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Active portfolio management with benchmarking: Adding a value-at-risk constraint

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Abstract

We examine the impact of adding a value-at-risk (VaR) constraint to the problem of an active manager who seeks to outperform a benchmark while minimizing tracking error variance (TEV) by using the model of Roll [1992. A mean/variance analysis of tracking error. *Journal of Portfolio Management* 18, 13–22]. We obtain three main results. First, portfolios on the constrained mean-TEV boundary still exhibit three-fund separation, but the weights of the three funds when the constraint binds differ from those in Roll's model. Second, the constraint mitigates the problem that when an active manager seeks to outperform a benchmark using the mean-TEV model, he or she selects an inefficient portfolio. Finally, when short sales are disallowed, the extent to which the constraint reduces the optimal portfolio's efficiency loss can still be notable but is smaller than when short sales are allowed.

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1. Introduction

As Roll (1992) and Cornell and Roll (2005) note, institutional investors often manage money against a benchmark. This has led to the practice by active portfolio managers (hereafter ‘managers’) of seeking to outperform the benchmark by a given percentage, subject to a limit on *tracking error variance*, or TEV.¹ However, this practice leads such managers to select portfolios that are mean-variance inefficient and under certain conditions have systematic risk that is greater than 1 when measured against the benchmark. Not surprisingly, large losses relative to the benchmark have occurred in some cases. A recent example involved the management of Unilever’s pension fund by Merrill Lynch, who in attempting to beat the FTSE All-Share Index by 1% per year, ended up as the defendant in a lawsuit due to underperforming the index by roughly 10% over a 15-month period.²

Two methods have been proposed for overcoming this tendency to invest in overly risky portfolios. Roll (1992) advocates constraining the portfolio’s beta, while Jorion (2003) advocates constraining the portfolio’s variance. In this paper we propose a third method that involves constraining the portfolio’s *Value-at-Risk*, or VaR.³

A VaR constraint is of particular interest for several reasons. First, as Jorion (2001, 2003) and Pearson (2002) note, the fund management industry is increasingly using VaR to: (1) allocate assets among managers, (2) set risk limits, and (3) monitor asset allocations and managers (these activities are often referred to as ‘risk budgeting’). Second, we show that Jorion’s result of bringing the optimal portfolio closer to the mean-variance efficient frontier with a variance constraint can also be obtained with a VaR constraint. Third, under certain conditions, the use of VaR as a risk measure is consistent with expected utility maximization (see Alexander and Baptista, 2002). Finally, VaR can be useful to reduce the regret of losses (see Shefrin, 2000).

We begin by examining the case when short sales are allowed. The set of portfolios that minimize TEV for various levels of expected return is referred to as the *mean-TEV boundary*, while the set of portfolios that do so given a VaR constraint is referred to as the *constrained mean-TEV boundary*. Like portfolios on the mean-TEV boundary, we find that portfolios on the constrained mean-TEV boundary exhibit three-fund separation, but the weights of the three funds when the constraint binds differ from those in its absence. Under certain conditions, we find that the constrained mean-TEV boundary consists of: (i) portfolios on the mean-variance boundary, (ii) portfolios on the mean-TEV boundary, and (iii) portfolios that do not belong to either of these boundaries. There are also conditions under which no portfolio on the mean-TEV boundary belongs to the constrained

¹A portfolio’s TEV is the variance of the difference between the returns on the portfolio and the benchmark.

²Merrill Lynch agreed to pay \$105 million to settle the case. A similar contract existed between Merrill Lynch and Sainsbury that also led to large losses. See *The Wall Street Journal*, December 7, 2001, p. C1.

³A portfolio’s VaR is the maximum loss at a given confidence level that the portfolio suffers over a time period.

mean-TEV boundary. Nevertheless, there are no conditions under which the constrained mean-TEV boundary includes more than two portfolios on the mean-variance boundary.

It is important to emphasize that the constrained mean-TEV boundary is related to the *constant-TEV mean-variance boundary* of Jorion (2003). A portfolio is on the constant-TEV mean-variance boundary if it satisfies a TEV constraint and there is no other portfolio with the same variance that satisfies the constraint and has a larger expected return. Under certain conditions, a portfolio on the constrained mean-TEV boundary with a given expected return is also on the constant-TEV mean-variance boundary for a TEV bound that depends on the required expected return and the VaR bound. Thus, the constrained mean-TEV boundary can be thought of as an extension of the constant-TEV mean-variance boundary.

Next, we show that a VaR constraint mitigates the problem that when a manager seeks to outperform a benchmark using the mean-TEV model, he or she selects a portfolio that is mean-variance inefficient. First, the constrained optimal portfolio dominates the unconstrained optimal portfolio according to both mean-variance and mean-VaR criteria. Second, under certain conditions, the constrained optimal portfolio dominates the benchmark according to both criteria. Third, there are no conditions under which the former is dominated by the latter according to either criteria.

Almazan et al. (2004) find that a large fraction of fund managers cannot engage in short sales. Hence, we also examine the case when short sales are disallowed. We find that when short sales are disallowed, the extent to which the constraint reduces the optimal portfolio's efficiency loss can still be notable but is smaller than when short sales are allowed.

Previous papers recognize that managers may have incentives to take actions that are not in the best interest of investors. First, these adverse incentives can be induced by compensation contracts that are explicitly based on the managers' performance relative to a benchmark. For example, Starks (1987) shows that managers spend a smaller amount of resources to produce superior returns than the amount that is optimal for investors. Admati and Pfleiderer (1997) show that when a manager has private information, he or she ends up selecting a portfolio that is suboptimal for investors. Grinblatt and Titman (1989) show that the option-like features in performance-based compensation give incentives for the manager to select a portfolio with a level of risk that may be suboptimal for investors.⁴ Carpenter (2000) shows that compensating a manager with a call option on the assets that he or she manages may lead to (but does not necessarily imply) greater risk seeking by the manager. Elton et al. (2003) provide empirical evidence that managers with performance-based compensation select riskier portfolios than managers whose compensation is not performance-based.

Second, adverse incentives can be induced implicitly by the relationship between fund inflows and performance. Sirri and Tufano (1998) document that investors

⁴See also Davanzo and Nesbitt (1987), Grinold and Rudd (1987), Record and Tynan (1987), and Bailey (1990).

flock to funds with the highest recent returns. Hence, when a manager's compensation is based on a fixed percentage of assets under management, he or she has incentives to change a fund's riskiness as a function of its performance. Chevalier and Ellison (1997) provide empirical evidence that managers respond to these incentives. Our work is related to these papers in that we explore the ability of a VaR constraint to curtail the tendency of managers who use the mean-TEV model to select overly risky portfolios.

Our paper proceeds as follows. Section 2 describes the model and characterizes the constrained mean-TEV boundary when short sales are allowed. Section 3 explains how to implement the VaR constraint. Section 4 presents an example to illustrate the portfolio selection implications of the constrained mean-TEV model. Section 5 examines the effect of disallowing short sales in our results. Section 6 presents an example to illustrate this effect. Section 7 concludes.

2. The model

Suppose that n securities are available for investment. Let μ denote the $n \times 1$ vector of security expected returns and Σ denote their $n \times n$ variance-covariance positive definite matrix. A portfolio is an $n \times 1$ vector w with entries that sum to one (the j th entry is security j 's weight). Note that short sales are allowed since each of these entries can be negative. Nevertheless, we later examine the case when short sales are disallowed. Let $E[r_w] \equiv w^\top \mu$ and $\sigma^2[r_w] \equiv w^\top \Sigma w$ denote, respectively, the expected return and variance of portfolio w .

2.1. The mean-variance boundary

Appendix A reviews a characterization of the mean-variance boundary by Merton (1972). The portfolio on this boundary with expected return E is given by

$$w(E) \equiv (1 - K)w_\sigma + Kw_A, \quad (1)$$

where K is a constant that depends on E as defined in Appendix A, w_σ is the minimum variance portfolio, and w_A is a certain portfolio on the mean-variance boundary.⁵ The portfolios on this boundary can be represented in $(E[r_w], \sigma^2[r_w])$ space by a parabola

$$\sigma^2[r_w] = 1/c + \frac{(E[r_w] - b/c)^2}{d} \quad (2)$$

for some constants a , b , c , and d defined in Appendix A. Note that b/c and $1/c$ are, respectively, the expected return and variance of w_σ . The asymptotic slope of the representation of the mean-variance boundary in $(E[r_w], \sigma[r_w])$ space is \sqrt{d} . A portfolio w is *efficient* if it lies on this boundary and $E[r_w] \geq b/c$. Otherwise, the portfolio is *inefficient*.

⁵Observe that E denotes the required expected return, while $E[\cdot]$ denotes the expectation operator.

2.2. The mean-TEV boundary

Let w_B be a benchmark. In practice, as Roll notes, the benchmark is not on the mean-variance boundary. Hence, suppose that w_B does not belong to the mean-variance boundary. Let $\sigma_e^2[r_w] \equiv (w - w_B)^\top \Sigma (w - w_B)$ denote the tracking error variance (TEV) of portfolio w .

Appendix B reviews a characterization of the mean-TEV boundary by Roll (1992). The portfolio on this boundary with expected return E is given by

$$w^e(E) \equiv w_B + K^e(w_A - w_\sigma), \quad (3)$$

where K^e is a constant that depends on E as defined in Appendix B. The portfolios on the mean-TEV boundary can be represented in $(E[r_w], \sigma^2[r_w])$ space by a parabola

$$\sigma^2[r_w] = 1/c + \frac{(E[r_w] - b/c)^2}{d} + \delta_B, \quad (4)$$

where $\delta_B > 0$ is the efficiency loss (measured in variance units) of any portfolio on this boundary.

2.3. The VaR constraint

A portfolio's VaR is the maximum loss at a given confidence level that the portfolio suffers over a time period. Assume that security returns have a multivariate normal distribution. This assumption is often made in practice to estimate VaR (see, e.g., Jorion, 2001). Moreover, it has been frequently used in the literature (see, e.g., Brennan, 1993; Jorion, 2003). Nevertheless, we extend our results to the case of non-normality in Appendix C.

Let $z_t \equiv \Phi^{-1}(t)$, where $\frac{1}{2} < t < 1$ and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Using the normality assumption, portfolio w 's VaR at the 100t% confidence level is

$$V[t, r_w] \equiv z_t \sigma[r_w] - E[r_w]. \quad (5)$$

While the expected return term in Eq. (5) is sometimes omitted when computing VaR, there are important reasons for not doing so. First, the omission of this term produces an error in the estimate of VaR that can be large for long holding periods. While financial institutions compute VaR for short holding periods such as a week, managers are typically interested in VaR calculations for longer holding periods such as a quarter (see Chow and Kritzman, 2001; Jorion, 2001, p. 408; Pearson, 2002, p. 158). For example, consider a holding period of a quarter and a portfolio w with $E[r_w] = 3\%$ and $\sigma[r_w] = 10\%$. Since $z_{0.95} = 1.64$, Eq. (5) implies that $V[0.95, r_w] = 13.45\%$. If the expected return term in Eq. (5) is omitted, then the estimate of w 's VaR is 16.45%. Hence, this estimate is about 22% $[= \frac{16.45}{13.45} - 1]$ larger than w 's VaR.

Second, omitting the expected return term results in identical estimates of VaR for portfolios with notably different expected returns but the same standard deviations (see Kupiec, 1999). For example, consider a holding period of a quarter and

portfolios \bar{w} and $\bar{\bar{w}}$ with $E[r_{\bar{w}}] = -E[r_{\bar{\bar{w}}}] = 3\%$ and $\sigma[r_{\bar{w}}] = \sigma[r_{\bar{\bar{w}}}] = 10\%$. Note that $V[0.95, r_{\bar{w}}] = 13.45\%$ and $V[0.95, r_{\bar{\bar{w}}}] = 19.45\%$. If the expected return term in Eq. (5) is omitted, then the estimate of VaR for both \bar{w} and $\bar{\bar{w}}$ is 16.45%. However, $\bar{\bar{w}}$'s VaR is about 45% $[= \frac{19.45}{13.45} - 1]$ larger than \bar{w} 's VaR.

Third, some researchers, including those who use simulation, do not omit the expected return term in the calculation of VaR (see, e.g., Artzner et al., 1999; Embrechts et al., 2002; Hull, 2006, pp. 438–440 and 448–449).

Consider a manager who is restricted to select a portfolio w such that

$$V[t, r_w] \leq V, \quad (6)$$

where V is the VaR bound. It follows from Eq. (5) that constraint (6) is equivalent to

$$E[r_w] \geq -V + z_t \sigma[r_w]. \quad (7)$$

Using Eq. (7), the portfolios that satisfy a VaR constraint lie on or above a line in $(E[r_w], \sigma[r_w])$ space with intercept $-V$ and slope z_t . The intercept increases if V decreases, and the slope increases if t increases. In either case, the constraint can be thought of as being ‘tightened.’

2.4. Intuition of the impact of the constraint on the mean-TEV boundary

A portfolio is on the *constrained mean-TEV boundary* if it satisfies the VaR constraint and there is no other portfolio with the same expected return that satisfies the constraint and has a smaller TEV. Provided that it exists, let $w^e(E, V)$ denote the portfolio on this boundary with expected return E .

Consider the joint application of the VaR constraint and the expected return constraint

$$E[r_w] = E. \quad (8)$$

Using Eqs. (7) and (8), we obtain the standard deviation constraint

$$\sigma[r_w] \leq \frac{V + E}{z_t}. \quad (9)$$

Thus, the joint application of VaR and expected return constraints is equivalent to the joint application of the same expected return constraint and a standard deviation constraint.

Suppose that: (1) $V[t, r_{w^e(E)}] > V$ and (2) $w^e(E, V)$ exists. It follows from Eq. (9) that

$$\sigma[r_{w^e(E, V)}] \leq \frac{V + E}{z_t} < \sigma[r_{w^e(E)}]. \quad (10)$$

Eq. (10) implies that the standard deviation of the portfolio on the constrained mean-TEV boundary with any level of expected return at which the constraint binds is *smaller* than that of the portfolio on the mean-TEV boundary with the same expected return.

2.5. A characterization of the constrained mean-TEV boundary

The following result is useful in our characterization of the constrained mean-TEV boundary.

Lemma 1. *If $V[t, r_{w^e(E)}] > V$ and $w^e(E, V)$ exists, then*

$$V[t, r_{w^e(E, V)}] = V \quad (11)$$

and

$$w^e(E, V) = (1 - X - Y)w_B + Xw_\sigma + Yw_A, \quad (12)$$

where X and Y are constants that depend on E and V as defined in Appendix D.

Lemma 1 says that when the VaR constraint binds, portfolios on the constrained mean-TEV boundary have two properties. First, they lie on the line representing the constraint in $(E[r_w], \sigma[r_w])$ space. Second, they exhibit three-fund separation like portfolios on the mean-TEV boundary, but the weights of funds w_B , w_σ , and w_A differ. Hence, Lemma 1 simplifies the constrained TEV minimization problem into a problem of wealth allocation among three funds.

It is important to emphasize that the constrained mean-TEV boundary is related to the *constant-TEV mean-variance boundary* of Jorion (2003). Consider the TEV constraint

$$\sigma_e^2[r_w] = T, \quad (13)$$

where T is the TEV bound. A portfolio is on the constant-TEV mean-variance boundary if it satisfies the TEV constraint and there is no other portfolio with the same variance that satisfies the constraint and has a larger expected return. Jorion shows that portfolios on this boundary can be represented by an ellipse in $(E[r_w], \sigma^2[r_w])$ space. He also shows that they exhibit three-fund separation like portfolios on the constrained mean-TEV boundary.

