Beyond Mean–Variance: Performance Measurement in a Nonsymmetrical World

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Most practitioners use the capital asset pricing model to measure investment performance. The CAPM, however, assumes either that all asset returns are normally distributed (and thus symmetrical) or that investors have mean–variance preferences (and thus ignore skewness). Both assumptions are suspect. Assuming only that the rate of return on the market portfolio is independently and identically distributed and that markets are "perfect," this article shows that the CAPM and its risk measures are invalid: The market portfolio is mean–variance inefficient, and the CAPM alpha mismeasures the value added by investment managers. Strategies with positively skewed returns, such as strategies limiting downside risk, will be incorrectly underrated. A simple modification of the CAPM beta, however, will produce correct risk measurement for portfolios with arbitrary return distributions, and the resulting alphas of all fairly priced options and/or dynamic strategies will be zero. The risk measure requires no more information to implement than the CAPM.

How can one determine whether an investment manager has added value to the funds the manager is handling relative to the risk the manager is taking? Correct performance assessment requires both good theory, to determine the proper measure of risk, and appropriate statistical techniques, to quantify risk magnitudes. This article focuses on measures of risk and the implications of those measures for investment performance evaluation.

Although some notable advances have recently been made in the theory of performance measurement, most practice is firmly rooted in the approach of the capital asset pricing model.[1] In the CAPM world, the appropriate measure of the risk of any asset or portfolio p is given by its beta:

$$\beta_p = \frac{\text{cov}(r_p, r_{mkt})}{\text{cov}(r_{mkt}, r_{mkt})}$$

$$= \frac{\text{cov}(r_p, r_{mkt})}{\text{var}(r_{mkt})}, \qquad (1)$$

where r_p and r_{mkt} are the random returns on, respectively, the portfolio and the market.

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In equilibrium, all assets and portfolios will have the same return after adjustment for risk, which implies the following formula for the expected return on the portfolio:

$$E(r_p) = r_f + \beta_p[E(r_{mkt}) - r_f], \qquad (2)$$

where r_f is the risk-free interest rate.

Superior performance in the CAPM world is measured by alpha, which is the incremental expected return resulting from applying managerial information (e.g., stock selection or market timing). The portfolio alpha can be represented formally as

$$\alpha_p = E(r_p \mid M) - E(r_p)$$
$$= E(r_p \mid M) - \beta_p[E(r_{mkt}) - r_f] - r_f, \qquad (3)$$

where $E(r_p \mid M)$ is the expected return to the portfolio conditioned by the information used by the manager, M.[2] In the CAPM equilibrium, alphas will be zero unless a manager has superior information. A portfolio with a positive alpha offers an expected return in excess of its equilibrium risk-adjusted level and, in this sense, has superior performance. A related, but not identical, performance measure is the Sharpe ratio (SR) of a portfolio. In it, $SR_p = [E(r_p \mid M) - r_f]/\sigma_p$. The Sharpe ratio provides an appropriate measure of investor welfare when the investor has mean–variance preferences and invests exclusively in the portfolio (and perhaps a

risk-free asset). Alpha, on the other hand, is a measure of performance when the portfolio is a small part of the investor's entire (fully diversified) portfolio of assets. A portfolio with a Sharpe ratio greater than the market's will have a positive alpha, but the converse does not necessarily hold.

Underlying the CAPM and its associated risk and performance measures are the strong assumptions that either (1) all asset returns are normally (and thus symmetrically) distributed or (2) investors care only about the mean and variance of returns (which implies that they view upside and downside risks with equal distaste). Neither assumption that justifies the CAPM approach is satisfactory: Portfolio returns are not, in general, normally distributed. And even if the underlying assets' returns were normal, the returns of portfolios that use options on those assets or use dynamic strategies would not be.[3] Furthermore, investors typically do distinguish between upside and downside risks.[4] For example, most investors have a preference for positively skewed returns, which implies that prices in equilibrium reflect more than mean and variance.[5]

Thus, the basic underpinnings of the CAPM are suspect, as is its risk measure—beta. And when beta does not correctly measure risk, estimates of alpha will be incorrect and the performance of portfolio managers will be mismeasured. Nevertheless, although these shortcomings have been cited in the academic literature, the CAPM is widely used by practitioners.[6]

This article goes beyond the mean–variance framework of performance measurement. It presents a simple risk measure that requires no more information to implement than the CAPM but correctly captures all elements of risk, including skewness, kurtosis, and other characteristics that further describe the shape of the return distribution. Thus, the results described here apply to nonsymmetrical return distributions.

The model requires only two assumptions:
1. Returns of the market portfolio are independently and identically distributed (IID) at each moment in time.
2. Markets are perfect. That is, the model considers no transaction costs or taxes, prices reflect perfect competition, and all relevant risks are traded in the market.

