The Index Number Problem: A Differential Geometric Approach.

A thesis presented

by

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Abstract

The first part of this thesis looks at issues in index number theory. By using techniques developed in differential geometry, it is shown that the so-called index number problem can be resolved by the development of a special economic derivative operator constructed for this purpose. This derivative is shown to give rise to a unique differential geometric index number which is then demonstrated to equal the Divisia index. It is then shown in the second chapter (co-authored with Eric Weinstein) that when placed under similar assumptions, this data based index equals the Konus index, previously asserted to be incalculable in the absence of a knowledge of preference maps. The third chapter deals with rural to urban migration. Based on the risk neutral Todaro model, it has been suggested that raising rural incomes will have the effect of controlling migration to urban labour markets. However, it is shown in this chapter that it in the case of decreasing absolute risk aversion it is possible for increasing income to have the perverse effect of increasing the desirability of migration.

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Acknowledgments

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The author benefited from a research assistantship working with Prof. **Amartya Sen** during the 1994-1995 academic year. Additionally, I have found Sen's discourse on index theory contained in the article *"Welfare Basis of Real Income Comparisons"* (cited here as Sen [32]) to be invaluable as a reference laying bare the otherwise implicit assumptions underlying the theory of index numbers.

I am grateful to **W. Erwin Diewert** for his thoughtful input and for freely sharing his comprehensive knowledge of index theory. Specifically, I received invaluable information on the state of the literature concerning, changing preferences, chaining arguments, and welfare discussions of the Divisia index, from a thorough letter which is cited here as Diewert [10]. Perhaps the most helpful resource in understanding the background and history of Index Number theory has proved to be his compilation of articles *Essays in Index Number Theory Vol. I* (cited here as Diewert and Nakamura [11]).

Many of the ideas contained in this dissertation are the result of an ongoing discussion, debate, and intellectual interchange with my co-author **Eric Weinstein.**

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To my parents **Esther and Hira Malaney**: You always expected the best from me, and it is only in striving to give it to you that I have made it this far. I am grateful to have had the opportunity to fulfill some of the dreams which when placed out of your reach, you put within mine. Thank you for all the love and support that have enabled me to achieve this.

I would like to dedicate this thesis to my intellectual inspiration and best friend, Eric: I have learned so much more than differential geometry from you along the way, and I know the journey continues to hold ever more.

Introduction

The theory of index numbers has been of interest to economists for over a century. This interest has been rejuvenated in the 1990s as the U.S. government grapples with the question of how to construct a true cost of living adjustment (C.O.L.A.), or how to correct for the problems of the current consumer price index (CPI). The inconsistencies between the hundreds of available algebraic index numbers, and their uncertain effects on welfare (when implemented as C.O.L.A) have led to extensive debate, and some pessimism about the ability of economic theory to answer the fundamental questions lying at the core of this issue.

The first chapter of this thesis takes a new approach to understanding the question of what the "correct" index number is. We first observe that all index numbers can be viewed as a combination of two components: an algebraic formula and a notion of constancy. A general implicit choice has been made using the ordinary calculus derivative as the notion of constancy. By using techniques in differential geometry, we construct here an economic derivative precisely to perform the separation of income and substitution effects necessary for an economic index. It is shown that when the choice of derivative is made explicit and the ordinary non-economic derivative is replaced by the differential geometric 'economic co-variant derivative', the discrepancy between all algebraic formulas for index numbers disappears, thereby resolving the index number problem. The unique index resulting from the use of the economic derivative in any of the previously developed index number formulas is then shown to equal the Divisia index.

The second chapter looks at the welfare implications of the Divisia index based on the new understanding of its mathematical underpinnings. We compare the welfare effects of the Divisia index with those of the theoretical Konus index. While the Konus is referred to as the 'true' economic index, it is asserted to be incalculable in the absence preference maps. The Divisia index is, however, entirely data based, and therefore computable. In comparing the two, however, it is found that there is a major difference in the assumptions upon which these indexes have been developed. The Konus index makes the implicit assumption of unchanging preferences. The Divisia makes no such assumption. It is essential therefore, when comparing the two, to uniformise the assumptions under which they are studied. It is shown in the first part of this chapter that if an individual is compensated using the Divisia price index as a cost of living adjustment, under constant preferences the Divisia price index is in fact equal to the Konus price index. The Divisia therefore provides us with a revealed preference method for finding the Konus index, previously thought to be unknowable in the absence of knowledge on preference maps. This is in fact an indication that the path dependent nature of the Divisia C.O.L.A. arises entirely from the failure of the assumption of unchanging preferences in the real world. In the second part of the chapter this unrealistic assumption is relaxed, and both indexes are analysed under the assumption of changing preferences. In a situation of unchanging preferences, two distinct Konus indexes are defined, the Laspeyres Konus or the Paasche Konus, depending on whether the indifference curve picked is that of the base time or the current time. Once preferences are allowed to change, one must select not only between base and current time indifference curves, but also base and current time preferences. We therefore define four Konus price indexes, with the various combinations of base and current indifference curves, and base and current preferences. It is proved that chaining these indexes also gives equivalence with the Divisia.

