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Option pricing methods: An overview

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This paper aims at giving an overview of option pricing methods. Its stress is on intuition rather than on replicating formulas; furthermore it does not presume pre-knowledge of option pricing. The paper considers the two major classes of option models (discrete and continuous trading) and mimicking and non-mimicking pricing methods.

Keywords: Mimicking, Arbitrage and first stochastic dominance, Binomial process, Diffusion process, Risk-neutral valuation relationship, Option bounds, Option pricing and aggregate data, Early exercise, Dividends, Market imperfections, Non-stationarity in interest rates and variance rates.

1. Introduction

Virtually any financial contract has option features or can be decomposed in options. Determining the economic value of the contract obligations is in many cases a matter of valuing the underlying options.

A great many option models have been developed relative to very different assets. Today the option literature knows how to price options on shares, bonds, foreign currency, futures, options, commodities, derivative assets. In addition it can handle many kinds of multi-asset options like the valuation of the option to exchange assets, multi-currency bond options, options on the minimum and maximum of assets, options on average asset values, multi-factor options.... Furthermore, much progress has been made in applying option techniques and results in valuation of real assets and contracts. For example, natural resource investments, general project evaluation, international trade and investments, the economic value of bond indenture provisions, the evaluation of convertibles, bank loan commitments, deposit insurance, portfolio insurance, agency problems, all have been fruitfully analysed with the help of option pricing techniques.

Given the multitude of results, this paper does not intend to reproduce, nor even to give an overview of all known pricing results. In fact overview articles of pricing results are available in the literature.

The present article does not consider the vast literature on empirical findings and estimation problems either. Overview articles are available here too.

This paper intends to give an overview of the main option pricing principles that can be applied in pricing contracts with option features. Its stress is on intuition rather than on replicating mathematical formulas.

The current option literature considers two main categories of option models (models that presume trading at discrete time intervals and those presuming continuous trading) and two different pricing methods (mimicking and non-mimicking). The mimicking approach seeks to construct a portfolio of assets which payoff is a carbon copy of the option's payoff; the value of the option is equal to the cost of this portfolio. It will be seen that when the construction of such a portfolio is impossible, two alternatives are available.

Under the first alternative the requirement of a unique option price is released and only bounds on option prices are derived. Under the second alternative a unique option price is determined by releasing the requirement of perfect mimicking according to criteria from utility theory and capital market equilibrium models.

For several reasons discrete and continuous trading models appear complementary. Discrete trading models may be the more realistic ones; however, continuous trading models fit the mimicking approach to pricing better and in some cases the solution from these models could serve as an approximation for the solution of discrete trading models. Conversely, it is not always possible to find an algebraic solution for the option price in a continuous trading model; in such cases the solution from a discrete trading model may serve as an approximation.

The paper is organised as follows: Section 2 contains definitions and the well-known put–call parity relation; Section 3 considers first stochastic dominance option pricing methods applied to simple options; Section 4 investigates alternative pricing approaches when first stochastic dominance principles fail; finally Section 5 considers extensions of pricing principles to more complicated options.

2. Options: Definitions

A call option is a contract giving the buyer the right to purchase a fixed amount of a specified financial asset at a fixed price at any time during the life of the option (= American call option) or solely at the time of expiration (= European call option).

Exercising the option is the act of purchasing the underlying asset. The exercise or striking price is the fixed contract price at which the underlying asset might be bought. The expiration date is the last day the option may be exercised. The premium or the call price is the market price of the option.

The converse of buying is writing a call, i.e., giving someone else the right to buy from you the asset at the striking price. At expiration, the payoff of a European call is therefore

$$C_T = \max[S_T - X, 0],$$

with C_T denoting the option value at the time of expiration T, and S_T the value of the underlying asset at time T; X is the exercise price.

A graphical representation of this payoff is given in Figure 1.

A put option is a contract giving the buyer the right to sell a fixed amount of a specified financial asset at a fixed price at any time during the life of the option (= American put option) or solely at the time of expiration (= European put option). Consequently, at expiration a European put has the following payoff:

$$P_T = \max[X - S_T, 0],$$

with P_T denoting the put price at time T.

Figure 2 depicts this payoff function

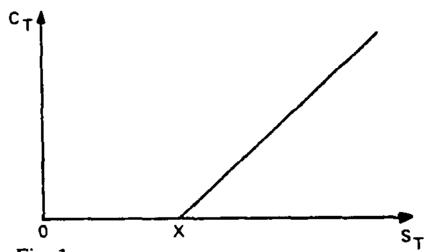

Fig. 1.

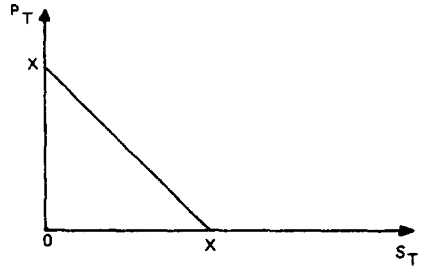

Fig. 2.

The calls and puts traded on Exchanges normally are of the American type. The implicit options in many contracts are of the European type.

European type options are easier to value. Therefore this paper will first discuss European options and extend the findings to American options.

Before going into the pricing of options in the next section, we will consider the very useful 'put–call parity'. Put–call parity gives a relationship between the price of a call and the price of a put with same exercise price and maturity. From this relationship we can immediately derive the value of a put as soon as we know the value of a corresponding call and vice versa. Hence we only need to price one type of option. This paper follows the tradition in finance and considers call pricing. Put pricing follows the same principles though.