Appendix E provides conditions such that when the VaR bound is V , the portfolio on the constrained mean-TEV boundary with expected return E is on the constant-TEV mean-variance boundary for some TEV bound T that depends on E and V . However, for fixed bounds V and T , the set of portfolios on the constrained mean-TEV boundary differs from the set of portfolios on the constant-TEV mean-variance boundary. As noted earlier, the subset of the former for which the VaR constraint binds can be represented by a line in $(E[r_w], \sigma[r_w])$ space, while the latter can be represented by an ellipse in $(E[r_w], \sigma^2[r_w])$ space. Thus, the constrained mean-TEV boundary can be thought of as an extension of the constant-TEV mean-variance boundary.

Next, we characterize the constrained mean-TEV boundary.⁶ We say that the confidence level is: (i) *low* if $t \leq \Phi(\sqrt{d})$ and (ii) *high* if $t > \Phi(\sqrt{d})$. Thus, we call $\Phi(\sqrt{d})$ the *threshold value*.

⁶This characterization differs from those of the constrained mean-variance boundary provided by Alexander and Baptista (2004, 2006) in two respects. First, constrained mean-TEV portfolios exhibit three-fund separation, while constrained mean-variance portfolios exhibit two-fund separation. Second,

2.5.1. Low confidence level

Suppose that the confidence level is low. Lemma 2 in Appendix D shows that there is at most a single portfolio on the mean-TEV (mean-variance) boundary at which the constraint binds, which we denote by w_V^e (w_V).

Proposition 1. (i) If $t < \Phi(\sqrt{d})$, then the constrained mean-TEV boundary consists of all portfolios given by either (a) Eq. (12) for $E[r_{w_V}] \leq E < E[r_{w_V^e}]$, or (b) Eq. (3) for $E \geq E[r_{w_V^e}]$. (ii) If $t = \Phi(\sqrt{d})$ and $V \leq -b/c$, then there is no portfolio on the constrained mean-TEV boundary. (iii) If $t = \Phi(\sqrt{d})$ and $V > -b/c$, then the constrained mean-TEV boundary consists of all portfolios given by either (a) Eq. (12) for $E[r_{w_V}] \leq E < E[r_{w_V^e}]$, or (b) Eq. (3) for $E \geq E[r_{w_V^e}]$.

Fig. 1 illustrates Proposition 1 by showing the impact of increasing the bound on the constrained mean-TEV boundary when the confidence level is low. First, panels (a) and (b), which represent case (i), show that when the confidence level is less than the threshold value, this boundary consists of two segments: (i) a line between the portfolios on the mean-variance and mean-TEV boundaries at which the constraint binds (i.e., $w_V = p_1$ and $w_V^e = p_2$ in panel (a) and $w_V = p_3$ and $w_V^e = p_4$ in panel (b)), and (ii) the portion of the mean-TEV boundary at and above the portfolio on this boundary at which the constraint binds (i.e., $w_V^e = p_2$ in panel (a) and $w_V^e = p_4$ in panel (b)). Note that portfolios on the mean-TEV boundary with large expected returns belong to the constrained mean-TEV boundary, while those with small ones do not.⁷ Moreover, there exists a unique portfolio on the constrained mean-TEV boundary that is also on the mean-variance boundary (i.e., $w_V = p_1$ in panel (a) and $w_V = p_3$ in panel (b)).

Second, panel (c), which represents case (ii), shows that when the confidence level is equal to the threshold value and the bound is small, the constrained mean-TEV boundary is empty.⁸

Finally, panel (d), which represents case (iii), shows that when the confidence level is equal to the threshold value and the bound is large, the results are similar to those when the confidence level is less than the threshold value (in panel (d), $w_V = p_5$ and $w_V^e = p_6$).

2.5.2. High confidence level

Suppose that the confidence level is high. Let w_t^e (w_t) denote the portfolio with minimum VaR among all portfolios on the mean-TEV (mean-variance) boundary. Lemma 3 in Appendix D shows that if $V > V[t, r_{w_t^e}]$, then there are two portfolios on the mean-TEV boundary, denoted by \underline{w}_V^e and \bar{w}_V^e , and two portfolios on the mean-

(footnote continued)

adding the constraint to the mean-TEV model causes the optimal portfolio to move closer to (but in general not to be on) the efficient frontier, while adding the constraint to the mean-variance model still results in the selection of a portfolio on the efficient frontier.

⁷For brevity, we omit the qualifier ‘relatively’ when referring to a ‘relatively small (or large) expected return’ and a ‘relatively small (or large) standard deviation.’

⁸For brevity, we omit the qualifier ‘sufficiently’ when referring to a ‘sufficiently small (or large) bound.’

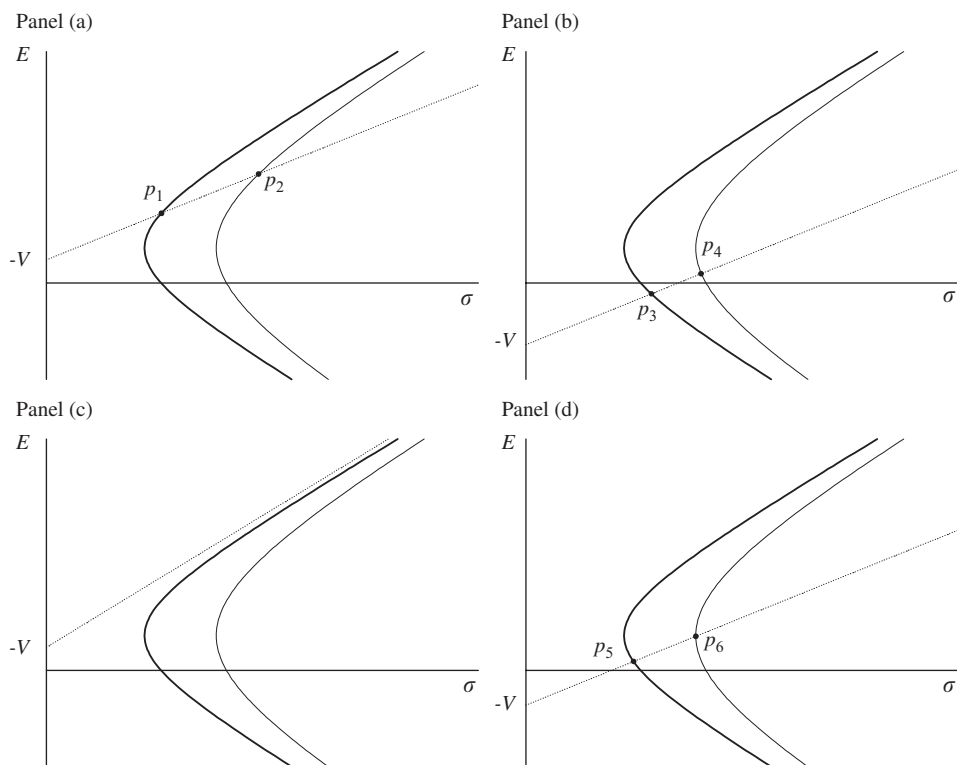


Fig. 1. The constrained mean-TEV boundary when the confidence level is low. The thick (thin) hyperbola represents the mean-variance (mean-TEV) boundary. The dotted line represents the VaR constraint. When the confidence level is less than a threshold value (panels (a) and (b)), or the confidence level is equal to the threshold value and the bound is large (panel (d)), the constrained mean-TEV boundary consists of two segments: (i) a line between the portfolios on the mean-variance and mean-TEV boundaries at which the constraint binds, and (ii) the portion of the mean-TEV boundary at and above the portfolio on this boundary at which the constraint binds. When the confidence level is equal to the threshold value and the bound is small, the constrained mean-TEV boundary is empty (panel (c)).

variance boundary, denoted by \underline{w}_V and \bar{w}_V , at which the constraint binds, where $E[r_{\underline{w}_V}] < E[r_{\underline{w}_V^e}] < E[r_{\bar{w}_V^e}] < E[r_{\bar{w}_V}]$.

Proposition 2. Suppose that $t > \Phi(\sqrt{d})$. (i) If $V < V[t, r_{w_t}]$, then there is no portfolio on the constrained mean-TEV boundary. (ii) If $V = V[t, r_{w_t}]$, then the constrained mean-TEV boundary consists of portfolio w_t . (iii) If $V[t, r_{w_t}] < V \leq V[t, r_{w_t^e}]$, then the constrained mean-TEV boundary consists of all portfolios given by Eq. (12) for $E[r_{\underline{w}_V}] \leq E \leq E[r_{\bar{w}_V}]$. (iv) If $V > V[t, r_{w_t^e}]$, then the constrained mean-TEV boundary consists of all portfolios given by either (a) Eq. (12) for $E[r_{\underline{w}_V}] \leq E < E[r_{\underline{w}_V^e}]$, or (b) Eq. (3) for $E[r_{\underline{w}_V^e}] \leq E \leq E[r_{\bar{w}_V^e}]$, or (c) Eq. (12) for $E[r_{\bar{w}_V^e}] < E \leq E[r_{\bar{w}_V}]$.

Fig. 2 illustrates Proposition 2 by showing the impact of increasing the bound on the constrained mean-TEV boundary when the confidence level is high. First,

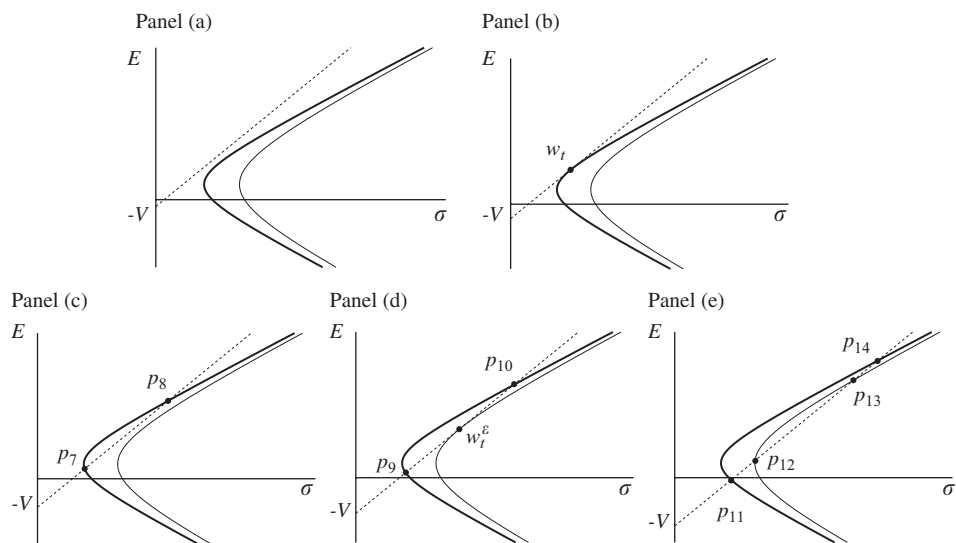


Fig. 2. The constrained mean-TEV boundary when the confidence level is high. The thick (thin) hyperbola represents the mean-variance (mean-TEV) boundary. The dotted line represents the VaR constraint. When the bound is small, the constrained mean-TEV boundary is empty (panel (a)). When the bound is equal to the VaR of the minimum VaR portfolio, this portfolio is the only one on the constrained mean-TEV boundary (panel (b)). When the bound is moderate (panels (c) and (d)), the constrained mean-TEV boundary consists of the line between the two portfolios on the mean-variance boundary at which the constraint binds. When the bound is large (panel (e)), the constrained mean-TEV boundary consists of three segments: (i) a line between the portfolios on the mean-variance and mean-TEV boundaries with the smallest expected return at which the constraint binds, (ii) the portion of the mean-TEV boundary between the two portfolios on this boundary at which the constraint binds, and (iii) the line between the portfolios on the mean-TEV and mean-variance boundaries with the largest expected return at which the constraint binds.

panel (a), which represents case (i), shows that when the bound is small, this boundary is empty. However, panels (b)–(e) show that it is non-empty when the bound is not small.

Second, panel (b), which represents case (ii), shows that when the bound is equal to w_t 's VaR, this portfolio is the only one on the constrained mean-TEV boundary.

Third, panel (c), which represents case (iii), shows that when the bound is moderate, the constrained mean-TEV boundary consists of the line between the two portfolios on the mean-variance boundary at which the constraint binds (i.e., $\underline{w}_V = p_7$ and $\bar{w}_V = p_8$). These two portfolios are the only ones on the constrained mean-TEV boundary that are also on the mean-variance boundary. Also, no portfolio on the mean-TEV boundary belongs to the constrained mean-TEV boundary. Panel (d), which represents the limit of case (iii), shows similar results when the bound is equal to the VaR of w_t^e with the exception that this portfolio belongs to both unconstrained and constrained mean-TEV boundaries.

Finally, panel (e), which represents case (iv), shows that when the bound is large, the constrained mean-TEV boundary consists of three segments: (i) a line between

the portfolios on the mean-variance and mean-TEV boundaries with the smallest expected return at which the constraint binds (i.e., $\underline{w}_V = p_{11}$ and $\underline{w}_V^e = p_{12}$), (ii) the portion of the mean-TEV boundary between the two portfolios on this boundary at which the constraint binds (i.e., $\underline{w}_V^e = p_{12}$ and $\bar{w}_V^e = p_{13}$), and (iii) the line between the portfolios on the mean-TEV and mean-variance boundaries with the largest expected return at which the constraint binds (i.e., $\bar{w}_V^e = p_{13}$ and $\bar{w}_V = p_{14}$). Note that portfolios on the mean-TEV boundary with moderate expected returns also belong to the constrained mean-TEV boundary, while those with small and large ones do not. As in the case when the bound is moderate, there exist only two portfolios on the constrained mean-TEV boundary that are also on the mean-variance boundary (i.e., $\underline{w}_V = p_{11}$ and $\bar{w}_V = p_{14}$).

2.6. Fraction of efficiency loss eliminated by the VaR constraint

As Figs. 1 and 2 show, any portfolio on the constrained mean-TEV boundary with an expected return E at which the VaR constraint binds is closer to the mean-variance boundary than the portfolio on the mean-TEV boundary with the same expected return. Of particular interest is the fraction of efficiency loss eliminated by the constraint. Eqs. (5) and (11) imply that $\sigma^2[r_{w^e(E,V)}] = [(V + E)/z_t]^2$. Using Eq. (2), $\sigma^2[r_{w(E)}] = 1/c + (E - b/c)^2/d$. As noted earlier, δ_B is the efficiency loss of any portfolio on the mean-TEV boundary. Consequently,

$$\phi(E) = 1 - \frac{\left(\frac{V + E}{z_t}\right)^2 - \left[1/c + \frac{(E - b/c)^2}{d}\right]}{\delta_B} \quad (14)$$

is the fraction of efficiency loss eliminated by the constraint.

2.6.1. Low confidence level

Suppose that the confidence level is low. Then, there is a unique level of expected return E_1 with $\phi(E_1) = 1$ (i.e., the efficiency loss is fully eliminated). For example, Fig. 1(d) shows that this occurs when E_1 is equal to p_5 's expected return. Moreover, $0 < \phi(E) < 1$ (i.e., the efficiency loss is reduced but *not* fully eliminated) for any level of expected return $E > E_1$ at which the constraint binds. The panel shows that this occurs for any level of expected return between those of p_5 and p_6 . Lastly, the efficiency loss remains unchanged for any level of expected return $E > E_1$ at which the constraint does not bind, in which case Eq. (14) is not applicable. The panel shows that this occurs for any level of expected return larger than that of p_6 .

2.6.2. High confidence level

Suppose that the confidence level is high. First, assume that $V = V[t, r_{w_t}]$. Then, there is a unique level of expected return E_2 with $\phi(E_2) = 1$. Fig. 2(b) shows that this occurs when E_2 is equal to w_t 's expected return.

Second, assume that $V > V[t, r_{w_t}]$. Then, there are two levels of expected return E_3 and E_4 with $E_3 < E_4$ such that $\phi(E_3) = \phi(E_4) = 1$. For example, Fig. 2(e) shows that

this occurs when E_3 and E_4 are equal to the expected returns of p_{11} and p_{14} . Moreover, $0 < \varphi(E) < 1$ for any level of expected return E with $E_3 < E < E_4$ at which the constraint binds. The panel shows that this occurs for any level of expected return between those of (i) p_{11} and p_{12} , or (ii) p_{13} and p_{14} . Lastly, the efficiency loss remains unchanged for any level of expected return E at which the constraint does not bind, in which case Eq. (14) is not applicable. The panel shows that this occurs for any level of expected return between those of p_{12} and p_{13} .