The first assumption, although clearly strong, underlies most econometric studies; therefore, the assumption is implicit in the current risk measures used by practitioners. (Relaxation of this assumption is considered later in the article.) The second assumption underlies the CAPM and most other equilibrium models of asset valuation as well as the model presented here.

In the limit, as the periods become infinitesimal in length, the first assumption implies that the market portfolio's returns are lognormally distributed over any finite interval.[7] In continuous time, the rate-of-return process is a diffusion with constant drift and volatility and, therefore, is consistent with the models of Black and Scholes (1973) and Merton (1973).[8]

Note that individual asset or portfolio returns are *not* assumed to be lognormal: The assumption of IID returns and the resulting lognormal return distribution refers only to the market portfolio.[9] Note also that no particular utility function is (directly) assumed to represent investor preferences.

Given the two assumptions, it will be shown that a valid risk measure exists for all portfolios that have *arbitrary* distributions of returns, including nonsymmetrical distributions. The correct risk measure will have the property that any portfolio strategy has zero measured excess return after adjustment for risk when that strategy can be implemented without superior information. Neither the CAPM alpha nor the Sharpe ratio possesses this property, as will be shown.

Problems with Mean–Variance Measures

In this section, a simple two-period IID binomial example shows that the market portfolio is mean–variance inefficient. A simple dynamic strategy is presented that does not require superior information to implement but has a higher Sharpe ratio than the market portfolio. The example contradicts the fundamental CAPM proposition that the market portfolio is mean–variance efficient.

Simple Binomial Example. Let the market portfolio increase by 25 percent or fall by 20 percent each year for a two-year period. The probability of an up move is 80 percent, which gives the market an annual expected return of 16 percent. The standard deviation of the market return in the two-year period is 29.71 percent. The annual risk-free rate is 5 percent, which implies a Sharpe ratio for the two-year period of $(1.16^2 - 1.05^2)/(0.2971) = 0.8182$. It can be easily shown that a static strategy that puts half its initial wealth in the market portfolio and half in bonds has the same Sharpe ratio with an expected return of 22.40 percent and a standard deviation of 14.85 percent for the two-year period.

Now, consider the following dynamic strategy: Start with a 60/40 stock-to-cash investment ratio. If the market rises in the first period, sell 44.8 percent of the market portfolio holding and convert it into cash. (Beginning the second period, 35.4 percent of

total holdings will be in the market and the remaining fraction will be in cash.) If the market falls in the first period, liquidate all cash and invest the proceeds in the market. (Beginning the second period, 100 percent of holdings will be in the market portfolio.) After two years, an initial wealth of $100 will become $131.13 in the (up, up) state, $112.50 in both the (up, down) and (down, up) states. Using the preceding binomial probabilities, this strategy will have an expected return of 22.80 percent and a standard deviation of 13.48 percent for the two-year period. The return is higher than and the volatility is lower than the 50/50 static strategy. The Sharpe ratio is 0.9310, which is substantially higher than that of the market or the 50/50 strategy. And a higher Sharpe ratio than the market implies a positive CAPM-measured alpha.

This dynamic strategy is multiperiod (and thus inconsistent with a single-period CAPM), but a *static* strategy using fairly priced options exists that yields exactly the same result as the dynamic strategy: Assuming the initial market value is normalized to $100, the option-based strategy would sell 0.794 at-the-money two-period call options on the market portfolio (with price, based on the binomial model, of $15.75). The strategy would invest the initial $100 wealth, plus the receipts of $12.50 from selling the call options, in the market portfolio. The strategy does not call for a subsequent change in portfolio holdings. It can be readily verified that this strategy using options yields the same payoffs as the dynamic strategy in each future state of the market.

By leveraging the dynamic strategy, or its options equivalent, the portfolio manager can obtain a higher expected return and lower risk than the market portfolio. The simple assumption of IID market returns, therefore, implies that the market portfolio is mean–variance inefficient in a perfect capital market![10]

Analysis of the Example. The mechanistic dynamic strategy appears to beat the market, and by traditional CAPM-based measures, it would achieve superior performance, although anyone could follow such a strategy.

The intuition underlying this example is as follows: Rubinstein (1976), Brennan (1979), and He and Leland (1993) showed that if the market portfolio's rate of return is IID and markets are perfect, the representative investor (whose preferences determine all prices) must have a power utility function.[11] The critical point about this utility function is that it has a positive third derivative, which implies skewness preference: Skewness will be positively valued by the market. Any investor can improve a

portfolio's performance in mean–variance terms by "selling" skewness (i.e., by accepting negatively skewed returns in return for improvements in mean and/or variance). This is exactly what the dynamic strategy created: negative skewness relative to the market return.

