asset weights until the weights converge to one over *N* times their beta with respect to the final portfolio, $\beta_{i,RP}$, in accordance with Equation A7. We also note that the quadratic formula allows for a "negative root" that is real. However, the positive root, specified in Equation 5, is the only one that maintains the budget constraint that the asset weights sum to one.

For a general K-factor risk model, the solution to the Risk Parity portfolio is

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2}\left[\left(\left(\sum_{k=1}^{K}\frac{\beta_{k,i}}{\gamma_k}\right)^2 + \frac{1}{N}\frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2}\right)^{1/2} - \sum_{k=1}^{K}\frac{\beta_{k,i}}{\gamma_k}\right] \tag{A19}$$

where
$$\gamma_k = \frac{2\sigma_{RP}^2}{\sum_{l=1}^{K}\beta_{RP,l}\, V_{k,l}}. \tag{A20}$$

The general *K*-factor Risk Parity portfolio is long-only by definition and can be used to inform a quick calculation routine similar to the one described above for the single-factor model.

Under the single-factor model, the implications of differences in beta and idiosyncratic risk on the relative magnitude of the weights in the Minimum Variance and Maximum Diversification portfolios is fairly evident from the algebraic form of Equations 3 and 4. The more complex form for Risk Parity portfolio weights in Equation 5 makes the impact of the different asset risk parameters less apparent, motivating formal calculus. The partial derivative of the Risk Parity asset weight in Equation 5 with respect to asset beta is

$$\frac{\partial w_{RP,i}}{\partial \beta_i} = \frac{-w_{RP,i}}{\gamma\left(\frac{\beta_i^2}{\gamma^2} + \frac{1}{N}\frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2}\right)^{1/2}} \tag{A21}$$

and thus always negative because weights in the long-only solution are positive, as is the square-root term in the numerator of Equation A21. The derivative is "partial" in the sense that

Equation A21 applies to the magnitude of asset weights compared to each other in a risk-balanced set. A full derivative would be more complex because a change in the risk parameter for any single asset will change the Risk Parity weights for the entire set.

The functional form of Equation 5 (i.e., squared terms and square-roots) for Risk Parity portfolios indicates that weights asymptotically approach zero with high beta, and increase with low beta, including negative betas if any. Indeed, under the assumption of homogenous idiosyncratic risk, Risk Parity weights form the positive side of a non-rectangular hyperbola, centered at the origin, with linear asymptotes of the X-axis for larger beta stocks, and a line given by

$$w(asymptote) = -\frac{\beta_{RP}\,\sigma_F^2}{\sigma_\varepsilon^2} \tag{A22}$$

for lower beta stocks. As an indication of the eccentricity of the hyperbolic curve for weights, an asset with a factor beta of exactly zero would have a weight of $\sigma_{RP} / \sigma_\varepsilon \sqrt{N}$.

Assets with a large negative beta with respect the single risk factor, β_i, may have a negative beta with respect to the final portfolio, $\beta_{i,RP}$. However, negative weight assets were likely not envisioned by those who formalized the Risk Parity condition. Specifically, an asset with a small positive $\beta_{i,RP}$ would have a relatively large positive weight according to the definition in Equation A7. But a slightly different asset with a small negative $\beta_{i,RP}$ would have a large negative weight, a nonsensical result for an asset with the potential to hedge the single risk factor. The lower limit for β_i on any single asset that still allows for a Risk Parity portfolio involving all investable assets is complex, although it can be shown that under the assumption of homogenous idiosyncratic risk, the lower limit is the arithmetic inverse of the hyperbolic asymptote in Equation A22. The number of iterations for convergence in the numerical process specified above for Risk Parity portfolios increases for investable sets that have betas which approach that limit, although betas of that magnitude were not encountered anywhere in the 528 months (1968 to 2011) for the 1000 largest U.S. stocks in the empirical part of this paper.

Exhibit 1

Performance of Risk-Based Portfolios from 1968 to 2011

	Market (Value-Weighted)	Equal Weighted	Risk Parity	Maximum Diversification	Minimum Variance
Average Excess Return	5.1%	7.2%	7.2%	5.5%	5.7%
Standard Deviation	15.6%	17.9%	16.7%	19.1%	12.5%
Sharpe Ratio	0.33	0.40	0.43	0.29	0.46
Compound Return	4.1%	6.0%	6.3%	3.8%	5.3%
Market Beta	1.00	1.09	1.01	0.94	0.52
Average Positions	1000.0	1000.0	1000.0	82.7	62.4
Effective N	138.5	1000.0	933.5	46.7	36.0