3. Implementing the VaR constraint

We now discuss setting the bound V when implementing the VaR constraint.

3.1. The unconstrained optimal portfolio

Consider a manager who selects the portfolio with minimum TEV among all portfolios with an expected return of $E[r_{w_B}] + G$, where $G > 0$ is the expected gain over the benchmark (hereafter ‘expected gain’). We refer to it as the *unconstrained optimal portfolio* and denote it by w^e . By construction,

$$E[r_{w^e}] = E[r_{w_B}] + G. \quad (15)$$

Since w^e is on the mean-TEV boundary, it follows from Eqs. (4) and (15) that

$$\sigma^2[r_{w^e}] = 1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d} + \delta_B \quad (16)$$

and that the unconstrained optimal portfolio is *inefficient*.

3.2. Maximum bound

Assume that the manager faces a VaR constraint. In this case, we refer to the manager’s optimal portfolio as the *constrained optimal portfolio*. Using Eqs. (5), (15), and (16),

$$V[t, r_{w^e}] = \bar{V} \equiv z_t \sqrt{1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d}} + \delta_B - E[r_{w_B}] - G. \quad (17)$$

Consequently, in order for the constraint to be binding, V has to be chosen so that $V < \bar{V}$. As Fig. 3(a) illustrates, if $V = \bar{V}$, then the unconstrained and constrained optimal portfolios coincide. Hence, the manager selects w^e .

3.3. Minimum bound

Let w be the efficient portfolio with

$$E[r_w] = E[r_{w_B}] + G. \quad (18)$$

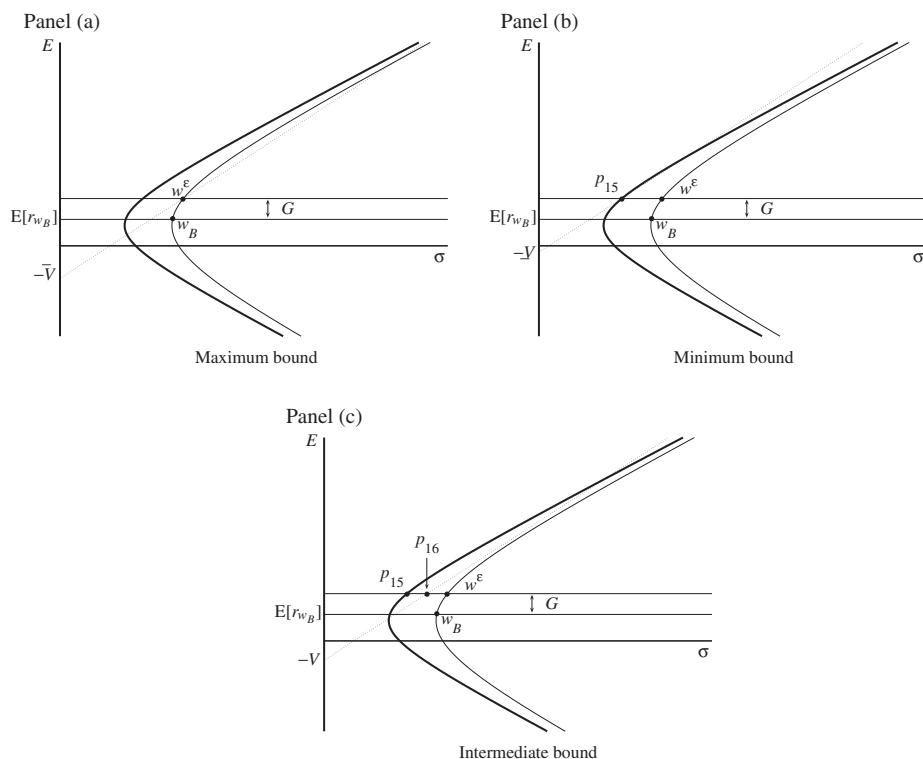


Fig. 3. Implementing the VaR constraint. The thick (thin) hyperbola represents the mean-variance (mean-TEV) boundary. The dotted line represents the VaR constraint, while the horizontal lines represent the expected return of the benchmark (w_B) and that plus the expected gain (G). Consider a manager who selects the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . Note that the unconstrained optimal portfolio (w^ε) is inefficient. Panel (a) illustrates that if a VaR constraint with bound \bar{V} is imposed where \bar{V} is the VaR of the unconstrained optimal portfolio, then the unconstrained and constrained optimal portfolios coincide. Panel (b) illustrates that if a VaR constraint with bound \underline{V} is imposed where \underline{V} is the VaR of the efficient portfolio with an expected return equal to that of the benchmark plus G , then the constrained optimal portfolio (p_{15}) is efficient. Panel (c) illustrates that if a VaR constraint with bound V is imposed where $\underline{V} < V < \bar{V}$, then the constrained optimal portfolio (p_{16}) is closer to the efficient frontier than the unconstrained optimal portfolio (w^ε).

Using Eqs. (2) and (18), we have

$$\sigma^2[r_w] = 1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d}. \quad (19)$$

It follows from Eqs. (5), (18), and (19) that

$$V[t, r_w] = \underline{V} \equiv z_t \sqrt{1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d}} - E[r_{w_B}] - G. \quad (20)$$

Since w is efficient, it has the minimum VaR among all portfolios with an expected return of $E[r_{w_B}] + G$. Consequently, in order for the constrained optimal portfolio to exist, V has to be chosen so that $V \geq \underline{V}$. As Fig. 3(b) illustrates, if $V = \underline{V}$, then p_{15} is the constrained optimal portfolio since this is the only portfolio that satisfies both the expected return and VaR constraints. Hence, the constrained optimal portfolio is *efficient*.

3.4. Intermediate bound

A constraint with $V = \overline{V}$ is not binding and thus does not reduce the efficiency loss. While a constraint with $V = \underline{V}$ fully eliminates it, the goal of selecting a portfolio with a small TEV is ignored. Hence, it is natural to also consider a constraint with $\underline{V} < V < \overline{V}$. As Fig. 3(c) illustrates, this constraint: (i) reduces the efficiency loss, but (ii) still allows the selection of a portfolio with a small TEV. This can be seen by noting that p_{16} lies between p_{15} and w^e .

3.5. A simple bound

Consider a VaR constraint with bound $V = V'$, where

$$V' \equiv V[t, r_{w_B}] - G. \quad (21)$$

Of particular interest is the question of whether $\underline{V} < V' < \overline{V}$. Suppose that $E[r_{w_B}] \geq b/c$. First, we show that $V' < \overline{V}$. Since w_B is on the mean-TEV boundary, Eqs. (4) and (5) imply that

$$V[t, r_{w_B}] = z_t \sqrt{1/c + \frac{(E[r_{w_B}] - b/c)^2}{d}} + \delta_B - E[r_{w_B}]. \quad (22)$$

It follows from Eqs. (17), (21), and (22) that $V' < \overline{V}$.

Second, we provide a condition so that $V' > \underline{V}$. For brevity, let

$$\delta_1 \equiv \frac{G^2 + 2(E[r_{w_B}] - b/c)G}{d}. \quad (23)$$

Using Eqs. (20)–(22), we have $V' > \underline{V}$ if and only if $\delta_B > \delta_1$.⁹

Assume that $\delta_B > \delta_1$ and let w' denote the constrained optimal portfolio as shown in Fig. 4(a). It follows from the expected return constraint that

$$E[r_{w'}] = E[r_{w_B}] + G. \quad (24)$$

Since the VaR constraint is binding, we have

$$V[t, r_{w'}] = V[t, r_{w_B}] - G. \quad (25)$$

⁹If $\delta_B < \delta_1$, then no portfolio satisfies both the expected return and VaR constraints. However, if $\delta_B \geq \delta_1$, then the constrained optimal portfolio exists. Since $\delta_1 \rightarrow 0$ as $G \rightarrow 0$, we have $\delta_B > \delta_1$ if G is small.

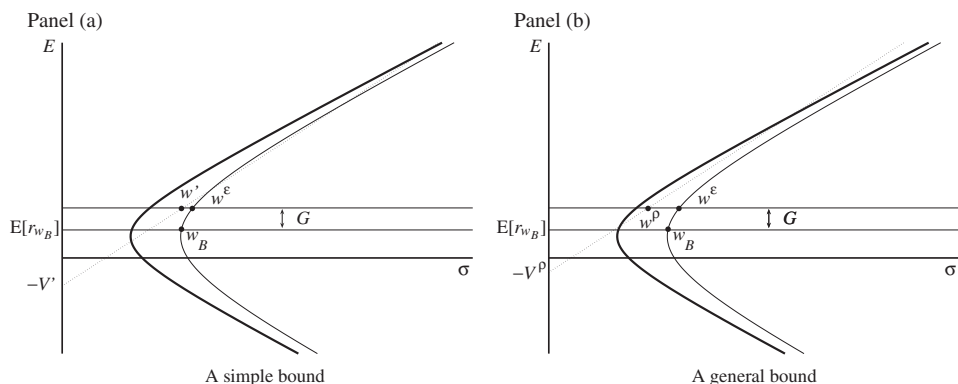


Fig. 4. A simple bound and a general bound. The thick (thin) hyperbola represents the mean-variance (mean-TEV) boundary. The dotted line represents the VaR constraint, while the horizontal lines represent the expected return of the benchmark (w_B) and that plus the expected gain (G). Consider a manager who selects the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . Note that the unconstrained optimal portfolio (w^e) is inefficient. Panel (a) illustrates that if a VaR constraint with bound V' is imposed where V' is the VaR of the benchmark minus G , then the constrained optimal portfolio (w') is closer to the efficient frontier than the unconstrained optimal portfolio (w^e). While imposing a VaR constraint with bound V' reduces the optimal portfolio's efficiency loss, the constrained optimal portfolio may still be inefficient. Panel (b) illustrates that if a VaR constraint with bound V^ρ is imposed where V^ρ is given by Eq. (30) and ρ is large, then the constrained optimal portfolio (w^ρ) is closer to the efficient frontier than that when a VaR constraint with bound V' is imposed (w').

Adding Eqs. (24) and (25) and using Eq. (5), we obtain

$$\sigma[r_{w'}] = \sigma[r_{w_B}]. \quad (26)$$

Hence, when $V = V'$, the joint application of the expected return and VaR constraints is equivalent to the joint application of the same expected return constraint and a standard deviation constraint with bound equal to w_B 's standard deviation. Jorion (2003) advocates the use of this standard deviation constraint to bring the optimal portfolio closer to the efficient frontier. As Fig. 4(a) illustrates, a VaR constraint with bound $V = V'$ can be used to obtain his result.

3.6. A general bound

We now derive the value of the bound that reduces the efficiency loss by proportion ρ , where $0 < \rho < 1$, which we denote by V^ρ . Let σ_V^2 denote the constrained optimal portfolio's variance when the bound is V . We need to find V^ρ so that

$$\frac{\sigma_V^2 - \sigma_{V^\rho}^2}{\sigma_V^2 - \sigma_{\underline{V}}^2} = \rho. \quad (27)$$

It follows from Eq. (5) and Lemma 1 that when $\underline{V} \leq V \leq \overline{V}$, we have

$$\sigma_V^2 = \left(\frac{V + E[r_{w_B}] + G}{z_t} \right)^2. \quad (28)$$

Using Eqs. (27) and (28), we obtain

$$(V^\rho)^2 + k_1 V^\rho + k_2 = 0, \quad (29)$$

where $k_1 \equiv 2(E[r_{w_B}] + G)$ and $k_2 \equiv -\{(1 - \rho)(\bar{V})^2 + \rho(\underline{V})^2 + 2[(1 - \rho)\bar{V} + \rho\underline{V}](E[r_{w_B}] + G)\}$. Suppose that $\underline{V} > 0$. Then, $V^\rho > 0$. Hence, it follows from Eq. (29) that

$$V^\rho = \sqrt{(k_1/2)^2 - k_2} - k_1/2. \quad (30)$$

As Figs. 4(a) and (b) illustrate, if ρ is large, then the constrained optimal portfolio when V^ρ is used (w^ρ) is closer to the efficient frontier than that when V' is used (w'). While V^ρ reduces a larger fraction of efficiency loss than V' , this comes at the cost of a larger TEV. Hence, ρ captures the trade-off between efficiency loss and TEV. Jorion (2003) examines this trade-off by using the constant-TEV mean-variance boundary.¹⁰ He shows that adding a variance constraint to the problem of expected return maximization subject to a TEV constraint brings the optimal portfolio closer to the efficient frontier. As this section shows, this result can also be obtained with a VaR constraint.

3.7. Comparison between unconstrained and constrained optimal portfolios

Suppose that $\underline{V} \leq V < \bar{V}$. By construction, the expected returns of the unconstrained and constrained optimal portfolios coincide. Moreover, the constrained optimal portfolio's variance is *smaller* than that of the unconstrained optimal portfolio. Thus, the VaR of the former is *smaller* than that of the latter. Hence, the constrained optimal portfolio dominates the unconstrained optimal portfolio according to both mean-variance and mean-VaR criteria.¹¹

3.8. Comparison between benchmark and constrained optimal portfolio

We now investigate whether the constrained optimal portfolio dominates the benchmark.

3.8.1. The case when $V = \underline{V}$

Suppose that $V = \underline{V}$. First, consider the mean-variance criterion. Since w_B is on the mean-TEV boundary, it follows from Eq. (4) that

$$\sigma^2[r_{w_B}] = 1/c + \frac{(E[r_{w_B}] - b/c)^2}{d} + \delta_B. \quad (31)$$

¹⁰For an examination of the asset allocation implications of using an objective function defined over expected return, variance, and TEV, see Chow (1995).

¹¹We say that \bar{w} dominates \underline{w} according to the mean-variance criterion if: (i) $E[r_{\bar{w}}] \geq E[r_{\underline{w}}]$, (ii) $\sigma^2[r_{\bar{w}}] \leq \sigma^2[r_{\underline{w}}]$, and (iii) at least one of these two inequalities is strict. The notion of dominance according to the mean-VaR criterion is similar, except that in (ii) we have $V[t, r_{\bar{w}}] \leq V[t, r_{\underline{w}}]$.

Eqs. (19), (23), and (31) imply that

$$\sigma^2[r_{w_B}] - \sigma^2[r_w] = \delta_B - \delta_1. \quad (32)$$

Using Eq. (32), whether the constrained optimal portfolio dominates the benchmark depends on δ_B . If $\delta_B < \delta_1$, then the former has a *larger* variance than the latter. Hence, the constrained optimal portfolio does not dominate the benchmark, nor does the benchmark dominate the constrained optimal portfolio since $E[r_{w_B}] < E[r_w]$. However, if $\delta_B \geq \delta_1$, then the constrained optimal portfolio has a variance *smaller than or equal* to that of the benchmark. In this case, the former dominates the latter.

Second, consider the mean-VaR criterion. Eqs. (20) and (22) imply that

$$\begin{aligned} V[t, r_{w_B}] - V[t, r_w] \\ = z_t \left[\sqrt{1/c + \frac{(E[r_{w_B}] - b/c)^2}{d}} + \delta_B - \sqrt{1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d}} \right] + G. \end{aligned} \quad (33)$$

Let

$$\delta_2 \equiv \left[\sqrt{1/c + \frac{(E[r_{w_B}] + G - b/c)^2}{d}} - \frac{G}{z_t} \right]^2 - 1/c - \frac{(E[r_{w_B}] - b/c)^2}{d}. \quad (34)$$

Using Eq. (33), whether the constrained optimal portfolio dominates the benchmark depends on δ_B . If $\delta_B < \delta_2$, then the former has a *larger* VaR than the latter. Hence, the constrained optimal portfolio does not dominate the benchmark, nor does the benchmark dominate the constrained optimal portfolio since $E[r_{w_B}] < E[r_w]$. However, if $\delta_B \geq \delta_2$, then the constrained optimal portfolio has a VaR *smaller than or equal* to that of the benchmark. In this case, the former dominates the latter.