If only mean and variance are assessed, the negatively skewed returns will *seem* to outperform.[12] Outperformance is a misnomer here, in the sense that the average investor would not prefer to sacrifice skewness to improve returns in terms of mean and variance only. Moreover, as discussed, the CAPM-based "outperformance" does not mean that the investment manager has added value through identification of undervalued assets or by informed market timing.

Strategies Using Options on the Market. An implication of the preceding discussion is that the performance of portfolios containing fairly priced option positions (or following equivalent dynamic strategies) will also be mismeasured by the CAPM, which relies on a mean–variance framework. Consider two types of option strategies: those in which a call option on the market is written against an underlying position in the market portfolio and those in which a put option is bought to protect an underlying position in the market portfolio. Option strike prices range from deep in the money to deep out of the money. Assume the market follows a logarithmic Brownian motion with annual expected return of 12 percent and annual volatility of 15 percent.[13] The risk-free rate is 5 percent. Because this is a Black–Scholes world, the option prices will be determined by the Black–Scholes formula.[14] It is straightforward to use these parameters and the lognormality of the market return to compute the expected returns, covariances with the market, and CAPM beta of any option-based strategy.

The first type of option strategy—holding the market portfolio and writing one-year covered calls on the market—creates payoffs that are a concave function of the market payoff and, in that way, reduces or "sells" skewness. The dynamic strategies equivalent to writing covered calls have the feature that they sell the market portfolio as its price rises and buy as its price falls, without superior information. This type can be (loosely) labeled "rebalancing" or "value" strategy. The first three columns of Panel A in **Table 1** give the annual expected returns, CAPM betas, and CAPM alphas of strategies that write one-year calls at various strike prices.

The second type of option strategy—holding the market portfolio and buying put options on the

Table 1. CAPM-Based β and α versus B and A

Strike Price	$E(r)$[a]	β[b]	α[c]	B[d]	A[e]
A. Rebalancing or value strategies: Long the market, short one call					
90	5.51%	0.038	0.24%	0.073	0
100	6.76	0.163	0.62	0.251	0
110	8.61	0.394	0.85	0.515	0
120	10.27	0.650	0.72	0.753	0
130	11.30	0.838	0.57	0.900	0
140	11.77	0.939	0.20	0.967	0
B. Momentum or portfolio insurance strategies: Long the market, long one put					
90	11.49%	0.962	−0.24%	0.927	0
100	10.24	0.837	−0.62	0.749	0
110	8.40	0.606	−0.84	0.485	0
120	6.73	0.351	−0.72	0.247	0
130	5.70	0.163	−0.44	0.101	0
140	5.24	0.062	−0.19	0.034	0

[a]Computed assuming a lognormal market portfolio with annual mean of 12 percent, standard deviation of 15 percent, and the distributions implied for portfolios and options.
[b]Computed by Equation 1 from the assumptions in the first column.
[c]Computed by Equation 3 with $E(r_p \mid M)$ equal to the data in the first column and a risk-free rate of 5 percent.
[d]Computed by Equation 7. Equation 8 implies $b = 3.63$.
[e]Computed by Equation 9.

market—creates convex payoffs and thus creates or "buys" additional skewness. These strategies can be (again, loosely) labeled "momentum" or "portfolio insurance" strategies. The equivalent dynamic strategy buys the market portfolio on strength and sells on weakness. The first three columns of Panel B in Table 1 give the expected returns, CAPM betas, and CAPM alphas of strategies that buy one-year put options at various strike prices.

When skewness is positively valued, mean–variance-based performance measures will overrate the rebalancing strategies, which reduce skewness, and underrate the momentum strategies, which buy skewness.[15] **Figure 1**, based on the first three columns of Table 1, plots the expected returns and CAPM betas of the two types of option strategies for various strike prices. The rebalancing strategies plot above the security market line. Momentum strategies plot below the security market line.[16] The CAPM-based alphas (from Table 1) are measured by the vertical distance between the point representing each portfolio and the security market line. The curved lines show that for strike prices near the money, alphas are substantially different from zero.[17]

Of course, if the alphas here were properly measured, they would be zero: Options are assumed to be purchased at a fair market price. The reason they are not zero is that the CAPM risk measure, beta, is incorrect and Equation 2 does not hold when the market is lognormally distributed. Although the manager has no additional informa-

tion (i.e., $E[r_p \mid M] = E[r_p]$), α_p in Equation 3 is nonzero. Note that any investment manager can "game" the CAPM performance measurement by selling options or rebalancing.

These examples consider only strategies buying or selling options on the market, but similar results are likely when individual security options are bought or sold because those actions will also affect the skewness of the managed portfolio relative to the market.