Put–call parity and the option pricing of Section 3 is based on the most basic condition for equilibrium in financial markets: the absence of arbitrage opportunities. In a perfect capital market [1] an arbitrage opportunity exists as soon as

[1] A perfect capital market presumes absence of transactions costs, perfect divisibility of assets, all relevant information freely available to all market participants, absence of taxes and frictions, price taking by market participants. A perfect capital market implies that lending and borrowing occur at the same rate.

two securities or portfolios with the same payoff for every future state trade at different market prices. If market participants' utility increases in wealth, arbitrage opportunities are inconsistent with market equilibrium. For our purposes it suffices to remark that the existence of arbitrage opportunities implies first-order stochastic dominance of one portfolio over another. [2]

Assume the following:
- options are of the European type;
- the capital market is perfect;
- the underlying asset pays no dividends between the present and expiration, nor does a direct investment in this asset involve holding costs (e.g. inventory costs).

Put–call parity considers the payoff of the following portfolio:
- buy a call which has still T time to 'live' with exercise price X;
- write a put with same maturity and exercise price as the call;
- invest X/R^T in a risk-free asset earning the yearly rate $R-1$ (T is expressed in years).

At expiration this portfolio's payoff is equal to:

	Value of portfolio at expiration	
	$S_T \leqq X$	$S_T > X$
Buy call	0	$S_T - X$
Write put	$S_T - X$	0
Invest risk-free X/R^T	X	X
Portfolio return	S_T	S_T

Hence the portfolio is a perfect substitute for holding the underlying asset directly. If we intend to hold this asset until time T, the equivalence is obvious; and adjusting the amount invested in the portfolio clearly is the substitute for trading in the asset before T.

It follows that at any time until expiration, the market value of this portfolio must be equal to the market value of the asset. Or for $t \in [0, T]$:

$$C_t - P_t + X/R^{T-t} = S_t. \tag{1}$$

Equation (1) is put–call parity. It shows for example how calls and puts allow investors to hold the underlying asset 'synthetically' through an equivalent portfolio. By solving (1) for say $P_t(C_t)$ put–call parity shows how to construct puts (calls) synthetically on the basis of the asset, the risk-free investment and calls (puts).

[2] The converse does not hold. For more details see Jarrow (1986b).

3. Call pricing through arbitrage

This section shows that options can be regarded as a portfolio of the underlying asset and the risk-free investment, and how this fact can be used to price them. In addition it is shown there exists a transformation of the underlying asset's return distribution with the property that option valuation is risk neutral (i.e. using the transformed return distribution, the value of the option is equal to the option's expected future payoff discounted to the present at the risk-free rate). Existence of such a risk-neutral transformation is a great advantage since risk-neutral valuation relationships depend only upon potentially observable parameters.

3.1. A simple one-period model

Suppose the three assumptions behind put–call parity are satisfied. Suppose there is only one more period in the life of a call option. The length of this period is T (the expiration date) $-$ 0 (the current date). Currently the underlying asset's value is equal to S_0. By the end of this period this value changes to either $g \cdot S_0$ or $b \cdot S_0$ with probability q and $1-q$, respectively. The risk-free lending and borrowing rate is $\hat{R}$ with $g > \hat{R} > b$. $\hat{R}$ remains constant over time and is fully earned over the period $T-0$. Since T is expressed in years, $\hat{R}^{1/T} = R$ (R is the yearly risk-free return). Denote the current market value of a call with exercise price X by C_0.

The mimicking or arbitrage approach determines C_0 by noticing the possibility of reproducing the call's payoff by a portfolio of the underlying asset and the risk-free investment. Simply buy s number of shares and invest y times the present value of the exercise price $X/\hat{R}$ in the risk-free asset such that the payoff equals

$$s \cdot g \cdot S_0 + y \cdot \hat{R} \cdot X/\hat{R} = C(g) \quad \text{and}$$

$$s \cdot b \cdot S_0 + y \cdot \hat{R} \cdot X/\hat{R} = C(b)$$

with $C(g) = \max[0,\ g \cdot S_0 - X]$ and $C(b) = \max[0,\ b \cdot S_0 - X]$.

It readily follows that by choosing $s = (C(g) - C(b))/((g-b) \cdot S_0)$ and $y = (g \cdot C(b) - b \cdot C(g))/((g-b) \cdot X)$, an investor holds a 'synthetic' call.

Furthermore, if there is to be no arbitrage, the time 0 cost of the portfolio must be equal to C_0. Or:

$$\begin{aligned} C_0 &= s \cdot S_0 + y \cdot X/\hat{R} \\ &= 1/\hat{R}\big[((\hat{R}-b)/(g-b)) \cdot C(g) \\ &\qquad + ((g-\hat{R})/(g-b)) \cdot C(b)\big] \\ &= 1/\hat{R}[p \cdot C(g) + (1-p) \cdot C(b)] \qquad (2) \end{aligned}$$

with $p = (\hat{R} - b)/(g-b)$.

Equation (2) is remarkable. It shows there exists a unique transformed probability distribution $(p, 1-p)$, relative to which the valuation of the option is risk neutral. [3] In addition, the pricing of the option is totally independent of the true probability distribution $(q, 1-q)$. To see why the valuation shows no relationship with specific attitudes of the investing public, it suffices to recall that the arbitrage argument is a first-degree stochastic dominance argument: only positive marginal utility of wealth is presumed; no information about how this marginal utility varies with wealth is needed. C_0 depends on q indirectly through S_0: the more favourable q, the higher S_0 in equilibrium and the higher C_0.

Risk-free valuation relationships are considered in depth in Rubinstein (1976), Ross (1978b), Jarrow (1986a, b) and others.