Exhibit 2
Risk-Based Portfolio Performance from 1968 to 2011

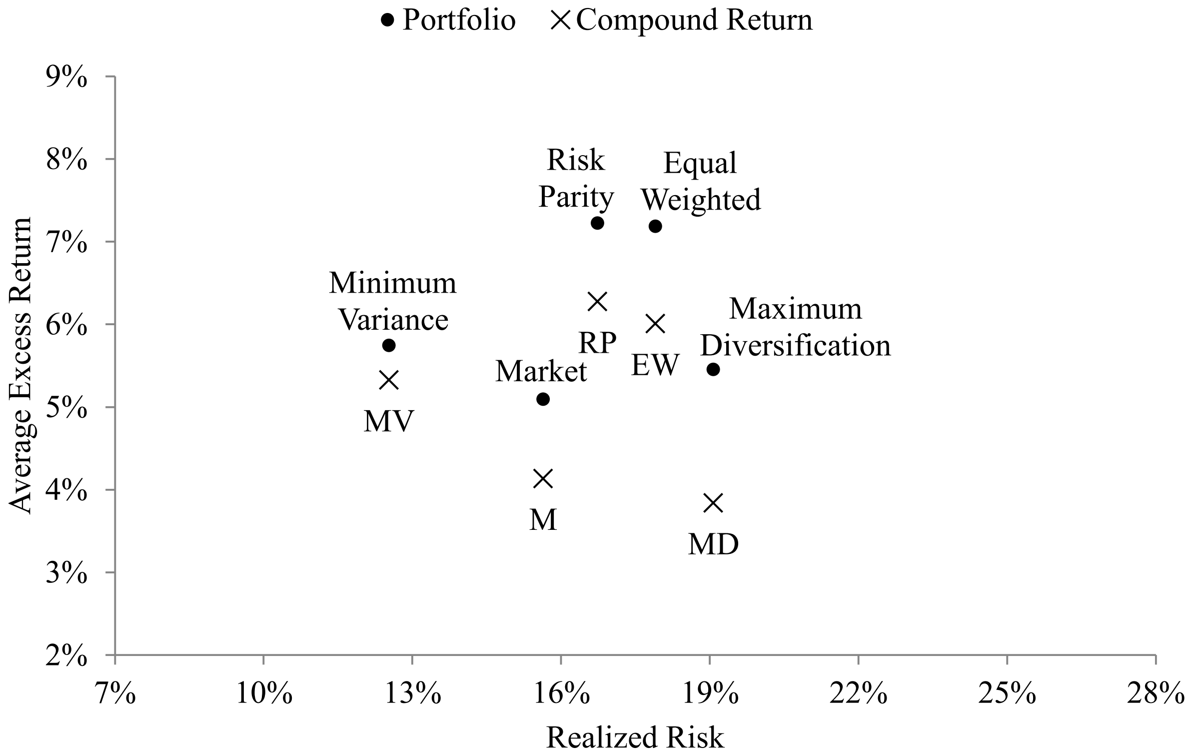


Exhibit 3
Risk-Quintile Portfolio Performance from 1968 to 2011