Correct Measures of Risk and Performance

The CAPM-based alpha systematically mismeasures performance when the market has IID returns because the CAPM-based beta does not capture skewness and other higher-order moments of the return distribution that investors value.[18] The first "patch" to fix this situation might be to incorporate skewness in the CAPM, as in Kraus and Litzenberger (1976). But this patch is insufficient because the power utility function consistent with a lognormally distributed market has nonzero derivatives of all orders. That is, kurtosis also matters to investors, as do all higher-order moments.[19] Any risk measure in this world must capture an infinite number of moments of the return distribution—a daunting task!

Fortunately, past research has dealt with a closely related problem. Rubinstein (1976) considered asset pricing in a model with power utility

Figure 1. CAPM Plot of Option Strategies
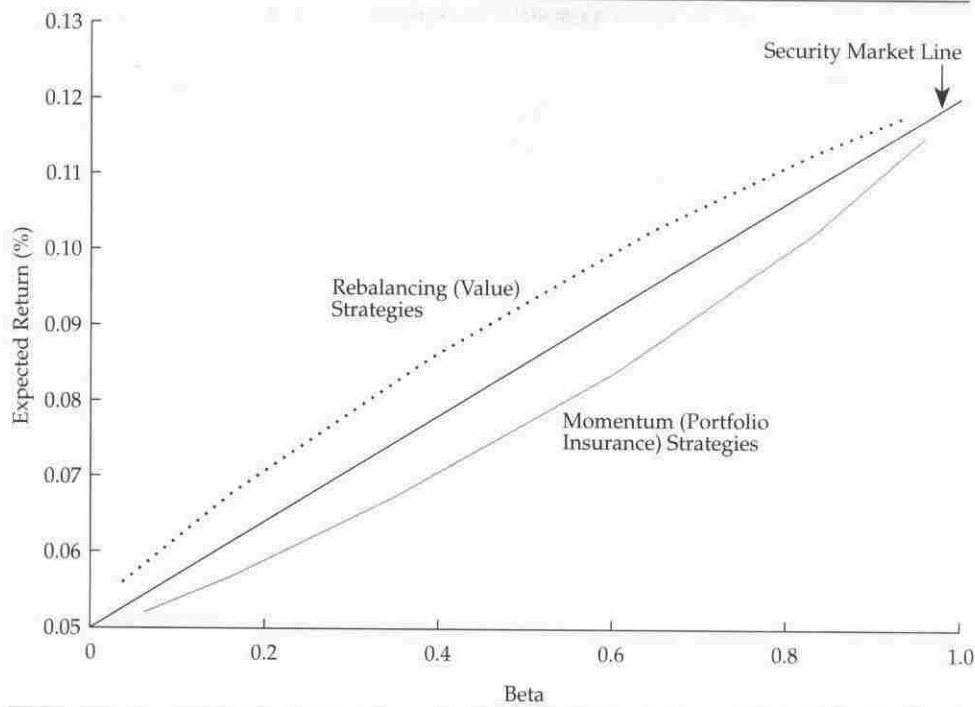
Notes: The security market line is the straight line joining the risk-free asset point ($\beta = 0$, $E[r] = 0.05$) with the market portfolio point ($\beta = 1$, $E[r] = 0.12$). The annual standard deviation of the market portfolio return is 15 percent. The points along the "Rebalancing (Value) Strategies" line range from a strike price of 90 in the lower left to a strike price of 140 in the upper right. The points along the "Momentum (Portfolio Insurance) Strategies" line range from a strike price of 140 in the lower left to a strike price of 90 in the upper right.

functions and lognormal returns for the market portfolio, both of which are implied by Assumptions 1 and 2. He derived the following equilibrium pricing equation (his Equation 3), which holds for assets with any returns over some time interval:[20]

$$P_0 = \frac{E[(1 + r_p)P_0] - \lambda \rho[(1 + r_p)P_0, -(1 + r_{mkt})^{-b}]}{1 + r_f}$$
$$\frac{\text{sd}[(1 + r_p)P_0]}{1 + r_f} \quad (4)$$

where P_0 is the price of any asset; r_p and r_{mkt} are, respectively, the returns to the portfolio and market over the time interval; $\rho[x, y]$ is the correlation of x and y; $-b < 0$ is the exponent of the marginal utility function of the average investor; sd is standard deviation, and

$$\lambda = \frac{\text{sd}[(1 + r_{mkt})^{-b}]}{E[(1 + r_{mkt})^{-b}]}. \quad (5)$$

Dividing both sides of Equation 4 by P_0, rearranging terms, and using the fact that this equation must also hold for the market portfolio gives

$$E(r_p) = r_f + B_p[E(r_{mkt}) - r_f], \quad (6)$$

where

$$B_p = \frac{\text{cov}[r_p, -(1 + r_{mkt})^{-b}]}{\text{cov}[r_{mkt}, -(1 + r_{mkt})^{-b}]}. \quad (7)$$

