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The Black-Scholes option pricing problem in mathematical finance: generalization and extensions for a large class of stochastic processes

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Résumé. — L'aptitude à quantifier le coût du risque et à définir une stratégie optimale de gestion de portefeuille dans un marché aléatoire constitue la base de la théorie moderne de la finance. Nous considérons d'abord le problème le plus simple de ce type, à savoir celui de l'option d'achat 'européenne', qui a été résolu par Black et Scholes à l'aide du calcul stochastique d'Ito appliqué aux marchés modélisés par un processus Log-Brownien. Nous présentons un formalisme simple et puissant qui permet de généraliser l'analyse à une grande classe de processus stochastiques, tels que les processus ARCH, de Lévy et ceux à sauts. Nous étudions également le cas des processus Gaussiens corrélés, dont nous montrons qu'ils donnent une bonne description de trois indices boursiers (MATIF, CAC40, FTSE100). Notre résultat principal consiste en l'introduction du concept de stratégie optimale dans le sens d'une minimisation (fonctionnelle) du risque en fonction du portefeuille d'actions. Si le risque peut être annulé pour les processus 'quasi-Gaussien' non-corrélés, dont le modèle de Black et Scholes est un exemple, cela n'est plus vrai dans le cas général, le risque résiduel permettant de proposer des coûts d'options "corrigés". En présence de très grandes fluctuations du marché telles que décrites par les processus de Lévy, de nouveaux critères pour fixer rationnellement le prix des options sont nécessaires et sont discutés. Nous appliquons notre méthode à d'autres types d'options, telles que 'asiatiques', 'américaines', et à de nouvelles options que nous introduisons comme les 'options à deux étages'... L'inclusion des frais de transaction dans le formalisme conduit à l'introduction naturelle d'un *temps caractéristique de transaction*.

Abstract. — The ability to price risks and devise optimal investment strategies in the presence of an uncertain "random" market is the cornerstone of modern finance theory. We first consider the simplest such problem of a so-called "European call option" initially solved by Black and Scholes using Ito stochastic calculus for markets modelled by a log-Brownian stochastic process. A simple and powerful formalism is presented which allows us to generalize the analysis to a large class of stochastic processes, such as ARCH, jump or Lévy processes. We also address the case of correlated Gaussian processes, which is shown to be a good description of three different market indices (MATIF, CAC40, FTSE100). Our main result is the introduction of the concept of an optimal strategy in the sense of (functional) minimization of the risk with respect to the portfolio. If the risk may be made to vanish for particular continuous uncorrelated 'quasi-Gaussian' stochastic processes (including Black and Scholes model), this is no longer the case for more general stochastic processes. The value of the residual risk is obtained and suggests the concept of risk-corrected option prices. In the presence of very large deviations such as in Lévy processes, new criteria for rational fixing of the option prices are discussed. We also apply our method to other types of options, 'Asian', 'American', and discuss new possibilities ('double-decker'...). The inclusion of transaction costs leads to the appearance of a natural characteristic trading time scale.

1. Introduction.

There are various reasons explaining why physicists might be interested in economy and finance. A first reason lies in the fact that market exchange is a typical example of complex system, where the apparently random fluctuations of market prices result from many causes, such as non-linear response of traders and speculators with highly inter-dependent behaviors, embedded in a time evolving - in a somewhat unpredictable way - environment. This difficulty in predicting the behavior of market prices has in fact been argued to be a fundamental property of "efficient" markets: any obvious predictable opportunity should rapidly be erased by the response of the market itself [1]. Present mathematics and economic theory of finance are developed from the perspective of stochastic models in which agents can revise their decisions continuously in time as a response to a variety of personal and external stimuli. It is the complexity of the agent expectations and interactions that provides the major difficulty in the study of finance. The development of new concepts and tools, in the theory of chaos, complexity and self-organizing (non linear) systems in the physics community in the past decades [2], make thus the modelling of market exchange particularly enticing, especially for physicists, as exemplified by quite a number of attempts: see e.g. [3-9] for recent discussions within this context. Another, more anecdotic reason, is that due to the job crisis, more and more physics students will probably end up working in finance. It is possible (although of course not certain) that this population will bring new concepts and methods, leading to a rapid evolution of the field.

There is also a historical reason: the French mathematician, Bachelier, discovered the theory of Brownian motion in connection with stock market fluctuations [10], five years before Einstein's classic 1905 paper. In particular, Bachelier proposed a formula for the price of an option (see below), based upon the idea that these fluctuations follow a Brownian process. This work laid somewhat dormant until the sixties, when many important ideas and methods were developed. This renewed activity paved the way to the seminal work of Black and Scholes [11] on the option pricing theory, which stands as a landmark in the development of mathematical finance: stochastic calculus was shown to be directly useful for an every day financial activity such as option pricing, which was, before Black and Scholes, only empirically (and not 'rationally') fixed. A considerable development of the field has then followed, both at fundamental [13-16] and commercial levels, and numerous softwares based on the Black-Scholes approach are now available.

In a nutshell, the simplest option pricing problem (the so called 'European call options') is the following: suppose that an operator wants to buy a given share, a certain time $t = T$ from now ($t = 0$), at a fixed 'striking' price x_c. If the share value at $t = T$, $x(T)$, exceeds x_c, the operator 'exercises' his option. His gain, when reselling immediately at the current price $x(T)$, is thus the difference $x(T) - x_c$. On the contrary, if $x(T) < x_c$ the operator does not buy the share. Symmetrically, a European put option gives the owner of a stock the right to sell his shares at time T at a preassigned price x_c. These possibilities given to the operator by -say- the "bank" are *options* and have obviously themselves a price. What is this price, and what trading strategy should be followed by the bank between now and T, depending on what the share value $x(t)$ actually does between $t = 0$ and $t = T$?

In a sense, the option problem can be considered as the elementary building block of the general problem of the evaluation of risks associated with market exchanges and more generally with human activity. Indeed, an option on a stock can be viewed as an insurance premium against the risk created by the uncertainty of share prices. When you insure your car against collisions, you are buying from the insurance company a 'put' option, i.e. an option to sell your car at a given price. That option will be worthless if you never have an accident (you pay the premium and collect nothing). But if your car is destroyed, you have the right to leave what remains of it with the insurance which is obliged to buy it and pay you the insured amount. The primary function of options is thus to give investors some control over how changes in the market will affect their portfolios. For a cost, buyers of options can limit losses with placing almost no limits on their profits. Sellers of options who expect little change in market prices can pocket an extra premium. In short, options satisfy the needs of both the prudent person and the speculator. Furthermore, pricing of much more complex financial securities essentially proceed along the same lines.

Black and Scholes' model gives an answer to both the above questions (price and strategy): assuming that the share value follows a log-Brownian process, they construct a strategy by which the bank can *exactly duplicate* the buyer's portfolio, in such a way that, for the bank, the whole process is *risk free* (the precise meaning of this statement will be discussed below). This construction is translated - using Ito's stochastic calculus [17] - into a partial differential equation, the solution of which containing both the option price and the bank trading strategy (see Appendix A).

The aim of this paper is to reformulate the problem in a more transparent and flexible way (at least in our eyes). The interest of our formulation is that 'generalized' Black-Scholes formulae can be readily obtained for a large class of stochastic processes, including general *correlated* Brownian processes. We propose to obtain generalized Black-Scholes strategies as those *minimizing* the risk in a functional sense - with the value of the residual risk as a by-product, which could allow one to propose *risk-corrected* option prices. We find that this residual risk is even zero for particular (continuous 'quasi-Gaussian') stochastic processes, which allows us to include Black and Scholes log-Brownian model in our general formalism. Lévy processes are also considered. For these extremely strongly fluctuating cases, new criteria for rationally fixing the option prices must be introduced.