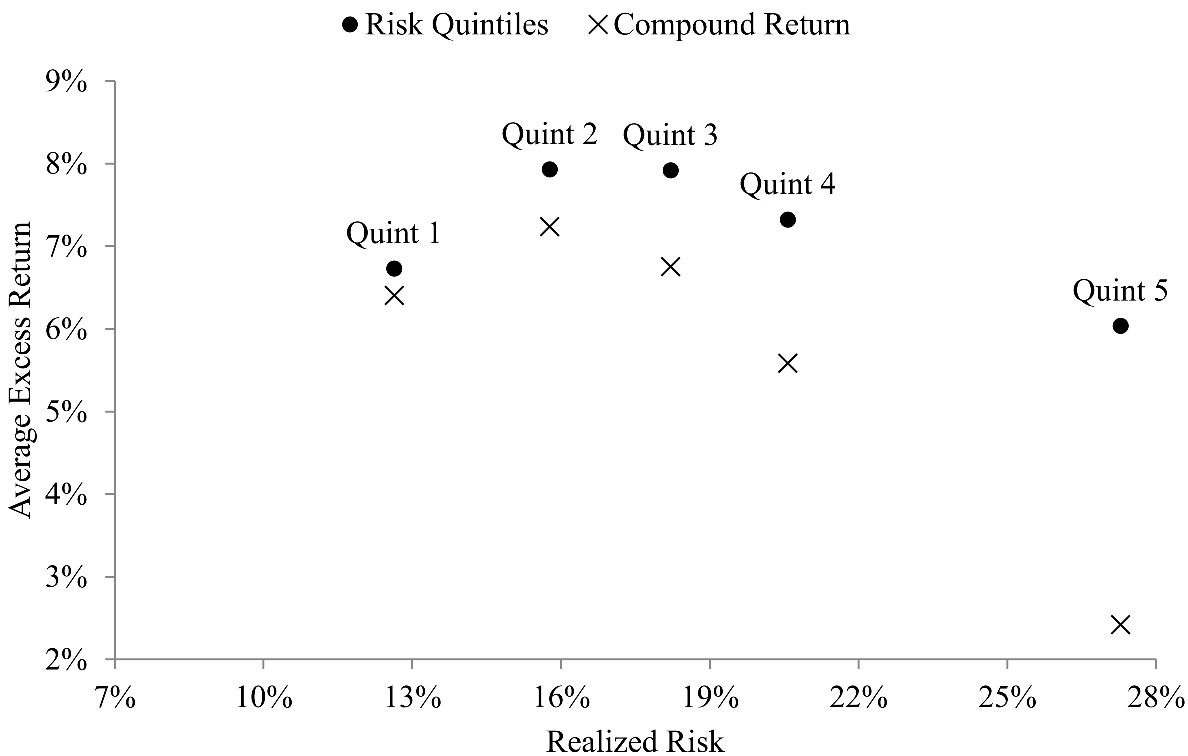


Exhibit 4
Beta and Idiosyncratic Risk for 1000 U.S. Stocks in 2012

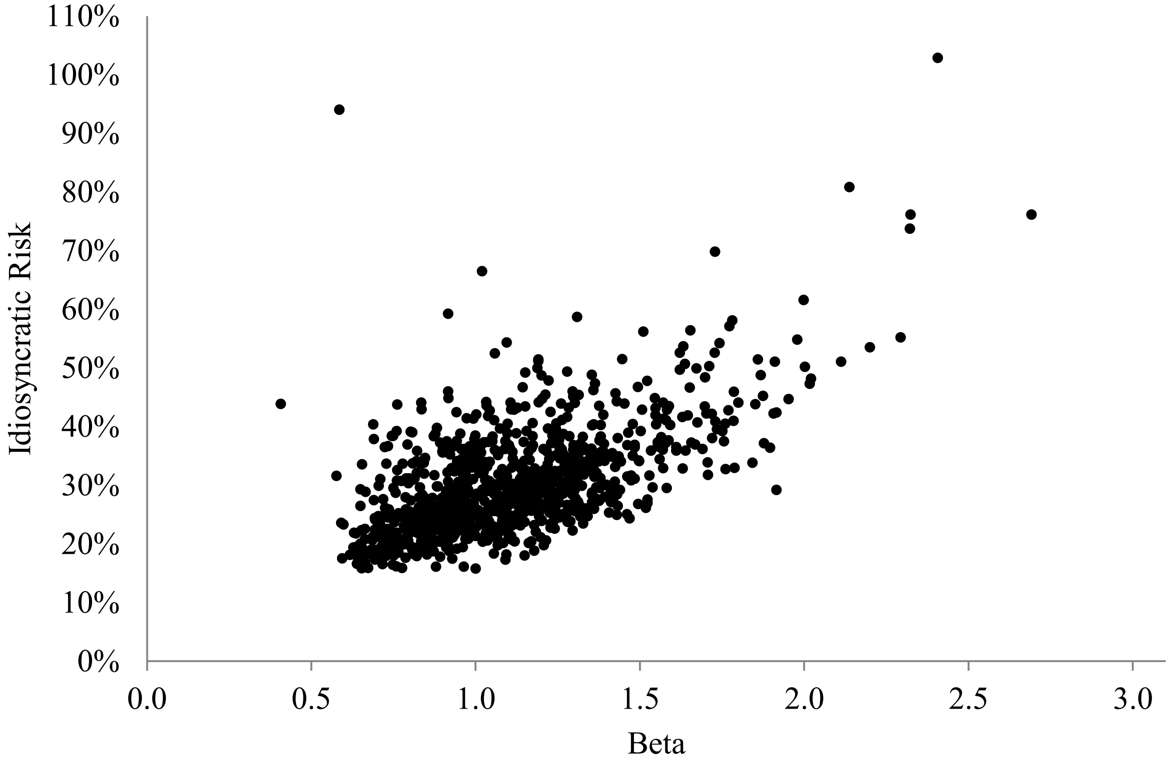
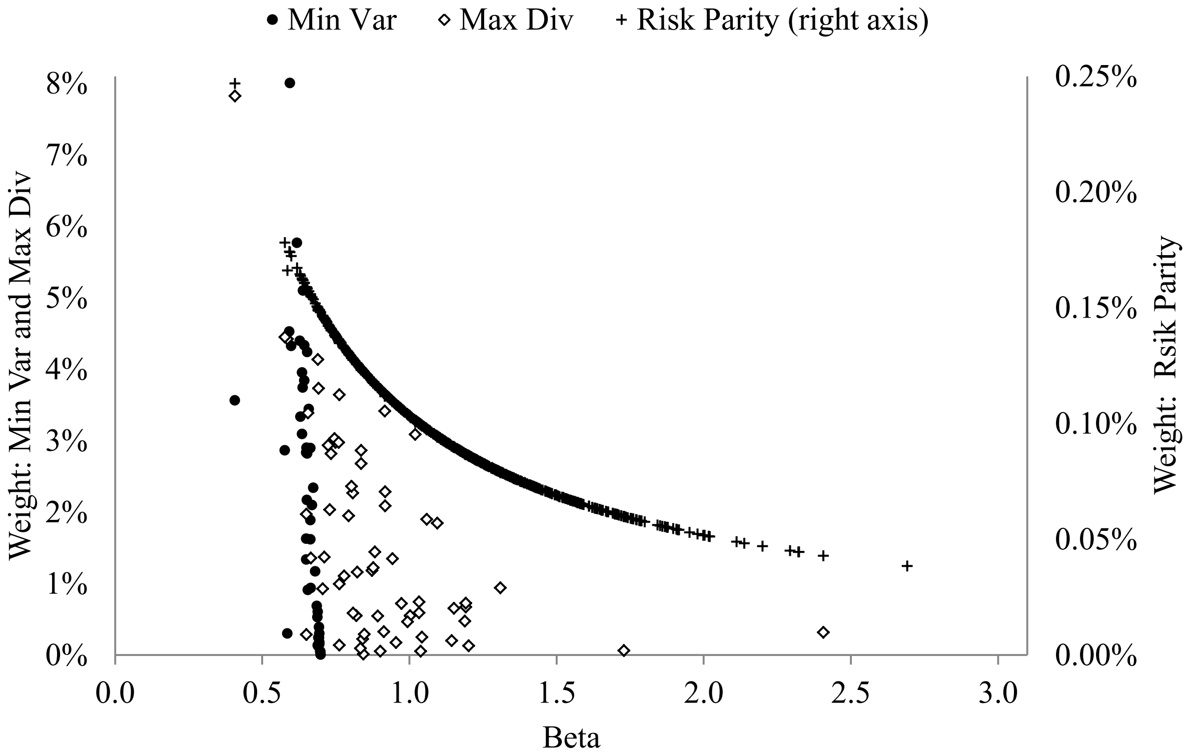