3.8.2. The case when $V = V'$

Suppose that $\delta_B \geq \delta_1$ and $V = V'$. Using Eqs. (24)–(26), the constrained optimal portfolio dominates the benchmark according to both mean-variance and mean-VaR criteria.

3.8.3. The case when $V = V^\rho$

Suppose that $V = V^\rho < V'$. Then, the constrained optimal portfolio's variance is *smaller* than when $V = V'$. Hence, the constrained optimal portfolio dominates the benchmark according to both mean-variance and mean-VaR criteria.

4. Example

We now present an example to illustrate our theoretical results. It involves solving a manager's problem of allocating wealth among eight asset classes.¹² Six asset classes involve stocks classified along the dimensions of: (a) large, mid, and small cap, and (b) value and growth. The other two asset classes are corporate and Treasury bonds. We use the Standard and Poor's stock indices and the Merrill Lynch bond indices to measure the returns on the asset classes. Sample return means, variances, and covariances that are associated with these indices were computed using quarterly data during the period January 1994–June 2002. These statistics were annualized and then were used as optimization inputs (see Table 1(a)).¹³ With these inputs, $\Phi(\sqrt{d}) = 90.23\%$. Hence, if $t \leq 90.23\%$, then Proposition 1 applies, but if $t > 90.23\%$, then Proposition 2 applies. For brevity, we assume that $t = 95\%$ and 99% , and thus focus on Proposition 2.

We consider three benchmarks. First, the *conservative* benchmark involves only bonds – corporate bonds (50%) and Treasury bonds (50%). Second, the *moderate* benchmark involves 50% stocks and 50% bonds – large/growth stocks (25%), large/value stocks (25%), corporate bonds (25%), and Treasury bonds (25%). Finally, the *aggressive* benchmark involves only stocks – large/growth stocks (50%) and large/value stocks (50%). Note that the expected return, standard deviation, and VaR of the conservative benchmark are smaller than those of the moderate benchmark, which in turn are smaller than those of the aggressive benchmark (see Table 1(b)).

4.1. Constrained mean-TEV boundary

Next, we illustrate Proposition 2 using the moderate benchmark and $t = 99\%$. The minimum VaR portfolios on the mean-variance and mean-TEV boundaries are characterized by

$$E[r_{w_{0.99}}] = 10.87\%, \quad \sigma[r_{w_{0.99}}] = 5.13\%, \quad V[0.99, r_{w_{0.99}}] = 1.07\%, \quad (35)$$

$$E[r_{w_{0.99}^e}] = 14.32\%, \quad \sigma[r_{w_{0.99}^e}] = 9.92\%, \quad V[0.99, r_{w_{0.99}^e}] = 8.75\%, \quad (36)$$

respectively. First, if $V < 1.07\%$, then there is no portfolio on the constrained mean-TEV boundary as Fig. 2(a) illustrates.

Second, if $V = 1.07\%$, then the constrained mean-TEV boundary consists simply of the minimum VaR portfolio w_t as Fig. 2(b) illustrates.

Third, suppose that $1.07\% < V < 8.75\%$, say $V = 5\%$. Eqs. (2) and (5) imply that $E[r_{w_V}] = 5.41\%$ and $E[r_{w_V^e}] = 19.85\%$. As Fig. 2(c) illustrates, the constrained mean-TEV boundary consists of all portfolios w such that Eq. (12) holds for $5.41\% \leq E \leq 19.85\%$. The portfolios on this boundary with expected returns of

¹²This is a common use of the mean-variance model as noted by, e.g., Sharpe (1987), Black and Litterman (1992), Michaud (1998), and Solnik and McLeavey (2003).

¹³This is sometimes done in applications (see Michaud, 1998, p. 12). The important issue here is that the optimization inputs are realistic, not how they are computed. Similar results are obtained using other realistic optimization inputs.

Table 1
Parameters used in the asset allocation example^a

<i>(a) Summary statistics associated with the asset classes</i>										
Asset class	Expected return (%)	Standard deviation (%)	Correlation coefficient							
			(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1) Large/growth stocks	12.83	20.26	1.00	0.78	0.83	0.55	0.78	0.53	−0.07	−0.26
(2) Large/value stocks	11.42	15.40		1.00	0.81	0.87	0.79	0.82	0.05	−0.19
(3) Mid/growth stocks	16.00	27.31			1.00	0.77	0.92	0.72	−0.10	−0.27
(4) Mid/value stocks	15.38	16.32				1.00	0.80	0.94	0.08	−0.11
(5) Small/growth stocks	9.88	22.39					1.00	0.82	−0.06	−0.25
(6) Small/value stocks	15.03	19.10						1.00	0.02	−0.18
(7) Corporate bonds	6.83	4.76							1.00	0.91
(8) Treasury bonds	6.64	4.98								1.00
<i>(b) Summary statistics associated with the benchmark portfolios^b</i>										
Benchmark	Expected return (%)	Standard deviation (%)	VaR (%)							
			95	99						
Conservative	6.74	4.76	1.09	4.33						
Moderate	9.43	8.42	4.42	10.16						
Aggressive	12.13	16.84	15.57	27.05						

^aThe table reports annualized expected returns and standard deviations. VaR is computed using the annualized expected returns and standard deviations.

^bThe *conservative* benchmark involves only bonds – corporate bonds (50%) and Treasury bonds (50%). The *moderate* benchmark involves 50% stocks and 50% bonds – large/growth stocks (25%), large/value stocks (25%), corporate bonds (25%), and Treasury bonds (25%). The *aggressive* benchmark involves only stocks – large/growth stocks (50%) and large/value stocks (50%).

5.41% and 19.85%, denoted by p_7 and p_8 , respectively, are also on the mean-variance boundary. However, the portfolios on the constrained mean-TEV boundary with expected returns strictly between 5.41% and 19.85% are neither on the mean-variance boundary nor on the mean-TEV boundary. In the limiting case when $V = 8.75\%$, the results are similar except that $w_{0.99}^e$ is also on the constrained mean-TEV boundary as illustrated by w_i^e in Fig. 2(d).

Finally, suppose that $V > 8.75\%$, say $V = 10\%$. Eqs. (2) and (5) imply that $E[r_{\underline{w}_V}] = 2.73\%$ and $E[r_{\bar{w}_V}] = 27.02\%$. Using Eqs. (4) and (5), we have $E[r_{\underline{w}_V^e}] =$

9.70% and $E[r_{w_B}^*] = 20.05\%$. As Fig. 2(e) illustrates, the constrained mean-TEV boundary consists of all portfolios w such that either (a) Eq. (12) holds for $2.73\% \leq E < 9.70\%$, or (b) Eq. (3) holds for $9.70\% \leq E \leq 20.05\%$, or (c) Eq. (12) holds for $20.05\% < E \leq 27.02\%$. The portfolios on this boundary with expected returns of 2.73% and 27.02%, denoted by p_{11} and p_{14} , respectively, are also on the mean-variance boundary. The portfolios on the constrained mean-TEV boundary with expected returns between 9.70% and 20.05% are also on the mean-TEV boundary. Finally, the portfolios on the constrained mean-TEV boundary with expected returns strictly between (a) 2.73% and 9.70%, or (b) 20.05% and 27.02% are neither on the mean-variance boundary nor on the mean-TEV boundary.

4.2. Change in the optimal portfolio's standard deviation

Consider a manager who selects the portfolio with minimum TEV among all portfolios with an expected return of $E[r_{w_B}] + G$, where $G > 0$. Since the optimal portfolio's TEV is increasing in G , a higher value of G is associated with a less TEV-averse manager. While $G = 1\%$ in the Merrill Lynch example discussed earlier, Roll (1992, Fig. 1) assumes that $G = 2\%$. Hence, we use these two values of G , but similar results hold when other reasonable values of G are used.

Let σ_{co} and σ_{un} denote, respectively, the standard deviations of the constrained and unconstrained optimal portfolios. Columns (1)–(3) of Table 2 provide the relative reduction in the optimal portfolio's standard deviation (i.e., $1 - \sigma_{co}/\sigma_{un}$). These columns show two main results when the constraint binds. First, the constrained optimal portfolio has a *smaller* standard deviation than the unconstrained optimal portfolio. Second, the decrease in standard deviation is larger when: (i) t is larger, (ii) V is smaller, (iii) $\sigma[r_{w_B}]$ is larger, or (iv) G is smaller. The reason for (i) and (ii) is that, as noted earlier, this corresponds to tightening the constraint. The reason for (iii) is that the constraint is relatively more restrictive in this case. Lastly, the reason for (iv) is that given bound V , the maximum standard deviation allowed is smaller when G is smaller (since E , which is equal to $E[r_{w_B}] + G$, is smaller in Eq. (9)).¹⁴

4.3. Fraction of efficiency loss eliminated by the VaR constraint

Columns (1)–(3) of Table 3 provide the fraction of efficiency loss eliminated by the constraint. Three main results can be seen when the constraint binds. First, not all of the efficiency loss is eliminated since $\underline{V} < V < \overline{V}$. Second, the fraction of efficiency loss eliminated is larger when: (i) t is larger, (ii) V is smaller, or (iii) $\sigma[r_{w_B}]$ is larger. The reason for (i) and (ii) is that this corresponds to tightening the constraint. The reason for (iii) is that the constraint is relatively more restrictive in this case. Third,

¹⁴As Fig. 2(e) shows, it is possible that the constraint is binding if G is large (i.e., if the required expected return is between those of p_{13} and p_{14}), but it is not binding if G is small (i.e., if the required expected return is between those of p_{12} and p_{13}). In this situation, the decrease in standard deviation is larger when G is larger.

Table 2

Relative reduction in the optimal portfolio's standard deviation arising from imposing a VaR constraint within a mean-TEV model when short sales are allowed^a

Confidence level and benchmark	Bound ^b					
	3%	5%	7%	\underline{V}	V'	V^p
	(1)	(2)	(3)	(4)	(5)	(6)
$G = 1\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	10.04	0.18	4.89
Moderate (%)	5.22	0.00	0.00	42.51	2.25	18.44
Aggressive (%)	42.44	35.30	28.16	63.17	1.13	24.65
(b) 99% Confidence level						
Conservative (%)	3.15	0.00	0.00	10.04	0.18	4.89
Moderate (%)	32.98	23.00	13.03	42.51	2.25	18.44
Aggressive (%)	59.30	54.25	49.21	63.17	1.13	24.65
$G = 2\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	9.48	2.86	4.62
Moderate (%)	1.11	0.00	0.00	39.28	5.07	17.27
Aggressive (%)	39.66	32.62	25.57	60.25	2.42	23.91
(b) 99% Confidence level						
Conservative (%)	0.00	0.00	0.00	9.48	2.86	4.62
Moderate (%)	30.08	20.39	10.70	39.28	5.07	17.27
Aggressive (%)	57.34	52.36	47.38	60.25	2.42	23.91

^aThe manager's goal consists of selecting the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . The relative reduction in the optimal portfolio's standard deviation is computed using the unconstrained optimal portfolio's standard deviation as the reference point. Hence, positive numbers indicate a decrease in the standard deviation.

^bThe bound \underline{V} is equal to the VaR of the efficient portfolio with an expected return equal to that of the benchmark plus G (see Eq. (20)). The bound V' is equal to the VaR of the benchmark minus G (see Eq. (21)). The bound V^p is set so that the fraction of efficiency loss eliminated by the constraint is 50% (see Eq. (30)).

whether the fraction of efficiency loss eliminated is larger when G is larger depends on t and w_B . This result can be understood by using Eq. (14) with $E = E[r_{w_B}] + G$. Note that the sign of $\partial\phi(E[r_{w_B}] + G)/\partial G$ depends on t and w_B .

4.4. Implementing the VaR constraint

We now illustrate implementing the VaR constraint with the bounds given in Section 3.

4.4.1. The case when $V = \underline{V}$

Suppose that $V = \underline{V}$. For example, $\underline{V} = 1.09\%$ when using the moderate benchmark, $G = 1\%$, and $t = 99\%$ (see Fig. 3(b)). Column (4) of Table 2 shows

Table 3

Fraction of efficiency loss eliminated by the VaR constraint when short sales are allowed^a

Confidence level and benchmark	Bound ^b					
	3% (1)	5% (2)	7% (3)	\underline{V} (4)	V' (5)	V^p (6)
$G = 1\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	100.00	1.85	50.00
Moderate (%)	15.18	0.00	0.00	100.00	6.63	50.00
Aggressive (%)	77.36	67.26	55.98	100.00	2.60	50.00
(b) 99% Confidence level						
Conservative (%)	32.50	0.00	0.00	100.00	1.85	50.00
Moderate (%)	82.28	60.82	36.38	100.00	6.63	50.00
Aggressive (%)	96.53	91.48	85.84	100.00	2.60	50.00
$G = 2\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	100.00	31.24	50.00
Moderate (%)	3.49	0.00	0.00	100.00	15.66	50.00
Aggressive (%)	75.53	64.84	52.98	100.00	5.67	50.00
(b) 99% Confidence level						
Conservative (%)	0.00	0.00	0.00	100.00	31.24	50.00
Moderate (%)	80.95	58.00	32.08	100.00	15.66	50.00
Aggressive (%)	97.15	91.81	85.88	100.00	5.67	50.00

^aThe manager's goal consists of selecting the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . A portfolio's efficiency loss is measured by the difference between its variance and that of the portfolio on the mean-variance boundary with the same expected return. The fraction of efficiency loss eliminated by the VaR constraint is obtained by using Eq. (14).

^bThe bound \underline{V} is equal to the VaR of the efficient portfolio with an expected return equal to that of the benchmark plus G (see Eq. (20)). The bound V' is equal to the VaR of the benchmark minus G (see Eq. (21)). The bound V^p is set so that the fraction of efficiency loss eliminated by the constraint is 50% (see Eq. (30)).

that the constrained optimal portfolio's standard deviation is *smaller* than that of the unconstrained optimal portfolio, and column (4) of Table 3 shows that the constraint *fully* eliminates the efficiency loss.

As noted earlier, the constrained optimal portfolio dominates the unconstrained optimal portfolio according to both mean-variance and mean-VaR criteria. Of particular interest is the question of whether the constrained optimal portfolio also dominates the benchmark according to these criteria. Using the moderate benchmark, $G = 1\%$, and $t = 99\%$, we have $\delta_B = 0.0050$, $\delta_1 = 0.0003$, and $\delta_2 = -0.0001$. Since $\delta_B > \delta_1$ and $\delta_B > \delta_2$, the constrained optimal portfolio dominates the benchmark according to both criteria. Similar results are obtained when we use either: (i) the aggressive or defensive benchmark, or (ii) $G = 2\%$, or (iii) $t = 95\%$.