Furthermore, Rubinstein (1976) and Breeden and Litzenberger (1978) showed how the exponent b is related to the excess return of the market when the market is lognormally distributed:

$$b = \frac{\ln[E(1 + r_{mkt})] - \ln(1 + r_f)}{\text{var}[\ln(1 + r_{mkt})]}. \quad (8)$$

This coefficient is a "market price of risk": the market's instantaneous excess rate of return divided by the variance of the market's instantaneous rate of return.[21]

Parallel to the CAPM-based alpha, the appropriate measure of excess returns, A_p, will be

$$A_p = E(r_p \mid M) - B_p[E(r_{mkt}) - r_f] - r_f. \quad (9)$$

Notice that A_p differs from α_p in Equation 3 only

because the modified measure of risk, B_p, differs from β_p. Clearly, β_p and B_p are related, however, as a comparison of Equations 1 and 7 will show. In addition, the estimates of A_p and B_p require no more raw data than the estimates of α_p and β_p.[22] The variable B_p depends on the covariance of the portfolio return and 1 + the market return raised to the $-b$ power. The coefficient b depends on the market return mean and variance and the risk-free rate, which are also required by the CAPM.

The correct B risk measures can be compared with betas in Table 1. Then, if β is replaced by B for each strike price, the alphas of the optioned portfolios become zero, as shown in the last column. That is, if the correct measure of risk is used, the result is correct: Managers who buy or sell fairly priced assets add no value!

As for the Sharpe ratio, it has no useful general substitute in the world of dynamic strategies or options, but previous studies by Leland (1980) and Brennan and Solanki (1981) offer some insights. Leland showed that an investor whose risk tolerance grows with wealth more quickly than the average investor's risk tolerance will want portfolio insurance (convexity); if risk tolerance grows less quickly than the market's, a rebalancing strategy (concavity) is optimal. Risk tolerance grows more quickly when the investor has a preference for greater skewness. Optimal strategies, therefore, are preference dependent, and no measure that depends on the distribution of portfolio returns alone will correctly rank all alternatives for all investors. Brennan and Solanki, however, derived an interesting partial result: If rankings are limited to the set of portfolios that have lognormal returns, the best of that set should maximize $(\mu_p - r_f)/\sigma_p$. Furthermore, among lognormal portfolios that could serve as the underlying portfolio for constructing nonlinear payoffs (through option or dynamic strategies), the best choice is the one that maximizes this ratio. The actual best nonlinear

strategy will, of course, be preference dependent.

As indicated, applying the Sharpe ratio to a portfolio with nonlognormal returns will, in general, produce nonsense as a measure of managerial ability. But this finding does not detract from the modified alpha, A_p, measure of performance because it can identify a manager's ability to select underpriced assets (or time the market correctly).

When Returns Are Lognormal

The appropriate measure of risk of any asset or portfolio when the market itself has lognormal returns is B, not β, and the difference between the two may be substantial when the asset or portfolio returns are distinctly skewed, as in option or dynamic strategies. Many portfolios and assets, however, including most equities, have returns that are approximately lognormal (although the return distributions' parameters may be quite different from the market portfolio's). If a manager uses β rather than B as the risk measure for such assets, is the manager making a major mistake? The answer is no—if the intervals over which the observations are made are one year or less. The appendix shows that the two risk measures are closely related when portfolio and market returns are jointly lognormal (that is, when the joint distribution of returns is multivariate lognormal).

Table 2 uses the results developed in the appendix to examine the difference between B_p and β_p for portfolios that are jointly lognormally distributed with the market. The differences between β and B are relatively small (B tends to be slightly closer to 1 than β). Consequently, the differences between α and A are small. And the differences become even smaller when the time interval of observations is less than one year.[23]

Therefore, whether one estimates B or β to assess the performance of assets or portfolios whose returns are (approximately) jointly lognormal with

Table 2. Values of B_p and β_p for Lognormally Distributed Assets
(computed B_p is the first of each pair; CAPM β_p in parentheses)

	$\rho_{p,\,mkt}$		
σ_p	0.25	0.50	0.75
0.15	0.256 (0.248)	0.508 (0.498)	0.756 (0.748)
0.25	0.415 (0.405)	0.819 (0.813)	1.213 (1.224)
0.35	0.561 (0.551)	1.103 (1.108)	1.625 (1.670)

Notes: $\rho_{p,\,mkt}$ is the correlation between the return to portfolio p and the market portfolio return, and σ_p is the standard deviation of portfolio p. This table assumes that the portfolio return and the market portfolio return are jointly lognormal. The market has an annual mean return of 12 percent and standard deviation of 15 percent; the annual risk-free rate is 5 percent.

the market return appears to matter little. Other estimation errors are likely to far outweigh the errors that result from this choice. Only when portfolios have distinctly skewed returns will an important difference be found between the CAPM and the modified technique in measuring performance.