It is therefore shown that when viewed in its natural mathematical setting, the Divisia index not only resolves the consistency issues that have heretofore plagued index number theory, but also answers the welfare concerns that have spurred the deep and longlasting economic interest in the issue.

The third chapter deals with issues of migration from rural to urban areas. Since the seminal work of Todaro in the 1960s, there has been much interest in the decision making process that leads to the migration of labour into urban areas which often suffer from high unemployment. The conclusion of the Todaro model was that the discrepancy in incomes between the low wage rural labour market and the high wage urban market was sufficient that it was possible for the present discounted value of the expected urban income to outweigh that of rural income, even with a high probability of unemployment in the urban market. The policy suggested by this conclusion has been to reduce the discrepancy in incomes either by raising the level of rural wages or by lowering the level of urban wages. In this chapter it is shown that these two approaches may in fact not be equivalent. The Todaro model makes two implicit assumptions which are unlikely to hold in reality. The first is that an individual maximises expected income as opposed to expected utility. This is equivalent to making the assumption of risk neutrality. It has been shown that in general individuals are risk averse, especially at the low levels of income being discussed here. The second assumption is that the decision making unit is the individual. Here too much evidence has been accumulated showing that migration decisions are often made at the level of the family unit and not solely by the individual. In this chapter we relax these two unrealistic assumptions. It is proved that when households facing migration decisions are characterised by decreasing absolute risk aversion, it is in fact possible for increasing rural wages to lead to increased migration to urban areas. This 'perverse' migration arises because with the increased security of higher wages for those family workers remaining in the rural labour market, it is possible for the family to 'gamble' more of its members on the high wage-low probability urban market. It is therefore essential that before using this policy as tool for stemming rural to urban migration, one ensures that wages are not within the range of perversity defined herein.

Chapter 1

The Index Number Problem: A Differential Geometric Approach

1.1 Introduction

> "A man with one watch knows what time it is; a man with two is never sure." -*Proverb*

The primary diagnostic measures of an economy are price and quantity index numbers. The plethora of existing indexes can therefore leave economists with an inability to evaluate accurately the most basic of our economic concepts. The most widely used index numbers, the Paasche and Laspeyres, can disagree not only in terms of the magnitude of change but also the direction. These contradictions in qualitative results (with regards to growth versus shrinkage or inflation versus deflation) caused by our index number problem, puts us as economists in the embarrassing position of being doctors unable to take a temperature without counseling the patient to get a second opinion. In the words of W.E. Diewert:

> "The relatively large divergence between the Paasche and Laspeyres indexes when a fixed base year is used is a source for some concern. In many national accounting systems (such as Canada's), the consumer price index is constructed using a fixed

> base Laspeyres index, while the implicit deflator for the consumption expenditures component of GNP is a fixed base Paasche Index. Given the importance of consumer price indexes in indexing wages and cost of living supplements, it is important that official price indexes be consistent with each other." -Diewert 1978 pp. 268

Or in the less academic record of Senate hearings on Social Security cost of living adjustments,

> "[Senator] Simpson stated ... Cost-of-living adjustments (COLAs) create huge transfers of wealth and about 45 percent of Federal receipts are affected by the CPI. ... [Senator] Conrad stated that what is at stake surrounding this issue is very significant – $1.3 trillion over the next seven years have to be cut in order to achieve a balanced budget. A 1 percent reduction in the CPI would achieve $280 billion in savings, over 21 percent of what is needed." -Report of Hearing before the Senate Finance Committee, June 6, 1995.

This points out that in practical terms, the social welfare of those who depend on cost of living adjustments is affected to a large extent by which of several gauges is currently in favor. To put it simply, consistency issues are pre-requisite to welfare analysis.

The fundamental nature of this problem has, over the last century, attracted the attention of many economists. Starting in the late 1800s the quest for the "true" index number led to the development of hundreds of different formulas, each offering a different result. The inability to evaluate the relative validity of any of these formulae led to the development of what is known as the test approach to index numbers. Since all index numbers agree in the case of a single good, an attempt was made to formulate plausible properties for index numbers in the one good case and test whether a proposed index satisfies them for many goods. There were systematic attempts made by Walsh, Fisher and other economists to develop tests to isolate the "true" index number. However the people developing the tests, not coincidentally, were often the people developing the indexes. Needless to say this lent an air of relativism to the entire process. In the 1930's, however, the test approach showed up an even bigger problem: Frisch proved an impossibility theorem showing that no bilateral index number could pass three basic tests[1]. This led to a pessimism as to the very existence of this "true" index that had been quested.