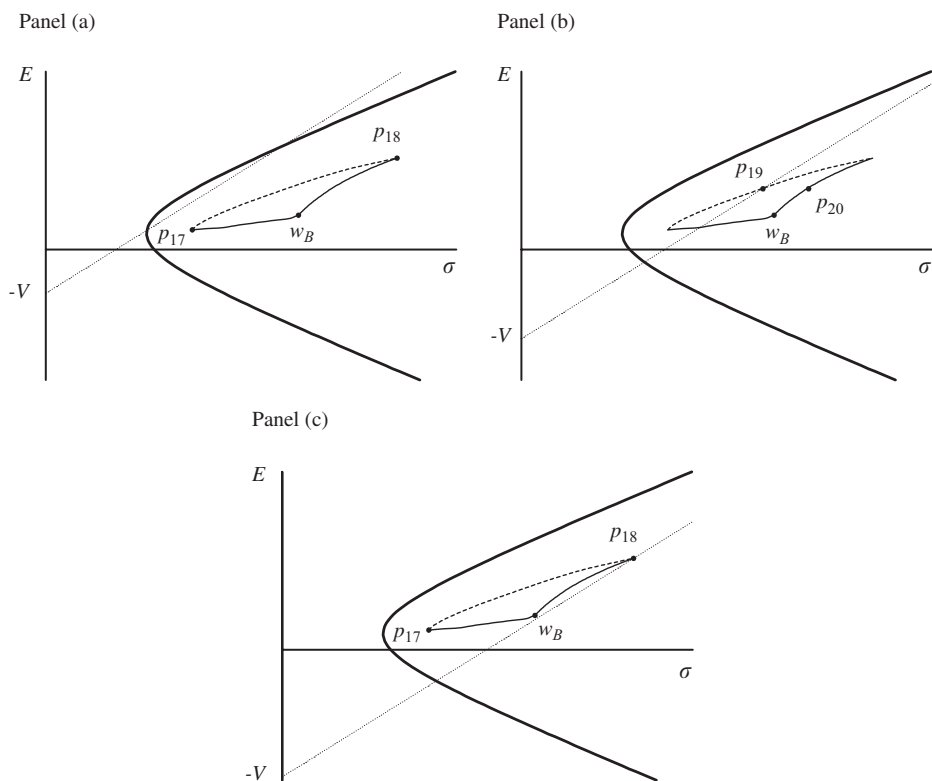


Fig. 5. Impact of a VaR constraint on the mean-TEV boundary when short sales are disallowed. The thick hyperbola (dashed curve) represents the mean-variance boundary when short sales are allowed (disallowed). The thin kinked curve represents the mean-TEV boundary when short sales are disallowed. The dotted line represents a VaR constraint. The case when short sales are disallowed differs from the case when short sales are allowed in three main respects. First, it is possible that there is no portfolio that satisfies a given VaR constraint when short sales are disallowed, but such a portfolio exists when short sales are allowed (panel (a)). Second, there is no VaR constraint that fully eliminates the efficiency loss (panel (b)). Note that there is no VaR constraint such that the portfolio on the constrained mean-TEV boundary with expected return equal to that of p_{19} is closer to the mean-variance boundary than p_{19} . Third, the maximum fraction of efficiency loss that can be eliminated by a VaR constraint depends on the required expected return (panel (c)). This fraction initially increases as we move up along the mean-TEV boundary away from p_{17} , but then decreases past the kink as we move closer to p_{18} .

4.4.2. The case when $V = V'$

Suppose that $V = V'$. For example, $V' = 9.16\%$ when using the moderate benchmark, $G = 1\%$, and $t = 99\%$ (see Fig. 4(a)). Column (5) of Table 2 shows that the constrained optimal portfolio's standard deviation is *smaller* than that of the unconstrained optimal portfolio. Observe that the relative decrease in standard deviation when $G = 2\%$ is larger than when $G = 1\%$. The explanation for this result is that the joint application of the expected return and VaR constraints results

in a standard deviation constraint with bound equal to the standard deviation of the benchmark. Thus, the VaR constraint is relatively more restrictive when $G = 2\%$. Accordingly, column (5) of Table 3 indicates that the fraction of efficiency loss eliminated by the constraint is larger when $G = 2\%$. While in some cases only a *small* portion of the efficiency loss is eliminated, the constrained optimal portfolio nevertheless dominates the unconstrained optimal portfolio and the benchmark according to both mean-variance and mean-VaR criteria.

4.4.3. The case when $V = V^\rho$

Suppose that $V = V^\rho$ where $\rho = 0.5$. For example, $V^\rho = 5.92\%$ when using the moderate benchmark, $G = 1\%$, and $t = 99\%$ (see Fig. 4(b)). Column (6) of Table 2 shows that the constrained optimal portfolio's standard deviation is *smaller* than that of the unconstrained optimal portfolio. Column (6) of Table 3 indicates that the constraint eliminates half of the efficiency loss, which is as expected given that $\rho = 0.5$.

Note that there is a larger (smaller) decrease in standard deviation and a larger (smaller) fraction of the efficiency loss that is eliminated if V^ρ is used instead of V' (\underline{V}) since the constraint is tighter (looser). Also, each constrained optimal portfolio dominates its respective unconstrained optimal portfolio and benchmark according to both mean-variance and mean-VaR criteria.

5. Disallowing short sales

Almazan et al. (2004) find that a large fraction of managers cannot engage in short sales. Hence, we now examine the case when short sales are disallowed.¹⁵ A portfolio is now an $n \times 1$ vector with non-negative entries that sum to one.

5.1. Effects of disallowing short sales on previous results

Since there are no closed-form solutions for portfolios on the mean-variance, mean-TEV and constrained mean-TEV boundaries when short sales are disallowed, graphical analysis is used. For brevity, suppose that all portfolios on the mean-variance boundary when short sales are disallowed differ from the portfolios with the same expected returns that lie on the mean-variance boundary when short sales are allowed. The results are similar if this assumption is not made, but the maximum fraction of efficiency loss that can be eliminated by a VaR constraint is possibly larger. Fig. 5 illustrates the effects of disallowing short sales on (1) the mean-variance boundary, (2) the mean-TEV boundary, and (3) the constrained mean-TEV boundary.

¹⁵For an examination of the mean-variance boundary when short selling is allowed but with certain restrictions attached to it, see, for example, Alexander (1993) and references therein.

5.1.1. Mean-variance boundary

We begin by examining the effect of disallowing short sales on the mean-variance boundary. The thick hyperbola (dashed curve) represents the mean-variance boundary when short sales are allowed (disallowed). Two results are worth noting. First, the range of expected returns for which there exists a portfolio on the mean-variance boundary is bounded when short sales are disallowed. Panel (a) illustrates that there is no portfolio on the boundary with an expected return that is lower (higher) than that of portfolio p_{17} (p_{18}). The idea of this result is straightforward. Let $E^l(E^h)$ denote the lowest (highest) expected return among those of all securities. When short sales are disallowed, there exists no portfolio with an expected return lower (higher) than $E^l(E^h)$. Hence, the range of expected returns for which there exists a portfolio on the mean-variance boundary is given by the interval $[E^l, E^h]$.

Second, the efficiency loss of portfolios on the mean-variance boundary when short sales are disallowed is larger for higher expected returns.¹⁶ This result is driven by: (i) securities with higher expected returns have typically higher variances; and (ii) when short sales are disallowed, portfolios on the mean-variance boundary with higher expected returns require an investment in securities with higher expected returns.

5.1.2. Mean-TEV boundary

Next, we examine the impact of disallowing short sales on the mean-TEV boundary.¹⁷ The thin kinked curve represents the mean-TEV boundary when short sales are disallowed. The case when short sales are disallowed differs from the case when short sales are allowed in four main respects. First, disallowing short sales imposes a restriction of the composition of benchmark and thus on its location relative to the mean-variance boundary.

Second, as in the case of the mean-variance boundary, the range of expected returns for which there exists a portfolio on the mean-TEV boundary when short sales are disallowed is bounded. Panel (a) illustrates that there is no portfolio on the boundary with an expected return that is lower (higher) than that of portfolio p_{17} (p_{18}). Hence, disallowing short sales places a restriction on the size of the expected gain over the benchmark G .

Third, there may exist portfolios on the mean-TEV boundary that are also on the mean-variance boundary when short sales are disallowed. Panel (a) illustrates that portfolios p_{17} and p_{18} are on both of these boundaries. Note that there exist portfolios on both boundaries if the following condition holds: there is a single security with an expected return equal to $E^l(E^h)$. Under this condition, there is a unique portfolio with an expected return equal to $E^l(E^h)$ when short sales are

¹⁶For simplicity, when short sales are disallowed, we still measure a portfolio's efficiency loss with respect to the portfolio with the same expected return that lies on the mean-variance boundary when short sales are allowed.

¹⁷For an examination of the effect of portfolio weight constraints on the mean-TEV boundary, see Bajeux-Besnainou et al. (2007).

disallowed. This portfolio has a weight of 100% in the security with an expected return equal to $E^l(E^h)$. By construction, this portfolio thus belongs to both the mean-variance and mean-TEV boundaries when short sales are disallowed.

Fourth, the representation of the portfolios on the mean-TEV boundary in $(\sigma[r_w], E[r_w])$ space when short sales are disallowed is possibly non-convex.¹⁸ Panel (a) illustrates the case when this representation is non-convex. Note that this case occurs if the following two conditions hold: (i) there is a single security with an expected return equal to $E^l(E^h)$; and (ii) the benchmark lies in $(E[r_w], \sigma[r_w])$ space below a line that connects the two securities with expected returns equal to E^l and E^h . Observe that the benchmark is on the mean-TEV boundary.¹⁹ Furthermore, if condition (i) holds, then the portfolio with a weight of 100% in the security with an expected return equal to $E^l(E^h)$ is also on the mean-TEV boundary. Since these three portfolios are on the mean-TEV boundary, condition (ii) implies that the representation of the portfolios on the mean-TEV boundary in $(\sigma[r_w], E[r_w])$ space is non-convex.²⁰

5.1.3. Constrained mean-TEV boundary

Finally, we examine the effect of disallowing short sales on the constrained mean-TEV boundary. The dotted line represents a VaR constraint. The case when short sales are disallowed differs from the case when short sales are allowed in three main respects. First, as panel (a) illustrates, it is possible that there is no portfolio that satisfies a given VaR constraint when short sales are disallowed, but such a portfolio exists when short sales are allowed. The intuition is that by assumption, the portfolios on the mean-variance boundary when short sales are disallowed have larger standard deviations than the portfolios with the same expected returns that lie on the mean-variance boundary when short sales are allowed. Hence, the former portfolios have larger VaRs than the latter. Thus, when the bound is small, there is no portfolio that satisfies the constraint when short sales are disallowed, but such a portfolio exists when short sales are allowed.

The fact that disallowing short sales may affect the existence of a VaR-constrained optimal portfolio is related to Best and Grauer (1991). They find that expected returns and standard deviations of portfolios on the mean-variance boundary when short sales are allowed (disallowed) are extremely (not very) sensitive to changes in security expected returns. Hence, the distance between the location of (1) portfolios on the mean-variance boundary when short sales are allowed and (2) portfolios on the mean-variance boundary when short sales are disallowed can be notable. Accordingly, disallowing short sales is more likely to affect the existence of a VaR-constrained optimal portfolio when this distance is large.

¹⁸Note that here we refer to the space $(\sigma[r_w], E[r_w])$ instead of the space $(E[r_w], \sigma[r_w])$ used in the figures.

¹⁹While Roll (1992) shows this result when short sales are allowed, it also holds when short sales are disallowed and the benchmark satisfies the short sales constraints.

²⁰Observe that there exist points on this representation where there are notable changes in the slope of the mean-TEV boundary. For example, this occurs in panel (a) at a level of expected return slightly below that of the benchmark. The changes in the slope are caused by changes in the number of securities with positive weights in the portfolios on the mean-TEV boundary.

Second, as panel (b) illustrates, there is no VaR constraint that fully eliminates the efficiency loss. Note that if the required expected return is equal to that of p_{19} , then the portfolio on the constrained mean-TEV boundary (p_{19}) is closer to the mean-variance boundary than the portfolio on the mean-TEV boundary (p_{20}). However, there is no VaR constraint such that the portfolio on the constrained mean-TEV boundary with expected return equal to that of p_{19} is closer to the mean-variance boundary than p_{19} . This result follows from the assumption that the portfolios on the mean-variance boundary when short sales are disallowed differ from the portfolios with the same expected returns that lie on the mean-variance boundary when short sales are allowed.

Third, as panel (c) illustrates, the maximum fraction of efficiency loss that can be eliminated by a VaR constraint depends on the required expected return (and thus on a manager's degree of TEV aversion). Note that this fraction initially increases as we move up along the mean-TEV boundary away from p_{17} , but then decreases past the kink as we move closer to p_{18} .

5.2. Implementing the VaR constraint

As in Section 3, consider a manager who selects the portfolio w^e with minimum TEV among all portfolios with an expected return of $E[r_{w_B}] + G$.

5.2.1. Maximum bound

Assume that the manager faces a VaR constraint. Let $\bar{V} = V[t, r_{w^e}]$. As Fig. 6(a) illustrates, in order for the constraint to be binding, V has to be chosen so that $V < \bar{V}$.

5.2.2. Minimum bound

Let $\underline{V} = V[t, r_w]$, where w denotes the portfolio with $E[r_w] = E[r_{w_B}] + G$ that lies on the efficient frontier when short sales are disallowed. As Fig. 6(b) illustrates, in order for the constrained optimal portfolio to exist, V has to be chosen so that $V \geq \underline{V}$. Moreover, a constraint with $V = \underline{V}$ eliminates the maximum fraction of efficiency loss that can be reduced when short sales are disallowed, which we denote by $\bar{\rho}$.

5.2.3. Intermediate bound

A constraint with $V = \bar{V}$ is not binding and thus does not reduce efficiency loss. While a constraint with $V = \underline{V}$ eliminates the maximum fraction of efficiency loss that can be reduced, the goal of selecting a portfolio with small TEV is ignored. Hence, it is natural to also consider a constraint with $\underline{V} < V < \bar{V}$. As Fig. 6(c) illustrates, this constraint: (i) reduces the efficiency loss, but (ii) still allows the selection of a portfolio with small TEV. This can be seen by noting that p_{22} lies between p_{21} and w^e .

5.2.4. A simple bound

Suppose that $V = V'$, where $V' = V[t, r_{w_B}] - G$. While the use of bound V' can be effective in reducing the efficiency loss, it is also possible that there may not exist a portfolio that satisfies the expected return, VaR, and short sales constraints with this

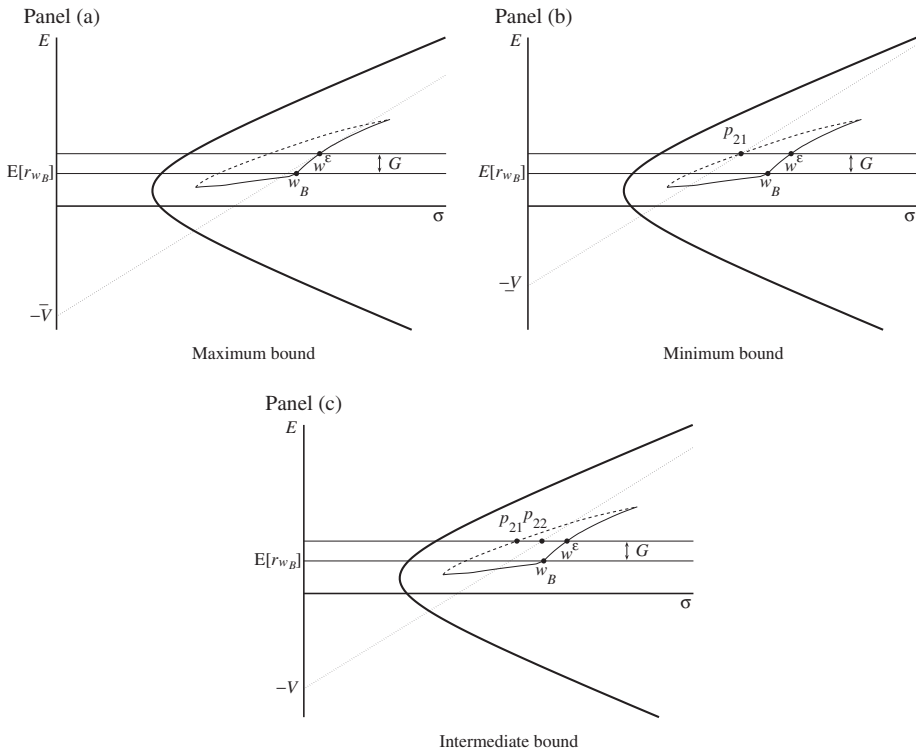


Fig. 6. Implementing the VaR constraint when short sales are disallowed. The thick hyperbola (dashed curve) represents the mean-variance boundary when short sales are allowed (disallowed). The thin kinked curve represents the mean-TEV boundary when short sales are disallowed. The dotted line represents a VaR constraint, while the horizontal lines represent the expected return of the benchmark (w_B) and that plus the expected gain (G). Consider a manager who selects the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . Note that the unconstrained optimal portfolio (w^e) is inefficient. Panel (a) illustrates that if a VaR constraint with bound \bar{V} is imposed where \bar{V} is the VaR of the unconstrained optimal portfolio, then the unconstrained and constrained optimal portfolios coincide. Panel (b) illustrates that if a VaR constraint with bound \underline{V} is imposed where \underline{V} is the VaR of the portfolio with an expected return equal to that of the benchmark plus G that lies on the efficient frontier when short sales are disallowed, then the constrained optimal portfolio (p_{21}) lies on the efficient frontier when short sales are disallowed. Panel (c) illustrates that if a VaR constraint with bound V is imposed where $\underline{V} < V < \bar{V}$, then the constrained optimal portfolio (p_{22}) is closer to the efficient frontier than the unconstrained optimal portfolio (w^e).

bound. For example, this can happen when the benchmark is close to the efficient frontier and G is large.