When the Market Return Is Not IID

The work of He and Leland suggests a way to extend this analysis to the case in which the market portfolio follows a diffusion process with drift and volatility components that may change with time and market level. (Examples would be constant elasticity of variance or Ornstein–Uhlenbeck mean-reverting processes.) He and Leland showed how to derive the representative investor's utility function that supports a given market stochastic process.

Knowledge of the representative utility function allowed Rubinstein (1973) to produce the result that the appropriate risk adjustment (or modified beta) for a portfolio is the ratio of the covariance of the portfolio's return with marginal utility to the expected covariance of the market portfolio's return with marginal utility.

It would be surprising if the market utility function derived from the market's stochastic process did not exhibit skewness preference (see Note 5). If it does exhibit skewness preference, the CAPM approach will still over- (under-) value strategies that exhibit negative (positive) co-skewness with the market return. Thus, the qualitative nature of the earlier results will hold in a much more general environment: Call-writing or rebalancing strategies will typically be overrated and portfolio insurance or momentum strategies will be underrated by CAPM performance measures. As before, the more pronounced the change in skewness relative to the market return, the worse the CAPM performance measures will perform.

Conclusion

In a world in which the market portfolio (but not necessarily individual securities) has identically and independently distributed returns, the market portfolio will be mean–variance inefficient, the CAPM beta will not properly measure risk, and the

CAPM alpha will mismeasure the value added by investment managers. The problem is particularly severe for portfolios using options or dynamic strategies. Strategies purchasing (writing) fairly priced options will be falsely accorded inferior (superior) performance when the CAPM alpha measure is used. With a proper risk measure, these strategies should be accorded zero alphas because they do not require additional managerial information about asset returns in order to be implemented.

The CAPM's failure to correctly assess performance results from the fact that when market portfolio returns are IID, skewness matters, and a skewness preference, in turn, implies that upside risks are less important to investors than downside risks.

The relatively straightforward modification of the CAPM beta presented here provides a valid risk measure for any asset, portfolio, or dynamic strategy. This modified beta requires no more data to estimate than does the CAPM beta. Other risk measures, such as the Sortino ratio or value at risk, are *ad hoc* attempts to incorporate the importance of downside risk. But unlike the risk measure described here, they totally ignore upside risk and are not based on an equilibrium model of asset-price formation.

For assets or portfolios whose returns are jointly lognormal with the market, the differences between the correct beta and the CAPM beta are small and the mismeasurement of alphas is similarly small. For portfolio or asset returns that are highly skewed, the correct beta differs substantially from the CAPM beta. Thus, using the correct beta is critical for correct performance measurement of investment strategies that use options, market timing, or other dynamic strategies. The approach described here can be modified to apply if the market return is not lognormally distributed and the manager can estimate the market's price process.

Notes

1. Sharpe, Alexander, and Bailey (1995) provide a good overview of current practice in their Chapter 25. Grinblatt and Titman (1989) review some key issues and provide extensions of traditional alpha measurement. Glosten and Jagannathan (1994) provide an elegant and general framework, but applications of their approach require assumptions similar to my framework (lognormal index returns and Black–Scholes option pricing) while also entailing greater complexity.

2. Measuring conditional expectations when managerial information is not directly observed is an important econometric challenge. Early CAPM-based studies (e.g., Jensen 1969) regressed portfolio excess return on market excess return. The constant term was interpreted as the alpha of Equation 3 in the text and the slope coefficient as beta in Equation 1. Roll (1978) indicated the unreliability of alpha measures when the market portfolio proxy is not mean–variance efficient. Difficulties in using alpha as a performance measure when managers are able to successfully time the market were discussed by Dybvig and Ross (1985); their results are closely related to the negative-state prices observed in the CAPM by Dybvig and Ingersoll (1982). Grinblatt and Titman (1989) proposed solving the problem by using positive period-weighting measures (i.e., state price densities), although their later empirical study (Grinblatt and Titman 1994) suggested that this approach makes little difference for evaluating mutual fund portfolios. Ferson and Schadt (1996) retained the CAPM framework but argued that beta should be estimated conditionally on a vector of relevant publicly available information variables that could change during the sample period.

3. Rubinstein and Leland (1981) elucidated the relationship between options and equivalent dynamic strategies. Henriksson and Merton (1981) examined the relationship between options and market-timing strategies.