3.2. n-period binomial valuation

The preceding arbitrage argument can be extended easily to a multi-period case. Suppose for example that during the time span $T - 0 = T$, the underlying asset's price can make a jump twice, once at $T/2$ and once at T.

Let $\hat{R}(2)$ denote the risk-free rate which is fully earned in each of the two subperiods, i.e. $[\hat{R}(2)]^2 = \hat{R}$ (the bracketed number 2 refers to the subdivision of T in 2 subperiods). Denote a subperiod's jump factors by $g(2)$ and $b(2)$. To extend the previous example, we take $[g(2)]^2 = g$ and $[b(2)]^2 = b$.

[3] Uniqueness follows from the fact that (s, y) is unique.

A call which matures at T has the following payoff structure:

$$\begin{array}{lll} & & C(gg,2) = \max[0, g(2)^2 \cdot S_0 - X] \\ & C(g,2) < & \\ C_0 < & & C(gb,2) = \max[0, g(2) \cdot b(2) \cdot S_0 - X] \\ & C(b,2) < & \\ & & C(bb,2) = \max[0, b(2)^2 \cdot S_0 - X] \end{array}$$

Fig. 3.

At time 1 only one more period is left in the life of the call and the mimicking problem is identical to the case already solved:

- if the call is in the state $C(g, 2)$ (and hence the underlying asset's price is $g(2) \cdot S_0$), choose $s(g, 2) = (c(gg, 2) - C(gb, 2))/[(g(2) - b(2)) \cdot g(2) \cdot S_0]$ and $y(g, 2) = (g(2) \cdot C(gb, 2) - b(2) \cdot C(gg,2))/[(g(2) - b(2)) \cdot X]$; this portfolio costs $C(g,2) = (1/\hat{R}(2)) \cdot [C(gg,2) \cdot (\hat{R}(2) - b(2))/(g(2) - b(2)) + C(gb, 2) \cdot (g(2) - \hat{R}(2))/(g(2) - b(2))]$;
- if the call is in the state $C(b, 2)$, choose $s(b, 2) = (C(gb,2) - C(bb,2))/[(g(2) - b(2)) \cdot b(2) \cdot S_0]$ and $y(b, 2) = (g(2) \cdot C(bb, 2) - b(2) \cdot C(gb, 2))/[(g(2) - b(2)) \cdot X]$, which costs $C(b, 2) = (1/\hat{R}(2)) \cdot [C(gb,2) \cdot (\hat{R}(2) - b(2))/(g(2) - b(2)) + C(bb, 2) \cdot (g(2) - \hat{R}(2))/(g(2) - b(2))]$.

At time 0 we need to construct a portfolio with time 1 payoff $C(g, 2)$ if the asset's price jumps to $g(2) \cdot S_0$ and payoff $C(b, 2)$ if it jumps to $b(2) \cdot S_0$. This can be achieved by taking $s(2) = (C(g, 2) - C(b, 2))/[(g(2) - b(2)) \cdot S_0]$ and $y(2) = (g(2) \cdot C(b, 2) - b(2) \cdot C(g, 2))/[(g(2) - b(2)) \cdot X/\hat{R}(2)]$, costing $C_0(2) = (1/\hat{R}(2)) \cdot [C(g,2) \cdot (\hat{R}(2) - b(2))/(g(2) - b(2)) + C(b,2) \cdot (g(2) - \hat{R}(2))/(g(2) - b(2))]$.

Normally $s(2) \neq s(g, 2)$ or $s(b, 2)$ and $y(2) \neq y(g, 2)$ or $y(b, 2)$. Hence, as time goes by, the mimicking portfolio is reshuffled. However, this reshuffling is fully self-financing: if at time 1 the asset's price moves to say $g(2) \cdot S_0$, we need to change our portfolio from $s(2)$ to $s(g, 2)$ and $y(2)$ to $y(g, 2)$; in that state the cost of such a portfolio is $C(g, 2)$; but this is exactly the amount our initial portfolio is worth (in that state). The same argument applies if the asset's market price moves to $b(2) \cdot S_0$

It can be shown that $s(2)$, $s(g, 2)$, $s(b, 2) \geq 0$ and $y(2)$, $y(b, 2)$, $y(g, 2) \leq 0$, or a call is a long position in the underlying asset, partially financed with borrowed funds.

The solution of the preceding recursive system when T is subdivided in n time intervals is

$$C_0(n) = \hat{R}(n)^{-n} \cdot \left[\sum_{j=0}^{n} [n!/(j! \cdot (n-j)!)] \cdot p(n)^j \cdot (1-p(n))^{n-j} \cdot \max\left[0,\ g(n)^j \cdot b(n)^{n-j} \cdot S_0 - x\right] \right],$$

where $p(n) = (\hat{R}(n) - b(n))/(g(n) - b(n))$. (3)

Expression (3) is the binomial option pricing formula. It gives a risk-neutral valuation for $C_0(n)$ relative to the transformed distribution $(p(n), 1 - p(n))$.

The binomial formula can also be brought into the following format:

$$C_0(n) = S_0 \cdot \Phi[a;\ n,\ p'(n)] - X \cdot R(n)^{-n} \cdot \Phi[a;\ n,\ p(n)] \tag{4}$$

with

$$\Phi[a;\ n,\ z] = \sum_{j=a}^{n} [n!/(j! \cdot (n-j)!)] \cdot z^j \cdot (1-z)^{n-j},$$

$$p'(n) = (g(n)/\hat{R}(n)) \cdot p(n),$$

a = the smallest non-negative integer greater than $\ln(X/S_0 \cdot b(n)^n)/\ln(g(n)/b(n))$; a is the minimal number of times $g(n)$ has to occur before exercising the option becomes profitable.