5.2.5. A general bound

Note that bound V^ρ can be chosen so that it reduces the efficiency loss by proportion ρ , where $0 < \rho \leq \bar{\rho}$. By construction, there always exists a portfolio that satisfies the expected return, VaR, and short sales constraints when bound V^ρ is used.

6. Example

We now illustrate the effect of disallowing short sales in the example of Section 4.

6.1. Change in the optimal portfolio's standard deviation

Columns (1)–(3) of Table 4 present the relative reduction in the optimal portfolio's standard deviation due to the VaR constraint. The results differ in three main respects from those in Table 2 where short sales are allowed. First, the dashes ('–') in the table indicate that there are cases in which there is no portfolio that satisfies the expected return, VaR, and short sales constraints. Also, the number of these cases is larger when: (i) t is larger, (ii) V is smaller, (iii) $\sigma[r_{w_B}]$ is larger, or (iv) G is larger. Second, there are four cases in which the reduction in standard deviation is larger than that when short sales are allowed. Two of these cases occur when using the moderate benchmark, $G = 1\%$, and either (1) $t = 95\%$ and $V = 3\%$, or (2) $t = 99\%$ and $V = 7\%$. The other two cases occur when using either (1) the moderate benchmark, $G = 2\%$, $t = 95\%$, and $V = 5\%$, or (2) the conservative benchmark, $G = 2\%$, $t = 99\%$, and $V = 3\%$. Third, in the latter two cases there is a reduction in standard deviation even though the constraint is not binding when short sales are allowed. Unlike Table 2, however, there are no cases in which the reduction in standard deviation is larger than 30%.

6.2. Fraction of efficiency loss eliminated by the VaR constraint

Columns (1)–(3) of Table 5 provide the fraction of efficiency loss eliminated by the VaR constraint. The results differ in three main respects from those in Table 3 where short sales are allowed. First, as in Table 4, the dashes in Table 5 indicate that there are cases in which there is no portfolio that satisfies the expected return, VaR, and short sales constraints. Also, the number of these cases is larger when: (i) t is larger, (ii) V is smaller, (iii) $\sigma[r_{w_B}]$ is larger, or (iv) G is larger. Second, there are cases in which the fraction of efficiency loss that is eliminated is larger than that when short sales are allowed. These cases are identical to those noted in Section 6.1 for which the reduction in standard deviation is larger than that when short sales are allowed. Third, there are cases in which a fraction of the efficiency loss is eliminated even though the constraint is not binding when short sales are allowed. These cases are identical to those noted in Section 6.1 for which there is a reduction in standard deviation even though the constraint is not binding when short sales are allowed. Unlike Table 3, however, there are no cases in which the fraction of efficiency loss that is eliminated is larger than 60%.

6.3. Implementing the VaR constraint

We now discuss implementing the VaR constraint with the bounds given in Section 5.2.

Table 4

Relative reduction in the optimal portfolio's standard deviation arising from imposing a VaR constraint within a mean-TEV model when short sales are disallowed^a

Confidence level and benchmark	Bound ^b					
	3%	5%	7%	\underline{V}	V'	V^p
	(1)	(2)	(3)	(4)	(5)	(6)
$G = 1\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	4.80	0.00	2.37
Moderate (%)	9.33	0.00	0.00	19.38	6.49	9.17
Aggressive (%)	–	–	27.30	28.55	0.00	13.10
(b) 99% Confidence level						
Conservative (%)	0.00	0.00	0.00	4.80	0.00	2.37
Moderate (%)	–	–	16.80	19.38	6.49	9.17
Aggressive (%)	–	–	–	28.55	0.00	13.10
$G = 2\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	3.17	–	1.57
Moderate (%)	–	1.81	0.00	12.08	–	5.85
Aggressive (%)	–	–	–	18.94	1.87	8.97
(b) 99% Confidence level						
Conservative (%)	1.93	0.00	0.00	3.17	–	1.57
Moderate (%)	–	–	–	12.08	–	5.85
Aggressive (%)	–	–	–	18.94	1.87	8.97

^aThe manager's goal consists of selecting the portfolio with minimum TEV among all portfolios with expected returns equal to that of the benchmark plus G . The relative reduction in the optimal portfolio's standard deviation is computed using the unconstrained optimal portfolio's standard deviation as the reference point. Hence, positive numbers indicate a decrease in the standard deviation. A dash (–) indicates that there is no portfolio that satisfies the expected return, VaR, and short sales constraints.

^bThe bound \underline{V} is equal to the VaR of the portfolio with an expected return equal to that of the benchmark plus G that lies on the efficient frontier when short sales are disallowed. The bound V' is equal to the VaR of the benchmark minus G (see Eq. (21)). The bound V^p is set so that the constraint eliminates 50% of the maximum fraction of efficiency loss that can be reduced by a VaR constraint.

6.3.1. The case when $V = \underline{V}$

Suppose that $V = \underline{V}$. Column (4) of Table 4 shows that the constrained optimal portfolio's standard deviation is *smaller* than that of the unconstrained optimal portfolio. However, column (4) of Table 5 shows that not all of the efficiency loss is eliminated. Note that the numbers in this column represent the maximum fraction of efficiency loss that can be eliminated by a VaR constraint when short sales are disallowed (i.e., \bar{p}).²¹ While \bar{p} is smaller when $G = 2\%$ (i.e., for less TEV-averse managers), the fraction of efficiency loss that is eliminated is still notable.

²¹Observe that \bar{p} depends on the variance–covariance matrix of the securities that are available. Intuitively, if short sales are disallowed, then the benchmark is likely to be closer to the mean-variance boundary in the case when, for example, all securities are positively correlated. Thus, \bar{p} is likely to be

Table 5
Fraction of efficiency loss eliminated by the VaR constraint when short sales are disallowed^a

Confidence level and benchmark	Bound ^b					
	3% (1)	5% (2)	7% (3)	\underline{V} (4)	V' (5)	V^p (6)
$G = 1\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	78.18	0.00	39.09
Moderate (%)	25.51	0.00	0.00	50.18	18.01	25.09
Aggressive (%)	–	–	54.75	56.85	0.00	28.42
(b) 99% Confidence level						
Conservative (%)	0.00	0.00	0.00	78.18	0.00	39.09
Moderate (%)	–	–	44.13	50.18	18.01	25.09
Aggressive (%)	–	–	–	56.85	0.00	28.42
$G = 2\%$						
(a) 95% Confidence level						
Conservative (%)	0.00	0.00	0.00	24.23	–	12.11
Moderate (%)	–	4.98	0.00	31.54	–	15.77
Aggressive (%)	–	–	–	40.80	4.42	20.40
(b) 99% Confidence level						
Conservative (%)	14.88	0.00	0.00	24.23	–	12.11
Moderate (%)	–	–	–	31.54	–	15.77
Aggressive (%)	–	–	–	40.80	4.42	20.40

^aThe manager's goal consists of selecting the portfolio with minimum TEV among all portfolios with an expected return equal to that of the benchmark plus G . A portfolio's efficiency loss is measured by the difference between its variance and that of the portfolio with the same expected return that lies on the mean-variance boundary when short sales are allowed. A dash (–) indicates that there is no portfolio that satisfies the expected return, VaR, and short sales constraints.

^bThe bound \underline{V} is equal to the VaR of the portfolio with an expected return equal to that of the benchmark plus G that lies on the efficient frontier when short sales are disallowed. Note that the numbers in this column represent the maximum fraction of efficiency loss that can be eliminated by a VaR constraint when short sales are disallowed (i.e., \bar{p}). The bound V' is equal to the VaR of the benchmark minus G (see Eq. (21)). The bound V^p is set so that the constraint eliminates 50% of the maximum fraction of efficiency loss that can be reduced by a VaR constraint. Note that the numbers in this column represent half of the maximum fraction of efficiency loss that can be eliminated by a VaR constraint when short sales are disallowed (i.e., $\bar{p}/2$).

6.3.2. The case when $V = V'$

Suppose that $V = V'$. Column (5) of Table 4 shows that when $G = 1\%$, the constraint binds if the moderate benchmark is used, but does not bind if either of the other two

(footnote continued)

smaller in this case than when some of the securities are negatively correlated. To investigate this case, we computed \bar{p} for the aggressive benchmark when only the six asset classes involving stocks are available. As Table 1 indicates, all asset classes involving stocks are positively correlated. Consistent with the aforementioned intuition, in the example \bar{p} is smaller when only stocks are available relative to when both stocks and bonds are available. Specifically, $\bar{p} = 35.14\%$ (37.29%) when only stocks are available and $G = 1\%$ (2%), but $\bar{p} = 56.85\%$ (40.80%) when both stocks and bonds are available and $G = 1\%$ (2%).

benchmarks is used. When $G = 2\%$, the constraint binds if the aggressive benchmark is used, but there is no portfolio that satisfies it if either of the other two benchmarks is used. Note that when the constraint binds and there is a portfolio that satisfies it, the relative reduction in the optimal portfolio's standard deviation is *small* and, as shown in column (5) of Table 5, only a *small* portion of the efficiency loss is eliminated.

6.3.3. The case when $V = V^\rho$

Suppose that $V = V^\rho$ where $\rho = \bar{\rho}/2$. Column (6) of Tables 4 and 5 show that the constrained optimal portfolio's standard deviation is *smaller* than that of the unconstrained optimal portfolio and that half of the efficiency loss that is reduced when $V = \underline{V}$ is eliminated, which is as expected given that $\rho = \bar{\rho}/2$.

7. Conclusion

Value-at-risk (VaR) is a popular risk management tool in the fund management industry. Moreover, active portfolio management is often conducted relative to a benchmark. We combine these two practices by examining the impact of adding a VaR constraint to the problem of a manager who seeks to outperform a benchmark by a given percentage. In doing so, we use the mean-tracking error variance (TEV) model of Roll (1992). We obtain three main results. First, portfolios on the constrained mean-TEV boundary still exhibit three-fund separation, but the weights of the three funds when the constraint binds differ from those in Roll's model. Second, the constraint mitigates the problem that when a manager seeks to outperform a benchmark using the mean-TEV model, he or she selects an inefficient portfolio. Finally, when short sales are disallowed, the extent to which the constraint reduces the optimal portfolio's efficiency loss can still be notable but is smaller than when short sales are allowed. Our results on the usefulness of a VaR constraint justify the observation of Jorion (2001, 2003) and Pearson (2002) that the fund management industry is increasingly using VaR to (1) allocate assets among managers, (2) set risk limits, and (3) monitor asset allocations and managers.

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Appendix A. The mean-variance boundary

A portfolio is on the *mean-variance boundary* if there is no portfolio with the same expected return that has a smaller variance. Let $a \equiv \mu^\top \Sigma^{-1} \mu$, $b \equiv \iota^\top \Sigma^{-1} \mu$,

$c \equiv \iota^\top \Sigma^{-1} \iota$, $d \equiv a - b^2/c$, where ι is the $n \times 1$ unit vector $(1, \dots, 1)$, denote the constants commonly used in the mathematics of the mean-variance boundary. Merton (1972) shows that the portfolio $w(E)$ on the mean-variance boundary with expected return E is given by Eq. (1) with $K = (E - E[r_{w_\sigma}])/(E[r_{w_A}] - E[r_{w_\sigma}])$, $w_\sigma \equiv (\Sigma^{-1}\iota)/c$, and $w_A \equiv (\Sigma^{-1}\mu)/b$. Thus, portfolios on the mean-variance boundary exhibit *two-fund separation*, where the two funds are w_σ and w_A .

Appendix B. The mean-TEV boundary

A portfolio is on the *mean-TEV boundary* if there is no portfolio with the same expected return that has a smaller TEV.²² Roll (1992) shows that the portfolio $w^e(E)$ on the mean-TEV boundary with expected return E is given by Eq. (3) with $K^e = (E - E[r_{w_B}])/(E[r_{w_A}] - E[r_{w_\sigma}])$. Thus, portfolios on the mean-TEV boundary exhibit *three-fund separation*, where the three funds are w_σ , w_A , and w_B . Eq. (3) has two implications. First, the portfolios on the mean-TEV boundary belong to the mean-variance boundary if and only if the benchmark belongs to the latter (i.e., $\delta_B = 0$). Second, the deviations from an initial position in w_B required to arrive at a final position in $w^e(E)$ (i.e., $K^e(w_A - w_\sigma)$) do not depend on w_B . However, $w^e(E)$ still depends on w_B since $w^e(E)$ is equal to the sum of w_B plus $K^e(w_A - w_\sigma)$.

Appendix C. Non-normality

Next, we extend our normality-based results to the case of non-normality.

C.1. Elliptically distributed returns

The multivariate elliptical distribution is a natural extension of the multivariate normal distribution (see, e.g., Fang et al., 1990). Hence, suppose that security returns have a multivariate elliptical distribution with finite first and second moments such as the multivariate t -distribution with more than two degrees of freedom.²³ Under this assumption, the distribution of the returns on any portfolio has a univariate elliptical distribution of the same type (see, e.g., Owen and Rabinovitch, 1983). Hence, portfolio w 's VaR is $V[t, r_w, e] = z_t^e \sigma[r_w] - E[r_w]$, where z_t^e denotes the quantile $1 - t$ of the corresponding standardized univariate distribution (see, e.g.,

²²Brennan (1993) shows that when some agents have a mean-TEV objective function, a two-beta equilibrium holds. Empirical evidence supporting such an equilibrium is presented by Gómez and Zapatero (2003). Assuming that the revenue function of an active manager is linear in the expected gain over the benchmark and TEV, Cornell and Roll (2005) derive an equilibrium in which a security's expected return depends on its betas computed against the market and the benchmark.

²³For example, Jorion (1996) presents evidence in favor of the t -distribution as a model for security returns. Tokat et al. (2003) examine the effect of fat tails on asset allocation using stable Paretian distributions. The stable Paretian distribution is a special case of the elliptical distribution whose first and second moments are possibly infinite.

Embrechts et al., 2002). Since portfolio w 's VaR is a linear function of $\sigma[r_w]$ and $E[r_w]$, our previous results still hold if security returns have a multivariate elliptical distribution, as $V[t, r_w, e]$ and z_t^e would simply be used instead of $V[t, r_w]$ and z_t , respectively.

C.2. O' Cinneide upper bound on VaR

There is an extensive literature recognizing that the mean-variance model is, at least as an approximation, consistent with expected utility maximization when no distributional assumption is made (see Markowitz, 1987, pp. 52–70). Furthermore, Berk (1997) provides joint conditions on utility functions and distributions that lead to mean-variance analysis. Hence, we now examine the case when the multivariate distribution of security returns is unknown, but has finite first and second moments.