4. As an *ad hoc* approach to recognizing the greater importance of downside risk, Sortino and van der Meer (1991) proposed that the Sharpe ratio be modified by replacing the variance of returns in the denominator with the lower semivariance of returns. A related risk measure is value at risk, the loss that could occur over a fixed time period with small probability (e.g., 1 percent). These approaches are not grounded in capital market equilibrium theory and may themselves spuriously identify superior or inferior managerial performance. See also Kahn and Stefek (1996).

5. A preference for skewness implies a positive third derivative of the investor's utility function, unlike the quadratic utility function, which has zero third (and higher-order) derivatives. An investor whose risky investments increase as wealth increases must have a positive third derivative (see Pratt 1964 and Arrow 1963). Furthermore, Dybvig and Ingersoll observed that quadratic utility implies that (very) high-return states will have negative marginal utility and thus negative state prices, which contradicts the no-arbitrage condition of equilibrium prices. Kraus and Litzenberger (1976) extended the CAPM to the case in which investors have a cubic utility function—hence, a skewness preference. This article demonstrates that when the market portfolio has identically and independently distributed returns, the average investor must have a preference for skewness.

6. This disparity between research and practice results partly because the results of empirical studies of alternatives to the CAPM (e.g., Kraus and Litzenberger; Grinblatt and Titman 1994) exhibit minimal differences from CAPM results when applied to typical stock portfolios. As the results to be described here show, substantial differences will be evident only for portfolios or assets with highly skewed return distributions.

7. The usual Central Limit Theorem conditions are required. In a recent empirical examination of 1928–96 market returns, Jackwerth (1997) found that daily market returns are not lognormal but for longer periods (e.g., three months), returns are quite "close" to lognormally distributed.

8. Lognormality results from a continuous diffusion process for the rate of return if both the drift and the volatility of the process are constant. Although requiring constant volatility, the Black–Scholes model does not require that the drift of the asset rates of return be constant; therefore, distributions other than the lognormal may be consistent with that model.

9. As is well known, a portfolio of assets with lognormal returns will not itself have lognormal returns. For this model, however, I am not assuming that lognormality holds for every asset, only for the market.

10. In the binomial model, it can be shown that the market is mean–variance efficient for each subperiod. (In a two-state world, any option on the market portfolio can be perfectly replicated by a static portfolio of the market and the risk-free asset.) But by the first assumption, the subperiods can be arbitrarily short (in the limit, the process becomes a logarithmic Brownian motion) and the market will always be mean–variance inefficient for any finite time interval.

11. In the continuous-time limit, markets are dynamically complete (Harrison and Kreps 1979) and a representative investor exists even when individual investors have heterogeneous utility functions (Constantinides 1982).

12. The example does not give the highest possible Sharpe ratio. In continuous time, assume the market rate-of-return process has drift μ and volatility σ. Consider an investor who has mean–variance preference (quadratic utility) with satiation wealth level equal to k. It can be shown that at any time t, the investor's optimal strategy is to invest an $\alpha(t)$ fraction of wealth $W(t)$ in the market portfolio, where $\alpha(t) = [(\mu - r)/\sigma^2][k/W(t) - 1]$, for $W(t) \leq k$. Bajeux-Besnainou and Portait (1995) further showed that when many risky securities exist, all dynamic mean–variance-efficient strategies are buy-and-hold combinations of two funds: a continuously rebalanced portfolio of these securities and a zero-coupon bond with maturity equal to the investor's horizon.

13. The lognormal distribution parameters are μ_{mkt} = 10.44 percent, σ_{mkt} = 13.33 percent.

14. Rubinstein (1976) showed that the Black–Scholes formula correctly prices options on the market in discrete time when market returns are lognormally distributed and the representative investor has power utility.

15. Dybvig and Ingersoll suggested that call options could be underpriced because of the negative marginal utility of the quadratic utility function at high levels of wealth, but my argument is that call options could be underpriced by the CAPM even if portfolio returns were bounded to levels of wealth less than the satiation level. Call options have greater skewness than the market and will be undervalued by CAPM measures, which ignore the positive value of skewness.

16. Bookstaber and Clarke (1985), although not providing analytical results, observed from simulations that option-based strategies seem to lie above or below the CAPM "market line."

17. When naked options on the market portfolio are considered, the mismeasurement becomes even more extreme. For example, a one-year call option on the market with a strike price 110 percent of the current market value (and parameters as in Table 1) has a CAPM beta of 17.88. A CAPM-based analysis of a naked option position (or a dynamic strategy replicating that position) would indicate a negative annual alpha of 25 percent!

18. See He and Leland for a discussion of the (unreasonable) stochastic process of the market that would be required for the CAPM to evaluate risk correctly.