Under appropriate distributional conditions and/or sufficiently large n $\Phi[a;\ n,\ p'(n)]$ is approximately equal to the number of shares required for a synthetic call. Similarly, $\Phi[a;\ n,\ p(n)^T]$ approximates the number of times the present value of the strike price needs to be borrowed (see also section 3.3).

3.3. Continuous time models as a limit to the binomial process

By reducing the time span between price movements to zero, the behaviour of the underlying asset's market value over time may converge to a continuous time distribution, and this depending upon the choice of $g(n)$, $b(n)$ and $q(n)$ [= the true probability of $g(n)$ occurring].

A growing part of the option literature studies the relationships between a set of definitions $(g(n), b(n), q(n))$ and corresponding limiting distributions.

As an illustration, consider the well-known case below. Define $S_n(n)$ as the underlying assets price after n jumps have occurred when T is subdivided in n intervals of length T/n. Consider the variable

$$\ln(S_n(n)/S_0) = j \cdot \ln(g(n)) + (n-j) \cdot \ln(b(n)) = j \cdot \ln(g(n)/b(n)) + n \cdot \ln(b(n)),$$

i.e. the log of the underlying asset's price n periods hence relative to the current price if $g(n)$ has occurred j times and $b(n)$ $(n-j)$ times. $\ln(S_n(n)/S_0)$ is equal to the continuously compounded rate of return on the underlying asset between time 0 and the nth jump; i.e., $S_0 \cdot \exp(\ln(S_n(n)/S_0)) = S_n(n)$.

Since $E(j) = n \cdot q(n)$ and $\sigma^2(j) = q(n) \cdot (1 - q(n)) \cdot n$, $\ln(S_n(n)/S_0)$ has expected value and variance:

$$E(\ln(S_n(n)/S_0)) = [q(n) \cdot \ln(g(n)/b(n)) + \ln(b(n))] \cdot n = \mu' \cdot n;$$

$$\sigma^2(\ln(S_n(n)/S_0)) = q(n) \cdot (1-q(n)) \cdot [\ln(g(n)/b(n))]^2 \cdot n = \sigma'^2 \cdot n.$$

Let:

$$g(n) = e^{\sigma \cdot \sqrt{(T/n)}},$$

$$b(n) = e^{-\sigma \cdot \sqrt{(T/n)}} \text{ and}$$

$$q(n) = 1/2 + (1/2) \cdot (\mu/\sigma)\sqrt{(T/n)},$$

with σ and μ some constants.

Upon substituting these definitions into the expected value and variance of $\ln(S_n(n)/S_0)$, we observe that as $n \to \infty$, $\mu' \cdot n \to \mu \cdot T$ and $\sigma'^2 \cdot n \to \sigma^2 \cdot T$. Furthermore, the distribution of $\ln(S_n(n)/S_0)$ converges to the normal with mean $\mu \cdot T$ and variance $\sigma^2 \cdot T$. Simultaneously dS/S converges to the Brownian motion

$$dS/S = \mu \cdot dt + \sigma \cdot dz(t),$$

with $dz(t)$ white noise and S the underlying asset's price at any time between 0 and T.

Meanwhile the binomial pricing formula converges to the celebrated Black–Scholes option pricing formula:

$$C_0 = S_0 \cdot \mathrm{N}(z) - X \cdot R^{-T} \cdot \mathrm{N}(z - \sigma\sqrt{T}), \tag{5}$$

with

$\mathrm{N}(\cdot)$ the cumulative normal distribution,
$z = \ln(S_0/(X \cdot R^{-T}))/(\sigma \cdot \sqrt{T}) + (1/2) \cdot \sigma \cdot \sqrt{T}$.

Equations (4) and (5) are strikingly similar. A basic difference between them is that equation (5) implicitly presumes continuous reshuffling of the mimicking portfolio whereas (4) only presumes trading at discrete intervals (which may be more realistic). Contrary to the coefficients Φ in (4), the coefficients N in (5) give the exact portfolio composition for a synthetic option.

The binomial pricing formula and its convergence to the Black–Scholes formula was first discussed in Cox, Ross and Rubinstein (1979); other convergence cases are discussed in Page and Sanders (1986), Cox and Rubinstein (1985).

3.4. Continuous trading models and stochastic calculus

This approach starts out from a continuous-time stochastic process right away and constructs a mimicking portfolio by continuously rebalancing the portfolio.

Here it is usually easier to mimick the risk-free investment instead of mimicking the option. To illustrate the procedure, consider the special case where $\mathrm{d}S/S$ follows the diffusion of Section 3.3. Then the call value is a function of S and T, $C_0 = V(S, T)$.

Denote by the subscripts S, SS or T partial derivatives.