An entirely different approach was developed by Konus in the 1920's which has come to be called the 'economic' approach to index numbers. The Konus index, a somewhat theoretical concept, is based upon economic optimisation under constraints. A Konus cost of living adjustment would compensate (or penalise) a person by the amount it would cost him to consume the least expensive basket of goods on his original indifference curve. While this in some sense captures what we as economists would like to think of as the "true" cost of living adjustment, there are several problems with this index and its interpretation.

Undoubtedly, the most obvious practical shortcoming of the Konus index is the general unavailability of information on preference maps. While this practical concern may limit its computation, there are even more challenging theoretical problems. One basic flaw is its unnatural assumption of unchanging preferences, without which the answers it returns can be more psychological than economic in nature[2]. Finally we once again are left with the decision of whether to use base or current year preferences, bringing us back to the original problem of consistency.

As has been noted by Sen and others, it is of the utmost importance in work on index theory to clearly identify the scope of inquiry and the implicit assumptions present. The focus of this paper will be on those price and quantity indices which are directly computable from time-dependent vectors $p(t)$, $q(t)$ of price and quantity data for an individual or group between two time periods $t = 0, 1$. We will refer to such indices as bilateral temporal indices. Since the disagreement between these indices can reduce what should be an objective measurement (viz. of inflation or growth) to a subjective choice of measurement instrument, the most vexing problem in the field is the consistency or so-called index number problem. The solution of this problem may in turn be thought of as the first step in the application of data-based indexes to the analysis of welfare. It is this problem to which this paper is addressed. It should be stressed that this is a theoretical paper attempting to further develop the logical foundations of index number theory.[3]

[1] Frisch 1930

[2] Sen (1979) addresses a broad range of issues underlying the assumption of unchanging preferences.

[3] For previous related results see: Richter 1966, Hulten 1973, Samuelson and Swamy

It is possible to look at any bilateral index number as a combination of two ingredients: an algebraic formula and a notion of constancy. We claim that given such a decomposition *there is no inherent problem with the algebraic formulas of any of the common bilateral index numbers.* The problem, when properly isolated, is seen to arise from a consistent implicit choice of an incorrect notion of constancy.

As the field of economics shifted in the last century to incorporate developments in calculus, economists have adopted the standard notion of derivatives: $df = 0$, as our notion of constancy. It was only later that mathematicians themselves refined their concept of derivatives to allow for them to be adapted to particular problems. The field of differential geometry was developed around work with these "adapted derivatives" or "connections". By using differential geometry to adapt our notion of constancy to fit the unique problem of index numbers, we are able to eradicate inconsistencies between all common bilateral index numbers.

Updating the field of index numbers to incorporate these developments in mathematics in this way enables us to cast age old issues into an entirely new framework. Just as differential calculus was the framework within which to think of optimisation, differential geometry is the correct framework within which to think of index numbers. As with previous mathematical developments, this framework immediately helps clarify the questions we are asking. Not only is it able to recover various scattered results from index theory in a cohesive picture, it enables us to prove new and valuable results.

We give here a plan of this chapter. In section 2 we use a simple toy example to show explicitly how a seemingly innocuous use of the ordinary derivative must be altered in order to recover the economics of constant purchasing power; this guiding analogy is intended to bring out the logical necessity of the differential geometric approach. In section 3 we will define the differential geometric set-up and show how it produces a preferred index which we refer to as the geometric index. A simplified 2-good economy is constructed in section 4 which exhibits the index number problem for the Paasche and Laspeyres indexes. We show in detail how the somewhat abstract geometric set-up is used to carry out an explicit computation producing agreement between the aforementioned offenders. In section 5 we show that this framework actually leads to a resolution of the index number problem for temporal bilateral indices. Section 6 contains a derivation and

1974, and Diewert and Nakamura 1993 and the references therein.

discussion of the Divisia index. In section 7 we summarize our results and discuss the merits of path dependence in the computations of C.O.L.A.s; we also discuss plans for future extensions. Finally, in Appendix A we give some of the terminology from differential geometry underlying the previous discussions.