Let x_t denote the quantile $1 - t$ of an unknown univariate distribution with mean μ_x and standard deviation σ_x . O' Cinneide (1990) shows that

$$|x_t - \mu_x| \leq \sigma_x \max \left\{ \sqrt{\frac{t}{1-t}}, \sqrt{\frac{1-t}{t}} \right\}. \quad (37)$$

Note that $\sqrt{t/(1-t)} > \sqrt{(1-t)/t}$ if $t > \frac{1}{2}$. Since we assume that $t > \frac{1}{2}$, Eq. (37) implies that $V[t, r_w] \leq y_t^O \sigma[r_w] - E[r_w]$, where $y_t^O \equiv \sqrt{t/(1-t)}$. Hence, O' Cinneide's inequality provides an upper bound on VaR, thereby allowing us to extend our normality-based results to the case when no distributional assumption is made.²⁴ For example, Propositions 1 and 2 characterize the constrained mean-TEV boundary with the threshold confidence level $t = d/(1+d)$. The constrained mean-TEV model based on the O' Cinneide approximation is useful to a manager who decides (or is required) to satisfy a VaR constraint but believes that it is not appropriate to impose a distributional assumption on security returns and wishes to be conservative in estimating VaR.

Using Chebyshev's inequality, Alexander and Baptista (2002, pp. 1178–1179) show that $V[t, r_w] \leq y_t^C \sigma[r_w] - E[r_w]$, where $y_t^C \equiv \sqrt{1/(1-t)}$. Hence, Chebyshev's inequality also provides an upper bound on VaR. Since $y_t^O < y_t^C$ (e.g., if $t = 0.95$, then $y_t^O = 4.3589$ and $y_t^C = 4.4721$), the O' Cinneide upper bound on VaR is tighter than the Chebyshev upper bound on VaR.

Appendix D. Proofs

Proof of Lemma 1. Assume $V[t, r_{w^e(E)}] > V$. First, we show that Eq. (11) holds. Suppose by way of a contradiction that $V[t, r_{w^e(E, V)}] < V$. Let $w^* \equiv \alpha w^e(E) + (1 - \alpha)w^e(E, V)$, where $\alpha > 0$ is arbitrarily small. Note that: (a) $E[r_{w^*}] = E$ since

²⁴For an examination of this bound, see Bertsimas et al. (2004).

$E[r_{w^e(E)}] = E[r_{w^e(E,V)}] = E$; (b) $\sigma_\varepsilon^2[r_{w^*}] < \sigma_\varepsilon^2[r_{w^e(E,V)}]$ since α is small and $\sigma_\varepsilon^2[r_{w^e(E)}] < \sigma_\varepsilon^2[r_{w^e(E,V)}]$; and (c) $V[t, r_{w^*}] < V$ since α is small and $V[t, r_{w^e(E,V)}] < V$. Observe that (a)–(c) contradict the fact that $w^e(E, V)$ is on the constrained mean-TEV boundary, which completes the first part of our proof.

Second, we show that Eq. (12) holds. We solve the problem

$$\min_{w \in \mathbb{R}^n} \frac{1}{2}(w - w_B)^\top \Sigma (w - w_B) \quad (38)$$

$$\text{s.t. } w^\top \iota = 1, \quad (39)$$

$$w^\top \mu = E, \quad (40)$$

$$z_t \sqrt{w^\top \Sigma w} - w^\top \mu = V. \quad (41)$$

The first-order conditions associated with the Lagrangian

$$\begin{aligned} L = & \frac{1}{2}(w - w_B)^\top \Sigma (w - w_B) - \gamma_1(w^\top \iota - 1) - \gamma_2(w^\top \mu - E) \\ & - \gamma_3(z_t \sqrt{w^\top \Sigma w} - w^\top \mu - V) \end{aligned} \quad (42)$$

and Lagrange multipliers γ_1 , γ_2 , and γ_3 are given by

$$\frac{\partial L}{\partial w} = \Sigma(w - w_B) - \gamma_1 \iota - \gamma_2 \mu - \gamma_3 \left(\frac{z_t \Sigma w}{\sqrt{w^\top \Sigma w}} - \mu \right) = 0, \quad (43)$$

$$\frac{\partial L}{\partial \gamma_1} = w^\top \iota - 1 = 0, \quad (44)$$

$$\frac{\partial L}{\partial \gamma_2} = w^\top \mu - E = 0, \quad (45)$$

$$\frac{\partial L}{\partial \gamma_3} = z_t \sqrt{w^\top \Sigma w} - w^\top \mu - V = 0. \quad (46)$$

Premultiplying both sides of Eq. (43) by Σ^{-1} , we obtain

$$w - w_B = \gamma_1(\Sigma^{-1} \iota) + \gamma_2(\Sigma^{-1} \mu) + \gamma_3 \left[\frac{z_t w}{\sqrt{w^\top \Sigma w}} - (\Sigma^{-1} \mu) \right]. \quad (47)$$

Eqs. (45) and (46) imply that

$$\sqrt{w^\top \Sigma w} = (V + E)/z_t. \quad (48)$$

Using Eqs. (47) and (48), we have

$$w = Xw_\sigma + Yw_A + Zw_B, \quad (49)$$

where $X \equiv \gamma_1 c/Q$, $Y \equiv (\gamma_2 - \gamma_3)b/Q$, $Z \equiv 1/Q$, and $Q \equiv 1 - \gamma_3[z_t^2/(V + E)]$. Premultiplying both sides of Eq. (49) by ι^\top and using the fact that w , w_σ , w_A , w_B are portfolios, we obtain:

$$X + Y + Z = 1. \quad (50)$$

Premultiplying both sides of Eq. (49) by μ^\top and using the fact that $E[r_{w_\sigma}] = b/c$, we have

$$(b/c)X + (a/b)Y + E[r_{w_B}]Z = E. \quad (51)$$

It follows from Eqs. (48) and (49) that

$$[Xw_\sigma + Yw_A + Zw_B]^\top \Sigma [Xw_\sigma + Yw_A + Zw_B] = [(V + E)/z_t]^2. \quad (52)$$

Using Eq. (52), we have

$$\begin{aligned} & [X^2 + 2X(Y + Z)](1/c) + Y^2(a/b^2) + Z^2\sigma^2[r_{w_B}] + 2YZ(E[r_{w_B}]/b) \\ & = [(V + E)/z_t]^2. \end{aligned} \quad (53)$$

We can now solve the system of three Eqs. (50), (51), and (53) for the three unknowns X , Y , and Z . This system can be used to obtain

$$k_3X^2 + k_4X + k_5 = 0, \quad (54)$$

where

$$\begin{aligned} k_3 & \equiv \frac{k_6^2a/b^2 + (cd)^2\sigma^2[r_{w_B}] + 2cdk_6E[r_{w_B}]/b}{(k_6 + cd)^2} - \frac{1}{c}, \\ k_4 & \equiv 2 \left\{ \frac{-k_6k_7a/b^2 + cdk_8\sigma^2[r_{w_B}] + (k_6k_8 - cdk_7)E[r_{w_B}]/b}{(k_6 + cd)^2} + \frac{1}{c} \right\}, \\ k_5 & \equiv \frac{k_7^2a/b^2 + k_8^2\sigma^2[r_{w_B}] - 2k_7k_8E[r_{w_B}]/b}{(k_6 + cd)^2} - \left(\frac{V + E}{z_t} \right)^2, \\ k_6 & \equiv b^2 - bcE[r_{w_B}], \\ k_7 & \equiv (E - E[r_{w_B}])bc, \\ k_8 & \equiv bcE - ac. \end{aligned}$$

Eqs. (50), (51), and (54) imply that $X = (-k_4 \pm \sqrt{k_4^2 - 4k_3k_5})/(2k_3)$, $Y = (k_7 - k_6X)/(k_6 + cd)$, and $Z = 1 - X - Y$, which completes the second part of our proof. \square

The following two lemmas are useful.

Lemma A. *The following holds along the mean-TEV boundary:*

- (i) If $t < \Phi(\sqrt{d})$, then $\partial V[t, r_w]/\partial E[r_w] < 0$, $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -\infty$, and $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$;
- (ii) If $t = \Phi(\sqrt{d})$, then $\partial V[t, r_w]/\partial E[r_w] < 0$, $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -\frac{b}{c}$, and $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$;
- (iii) If $t > \Phi(\sqrt{d})$, then $\partial V[t, r_w^\varepsilon]/\partial E[r_w^\varepsilon] = 0$ and $\partial^2 V[t, r_w]/\partial E[r_w]^2 > 0$.

Proof. First, we show (i). Assume $t < \Phi(\sqrt{d})$. Then, $z_t < \sqrt{d}$. Using Eq. (5),

$$\frac{\partial V[t, r_w]}{\partial E[r_w]} = z_t \frac{\partial \sigma[r_w]}{\partial E[r_w]} - 1. \quad (55)$$

Eq. (4) implies that

$$\frac{\partial \sigma[r_w]}{\partial E[r_w]} < \frac{1}{\sqrt{d}}, \quad (56)$$

$$\lim_{E[r_w] \rightarrow +\infty} \frac{\partial \sigma[r_w]}{\partial E[r_w]} = \frac{1}{\sqrt{d}}, \quad (57)$$

$$\lim_{E[r_w] \rightarrow -\infty} \frac{\partial \sigma[r_w]}{\partial E[r_w]} = -\frac{1}{\sqrt{d}}. \quad (58)$$

Since $0 < z_t < \sqrt{d}$, it follows from Eqs. (55) and (56) that $\partial V[t, r_w]/\partial E[r_w] < 0$. Using Eqs. (55) and (57), $\lim_{E[r_w] \rightarrow +\infty} \partial V[t, r_w]/\partial E[r_w] < 0$ since $0 < z_t < \sqrt{d}$. Eqs. (55) and (58) imply that $\lim_{E[r_w] \rightarrow -\infty} \partial V[t, r_w]/\partial E[r_w] < 0$. Hence, $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -\infty$ and $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$, which completes the first part of our proof.

Second, we show (ii). Assume $t = \Phi(\sqrt{d})$. Then, $z_t = \sqrt{d}$. Using Eqs. (4) and (5), $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = z_t \sigma[r_w] - \{b/c + \sqrt{d} \sigma[r_w]\} = -b/c$ along the mean-TEV boundary. Eqs. (55) and (58) imply that $\lim_{E[r_w] \rightarrow -\infty} \partial V[t, r_w]/\partial E[r_w] < 0$. Hence, $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$, which completes the second part of our proof.

Third, we show (iii). Assume $t > \Phi(\sqrt{d})$. Then, $z_t > \sqrt{d}$. Since w_t^e minimizes VaR along the mean-TEV boundary, $\partial V[t, r_{w_t^e}]/\partial E[r_{w_t^e}] = 0$ along this boundary. Let w be a portfolio on the mean-TEV boundary. Eqs. (4) and (5) imply that

$$\frac{\partial V[t, r_w]}{\partial E[r_w]} = z_t \frac{\frac{E[r_w] - b/c}{d}}{\sqrt{\frac{1}{c} + \frac{(E[r_w] - b/c)^2}{d}} + \delta_B} - 1. \quad (59)$$

Using Eq. (59), we obtain

$$\frac{\partial^2 V[t, r_w]}{\partial E[r_w]^2} = \frac{\frac{z_t(1/c + \delta_B)}{d}}{\left[\frac{1}{c} + \frac{(E[r_w] - b/c)^2}{d} + \delta_B\right]^{3/2}} > 0,$$

which completes the third part of our proof. \square

Lemma B. The set $\{w : V[t, r_w] \leq V\}$ is non-empty if and only if one of the following conditions hold: (i) $t < \Phi(\sqrt{d})$; (ii) $t = \Phi(\sqrt{d})$ and $V > -b/c$; and (iii) $t > \Phi(\sqrt{d})$ and $V \geq V[t, r_{w_t}]$.

Proof. First, we show the “if part.” Assume $t < \Phi(\sqrt{d})$. Then, $z_t < \sqrt{d}$. Let $V \in \mathbb{R}$ be arbitrarily chosen. Using arguments similar to those used in the proof of Lemma A, we have $\partial V[t, r_w]/\partial E[r_w] < 0$ and $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -\infty$ along the mean-variance boundary. Hence, any portfolio on the mean-variance boundary with a sufficiently large expected return satisfies the constraint, that is, the set $\{w : V[t, r_w] \leq V\}$ is non-empty.

Assume $t = \Phi(\sqrt{d})$ and $V > -b/c$. Then, $z_t = \sqrt{d}$. Using arguments similar to those used in the proof of Lemma A, we have $\partial V[t, r_w]/\partial E[r_w] < 0$ and $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -b/c$ along the mean-variance boundary. Since $V > -b/c$, any portfolio on this boundary with a sufficiently large expected return satisfies the constraint. Hence, the set $\{w : V[t, r_w] \leq V\}$ is non-empty.

Assume $t > \Phi(\sqrt{d})$ and $V \geq V[t, r_{w_t}]$. The set $\{w : V[t, r_w] \leq V\}$ is non-empty since w_t belongs to it.

Second, we show the “only if part.” If $t = \Phi(\sqrt{d})$ and $V \leq -b/c$, then the set $\{w : V[t, r_w] \leq V\}$ is empty since $V[t, r_w] > -b/c$ for any portfolio w . If $t > \Phi(\sqrt{d})$ and $V < V[t, r_{w_t}]$, then this set is empty by definition of w_t . This completes the second part of our proof. \square

The following lemma is useful in the proof of Proposition 1.

Lemma 2. (i) If $t < \Phi(\sqrt{d})$, then there is a unique portfolio w_V^e on the mean-TEV boundary with $V[t, r_{w_V^e}] = V$. (ii) If $t = \Phi(\sqrt{d})$ and $V \leq -b/c$, then there is no portfolio on the mean-TEV boundary at which the constraint binds. (iii) If $t = \Phi(\sqrt{d})$ and $V > -b/c$, then there is a unique portfolio w_V^e on the mean-TEV boundary with $V[t, r_{w_V^e}] = V$.

Proof of Lemma 2. First, we show (i). Assume $t < \Phi(\sqrt{d})$. Let $V \in \mathbb{R}$ be arbitrarily chosen. Using (i) in Lemma A, $\partial V[t, r_w]/\partial E[r_w] < 0$, $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -\infty$, and $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$ along the mean-TEV boundary. Since $V[t, r_w]$ is continuous in $E[r_w]$ along this boundary, there exists w_V^e on it with $V[t, r_{w_V^e}] = V$. The uniqueness of w_V^e follows from the fact that $\partial V[t, r_w]/\partial E[r_w] < 0$ along the mean-TEV boundary. This completes the first part of our proof.

Second, we show (ii). Assume $t = \Phi(\sqrt{d})$ and $V \leq -b/c$. The desired result follows from Lemma B. This completes the second part of our proof.

Third, we show (iii). Assume $t = \Phi(\sqrt{d})$ and $V > -b/c$. Using (ii) in Lemma A, $\partial V[t, r_w]/\partial E[r_w] < 0$, $\lim_{E[r_w] \rightarrow +\infty} V[t, r_w] = -b/c$ and $\lim_{E[r_w] \rightarrow -\infty} V[t, r_w] = +\infty$ along the mean-TEV boundary. Since $V[t, r_w]$ is continuous in $E[r_w]$ along the mean-TEV boundary, there exists w_V^e on the mean-TEV boundary with $V[t, r_{w_V^e}] = V$. The uniqueness of w_V^e follows from the fact that $\partial V[t, r_w]/\partial E[r_w] < 0$ along the mean-TEV boundary. This completes the third part of our proof. \square

The following lemma is useful in the proof of Proposition 2.

Lemma 3. Suppose that $t > \Phi(\sqrt{d})$. (i) If $V < V[t, r_{w_t^e}]$, then there is no portfolio on the mean-TEV boundary at which the constraint binds. (ii) If $V = V[t, r_{w_t^e}]$, then w_t^e is the unique portfolio on the mean-TEV boundary with a VaR of V . (iii) If $V > V[t, r_{w_t^e}]$, then there exist two portfolios \underline{w}_V^e and \bar{w}_V^e on the mean-TEV boundary with $V[t, r_{\underline{w}_V^e}] = V[t, r_{\bar{w}_V^e}] = V$, where $E[r_{\underline{w}_V^e}] < E[r_{w_t^e}] < E[r_{\bar{w}_V^e}]$.