We discuss several different extensions of our approach, in particular to other types of options ('Asian', 'American'), or to include the cost of transactions (market friction). We show in particular how a characteristic trading time scale naturally appears when this 'friction' is taken into account.

We also give evidence, based on a statistical analysis of three different indices (MATIF, CAC40, FTSE100), that the fluctuations are quite well described by a *correlated Brownian process*, which justifies the interest of our generalization of the Black-Scholes formula.

2. Solution of the option problem using a risk minimisation procedure : generalization of Black and Scholes.

In appendix 1, we give a brief summary of the results and method used by Black and Scholes, and most subsequent workers in this field. This may be useful in order to connect our approach with those developed in the mathematical finance literature. As already stated, their approach relies heavily on Ito stochastic calculus [17] and it is not straightforward to understand the underlying principles at the origin of their solution. The method that we now present relies on a completely different point of view and uses a much simpler formalism, which only requires basic knowledge in probability theory.

We shall call $x(t)$ the value of the underlying share of stock at time t, and consider the series $\{x(t)\}_{t=0,T}$ as a stochastic process, described by a certain probability density $P(x,t|y,t')$, giving that the value of x at t, knowing that it was y at $t' < t$, occurs (within $\mathrm{d}x$) with probability $P(x,t|y,t')\mathrm{d}x$.

The starting point is to express the total variation of the bank's wealth ΔW between $t=0$ and $t=T$ taking into account the existence of the option and the underlying share. Note that the existence of many other stocks in the market has no bearing on the option pricing problem. This justifies the book-keeping of only the share underlying the option. The fair price of the option will then be determined by the condition that the average of ΔW be zero (not net gain or loss on average for the bank and for the option buyer).

The total variation of the bank's wealth ΔW is the sum of three contributions : 1) the gain from pocketing from the buyer the option price, that shall be noted $\mathcal{C}(x_0, x_c, T)$ for a call option on a share of initial price x_0 starting at $t=0$, of striking price x_c and maturing at time T; 2) the potential loss equal to $-(x(T) - x_c)$ if $x(T) > x_c$ (stemming from the fact that the bank must, in this case, produce the share at $t = T$) and zero otherwise; 3) the gain or loss incurred due to the variation of stock prices during the time period extending from $t=0$ to $t=T$. This last term depends on the number of shares $\phi(x,t)$ at time t (at which the share price is x) held by the bank. The fact that this third term must be taken into account can simply be illustrated by considering the following scenario : suppose that the price of the share was to increase with certainty between $t=0$ and $t=T$. Then, it is clear that the bank would then have advantage in buying a share at $t=0$ ($\phi(x_0, t=0) = 1$) and in holding it until $t=T$ ($\phi(x, 0<t<T) = 1$), at which time, it will give it to the buyer for the price x_c. This simple example suggests that, more generally for an arbitrary realization of the share price $x(t)$, holding a certain amount of shares prior to the exercice time can be favorable for the bank. $\phi(x,t)$ has the meaning of the number of shares per option, taken in the limit where a large number of options and shares are traded simultaneously. It is of course in this limit that continuous trading makes sense. Then, if the bank has, at time t, $\phi(x,t)$ shares, the true variation of its wealth W (shares + other assets) between t and $t+\mathrm{d}t$ is only due to the fluctuations of the share price, i.e.: $\dfrac{\mathrm{d}W}{\mathrm{d}t} \equiv \phi(x,t)\dfrac{\mathrm{d}x}{\mathrm{d}t}$. (Note that the term $\dfrac{\mathrm{d}\phi(x,t)}{\mathrm{d}t}x$ describes conversion of shares into other assets or the reverse, but not a real change of wealth).

Hence the total variation of wealth of the bank taking into account the three above contributions is for a given realisation of the process $\{x(t)\}$ and a given strategy $\phi(x,t)$:

$$\Delta W = \mathcal{C}(x_0, x_c, T) - \theta(x(T) - x_c) + \int_0^T \phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \tag{1}$$

with $\theta(u) = u$ for $u > 0$ and zero otherwise ($\theta(u)$ is equal to u times the Heaviside function). The last term in the r.h.s. of equation (1) quantifies the effect of the trading between $t=0$ and $t=T$.

2.1 INDEPENDENT INCREMENTS OF THE SHARE VALUE. — Let us suppose for a while that the local slopes $\frac{\mathrm{d}x}{\mathrm{d}t}$ are statistically independent for different times. In all that follows, we shall always implicitly refer to a time discretized problem, and an expression like $\phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}$ in fact means $\phi(x_i)(x_{i+1}-x_i)$. For convenience, we shall however use the continuum notation. Then, denoting by $\langle\, ..\rangle$ the average over different realisations of the process $\{x(t)\}$, one has $\langle\phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\rangle \equiv \phi(x,t)\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle$, because $\frac{\mathrm{d}x}{\mathrm{d}t}$ is always posterior, and thus uncorrelated to the value of $x(t)$. The 'fair game' condition $\langle\Delta W\rangle = 0$ gives the European option pricing formula, which in the unbiased case [18], $\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle = 0$, reads $\mathcal{C}(x_0, x_c, T) = \langle\theta(x(T) - x_c)\rangle$, i.e.

$$\mathcal{C}(x_0, x_c, T) = \int_{x_c}^{\infty} \mathrm{d}x'(x' - x_c)P(x', T|x_0, 0) \tag{2}$$

This recovers the well-known Black and Scholes pricing formula (see Eq.(A2) in Appendix 1) which is always explicited for the Log-Brownian case, for which $P(x,T|x_0,0)$ is given by equation (A1) of Appendix 1 [11, 13]. It is important to note that this formula (2) holds generally, without any assumption on the specific form of $P(x,T|x_0,0)$ (although the assumption of independent increments is important - see below, Eq. (12)). The interpretation of equation (2) is straightforward: this formula simply says that the theoretical option price $\mathcal{C}$ is such that, on average, the total expected gain of both parties is zero: risk-free profit should not exist. The expected gain of the operator is indeed given by the right hand side of equation (2), since its gain is either $x' - x_c$ if $x' > x_c$ and zero otherwise, which must be weighted by the corresponding probability. Note that the option price is independent of the portfolio strategy of the bank, i.e. of the specific choice of the number $\phi(x,t)$ of shares per option. This is in contrast with the point of view of Black and Scholes and subsequent authors, for whom the option price is deeply linked with the underlying strategy.

In the Gaussian case, $P(x,T|x_0,0)$ reads:

$$P_{\mathrm{G}}(x, T|x_0, 0) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{(x - x_0)^2}{2Dt}\right) \tag{3}$$

where D is the "volatility" of the underlying stock (i.e. diffusion coefficient of the stock price) [28]. Introducing the complementary error function $\mathrm{erfc}(u) \equiv \frac{2}{\sqrt{\pi}}\int_u^{\infty} \mathrm{d}v \exp(-v^2)$, one finds $\mathcal{C}(x_0, x_c, T) = \sqrt{Dt}C(X_c)$ with $X_c \equiv \frac{x_c - x_0}{\sqrt{2Dt}}$ and

$$C(X_c) = \frac{1}{\sqrt{2}}[\frac{\exp(-X_c^2)}{\sqrt{\pi}} - X_c \mathrm{erfc}(X_c)] \tag{4}$$

Equation (4) leads to the following asymptotic behaviour: $C(0) = (2\pi)^{-1/2}$,

$$C(X_c \longrightarrow +\infty) \simeq (2\pi)^{-1/2}X_c^{-2}\exp(-X_c^2)$$

and

$$C(X_c \longrightarrow -\infty) \simeq -\sqrt{2}X_c + (8\pi)^{-1/2}X_c^{-2}\exp(-X_c^2)$$

which means that, as expected, $\mathcal{C}(x_0, x_c, T)$ is of order $\sqrt{Dt}$ for $x_c \simeq x_0$, very small if x_c is much greater than the initial price x_0, and equal to the quasi certain gain of $x_0 - x_c$ in the other limit.