An instantaneous risk-free investment is created by buying one call at the price $V(S, T)$ and financing it partially by a short sale of V_S shares. This portfolio is risk-free and requires a net investment of $V - V_S \cdot S_0$. Hence if there is no arbitrage, this portfolio earns over the time interval $\mathrm{d}T$ the risk-free rate r (r is the continuously compounded risk-free rate and satisfies $e^{rT} = R^T$). Or:

$$[\mathrm{d}V - V_S \cdot \mathrm{d}S_0]/[V - V_S \cdot S_0] = r \cdot \mathrm{d}T.$$

Applying Ito's lemma to $\mathrm{d}V$ and using the fact that $(\mathrm{d}S_0)^2 = \sigma^2 \cdot S_0^2 \cdot \mathrm{d}T$, this condition can be transformed into the following partial differential equation:

$$(1/2) \cdot \sigma^2 \cdot S_0^2 \cdot V_{SS} + r \cdot S_0 \cdot V_S + V_T - r \cdot V = 0. \tag{6}$$

With the boundary conditions given just below, the solution is equation (5):

$$\text{at } T = 0,\ V = \max[0,\ S_0 - X], \tag{7a}$$

i.e., at expiration the value of the call is equal to its exercise value;

$$V(S_0 = 0) = 0. \tag{7b}$$

As mentioned earlier, the stochastic calculus approach has the disadvantage that it presumes continuous trading. In addition, it is not so easy to model instationarity with it. Also for many types of options the equation of motion has no algebraic solution. In that case we may approximate the solution either by numerical methods or by an appropriately defined by binomial price process [see for example Omberg (1987)].

However, sometimes the continuous representation can serve as an approximation to a discrete process, especially when the underlying asset's price movement does not allow pricing through mimicking, but the continuous approximation does. For example, suppose we feel that the true discrete time movement of the underlying asset's price in every subperiod of length T/n is the following one:

$$S \begin{cases} g \cdot S & \text{with probability } q_1 \\ S & \text{with probability } 1 - q_1 - q_2 \\ b \cdot S & \text{with probability } q_2 \end{cases}$$

Fig. 4.

Clearly no mimicking is possible. In fact, all arbitrage arguments can do in this case is derive bounds on C_0 (see below).

However, with the subsequent definitions $\mathrm{d}S/S$ converges to the diffusion process from Section 3.3:

$g = e^{\sigma\sqrt{(2T/n)}}$, $b = 1/g$,

$q_1 = 1/4 + (1/4) \cdot (\mu/\sigma) \cdot \sqrt{(2T/n)}$ and

$q_2 = 1/4 + (1/4) \cdot (\mu/\sigma) \cdot \sqrt{(2T/n)}$.

Since the Black–Scholes price is a unique solution to the system (6), (7a), (7b), it follows that as

$n \to \infty$, the set of option prices compatible with absence of arbitrage reduces to a single element.

To see intuitively why diffusions allow mimicking while the discrete representation of the process may not have this property, consider the following two-stage price process where each stage has length $T/2n$:

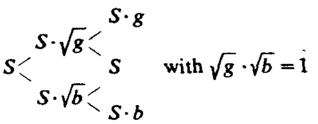

Fig. 5.

The process of Figure 5 allows mimicking.

Although in the discrete case this process may not model exactly the real phenomenon (the correct modelling is the one in Figure 4), the fact that it doesn't becomes less and less important as n grows large. In the limit, the two processes are indistinguishable.

4. Semi- and non-mimicking approaches to valuing options

As mentioned already in the introduction, arbitrage does not always suffice to uniquely determine option prices. When arbitrage fails, several possibilities remain open. On the one hand, if we contend ourselves with bounds on option prices, we can still use first-degree stochastic dominance arguments; these bounds can be tightened by adding more information concerning utility functions and/or distributions (see Section 4.1). On the other hand, we can still arrive at unique prices by loosening the requirement of perfect mimicking according to certain criteria (see Section 4.2).

4.1. Stochastic dominance and bounds on option prices

When arbitrage fails to produce unique prices, it can still be used to develop boundaries on market values. To see what type of bounds can be derived, reconsider the three-outcome discrete process of Section 3.4. In a one-period world the underlying asset's price may jump to $g \cdot S$, to $b \cdot S$ or stay at its current level S with respective probabilities q_1, q_2, $1 - q_1 - q_2$. A call option with exercise price X has payoffs $C(g)$, $C(b)$ and $C(1)$.

The portfolio which perfectly mimicks the payoff of this call if state $g \cdot S$ or S occurs contain $(C(g) - C(1))/((g-1) \cdot S)$ shares and invests $(g \cdot C(1) - C(g))/((g-1) \cdot X)$ times the present value of X in the risk-free asset. The cost of this portfolio (= risk-neutral valuation) is

$$C_0^* = (1/\hat{R}) \cdot \left[p_1^* \cdot C(g) + (1 - p_1^* - 0) \cdot C(1) + 0 \cdot C(b) \right]$$

with $p_1^* = (\hat{R} - 1)/(u - 1)$ and $p_2^* = 0$.

It is easy to check that this portfolio's payoff in the state $b \cdot S$ is equal to or below the real option's payoff. Hence to avoid arbitrage $C_0^* \leqq C_0$. Similarly, the portfolio mimicking the call's payoff in states S and $b \cdot S$ performs worse than the real call in state $g \cdot S$. Hence $C_0^{**} \leqq C_0$ with

$$C_0^{**} = (1/\hat{R}) \cdot \left[0 \cdot C(g) + (1 - 0 - p_2^{**}) \cdot C(1) + p_2^{**} \cdot C(b) \right] \text{ and}$$

$$p_2^{**} = (\hat{R} - b)/(1 - b).$$

The portfolio mimicking the option in states $g \cdot S$ and $b \cdot S$ may perform better or worse than the real option in state S and therefore provides no arbitrage bound.

The conclusion from all this is that any weighted average of the distributions $(p_1^*, 0, 1 - p_1^* - 0)$ and $(0, p_2^{**}, 1 - 0 - p_2^{**})$ is consistent with arbitrage pricing. Narrowing the bounds requires additional information on preferences and/or the structure of the financial market.

First-degree stochastic dominance bounds are treated in detail in Merton (1973a), Garman (1976), Levy (1985), Cox and Rubinstein (1985) and others; Gerber and Shiu (1988) consider non-uniqueness of risk-neutral valuations.