Proof of Lemma 3. First, we show (i). The desired claim follows from the definition of w_t^e . This completes the first part of our proof.

Second, we show (ii). Assume $t > \Phi(\sqrt{d})$ and $V = V[t, r_{w_t^e}]$. The desired claim follows from the uniqueness of w_t^e . This completes the second part of our proof.

Third, we show (iii). Assume $t > \Phi(\sqrt{d})$ and $V > V[t, r_{w_t^e}]$. Using (iii) in Lemma A, we have $\partial V[t, r_{w_t^e}]/\partial E[r_{w_t^e}] = 0$ and $\partial^2 V[t, r_w]/\partial E[r_w]^2 > 0$ along the mean-TEV boundary. Hence, there exist two portfolios \underline{w}_V^e and \bar{w}_V^e on the mean-TEV boundary with $V[t, r_{\underline{w}_V^e}] = V[t, r_{\bar{w}_V^e}] = V$, where $E[r_{\underline{w}_V^e}] < E[r_{w_t^e}] < E[r_{\bar{w}_V^e}]$. This completes the third part of our proof.²⁵ \square

Proof of Proposition 1. First, we show (i). Assume $t < \Phi(\sqrt{d})$. Using arguments similar to those used in the proof of Lemma A, we have $\partial V[t, r_w]/\partial E[r_w] < 0$ along the mean-variance boundary. Hence, $V[t, r_w] > V$ for any portfolio w with $E[r_w] < E[r_{w_V}]$. Thus, there is no portfolio w on the constrained mean-TEV boundary with $E[r_w] < E[r_{w_V}]$. Since $\partial V[t, r_w]/\partial E[r_w] < 0$ along the mean-variance and mean-TEV boundaries, we have: (I) for any E with $E[r_{w_V}] \leq E < E[r_{w_V^e}]$, there is a portfolio w such that $E[r_w] = E$ and $V[t, r_w] \leq V$; and (II) $V[t, r_w] > V$ for any portfolio w on the mean-TEV boundary with $E[r_w] < E[r_{w_V^e}]$. Using Lemma 1, any portfolio given by Eq. (12), where $E[r_{w_V}] \leq E < E[r_{w_V^e}]$, is on the constrained mean-TEV boundary. Since $\partial V[t, r_w]/\partial E[r_w] < 0$ along the mean-TEV boundary, any portfolio given by Eq. (3), where $E \geq E[r_{w_V^e}]$, satisfies the VaR constraint and thus is on the constrained mean-TEV boundary. This completes the first part of our proof.

Second, we show (ii). Assume $t = \Phi(\sqrt{d})$ and $V \leq -b/c$. The desired result follows from Lemma B. This completes the second part of our proof.

Third, we show (iii). Assume $t = \Phi(\sqrt{d})$ and $V > -b/c$. The proof is similar to the proof of (i) and therefore omitted. This completes the third part of our proof. \square

Proof of Proposition 2. Suppose that $t > \Phi(\sqrt{d})$. First, we show (i). Assume $V < V[t, r_{w_t}]$. The desired result follows from Lemma B. This completes the first part of our proof.

Second, we show (ii). Assume $V = V[t, r_{w_t}]$. The desired result follows from the uniqueness of w_t . This completes the second part of our proof.

Third, we show (iii). Assume $V[t, r_{w_t}] < V \leq V[t, r_{w_t^e}]$. Using arguments similar to those used in the proof of Lemma A, we have $\partial V[t, r_{w_t}]/\partial E[r_{w_t}] = 0$ and $\partial^2 V[t, r_w]/\partial E[r_w]^2 > 0$ along the mean-variance boundary. Hence, $V[t, r_w] > V$ for any portfolio w with $E[r_w] < E[r_{\underline{w}_V}]$ or $E[r_w] > E[r_{\bar{w}_V}]$. Thus, there is no portfolio w on the constrained mean-TEV boundary with $E[r_w] < E[r_{\underline{w}_V}]$ or $E[r_w] > E[r_{\bar{w}_V}]$. Note that: (I) for any E with $E[r_{\underline{w}_V}] \leq E \leq E[r_{\bar{w}_V}]$, there is a portfolio w such that $E[r_w] = E$ and $V[t, r_w] \leq V$; (II) if $V < V[t, r_{w_t^e}]$, then $V[t, r_w] > V$ for any portfolio w on the

²⁵As noted earlier, the mean-TEV boundary coincides with the mean-variance boundary when w_B is on the latter boundary. Thus, results similar to Lemmas 2 and 3 also hold for the mean-variance boundary with w_t^e being replaced by w_t . Since portfolios on the mean-TEV boundary are located to the right of portfolios on the mean-variance boundary in $(E[r_w], \sigma[r_w])$ space, we have $E[r_{\underline{w}_V}] < E[r_{w_t^e}] < E[r_{\bar{w}_V}] < E[r_{w_V}]$.

mean-TEV boundary with $E[r_{w_V}] \leq E[r_w] \leq E[r_{\bar{w}_V}]$; and (III) if $V = V[t, r_{w_t^e}]$, then $V[t, r_w] > V$ for any portfolio w on the mean-TEV boundary with $E[r_{w_V}] \leq E[r_w] \leq E[r_{\bar{w}_V}]$ and $E[r_w] \neq E[r_{w_t^e}]$. Using Lemma 1, any portfolio given by Eq. (12), where $E[r_{w_V}] \leq E \leq E[r_{\bar{w}_V}]$, is on the constrained mean-TEV boundary. This completes the third part of our proof.

Fourth, we show (iv). Assume $V > V[t, r_{w_t^e}]$. Using arguments similar to those of the third part of this proof there is no portfolio w on the constrained mean-TEV boundary with $E[r_w] < E[r_{w_V}]$ or $E[r_w] > E[r_{\bar{w}_V}]$. Note that: (I) for any E with $E[r_{w_V}] \leq E < E[r_{w_t^e}]$ or $E[r_{\bar{w}_V}] < E \leq E[r_{\bar{w}_V}]$, there is a portfolio w with $E[r_w] = E$ and $V[t, r_w] \leq V$; and (II) $V[t, r_w] > V$ for any portfolio w on the mean-TEV boundary with $E[r_{w_V}] \leq E[r_w] < E[r_{w_t^e}]$ or $E[r_{\bar{w}_V}] < E[r_w] \leq E[r_{\bar{w}_V}]$. Using Lemma 1, any portfolio given by Eq. (12), where $E[r_{w_V}] \leq E[r_w] < E[r_{w_t^e}]$ or $E[r_{\bar{w}_V}] < E[r_w] \leq E[r_{\bar{w}_V}]$, is on to the constrained mean-TEV boundary. Since $\partial^2 V[t, r_w] / \partial E[r_w]^2 < 0$ along the mean-TEV boundary, any portfolio w on this boundary with $E[r_{w_V}] \leq E[r_w] \leq E[r_{\bar{w}_V}]$ satisfies the VaR constraint and thus is on the constrained mean-TEV boundary. This completes the fourth part of our proof. \square

Appendix E. Relation between Lemma 1 and Jorion (2003)

Next, we relate Lemma 1 to the following problem examined by Jorion (2003)

$$\max_{w \in \mathbb{R}^n} (w - w_B)^\top \mu \quad (60)$$

$$\text{s.t. } (w - w_B)^\top l = 0, \quad (61)$$

$$(w - w_B)^\top \Sigma (w - w_B) = T, \quad (62)$$

$$w^\top \Sigma w = \sigma^2. \quad (63)$$

Jorion shows that the solution to this problem is given by

$$w(T, \sigma^2) = w_B - \frac{\Sigma^{-1}(\mu + \lambda_1 l + \lambda_3 \Sigma w_B)}{\lambda_2 + \lambda_3}, \quad (64)$$

where λ_1 , λ_2 , and λ_3 are Lagrange multipliers associated with constraints (61)–(63). Let $E(T, \sigma^2)$ and $V(T, \sigma^2)$ denote, respectively, the expected return and VaR of $w(T, \sigma^2)$.

As shown in the proof of Lemma 1, if $V[t, r_{w^e(E)}] > V$ and $w^e(E, V)$ exists, then $w^e(E, V)$ solves problem (38) subject to constraints (39)–(41). Let $T(E, V)$ and $\sigma^2(E, V)$ denote, respectively, the TEV and variance of $w^e(E, V)$.

Note that problem (38) subject to constraints (39)–(41) is closely related to problem (60) subject to constraints (61)–(63). First, constraint (39) is equivalent to constraint (61). Second, if $\sigma = (V + E)/z_t$, then the joint application of constraints (40) and (41) results in constraint (63). Third, if expected return E and TEV bound T

are appropriately chosen, then minimizing TEV given expected return E is equivalent to maximizing expected gain given TEV bound T .

Let E_σ be the expected return of the portfolio with minimum TEV among all portfolios with variance less than or equal to σ^2 . Fix an expected return E and a bound V such that (1) $V[t, r_{w^e(E)}] > V$, (2) $w^e(E, V)$ exists, and (3) $E \geq E_{\sigma(E, V)}$. Then,

$$w^e(E, V) = w(T(E, V), \sigma^2(E, V)). \quad (65)$$

That is, the portfolio on the constrained mean-TEV boundary with expected return E when the VaR bound is V coincides with the portfolio on the constant-TEV mean-variance boundary with variance $\sigma^2(E, V)$ when the TEV bound is $T(E, V)$. It follows from Eqs. (12), (64), and (65), and the definitions of w_σ and w_A that $X = -\lambda_1 c / (\lambda_2 + \lambda_3)$ and $Y = -b / (\lambda_2 + \lambda_3)$. Eq. (65) implies that $E(T(E, V), \sigma^2(E, V)) = E$ and $V(T(E, V), \sigma^2(E, V)) = V$.

References

- Admati, A.R., Pfleiderer, P., 1997. Does it all add up? Benchmarks and the compensation of active portfolio managers. *Journal of Business* 70, 323–350.
- Alexander, G.J., 1993. Short selling and efficient sets. *Journal of Finance* 48, 1497–1506.
- Alexander, G.J., Baptista, A.M., 2002. Economic implications of using a mean-VaR model for portfolio selection: a comparison with mean-variance analysis. *Journal of Economic Dynamics and Control* 26, 1159–1193.
- Alexander, G.J., Baptista, A.M., 2004. A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. *Management Science* 50, 1261–1273.
- Alexander, G.J., Baptista, A.M., 2006. Does the Basle Capital Accord reduce bank fragility? An assessment of the value-at-risk approach. *Journal of Monetary Economics* 53, 1631–1660.
- Almazan, A., Brown, K.C., Carlson, M., Chapman, D.A., 2004. Why constrain your mutual fund manager? *Journal of Financial Economics* 73, 289–321.
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. *Mathematical Finance* 9, 203–228.
- Bailey, J.V., 1990. Some thoughts on performance-based fees. *Financial Analysts Journal* 46, 31–40.
- Bajeux-Besnainou, I., Belhaj, R., Maillard, D., Portait, R., 2007. Portfolio optimization under tracking error and weights constraints. Working paper, The George Washington University.
- Berk, J.B., 1997. Necessary conditions for the CAPM. *Journal of Economic Theory* 73, 245–257.
- Bertsimas, D., Lauprete, G.J., Samarov, A., 2004. Shortfall as a risk measure: properties, optimization and applications. *Journal of Economic Dynamics and Control* 28, 1353–1381.
- Best, M.J., Grauer, R.R., 1991. On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results. *Review of Financial Studies* 4, 315–342.
- Black, F., Litterman, R.B., 1992. Global portfolio optimization. *Financial Analysts Journal* 48, 28–43.
- Brennan, M., 1993. Agency and asset pricing. Working paper, UCLA.
- Carpenter, J.N., 2000. Does option compensation increase managerial risk appetite? *Journal of Finance* 55, 2311–2331.
- Chevalier, J., Ellison, G., 1997. Risk taking by mutual funds as a response to incentives. *Journal of Political Economy* 105, 1167–1200.
- Chow, G., 1995. Portfolio selection based on return, risk, and relative performance. *Financial Analysts Journal* 51, 54–60.
- Chow, G., Kritzman, M., 2001. Risk budgets. *Journal of Portfolio Management* 27, 56–60.
- Cornell, B., Roll, R., 2005. A delegated-agent asset-pricing model. *Financial Analysts Journal* 61, 57–69.
- Davanzo, L.E., Nesbitt, S.L., 1987. Performance fees for investment management. *Financial Analysts Journal* 43, 14–20.

- Elton, E.J., Gruber, M.J., Blake, C.R., 2003. Incentive fees and mutual funds. *Journal of Finance* 58, 779–804.
- Embrechts, P., McNeil, A.J., Straumann, D., 2002. Correlation and dependence in risk management: properties and pitfalls. In: Dempster, M. (Ed.), *Risk Management: Value at Risk and Beyond*. Cambridge University Press, Cambridge.
- Fang, K.-T., Kotz, S., Ng, K.-W., 1990. *Symmetric Multivariate and Related Distributions*. Chapman & Hall, London.
- Gómez, J.-P., Zapatero, F., 2003. Asset pricing implications of benchmarking: a two-factor CAPM. *European Journal of Finance* 9, 343–357.
- Grinblatt, M., Titman, S., 1989. Adverse risk incentives and the design of performance-based contracts. *Management Science* 35, 807–822.
- Grinold, R., Rudd, A., 1987. Incentive fees: Who wins? Who loses? *Financial Analysts Journal* 43, 27–38.
- Hull, J.C., 2006. *Options, Futures, & Other Derivatives*. Pearson Prentice-Hall, Upper Saddle River, NJ.
- Jorion, P., 1996. Risk²: measuring the risk in value at risk. *Financial Analysts Journal* 52, 47–56.
- Jorion, P., 2001. *Value at Risk: The New Benchmark for Controlling Market Risk*. McGraw-Hill, New York, NY.
- Jorion, P., 2003. Portfolio optimization with constraints on tracking error. *Financial Analysts Journal* 59, 70–82.
- Kupiec, P.H., 1999. Risk capital and VaR. *Journal of Derivatives* 7, 41–52.
- Markowitz, H.M., 1987. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Basil Blackwell, New York, NY.
- Merton, R.C., 1972. An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis* 7, 1851–1872.
- Michaud, R.O., 1998. *Efficient Asset Allocation: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. Harvard Business School Press, Boston, MA.
- O’Cinneide, C.A., 1990. The mean is within one standard deviation of any median. *The American Statistician* 44, 292–294.
- Owen, R., Rabinovitch, R., 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice. *Journal of Finance* 38, 745–752.
- Pearson, N.D., 2002. *Risk Budgeting: Portfolio Problem Solving with Value-at-Risk*. Wiley, New York, NY.
- Record, E.E., Tynan, M.A., 1987. Incentive fees: the basic issues. *Financial Analysts Journal* 43, 39–43.
- Roll, R., 1992. A mean/variance analysis of tracking error. *Journal of Portfolio Management* 18, 13–22.
- Sharpe, W.F., 1987. *Asset Allocation Tools*. The Scientific Press, Redwood City, CA.
- Shefrin, H., 2000. *Beyond Greed and Fear*. Harvard Business School Press, Boston, MA.
- Sirri, E.R., Tufano, P., 1998. Costly search and mutual fund flows. *Journal of Finance* 53, 1589–1622.
- Solnik, B., McLeavey, D., 2003. *International Investments*. Pearson Addison-Wesley Publishing Company, Boston, MA.
- Starks, L.T., 1987. Performance incentive fees: an agency theoretic approach. *Journal of Financial and Quantitative Analysis* 22, 17–32.
- Tokat, Y., Rachev, S.T., Schwartz, E.S., 2003. The stable non-Gaussian asset allocation: a comparison with the classical Gaussian approach. *Journal of Economic Dynamics and Control* 27, 937–969.