What should be the optimal portfolio strategy of the bank, i.e. the best function $\phi^*(x,t)$ giving the number of shares per option as a function of time between 0 and T? A first plausible idea would be to maximize profit, i.e. find $\phi(x,t)$ such that ΔW given by equation (1) be maximum. However, only the third term in the r.h.s. of equation (1) depends on $\phi(x,t)$, which means that the maximization of ΔW is equivalent to that of $\int_0^T \phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$. This term is linear in $\phi(x,t)$ and, in absence of correlations between the increments $\frac{\mathrm{d}x}{\mathrm{d}t}$ at different times, cannot be maximized without knowing in advance the specific realization of $x(t)$ (on average, this term vanishes and thus maximization cannot be performed either on the average).

The next natural strategy for the bank is to attempt to minimize its risk. Since its average gain $\langle\Delta W\rangle$ is zero, the risk $\mathcal{R}$ is measured by the *fluctuations* of ΔW around its (zero) average, i.e. $\mathcal{R} = \langle\Delta W^2\rangle$, and thus explicitly depends on the strategy $\phi(x,t)$. We determine the optimal strategy $\phi^*(x,t)$ so that the risk is functionally minimized:

$$\frac{\delta\mathcal{R}}{\delta\phi(x,t)}|_{\phi=\phi^*} = 0 \tag{5}$$

For independent (but not necessarily Gaussian) increments such that $\langle\frac{\mathrm{d}x}{\mathrm{d}t}|_t\frac{\mathrm{d}x}{\mathrm{d}t}|_{t'}\rangle = \mathcal{D}(x)\delta(t-t')$, one finds:

$$\begin{aligned}\mathcal{R} = \mathcal{R}_c &+ \int_0^T \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x\ \mathcal{D}(x)P(x,t|x_0,0)\phi^2(x,t)\\ &-2\int_0^T \mathrm{d}t\int_{x_c}^{\infty}\mathrm{d}x'\int_{-\infty}^{+\infty}\mathrm{d}x\ \phi(x,t)(x'-x_c)\\ &\times\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle_{(x,t)\longrightarrow(x',T)}P(x,t|x_0,0)P(x',T|x,t)\end{aligned} \tag{6}$$

where $\mathcal{R}_c$ is the "bare" risk which would prevail in the absence of trading ($\phi(x,t)\equiv 0$):

$$\mathcal{R}_c = \left[\int_{x_c}^{\infty}\mathrm{d}x(x-x_c)^2P(x,T|x_0,0)\right] - [\mathcal{C}(x_0,x_c,T)]^2 \tag{7}$$

The term $\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle_{(x,t)\longrightarrow(x',T)}$ is the mean instantaneous increment conditioned to the initial condition (x,t) and a final condition (x',T). It is non-vanishing, contrary to the unconditionned increment $\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle = 0$ (neglecting the average interest rate by a suitable change of frame [18]) Equation (6) contains a positive term proportional to ϕ^2 and a negative term linear in ϕ, showing that an optimal solution exists, and leads to a reduced risk (compared to the bare one). The general solution reads:

$$\phi^*(x,t) = \int_{x_c}^{\infty}\mathrm{d}x'\langle\frac{\mathrm{d}x}{\mathrm{d}t}\rangle_{(x,t)\longrightarrow(x',T)}\frac{(x'-x_c)}{\mathcal{D}(x)}P(x',T|x,t) \tag{8}$$

and

$$\mathcal{R}^* = \mathcal{R}_c - \int_0^T \mathrm{d}t\int_{-\infty}^{+\infty}\mathrm{d}x\mathcal{D}(x)P(x,t|0,0)\phi^{*2}(x,t) \tag{9}$$

which are valid for an *arbitrary uncorrelated stochastic process*, including 'jump', or discrete-time, processes. The process can furthermore be explicitly time-dependent, with a variance

$\mathcal{D}(x,t)$ which is a function of time, as for the much studied 'ARCH' processes (see, e.g. [16]), which are Gaussian processes with a t-dependent variance $\mathcal{D}(t)$ [19]. ('ARCH' stands for Auto-Regressive Conditional Heteroscedasticity.)

Formulae (8-9) can be simplified in several cases. Suppose first that the increments $\frac{\mathrm{d}x}{\mathrm{d}t}$ are identical independent random variables of zero mean and variance given by D (The case where $D = \infty$ will be addressed below). Equation (8) then reads:

$$\phi^*(x,t) = \frac{1}{D(T-t)} \int_{x_c}^{\infty} \mathrm{d}x'(x'-x_c)(x'-x)P(x',T|x,t) \tag{8'}$$

If furthermore $\frac{\mathrm{d}x}{\mathrm{d}t}$ are Gaussian variables, then expression (3) will exactly hold for all time delays $T-t$, and one may transform equation (8) into :

$$\phi^*_{\mathrm{G}}(x,t) = \int_{x_c}^{\infty} \mathrm{d}x'(x'-x_c)\frac{\partial P_{\mathrm{G}}(x',T|x,t)}{\partial x} \tag{8''}$$

This expression is exactly the result obtained by Black and Scholes [11, 13] using a completely different formalism (see Appendix 1), when replacing $P_G(x',T|x,t)$ in equation (8") by the log-Brownian expression (A1). In fact, one can show that equation (8") holds both for the Brownian and log-Brownian models and more generally for 'quasi-Gaussian' models such that $\frac{\mathrm{d}x}{\mathrm{d}t} = g(x)\eta(t)$, where $g(x)$ is an arbitrary function and η a Gaussian noise.

What Black and Scholes were aiming at was to find a strategy such that the risk be exactly zero, for all realizations $x(t)$ of the stock prices. We do recover this result for the Brownian, the Log-Brownian models and more generally for 'quasi-Gaussian' models. Namely, we find that the residual risk reads:

$$\mathcal{R}^* = \mathcal{R}_c - D\int_0^T \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x P_{\mathrm{G}}(x,t|0,0)\phi^{*2}_{\mathrm{G}}(x,t) \tag{9'}$$

which vanishes exactly for Gaussian processes, due to the following identity:

$$\int_0^T \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x P_{\mathrm{G}}(x,t|x_0,0)\frac{\partial P_{\mathrm{G}}(x_1,T|x,t)}{\partial x}\frac{\partial P_{\mathrm{G}}(x_2,T|x,t)}{\partial x} =$$

$$= P_{\mathrm{G}}(x_1,T|x_0,0)\delta(x_1-x_2) - P_{\mathrm{G}}(x_1,T|x_0,0)P_{\mathrm{G}}(x_2,T|x_0,0) \tag{10}$$

Indeed, using the identity (10), one can show that the integral in the right hand side of equation (9') is *exactly* equal to $\mathcal{R}_c$, thus leading to a *zero residual risk* - for all 'quasi-Gaussian models'.

This equality (10) will however *not hold* for an arbitrary stochastic process [23] and thus the residual risk will not vanish in general. This is the main difference between the present approach and that of Black and Scholes and subsequent workers : 1) we find that a vanishing residual risk cannot be achieved in the general case; 2) however, this does not imply that an optimal strategy does not exist. We have indeed found an optimal $\phi^*(x,t)$ which minimize the risk, given by expression (9) and which is a simple generalization of Black and Scholes result.