19. Indeed, as can be readily observed, the derivatives of the power utility function alternate in sign. Thus, mean, skewness, and higher odd-numbered moments of the distribution are always positively valued by investors; variance, kurtosis, and higher even-numbered moments are negatively valued.

20. There is a misprint in Rubinstein's equation: The numerator contains a covariance that should be a correlation. Rubinstein's Equation 2, from which his Equation 3 was derived, has the correct term. The Rubinstein (1976) result is closely related to the general single-period result derived in Rubinstein (1973).

21. In continuous time, $b = (\mu_{mkt} - r_f)/\sigma_{mkt}^2$, where the market portfolio process is $dM/M = \mu_{mkt}dt + \sigma_{mkt}dz$.

22. Note that many of the econometric problems related to estimating α_p will also be relevant to estimating A_p, including finding an appropriate proxy for the market portfolio.

23. Empirical studies of equity portfolio betas undertaken by Aamir Sheikh of BARRA confirmed that β and *B* coefficients for three-month and zero-month measurement periods are practically identical. Grinblatt and Titman (1994) also found that performance evaluations of mutual fund returns are relatively insensitive to whether the CAPM or power (marginal) utility approach is used.

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Appendix. B_p and β_p for Lognormally Distributed Assets

Recall that B_p is defined as

$$
\begin{aligned}
B_p &= \frac{\text{cov}(r_p, R_M^{-b})}{\text{cov}(r_{mkt}, R_M^{-b})} \\[2mm]
&= \frac{\text{cov}(R_p, R_M^{-b})}{\text{cov}(R_M, R_M^{-b})} \\[2mm]
&= \frac{E(R_p, R_M^{-b}) - E(R_p)E(R_M^{-b})}{E(R_M, R_M^{-b}) - E(R_M)E(R_M^{-b})},
\end{aligned}
$$

where $R_M = (1 + r_{mkt})$ and $R_p = (1 + r_p)$.

If R_M and R_p are jointly lognormal, with

$$
\begin{aligned}
E[\log(R_M)] &= \mu_M, \quad E\{[\log(R_M) - \mu_M]^2\} = \sigma_M^2 \\
E[\log(R_p)] &= \mu_p, \quad E\{[\log(R_p) - \mu_p]^2\} = \sigma_p^2 \\
\text{cov}[\log(R_m), \log(R_p)] &= \sigma_{pM},
\end{aligned}
$$

then

$$
B_p = \frac{\exp\left[-b\mu_M + \mu_p + 0.5(b^2\sigma_M^2 - 2b\sigma_{pM} + \sigma_p^2)\right] - \exp\left(\mu_p + 0.5\sigma_p^2\right)\exp\left(-b\mu_M + 0.5b^2\sigma_M^2\right)}{\exp\left[-b\mu_M + \mu_M + 0.5\left(b^2\sigma_M^2 - 2b\sigma_M + \sigma_M^2\right)\right] - \exp\left(\mu_M + 0.5\sigma_M^2\right)\exp\left(-b\mu_M + 0.5b^2\sigma_M^2\right)}.
$$

Factoring the numerator and denominator gives

$$
B_p = \frac{\exp\left(-b\mu_M + 0.5b^2\sigma_M^2 + \mu_p + 0.5\sigma_p^2\right)\left[\exp\left(-b\sigma_{pM}\right) - 1\right]}{\exp\left[\left(1 - b\right)\mu_M + 0.5\left(1 + b^2\right)\sigma_M^2\right]\left[\exp\left(-b\sigma_M^2\right) - 1\right]}
$$

$$
= \exp\left(\mu_p - \mu_M + 0.5\sigma_p^2 - 0.5\sigma_M^2\right)\left[\frac{\exp(-b\sigma_{pM}) - 1}{\exp(-b\sigma_M^2) - 1}\right].
$$

Now, $\beta_p = \text{cov}(r_p, r_M)/\text{var}(r_M)$ is simply the preceding expression when $b = -1$.

Therefore, after simplification,

$$
\frac{B_p}{\beta_p} = \left[\frac{\exp(-b\sigma_{pM}) - 1}{\exp(-b\sigma_M^2) - 1}\right]\left[\frac{\exp(\sigma_M^2) - 1}{\exp(\sigma_{pM}) - 1}\right].
$$

To a first-order Taylor series expansion, $\exp(x) = 1 + x$. It immediately follows that to the first order,

$$
\frac{B_p}{\beta_p} = \left(\frac{-b\sigma_{pM}}{-b\sigma_M^2}\right)\left(\frac{\sigma_M^2}{\sigma_{pM}}\right)
$$

$$
= 1.
$$

Thus, over relatively short time periods (when volatilities are small), both techniques will yield identical estimates for "beta." For longer time periods, the two techniques will not give identical results (see Table 2).