Levy (1985) reports on second-degree stochastic dominance bounds. Bounds requiring additional restrictions on distributions and/or preferences are reported in Perrakis and Ryan (1984), Ritchken (1985), Lo (1987) and others.

4.2. Exact option pricing and general market restrictions

As mentioned earlier, the advantage of a risk-neutral valuation relationship is the fact that it depends only on potentially observable parameters. Not surprisingly attention has been devoted to developing pricing models with a unique risk-neutral valuation relationship in cases where arbitrage arguments alone do not lead to a single price anymore. Since aggregate data have the advantage of measurability, it is natural to impose restrictions which allow to establish a functional relationship between aggregate data and the option value.

The problem of finding a risk-neutral valuation relationship is more difficult for discrete trading models than for continuous trading models. The reason is that because portfolios can be adjusted continuously in the latter case, it is easier to obtain decent approximations with synthetic assets. In addition, in continuous trading models option returns through time tend to have the same type of distribution as the underlying asset returns. This is generally not the case when trading can take place only at discrete intervals. Let us first consider the pricing problem in a discrete model and then look at a continuous trading model.

4.2.1. Option pricing in a one-period trading model

Let $C(S_T)$ denote the end-of-period payoff on a call option as a function of S_T (the end-of-period value of the underlying asset). Let $f(S_T \mid S_0)$ denote the density function of the end-of-period asset price conditional on its initial price, S_0. Define $\underline{f}(S_T \mid S_0)$ as a density function whose location parameter is chosen so that the mean of the distribution is $S_0 \cdot \hat{R}$, while the other parameters of the distribution are identical to those of $f(S_T \mid S_0)$. Suppose trading is only possible at the beginning of the period. In Brennan (1979) a risk-neutral valuation relationship is said to exist if the subjective equilibrium valuation of all individual investors has the following unique representation:

$$C_0 = \hat{R}^{-1} \int_{-\infty}^{+\infty} C(S_T) \cdot \underline{f}(S_T \mid S_0) \cdot \mathrm{d}S_T.$$

In addition, this parameter-adjusted distribution should be independent of the individual characteristics of any investor. To meet the latter condition, we need an economy in which the aggregation problem has been solved. The aggregation problem is solved if security prices are independent of the allocation of wealth across investors.[4] The two main assumptions which guarantee this independence are homogeneous expectations and linear risk tolerance utility functions[5] [see Brennan and Kraus (1978)].

The consequence of solving the aggregation problem is that security prices are determined as though there existed only identical representative investors.

Under these circumstances the current market price of the call (or any other security) satisfies

$$C_0 = \hat{R}^{-1} \cdot \mathrm{E}\left[C(S_T) \cdot u\right],$$

with

$u = U_w(w_T)/\mathrm{E}[U_w(w_T)]$, and

U_w = marginal utility of wealth of the representative investor at time T;

w_T = aggregate wealth at time T.

A risk-neutral valuation exists if $f(S_T \mid S_0)$ and U_w can be chosen such that

$$\begin{aligned} C_0 &= \hat{R}^{-1} \cdot \int_{-\infty}^{+\infty} C(S_T) \cdot u \cdot f(S_T \mid S_0)\, \mathrm{d}S_T \\ &= \hat{R}^{-1} \int_{-\infty}^{+\infty} C(S_T) \cdot \underline{f}(S_T \mid S_0) \cdot \mathrm{d}S_T. \end{aligned}$$

Bayesian statisticians will recognize in this the problem of finding conjugate families whereby U_w plays the role of the likelihood function and $f(S_T \mid S_0)$ that of the prior. Here there is one additional restriction, however: only the mean of the distribution may shift.

Solutions have been found for f being normal, lognormal and multinomial. The 'conjugate family' utility functions prove to be subsets of the class of linear risk tolerance functions. See Rubinstein (1976), Brennan (1979), Stapleton and Subrahmaniam (1984a, b).

4.2.2. Continuous trading models

Sometimes lack of hedging opportunities can be remedied by assuming some specific intertemporal security pricing model. For illustration, suppose assets are priced according to Merton's (1973)

[4] The aggregation problem is part of the theoretical foundation of many equilibrium pricing models such as the Sharpe–Treynor Capital Asset Pricing Model (CAPM) and Merton's (1973) Intertemporal Capital Asset Pricing Model (ICAPM).

[5] Linear risk tolerance utility functions satisfy the following condition: $-U_w(w)/U_{ww}(w) = a + bw$ with $U(w)$ = utility of wealth w and subscripts denoting partial derivatives; a and b are constants.

Intertemporal Capital Asset Pricing Model (ICAPM). The ICAPM has the implication that the instantaneous risk premium on any asset is proportional to the instantaneous covariance of the returns of the asset with the returns of all other assets in the economy (the market portfolio). Hence only 'covariance' or systematic risk receives a risk premium; the 'non-covariance' or idiosyncratic risk is priced risk-neutrally. The idea in pricing the option is to replace S and its distribution by its 'covariance' part. Hence, instead of mimicking perfectly, only the systematic part of the risk is 'copied' by the synthetic option.

This solution has been applied to price options written on assets which return follows a diffusion-jump process [e.g. Merton (1976), Cox and Rubinstein (1985), Shimco (1986)]. The same approach has also been used to value investments in real assets [e.g. McDonald and Siegel (1985), Constantinides (1978)]. [6]

5. Some extensions of option pricing

This section considers the impact of early exercise, dividend payments, taxes, transactions costs and margin requirements, changing risk-free rates and changing volatility on the applicability of arbitrage pricing.