These findings are relevant to various concrete situations. In particular:

a) strong deviations from a Gaussian behaviour ("leptokurtosis") are expected when the time delay $T-t$ is not large. In this case, our formula (9) allows one to estimate the residual risk and correct the option price accordingly.

b) More importantly, the transaction costs (market friction) prevents to trade continuously with time. Rather, trading will be restricted to a finite number of occurrences, separated by a non-zero time interval τ. The risk $\mathcal{R}^*$ will again be non zero in that case; in fact, it can be estimated using the Euler-McLaurin formula, which gives the difference between a continuous integral and its finite sum approximant. We find:

$$\mathcal{R}^* = \frac{D\tau}{2}\mathcal{P}(1-\mathcal{P}) + O(\tau^2) \tag{11}$$

where $\mathcal{P} \equiv \int_{x_c}^{\infty} \mathrm{d}x' P_\mathrm{G}(x',T|x_0,0)$ is the total probability of realization of the option. Note that $\mathcal{R}^*$ is maximum for $\mathcal{P} = 0.5$, i.e. when $x_c = x_0 = \langle x(T)\rangle$. Formula (11) will be of importance below: we shall show how it can be used to determine self consistently an optimal value of τ when market friction is taken into account.

Let us end this section by stressing the implications of the remarkable result that the risk can be made to vanish for 'quasi-Brownian' processes. This 'Black-Scholes' result would still hold if one replaces the function $\theta(x(T)-x_c)$ by an arbitrary function $\psi(x(T))$ of the final value $x(T)$: if $\{x(t)\}$ is a Gaussian process, it is always possible to choose C_ψ and $\varphi(x,t)$ so that $\psi(x(T)) = C_\psi(x_0,x_c,T) + \int_0^T \varphi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$ for any realisation of the random series $\{x(t)\}$. More precisely,

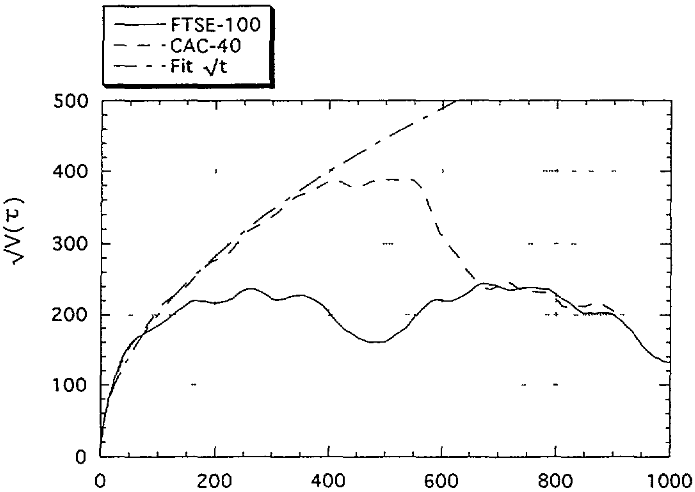
$$C_\psi(x_0,x_c,T) = \int_{-\infty}^{+\infty} \mathrm{d}x'\psi(x')P_\mathrm{G}(x',T|x_0,0)$$

and

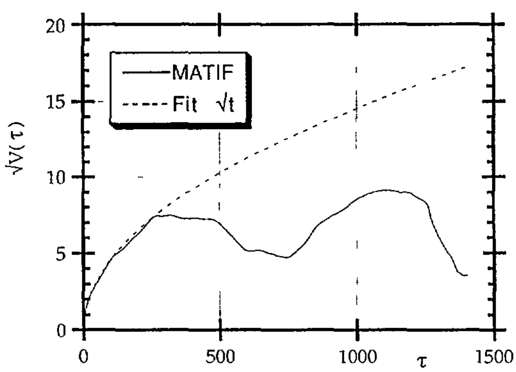
$$\varphi(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}x'\psi(x')\frac{\partial P_\mathrm{G}(x',T|x,t)}{\partial x}$$

These results allow one to express the arbitrary (non-linear) function $\psi(x(T))$ of the random variable $x(T)$ as a *path-independent* sum over past values of $x(t)$.

2.2 CORRELATED GAUSSIAN PROCESS. — Let us now assume that we have a general correlated Gaussian process, such that the difference $x(t) - x(t')$ is a Gaussian variable with zero mean and variance $V(t-t')$, where $V(.)$ is a given function. In the case of uncorrelated increments, $V(\tau) = D|\tau|$; the generalization to a power law behaviour $V(\tau) = D\tau^{2H}$ was proposed by Mandelbrot and Van Ness (see [24] b) under the name of 'fractional Brownian motion', as a way to model 'persistent' or 'anti-persistent' evolution of share prices. This case is outside the domain of validity of the Black and Scholes formalism, whereas it turns out to be exactly soluble for arbitrary $V(.)$ within our approach. The interest of this model is also motivated by the statistical analysis of some charts, which we now describe. We first determined the function $V(\tau)$ by computing the average $\langle[x(t_0+\tau)-x(t_0)]^2\rangle_{t_0}$, where $\langle...\rangle_{t_0}$ denotes a sliding average over the choice of the 'initial' time t_0 and $x(t)$ is the daily (closing) value of the London FTSE-100, the Paris CAC-40 and the French MATIF, in the period 1987-1992, *corrected by the average trend.* The behaviour of the function $V(\tau)$ is reproduced in figures 1a, b. Quite interestingly, $V(\tau)$ first grows linearly with τ as a standard uncorrelated Gaussian process until τ reaches τ_c, beyond which strong departures are observed. τ_c is found to be $\simeq$ 100, 350, 250 days for the FTSE, CAC-40, and MATIF respectively (250 days correspond to a year in real time). Then, $V(\tau)$ saturates and, for the FTSE and MATIF, *decreases* to a minimum around $\tau = 500-700$ days, corresponding to a two-year statistical pseudo-cycle (A true cycle of period T^* would correspond to $V(T^*) = 0$). We have checked that a certain degree of stationarity holds: similar conclusions are valid on the restricted periods 87-90, 89-92.



a)



b)

Fig. 1. — a) Behaviour of the root mean square fluctuations $\sqrt{V(\tau)}$ as a function of the time delay τ (in days) for the FTSE and the CAC-40. Note that the initial part of the curve is very well fitted by $V(\tau) \propto \tau$, which corresponds to an uncorrelated random walk, before a 'saturation' regime is rather sharply reached. b) Same as figure 1a, but for the MATIF. The CAC-40 (CAC: compagnie des Agents de Change) (resp. FTSE-100) is a French (resp. British) market index calculated from a weighted average of 40 (resp. 100) stock prices of the main sectors of activity of the country. MATIF stands for Marché à Terme International de France (French futures and options exchange).

Next, the approximate Gaussian character of $x(t_0)-x(t_0+\tau)$ was established by constructing the histogram of the rescaled excursion $y = \dfrac{x(t_0) - x(t_0 + \tau)}{\sqrt{V(\tau)}}$, for different values of τ between 50 and 800 days. As shown in figure 2, all the histograms are (within statistical errors) superimposed. The average of all these curves is reproduced in figure 3; we find that as long as the fluctuations are not too large the probability is quite well described by a simple Gaussian, although a slightly 'fatter' tail appears for larger deviations. Thus, our Gaussian correlated model seems to be a good description of the charts for not too small time delays (a