Exhibit 5
Risk-Based Portfolio Asset Weights and Market Betas in 2012



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[1] The inverse volatility property of Maximum Diversification portfolio weights was also demonstrated by Maillard, Roncalli, and Teiletche [2010] given "constant correlation" in the asset covariance matrix, a simplifying assumption for risk-model analysis first introduced by Elton and Gruber [1973]. In fact, under the simplifying assumption of constant correlation, the Maximum Diversification portfolio is equivalent to the Risk Parity portfolio, as discussed in Choueifaty and Coignard [2008].

[2] The "shrinkage" of historical betas for purposes of risk prediction is similar to the Bloomberg rule of adjusting historical beta values ⅓ towards one. We use ½ instead of ⅓ based on observed values of the coefficient in cross-sectional regressions of 60-month realized betas on 60-month historical betas for 1000 stocks. The choice of ⅓ shrinkage towards the mean for log idiosyncratic risk is also based on historical regression values. We shrink using logs because idiosyncratic risk is by definition a positively valued variable and thus highly skewed.

[3] Excess compound returns are calculated as the compound total return minus the compound Treasury bill return from 1968 to 2011. A common rule of thumb, depending on the distribution of returns, is that compound returns are equal to the arithmetic average return minus half the return variance. The compound returns reported in this study are fairly consistent with that rule.

[4] Note that the initial identification of the low risk anomaly by Ang et. al. used a short-horizon risk estimate (daily returns for the prior month), but quantitative portfolio managers generally attempt to exploit the anomaly using longer-horizon risk estimates (monthly returns for the past 3 to 5 years.

[5] Consistent with Choueifaty and Coingnard [2008], we find in sensitivity analysis that the Maximum Diversification portfolio fares better using the top 500 (i.e., S&P 500) rather than the top 1000 (e.g., Russell 1000), although still not as well as the other two portfolios, similar to the findings of Linzmeier [2011]. We also note that correlations, which are critical to Maximum Diversification portfolio structure, may be better estimated with a multi-factor risk model.

[6] Quantitative portfolio management has a long tradition of shrinking forecasted or expected returns, for example using the ex-ante Information Coefficient in the Grinold-Kahn framework or equilibrium expected returns in the Black-Litterman approach. As portfolio construction techniques that rely solely on risk parameters become more common, managers are learning to similarly shrink the cross-sectional spread of historical risk. A formal statistical approach using the Bayesian theory of Ledoit and Wolf [2004] varies around ½, but we choose to leave the shrinkage parameter at exactly ½ over time for ease of study replication.