5.1. Early exercise

Early exercise is not problematic for binomial option valuation. Working backwards, one simply computes $C(\cdot,\ n)$ at every node by the mimicking approach and compares this value with the income received by exercising the option at that node. As long as this income is smaller than the 'living' option value, the option should not be exercised. When the reverse holds, the option should be exercised, and that node's $C(\cdot,\ n)$ is equal to the income from exercising; from there on one simply continues to work backwards.

Early exercise is very problematic for the stochastic calculus approach. In fact, when early exercise is optimal, generally no algebraic solution exists for the option's equation of motion and boundary conditions (e.g., no algebraic solution for the American put is available). Then the option value may be approximated either by numerical methods or by a binomial process.

Much of the option literature is concerned with determining the conditions under which early exercise is optimal. It has been shown, for example, that early exercise is sub-optimal for calls as long as the underlying asset pays no dividends; with dividends exercise may be optimal just before the payout. For puts early exercise may be optimal, even without dividends; with dividends, the optimal exercise time comes after the payout. [See for example Cox and Rubinstein (1985), Geske and Shastri (1985). Margrabe (1978), Stultz (1982) and others solved the optimal exercise problem for more complex options.] [7]

5.2. Dividends

Options, unprotected against dividend payout (even European ones) may be impossible to price by mimicking.

To see how this may come about, consider the following simple example concerning a European call option with only one more period to go:

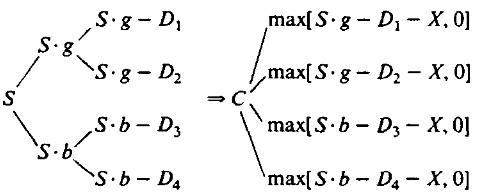

Fig. 6.

[6] Remark that the solution of 4.2.2 can be shown to be a special case of the solution of 4.2.1. Under the assumptions of the ICAPM the aggregation problem is solved; also the virtually equivalent problem of 'portfolio separation' is solved. Portfolio separation occurs when investors' portfolio choices over risky securities are independent of their wealth positions. A requirement for this to happen is that either all investors' utility functions belong to the linear risk tolerance class or that the return distributions of all assets belong to a number of classes (including the normal distribution). Because in the ICAPM returns are normally distributed the linear risk tolerance requirement is not needed. Even if in the discrete trading case normal return distributions for the underlying assets are assumed, the linear risk tolerance requirement can not be dropped because the option's return distribution is not normal. Portfolio separation is considered in Ross (1978a), Mossin (1977), Brennan and Kraus (1976).

[7] It should be mentioned here too that as soon as early exercise is optimal, put–call parity (see Section 2) generally does not exactly hold anymore; an evaluation of this relationship under 'real life' circumstances can be found in Merton (1973b), Stoll (1973), Klemkosky and Resnick (1979) and Brenner and Galai (1986).

By the end of the period, the value of the underlying asset jumps either to $S \cdot g$ or to $S \cdot b$. However, just before the call can be exercised a lump sum dividend D is paid. At the beginning of the period the amount of the dividend is uncertain.

It is clear that unless $D_1 = D_2$ and $D_3 = D_4$ (i.e., D is a deterministic function of the asset's value), or unless a third asset, perfectly correlated with the dividend, can be traded in, [8] mimicking is impossible. Under these circumstances even subdividing the interval and taking limits won't lead to a continuous trading model which allows pricing by arbitrage. The problem is that in the stochastic calculus approach, unless the dividend is a deterministic function of a tradable asset D_1, D_2, D_3, D_4 each would need a separate boundary condition; every one of these boundary conditions would lead to a different solution; and arbitrage arguments alone could not show how these solutions would have to be combined into a single price.

Whenever dividends can be regarded as 'known' or as a deterministic function of a tradable asset's value, binomial pricing is possible. Then also the stochastic calculus approach produces an algebraic solution for American calls [see Roll (1977), Geske (1977a, b), Whaley (1981)]. This work shows that an (unprotected against dividend payout) American call on dividend paying assets can be regarded as a portfolio of simpler options.

5.3. Transaction costs, taxes and margin requirements

Leland noted in (1985) that for the whole class of option-diffusion models, the implicit assumption of continuous portfolio adjustment is awkward in the presence of transactions costs. Since diffusion processes have infinite variation, continuous trading would be ruinously expensive. Leland shows that with proportional transactions costs, for T/n small enough, a Black–Scholes synthetic option, but based on a modified variance, replicates the real option's payoff with small error. The modified variance is equal to the real variance multiplied by one plus a transaction costs term.

Taxes seem less hard to handle than transactions costs. In fact, under a number of conditions no change would be needed in the approach of the previous sections, although of course the pricing formula would reflect the taxes.

More details about this issue can be found in Cox and Rubinstein (1985), Garman (1976), Dammon and Green (1987) and others.

A nasty little wrinkle in arbitrage pricing models is the existence of so called 'doubling' strategies. These strategies may be structurally present in many models and generate a sure dollar for a zero investment, thereby violating the premise of no arbitrage opportunities. A discrete version of a doubling strategy applied to betting on the flip of a fair coin is as follows:

(a) Bet \$1 on heads at toss number 1. If win, stop. Otherwise, bet \$2 on heads at toss number 2.
(b) At toss number n, if all previous bets are lost, bet $\$(2)^{n-1}$ on heads. If win, stop. Winnings will cover previous losses plus \$1. If lose, bet $\$2^n$ on head in next flip.