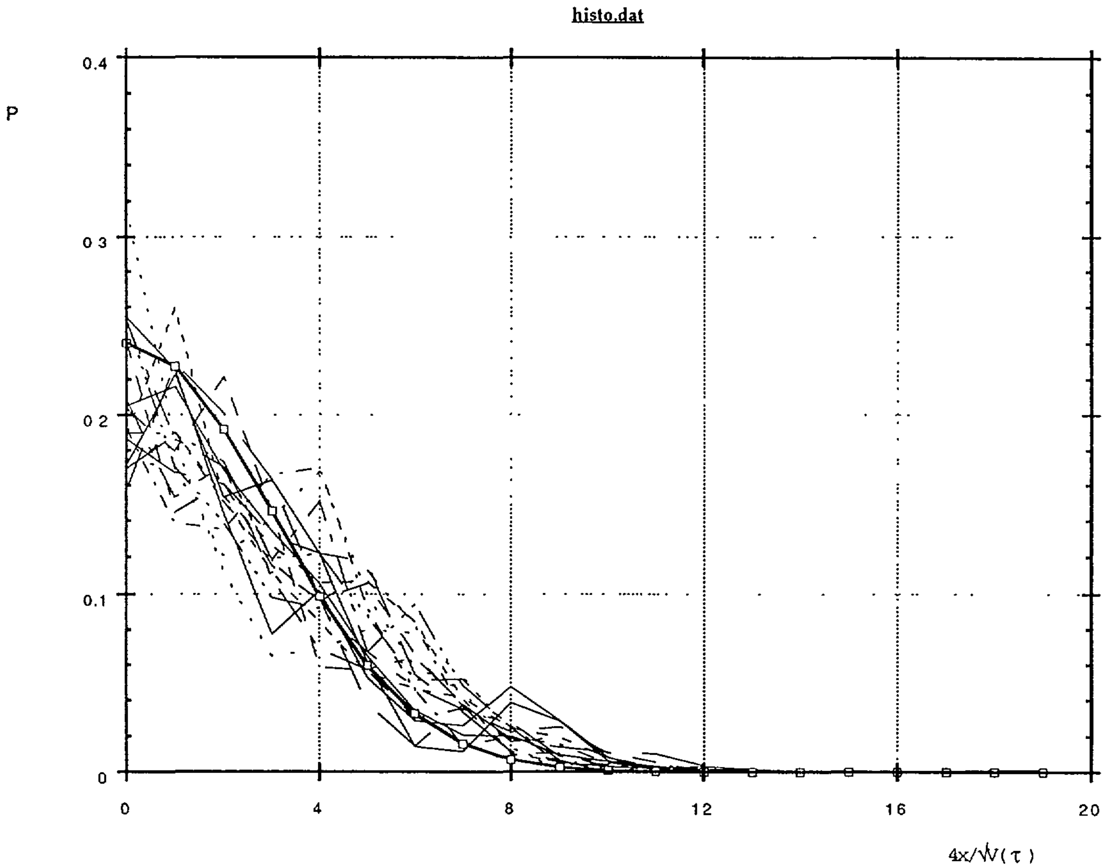


Fig. 2. — Histograms of the rescaled variable $y = |x(t_0) - x(t_0+\tau)|/\sqrt{V(\tau)}$, for different values of $\tau = 50, 100, 150, .. 800$. Although noisy, these different curves superimpose satisfactorily. A Gaussian is shown for comparison in thicker line (see Fig 3).

large kurtosis, corresponding to a rescaled fourth moment larger than the Gaussian value 3, is observed for small time delays).

Turning now to the European call option problem for such a process, one can perform the same calculation as above, although the presence of correlations slightly complicates the matter. As for the uncorrelated case, we start again from equation (1). The novel feature comes from the fact that one can show that, even in the case $\langle \frac{\mathrm{d}x}{\mathrm{d}t} \rangle = 0$, $\langle \phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t} \rangle$ is no more vanishing. It is given by $(\int_{0+}^{t} \mathrm{d}t' \frac{\mathrm{d}^2 V(t')}{\mathrm{d}t'^2})(2\pi V(t))^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}x \frac{\partial \phi(x,t)}{\partial x} \exp -[\frac{x^2}{2V(t)}]$. Hence, holding a certain portfolio of correlated shares leads to a *non-zero* average gain (or loss) $\mathcal{G}_\phi \equiv \int_0^T \mathrm{d}t \langle \phi(x,t)\frac{\mathrm{d}x}{\mathrm{d}t} \rangle$. The price $\mathcal{C}_c$ of the option on the correlated stock prices is thus given by $\mathcal{C}_c = \mathcal{C} - \mathcal{G}_\phi$, where $\mathcal{C}$ is still given by equation (2). The price of the option thus depends on the optimal strategy, to be determined through the minimization of the risk. After manipulations of Gaussian integrals, we obtain an integral equation determining the optimal strategy $\phi^*(x,t)$, or more precisely its

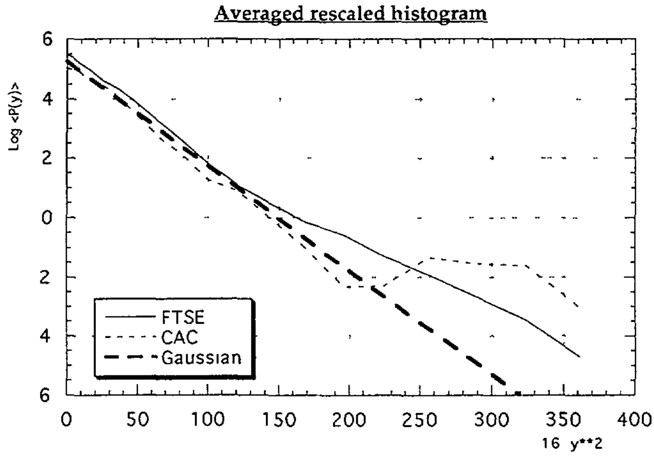


Fig. 3. — Average of the histograms plotted in figure 3 over τ, both for the FTSE and CAC (the MATIF gives similar results). We have shown here the logarithm of this average as a function of y^2, which should be a straight line for Gaussian distributions.

Fourier transform $\phi^*(\lambda,t) = \int_{-\infty}^{\infty} \mathrm{d}x e^{i\lambda x}\phi^*(x,t)$. This equation has the following form:

$$\int_0^T \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\lambda' \phi^*(\lambda',t')K(\lambda,t,\lambda',t') = F(\lambda,t) - H(\lambda,t)\mathcal{C}_c \tag{12}$$

where the kernel K and the functions F, H are given in Appendix 2. Let us note that solving this equation requires to invert K and to Fourier transform back to obtain $\phi^*(x,t)$, with $\mathcal{G}_{\phi^*}$ (and thus $\mathcal{C}_c$) determined self-consistently. Such a procedure can be implemented numerically once $V(\tau)$ is determined, and would be rather important for long term options, where the effect of correlations becomes crucial (see Figs. 1a, b).

2.3 Rare events and Lévy processes. — There might be interesting cases where the fluctuations are so strong that the notion of variance (or even average) loses its meaning - at least formally. This is the case of Lévy processes, which have been argued by many authors [24] to be adequate models for short enough time lags, when the kurtosis is large. Suppose then that the probability distribution of the stock price x at time T is given by:

$$P(x,T|x_0,0) = \frac{1}{(ZT)^{\frac{1}{\mu}}} L_\mu \left(\frac{(x-x_0)}{(ZT)^{\frac{1}{\mu}}} \right) \tag{13}$$

where $\mu < 2$ is the characteristic exponent of the Lévy process, $L_\mu(u)$ the corresponding Lévy distribution and Z the generalization of the 'volatility', which one might call 'hypervolatility': $(ZT)^{\frac{1}{\mu}}$ is the typical excursion of the share price during time T, just as $\sqrt{Dt}$ in the case of an uncorrelated Brownian process. Note that $L_\mu(u)$ decays, for $u \longrightarrow \infty$, as $\frac{C_\mu}{u^{1+\mu}}$, where C_μ is a μ-dependent number. (cf., e.g. [25]). It is interesting to discuss the option pricing formula, equation (2), in the limit $x_c - x_0 \gg (ZT)^{\frac{1}{\mu}}$. One finds:

$$\mathcal{C} = \frac{ZTC_\mu}{\mu} \int_{x_c}^{\infty} \frac{\mathrm{d}v}{v^\mu} \tag{14}$$

One must thus distinguish two cases: $\mu < 1$ and $\mu > 1$.

i) $\mu < 1$. In this case, one finds that the integral in equation (14) diverges, and hence that $\mathcal{C} = \infty$! Is the option pricing impossible in that case ? Yes and no: if the process was really described by a Lévy distribution, even far in the tails, the notion of average would be meaningless when $\mu < 1$, and the price of the option should be fixed using a different criterion. A possibility would be, for example, to demand that a loss (for the bank) greater than a certain acceptable level $\mathcal{L}$ should have a small probability p to occur, giving:

$$\int_{\mathcal{L}+\mathcal{C}+x_c}^{\infty} \mathrm{d}x P(x,T|y,0) = p \tag{15}$$

or, using the asymptotic form of L_μ, $\mathcal{C} \simeq (\frac{C_\mu ZT}{\mu p})^{\frac{1}{\mu}} - \mathcal{L} - x_c$ (for small p). However, no 'experimental' process is truly of the Lévy type far in tails: there always exists a cut-off $x_{\max}(T)$ beyond which the fluctuations are truncated due to a physical, or economical, mechanism (see e.g. [25, 26]). In this case, formula (14) becomes well defined, but in the case considered here ($\mu < 1$), the price of the option is extremely sensitive to the 'rarest events': $\mathcal{C} = \frac{ZTC_\mu}{(1-\mu)\mu}[x_{\max}(T)]^{1-\mu}$.

ii) $2 > \mu > 1$, corresponding to values of μ often quoted in the literature [24]. In this case, equation (14) converges and gives, quite independently of the value of $x_{\max}$, $\mathcal{C} = \frac{ZTC_\mu}{(\mu-1)\mu} x_c^{1-\mu}$ However, the criterion (Eq. (5)) fixing the optimal strategy, based on a minimization of the variance (which is infinite for Lévy processes), is still ill-defined. A possibility would be to study the tails of the distribution of the wealth variation ΔW, which decays as $\frac{W_0^\mu}{\Delta W^{1+\mu}}$ for large losses ($\Delta W \longrightarrow -\infty$), with W_0 depending on $\phi(x,t)$. $\phi^*(x,t)$ would then be determined so that $\frac{\delta W_0}{\delta\phi(x,t)} = 0$ - corresponding to a minimization of the 'catastrophic' risks, since W_0 controls the scale of the distribution of losses, i.e. their order of magnitude. We leave this problem for future work.

3. Extension to other types of options. Discussion.

As stated in the introduction, the European option is the simplest type of option. Many more complicated scenarii exist (the only limitation lies in the imagination of the banks and in their ability to price the new products and devise optimal investment stategies). Let us discuss two possibilities:

• Asian options. For technical reasons, it is often not the final value of the stock $x(T)$ which is taken as a basis, but rather an average value of $x(t)$ over the last few days. We shall here consider a slightly generalized version of this problem, and define the operator profit as $\theta(\tilde{x} - x_c)$, where $\theta(u)$ is still u for positive u and zero otherwise. $\tilde{x}$ is defined as:

$$\tilde{x} = \int_0^T \mathrm{d}t\ w(t)\ x(t) \tag{16}$$

$w(t)$ is an arbitrary weight function. Asian options correspond to $w(t) = 1/M$ for $t = T-M+1, T-M+2, .\ , T$, and $w(t) = 0$ otherwise, but smoother functions could be imagined. ($w(t) = 1/T$ would correspond to the option on the mean) . This problem can be completely solved using the method introduced above in the case of an arbitrary correlated Gaussian process; the formulae are simpler in the absence of correlations. We find in this case:

$$\mathcal{C}(x_0, x_c, T) = \int_{x_c}^{\infty} \mathrm{d}x' \frac{(x' - x_c)}{\sqrt{2\pi D[\Omega^2](T)}} \exp - \left[\frac{(x' - x_0)^2}{2D[\Omega^2](T)}\right] \tag{17}$$

and

$$\phi^*(x,t) = \frac{\Omega(t)}{\sqrt{2D\pi([\Omega^2](T) - \frac{[\Omega]^2(t)}{t})}}$$

$$\times \int_{x_c}^{\infty} \mathrm{d}x' \exp - \left[\frac{(x - x' \frac{[\Omega](t)}{[\Omega^2](T)})^2}{2D(t - \frac{[\Omega]^2(t)}{[\Omega^2](T)})} + \frac{x'^2}{2D[\Omega^2](T)} - \frac{x^2}{2Dt}\right] \tag{18}$$

where we have introduced the notation $\Omega(t) \equiv \int_t^T w(t')\mathrm{d}t'$ and $[f](t) \equiv \int_0^t \mathrm{d}t' f(t')$, and taken the origin of share prices at x_0 in equation (18). Since $\Omega(t) \leq 1$, we find from equation (17) that Asian options are less costly than European ones (which correspond to $w(t) = \delta(t-T)$ and $\Omega(t) = 1$). The residual risk is still given by equation (9). Note that since the Gaussian and log-Brownian models (see Appendix 1) coincide in the limit $\frac{x(T) - x(0)}{x(0)} \ll 1$, i.e. $\sigma T \ll \sqrt{\sigma T} \ll 1$, with the identification $D \equiv \sigma x_0^2$, the calculations presented here could be useful to check some results obtained within the log-Brownian model [27].

• American options. A more complicated problem arises when one considers 'American' options. In this case, the operator may exercise his option *at any time* between $t = 0$ and $t = T$. Even the price of the option $\mathcal{C}$ is difficult to determine since one first has to know what the operator will actually do. The only 'easy' result is to show [13] that the American option price is at least larger than the European option price, since the 'optimal' strategy for the operator, in the sense of leading to the largest average profit, cannot be worse than just waiting until 'maturity' $t = T$. An interesting way to attack this problem is to consider what we shall call 'double-decker' options, which can be exercized either at time T_1 and price x_{c1} or at time $T_2 > T_1$ and price x_{c2}. Let us call $f(x_1)$ the probability that the operator decides to exercize his option at time T_1, knowing that the stock has reached x_1. It is then a simple matter to show that the expected gain of the operator is given by:

$$\mathcal{C}(x_0, x_{c2}, T_2) + \int_{x_{c1}}^{\infty} \mathrm{d}x_1 P(x_1, T_1|x_0, 0) f(x_1)[x_1 - x_{c1} - \mathcal{C}(x_1, x_{c2}, T_2 - T_1)] \tag{19}$$

where $\mathcal{C}$ is the European call option price discussed above (Eq. (2)). The optimal strategy $f(x_1)$ is:

$$f(x_1) = \begin{cases} 1 & \text{for } x_1 > x_{c1} + \mathcal{C}(x_1, x_{c2}, T_2 - T_1) \\ 0 & \text{for } x_1 < x_{c1} + \mathcal{C}(x_1, x_{c2}, T_2 - T_1) \end{cases} \tag{20}$$

since this ensures that only the positive contributions of the integrand are kept in the integral in equation (19). Let us first discuss the usual 'pre-American' option case where $x_{c1} = x_{c2} = x_c$. Since $\mathcal{C}(x_0, x_c, T) + (x_c - x_0) > 0$ for all finite x_c, one has $f(x_1) \equiv 0$: the optimal strategy is to exercize the option at maturity, which thus becomes *de facto* a European option. Extending the argument to a continuous set of choices at the same striking price, i.e. to the American option, we thus find that its rational price is the same as that for a European option [13, 11]. This result is somewhat non intuitive and paradoxical: giving more freedom does not increase the average gain ! In fact, the difference betwen American and

European options lies in the probability distribution of gains. One can for instance show that the probability of a strickly positive gain (for the buyer) in the 'double-decker' case is given by $1 - \int_{-\infty}^{x_{c1}} \mathrm{d}x_1 P(x_1, T_1|x_0, 0) \int_{-\infty}^{x_{c2}} \mathrm{d}x_2 P(x_2, T_2|x_1, T_1)$ which is greater than or equal to the probability of positive gain in the European case, given by $1 - \int_{-\infty}^{x_{c2}} \mathrm{d}x_2 P(x_2, T_2|x_0, 0)$. This rationalizes the expectation that American options are more 'attractive' than European options - one gains more often (but smaller amounts such that the average gain is less).