It is clear that

$$\text{Prob(winning \$1)} = 1 - \text{Prob(lose all bets)} = 1 - \lim_{n \to \infty} (1/2)^n = 1.$$

When we write out the profits and losses of consecutive tosses, each weighted by their probability, we arrive at the following series:

$$\$[2^{-1} \cdot 1 + 2^{-1} \cdot (-1) + 2^{-2} \cdot 2 + 2^{-2} \cdot (-2) + 2^{-3} \cdot 4 + 2^{-3} \cdot (-4) + \ldots$$
$$= \$[0.5 - 0.5 + 0.5 - 0.5 + \ldots .$$

This series consists of positive and negative terms with the property that both the subseries of positive and of negative terms diverges. It is well known that the sum of the total series may converge to any number on the real line simply by changing the order in which the terms are added [9] (= by changing our betting policy). Hence its value is not well defined and arbitrage pricing breaks down. Clearly such problems would be ruled out if

[8] Such an asset usually does not exist; this is known as the problem of incomplete markets. See Mossin (1977).

[9] See for example W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 3rd edition, 1976, theorem 3.54, pp. 76–77.

we only allowed series with a converging subseries of negative terms. In the context of our model it would suffice to require that the wealth position of the better, at all times, should be above some lower limit L. One way to assure this is to impose margin requirements (i.e., these limit borrowing opportunities). It has been shown in Heath and Jarrow (1987) that the existence of margin requirements does not invalidate Black–Scholes option pricing.

The doubling strategy was discovered by Harrison and Kreps (1979) and further studied by Harrison and Pliska (1981), Heath and Jarrow (1987) and others.

5.4. Changing risk-free rates

If interest rates change over time in a perfectly predictable way, all of the preceding analysis continues to hold. In applying the binomial method one simply has to use the appropriate interest rate for each (sub)period. To the extend that the option price's equation of motion continues to have an algebraic solution, deterministically changing instantaneous interest rates do not affect the continuous trading models either.

In discrete trading models, arbitrage pricing becomes quite cumbersome when interest rates are stochastic. Even if the spot interest rate and the underlying asset price each follow a (correlated) binomial, bonds of three different maturities and the underlying asset are needed for the mimicking. With continuous trading only one bound and the asset are required. To see why this is the case consider Figures 7 and 8.

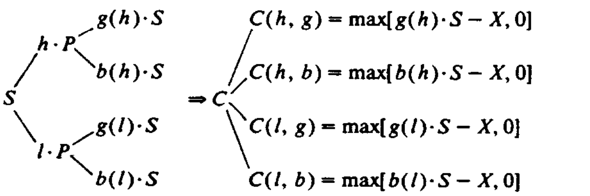

Fig. 7.

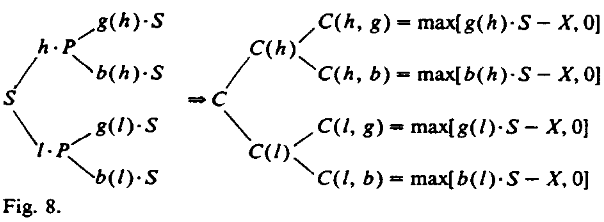

Fig. 8.

The left-hand sides of the preceding pictures express the price changes of the underlying asset and of a discount bond within any one of the n periods of a call's life. The change in the spot interest rate results in a jump to $h \cdot P$ or $l \cdot P$ from the bond's beginning of period price P. [10] The change in the spot interest rate also affects the jump in the asset's price; therefore the jump in S is also a function of the jump h or l. The right-hand sides of Figures 7 and 8 depict the corresponding price changes of the call.

The difference between the figures is that in the first one, trading is only possible after both P and S have moved, whereas in the second one, trading may occur a first time after P has changed but before S has moved and a second time after S has changed also. Figure 7 correctly depicts the phenomenon which we wish to model; however, mimicking requires trading in three different maturity discount bonds and in the underlying asset. Mimicking in Figure 8 requires trading in only one bond and the underlying asset. Although Figure 8 gives a wrong representation in a discrete trading model, it may become indistinguishable from Figure 7 when the length of every subperiod becomes infinitely small (see also Section 3.4).

Using the stochastic calculus approach, Merton (1973a) develops an expression for a European call very similar to the Black–Scholes formula. The main difference is that S_0 and $X \cdot R^{-T}$ are rescaled by $X \cdot P_0$ with P_0 the market value of a discount bond which promises to pay one money unit at its maturity T. Extensions to multi-asset options are reported in Cheng (1987).

Closely related is the work on bond options, options on futures of interest bearing securities and work on the term structure of interest rates. The bibliography of Cox and Rubinstein (1985) refers to a host of articles on these subjects.

5.5. Changing volatility

Similar remarks as in Section 5.3 apply when volatility changes: unless moves in σ are perfectly predictable, or a deterministic function of S or unless a third asset, perfectly correlated with σ, can be traded in, arbitrage pricing breaks down.

[10] Because the bond price has to converge to its nominal value at the bond's maturity, h and l cannot be chosen independent of time to maturity.

A well-known model among those with varying volatility is the one in which dS/S follows a diffusion with σ related to S according to:

$\sigma = m \cdot S^k$ with m and k some constants.

The solution of this model is the constant elasticity of variance option pricing formula. It takes into account the empirical observation that σ^2 tends to vary inversely with stock prices [see Cox and Ross (1967), Cox and Rubinstein (1985)].

Work applying the approach of Section 4.2 can be found in Hull and White (1987), Cox, Ingersoll and Ross (1985) and others.

Since σ is the hardest to estimate input in the Black–Scholes model, a large literature on the empirical estimation of σ has emerged. The bibliography in Cox and Rubinstein (1985) refers to work in this field.

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