A much more interesting situation occurs where the exercice price is not constant, more precisely, when $x_{c1} < x_{c2}$. In that case, the equation $x^* - x_{c1} = \mathcal{C}(x^*, x_{c2}, T_2 - T_1)$ has a non trivial solution $x^*(x_{c1}, x_{c2}, T_2 - T_1)$, and the option is exercized prematurely whenever $x_1 > x^*$. This leads to an increased expected gain for the operator, and thus, correspondingly, an increase in the option price which can be explicitly computed by replacing x_{c1} with x^* and $f(x_1)$ with 1 in equation (19). Once x^* is known, the optimal strategy for the bank - in the sense, as above, of minimizing the risk - can also be obtained. This line of thought may of course be generalized to 'Treble-Decker' or 'p-Decker' options, with an arbitrary sequence of exercice prices $x_{c1}, ..., x_{cp}$, which could be fixed, e.g., by the operator himself. This could be an interesting new family of financial products.

• Let us finally turn to a slightly different theme, which is the inclusion of the 'market friction', in other words the fact that trading by itself induces costs. A realistic way to model this is to add to equation (1), in the case of European options, a term like $-\gamma_0 \int_0^T \mathrm{d}t |\frac{\partial \phi(x,t)}{\partial t}| - \gamma_1 M(T)$. $M(T)$ is the total number of operations, i.e. times when $\frac{\mathrm{d}\phi}{\mathrm{d}t} \neq 0$. These terms mean that any operation on the stock (buying or selling) costs a certain fraction γ_0 of the transaction *plus* a fixed price γ_1. If the time interval between trading is τ, the order of magnitude of these transaction costs is:

$$\frac{T}{\tau}(\gamma_0 \phi \sqrt{\frac{\tau}{T}} + \gamma_1) \tag{21}$$

showing that, not unexpectedly, the γ_1 term dominates for small τ. On the other hand, as we have shown in section 2, the residual risk for a Brownian or Log-Brownian process grows proportionally to τ itself (Eq. (11)). Hence, the extra cost of risk + transactions leads to a modified option price:

$$\mathcal{C} \longrightarrow \mathcal{C} + \gamma_1 \frac{T}{\tau} + \sqrt{\frac{D\tau}{2}\mathcal{P}(1-\mathcal{P})} \tag{22}$$

showing that an optimal trading time τ^* appears, which leads to a minimized extra cost $\Delta\mathcal{C}$. In order of magnitude, we find:

$$\tau^* \simeq \left[\frac{2\gamma_1^2 T^2}{D\mathcal{P}(1-\mathcal{P})}\right]^{1/3} \tag{23}$$

and

$$\Delta\mathcal{C} \simeq [4\mathcal{P}(1-\mathcal{P})\gamma_1 DT]^{1/3} \tag{24}$$

Taking typically $D \simeq x_0^2\ 10^{-4}$ per day (that is 1 per cent variation per day of the stock value x_0), $\gamma_1 \simeq x_0\ 10^{-3}$, $T = 100$ days and $\mathcal{P} = 1/2$, we find $\tau^* \simeq 9$ days and $\Delta\mathcal{C} = 2.2 \times 10^{-2} x_0$.

4. Conclusion.

Using the language of physicists, we have analyzed the option pricing problem, which constitutes the cornerstone of the modern theory of finance and more precisely of the general problem of the evaluation of risks associated with market exchanges or human activity. We have introduced a simple and powerful formalism which has allowed us to solve the pricing problem for a large class of stochastic processes, such as ARCH, Lévy and correlated Gaussian processes. Our main result is the introduction of the concept of an optimal strategy in the sense of minimization of the risk as a function of portfolio. The remarkable result, that the risk may be made to vanish for particular continuous 'quasi-Gaussian' stochastic processes, which includes Black and Scholes log-Brownian model, is shown to be wrong for the case of more general stochastic processes. In the presence of very large deviations such as in Lévy processes, we have discussed new criteria for rationaly fixing the option prices. We have also applied our method to other types of options, 'Asian', 'American', as well as to novel ones, 'double-decker'... Many other options exist or can be invented, whose valuation and underlying porfolio strategy are amenable within our formalism. Examples which will be presented elsewhere comprise so-called explosive options ("caps", "floors", "collars"..), options on the maximum or options in which the weight $w(t)$ is determined in real time... Furthermore, the idea of risk minimization could lead to interesting numerical developments using Monte-Carlo methods and simulated annealing for cases not amenable to analytical treatments.

Much remains to be done in order to understand and model the full complexity of financial markets and liabilities. However, in accord to the standard trend in the mathematical theory of finance, a theory of options is a necessary first step before generalizing to more complex financial products, generally shown or believed to be combinations of options [12]. The theory of options is thus deeply linked with the branch of finance, called contingent-claim analysis, whose applications range from the pricing of complex financial securities to the evaluation of corporate capital budgeting and strategic decisions. Finally, it is often claimed that the trading of options, i.e. *insurance premium*, contribute to the stability of markets, their efficiency and liquidity.

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Appendix 1.

The Black-Scholes results.

Here, we summarize the seminal result obtained by Black and Scholes for the option pricing problem and the determination of the portfolio strategy that the bank must follow. The fundamental idea underlying their treatment is that a suitable strategy for the bank should make the risk vanish. Expressing this condition using Ito stochastic calculus, they obtain in the same token from the solution of a Fokker-Planck equation the option price and the strategy.

Recall that $x(t)$ denotes the value of the underlying share of stock at time t. Furthermore, we consider the series $\{x(t)\}_{t=0,T}$ as a stochastic process, described by a certain probability density $P(x,t|y,t')$, giving that the value of x at t, knowing that it was y at $t' < t$, occurs (within $\mathrm{d}x$) with probability $P(x,t|y,t')\mathrm{d}x$.

The log-Brownian model studied by Black and Scholes is the most used in the mathematical finance literature [12]: it assumes that interest rates are independent random Gaussian variables. As a consequence, the share value $x(t)$ is distributed according to a log-normal law :

$$P_{\mathrm{BS}}(x,t|y,t') = \frac{1}{x\sqrt{2\pi\sigma(t-t')}} \exp\left(-\frac{[\log(x/y) - (m-\sigma/2)(t-t')]^2}{2\sigma(t-t')}\right) \tag{A1}$$

where m is the risk free interest rate, which in the following will be set to zero for simplicity [18]. σ is called the 'volatility' (i.e. "diffusion coefficient" of the interest rate) [28]. With this notation, the results of Black and Scholes are as follows. If one denotes $\mathcal{C}(x_0,x_c,T)$ the price of the option (where $x_0 \equiv x(t=0)$ and x_c is the striking price) the 'Black-Scholes' formulae can be written as [15]:

$$\mathcal{C}(x_0,x_c,T) = \int_{x_c}^{\infty} \mathrm{d}x'(x'-x_c)P_{\mathrm{BS}}(x',T|x_0,0) \tag{A2}$$

This expression (A2) can be shown to be equivalent to

$$\mathcal{C}(x_0,x_c,T) = \int_{x_c}^{\infty} \mathrm{d}xR(x,x_0,T) \tag{A3}$$

using the repartition function $R(x,x_0,T) = \int_x^{\infty} \mathrm{d}x'P_{\mathrm{BS}}(x',T|x_0,0)$. Using the Log-Brownian model (A1) in equation (A3) yields the most often quoted form of the Black-Scholes pricing formula. Note that when the relative price fluctuations of the underlying stock are small, i.e. $\frac{x-x_0}{x_0} \ll 1$, or $\sigma T \ll \sqrt{\sigma T} \ll 1$, the log-Brownian model coincides - to leading order - with the Brownian model discussed in the main text (see Eq. (4)), with the identification $D \equiv \sigma x_0^2$.

Black and Scholes give at the same time the number $\phi^*(x,t)$ of shares per emitted option that the bank must possess at time t if the observed price is x:

$$\phi^*(x,t) = \frac{\partial}{\partial x}\int_{x_c}^{\infty} \mathrm{d}x'(x'-x_c)P_{\mathrm{BS}}(x',T|x,t) \equiv \frac{\partial\mathcal{C}(x,x_c,T-t)}{\partial x} \tag{A4}$$

(we have used the stationarity of the process, i.e. the fact that only $t-t'$ enters expression (A1)).

The argument of Black-Scholes and subsequent authors [11, 15, 16] is the following (although these authors use a different language): if the bank has, at time t, $\phi(x,t)$ shares, then the true variation of its total wealth W (shares + other assets) between t and $t+\mathrm{d}t$ is only due to the fluctuations of the share price, i.e.: $\frac{\mathrm{d}W}{\mathrm{d}t} \equiv \phi^*(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}$. (Note that the term $\frac{\mathrm{d}\phi^*(x,t)}{\mathrm{d}t}x$ describes conversion of shares into other assets or the reverse, but not a real change of wealth). Hence the variation of wealth due to trading between $t=0$ and $t=T$ is, according to equation (A4):

$$\Delta W = \int_0^T \mathrm{d}t\phi^*(x,t)\frac{\mathrm{d}x}{\mathrm{d}t} = \int_0^T \mathrm{d}t\frac{\partial\mathcal{C}(x,x_c,T-t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{C}(x(T),x_c,0) - \mathcal{C}(x_0,x_c,T) \tag{A5}$$

and thus, taking into account the initial amount $\mathcal{C}(x_0,x_c,T)$ paid by the operator, the extra wealth available to the bank at $t=T$ is simply $\Delta W + \mathcal{C}(x_0,x_c,T) = \mathcal{C}(x(T),x_c,0)$.

From the definition (A2) of the function $\mathcal{C}(x_0, x_c, T)$, $\mathcal{C}(x(T), x_c, 0) = x(T) - x_c$ if $x(T) > x_c$ and zero otherwise (we have used the property $P_{\rm BS}(x, T|y, T) = \delta(x-y)$). Hence equations (A4, A5) mean that *for all realisations* of the process $\{x(t)\}$ (and not only on average!), the wealth available to the bank exactly compensates the losses incurred if the operator makes a profit - and thus the risk is zero. In other words, in this idealized market situation, the possession of the portfolio $\phi^*(x,t)$ given by equation (A4) is completely equivalent to the possession of the option.

There is however a subtlety in the sequence of transformations in equation (A5), which would certainly be valid if the series $\{x(t)\}$ were sufficiently 'smooth'. If, as is the case in reality, $\{x(t)\}$ jumps discontinuously for certain times t_i, then the meaning of the integrals must be specified more carefully. Since the strategy $\phi^*(x,t)$ must be determined *before* (and *not* simultaneously to) the price variation, the equality $\int_0^T \mathrm{d}t \frac{\partial f(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(T),T) - f(x_0,0)$ (where f is an arbitrary function) does *not* hold in general - there is a correction term (called the Ito correction) [20]. For Log-Brownian processes however (as well as for more general 'quasi-Brownian' processes), one can show that in fact $\mathcal{C}(x, x_c, t)$ given by equation (A2) obeys a diffusion equation $\frac{\partial \mathcal{C}(x, x_c, T-t)}{\partial t} = -\frac{1}{2}\frac{\partial^2 \langle \frac{\mathrm{d}x^2}{\mathrm{d}t}\rangle \mathcal{C}(x, x_c, T-t)}{\partial x^2}$ This is easily verified using the fact that $\frac{\partial P_{\rm BS}(x', T|x, 0)}{\partial x} = -\frac{\partial P_{\rm BS}(x', T|x, 0)}{\partial x'}$ This ensures that the Ito correction vanishes and that equation (A5) is correct. This was indeed the very condition used by Black and Scholes to fix $\mathcal{C}(x_0, x_c, T)$!

The derivation of these results in the framework of stochastic calculus is rigorous and well-established [12]. However, it seems strange and counter-intuitive. Indeed, the notion of zero risk is completely orthogonal to the intuition developed in the study of stochastic processes in physics : to be able to transform a stochastic process into something certain sounds strange ! Our initial naive intuition was in fact that this result could be wrong and could have resulted from the use of the wrong stochastic calculus prescription (Ito or Stratonovitch) [21, 22] - tantamount to an incorrect discretized version of the problem - which can lead to totally erroneous conclusions.

Our different approach exposed in the main text in fact conforts these results, albeit in a more transparent way. Furthermore, we show how the notion of zero risk can be straightforwardly generalized to the concept of a minimization of the risk, in order to encompass more general stochastic processes and also different types of options.

Appendix 2.

Formulae for K, H and F.

We give here, for completeness, the expressions of the kernel K and the functions F, H introduced in the case of correlated Brownian processes. These results were obtained by using standard properties of Gaussian integrals. Let us first introduce the correlation function $C(t'-t'')$ such that $V(t) \equiv \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' C(t'-t'')$ and define the function β as:

$$\beta(t_1, t_2) = \int_0^{t_1} \mathrm{d}t' \int_0^{t_2} \mathrm{d}t'' C(t'-t'') \tag{B1}$$

Then one has:

$$F(\lambda,t) = \frac{i}{2\lambda}\frac{\partial}{\partial t}\int_{x_c}^{\infty} \mathrm{d}x' \frac{(x'-x_c)}{\sqrt{2\pi\beta(T,T)}}$$

$$\times \exp\left[\frac{1}{2\beta(T,T)}\{\lambda^2(\beta^2(T,t)-\beta(t,t)\beta(T,T)) + 2i\lambda x'\beta(T,t) - x'^2\}\right] \tag{B2}$$

$$H(\lambda,t) = i\lambda[\int_0^{t^-} \mathrm{d}t' C(t-t')] \exp -[\frac{\lambda^2\beta(t,t)}{2}] \tag{B3}$$

and

$$K(\lambda,t,\lambda',t') = \frac{1}{4}[\lambda^2 \frac{\mathrm{d}\beta(t,t)}{\mathrm{d}t}\frac{\partial\beta(t,t')}{\partial t'} + \lambda'^2\frac{\mathrm{d}\beta(t',t')}{\mathrm{d}t'}\frac{\partial\beta(t,t')}{\partial t}$$

$$+2\lambda\lambda'\frac{\partial\beta(t,t')}{\partial t}\frac{\partial\beta(t,t')}{\partial t'} - \frac{\partial^2\beta(t,t')}{\partial t\partial t'}]$$

$$\times \exp -\frac{1}{2}[\lambda^2\beta(t,t) + \lambda'^2\beta(t',t') + 2\lambda\lambda'\beta(t,t')] \tag{B4}$$

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[18] The case of a non zero biais $m \neq 0$ can be very easily treated if m is the same for all assets, without forgetting the fact that the initial amount paid by the buyer will itself price up. When m is greater than the risk-free expected return, due, say, to dividends, then slightly more involved calculations follow much along the lines leading to equation (12).

[19] The result for ϕ^* (Eq. (8')) for ARCH processes is simply obtained by replacing $D(T-t)$ by $\int_t^T \mathcal{D}(t')\mathrm{d}t'$.

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[28] In the finance literature, the term "volatility" is often used for the standard deviation $\sqrt{\sigma}$ of the interest rate. We use here the term in a slightly broader sense.