1.2 Adapted Derivatives and Differing Notions of Constancy

Each index number can be viewed as a combination of two elements :

1. An algebraic formula
2. A notion of constancy.

What do we mean by a notion of constancy? Recall from differential calculus that when a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $\frac{df}{dx} = 0$ we conclude that the function is constant. This suggests that the derivative is at the heart of our notion of constancy. Economists are comfortable with the use of the ordinary derivative to express the notion of constancy. Since the days of Newtonian calculus, however, mathematicians have discovered that this is but one derivative amongst many. Each derivative represents a particular notion of constancy. The use of any derivative therefore represents a choice of the notion of constancy, and must be justified on the basis of the economic question that is being asked. The incorrect choice can lead us to at best wrong and at worst nonsensical economic answers.

Let us go back to the simple one good case to illustrate this. Consider a person who buys only one good whose price at time t is P(t). Let us say he has a salary $S(t)$ which at time $t = 0$ is \$100. Our standard derivative $\frac{dS(t)}{dt}$ would tell us that his salary was constant if it stayed at \$100 , even if the price of the good were rising. Clearly as economists this is not the notion of constancy we would want. It is a constant purchasing power salary that we want to recognise as constant. That is, for any scalar $m \in \mathbb{R}$ we would want a derivative which sees $S(t) = mP(t)$ as 'constant'. While in the one good case it may be easy to find the answer we are looking for, we would like to use this simple example to illustrate an important point: it is possible to actually develop a derivative that *sees* the constant purchasing power salary as being constant. Let us see how we would adapt our derivative in this simple case to give us the correct notion of constancy.

Consider a derivative operator $\frac{\delta(\cdot)}{\delta t}$ that sends a salary with constant purchasing power to zero.

$$\frac{\delta S(t)}{\delta t} = \frac{\delta m P(t)}{\delta t} = 0 \tag{1.1}$$

Let us posit:

$$\frac{\delta(\cdot)}{\delta t} = \frac{d(\cdot)}{dt} + g(t)(\cdot). \tag{1.2}$$

With this ansatz let us solve for g(t):

$$0 = \frac{\delta S(t)}{\delta t} = m\frac{\delta P(t)}{\delta t} = m(\frac{dP(t)}{dt} + g(t)P(t)) \tag{1.3}$$

$$\Rightarrow g(t) = -\frac{1}{P(t)}\frac{dP(t)}{dt} = -\frac{dln(P(t))}{dt} \tag{1.4}$$

$$\Rightarrow \frac{\delta}{\delta t} = \frac{d}{dt} - \frac{dln(P(t))}{dt} \tag{1.5}$$

This is our adapted derivative, i.e. The derivative which recognizes a constant purchasing power salary as constant. We will call this adapted derivative a **covariant derivative** or **connection**.

Thus we see that we have a choice between two different derivatives where we may have believed that only one existed. It is plain to see that *failure to select the correct derivative leads to wrong economics.* Fortunately, in the one good case it is easy to recognise the problem and it can be corrected for in several ways; this contrasts markedly with the more complicated situations. Our claim in this paper is that the so-called "index number problem" arises from a consistent improper choice made from an analogous pair of derivatives. All bilateral index numbers implicitly choose to use the ordinary derivative as their notion of constancy; what we propose to show in this paper, is that when instead of the ordinary derivative, we use the derivative uniquely adapted to our problem, the discrepancies between the various bilateral index numbers disappear.

1.3 The Differential Geometric Index

1.3.1 Notation and Introduction to the Index

Let V^n be a space of n goods and services under consideration. Let V^* be the space of pricing systems of V^n. (V and V^* can be thought of as column and row vectors respectively.) Then if $q \in V^n$ and $p \in V^*$, we have

$$p(t) \cdot q(t) = \text{value of q in pricing system p at time t.} \tag{1.6}$$

Assume $p(t) \cdot q(t) \neq 0 \ \ \forall t$, i.e our basket is never worthless. Any combination of goods and services $w \in V$ can be uniquely described by a sum

$$w(t) = \lambda(t)q(t) + b(t) \tag{1.7}$$

where $p(t) \cdot b(t) = 0$. Here $b(t) \in \beta(t)$ where $\beta(t)$ is the space of *barters* at time t, i.e. all baskets which our pricing system at time t sends to zero. Then if $w(t) = \frac{dv(t)}{dt} = \lambda(t)q(t) + b(t)$ with $\lambda(t) = 0$ then $v(t)$ is "changing without growing" because $v(t)$ changes only by barters, the values of which are 0 at time t.

Exactly what we are looking for in an index is the freedom to allow change without growth. Let us say we are trying to develop a price index to evaluate inflation from biblical times, when people's baskets included frankincense and myrrh, to current times, when we are consuming computers and cars. Neither the Laspeyres, which would evaluate frankincense in today's prices, nor the Paasche, which would attempt to evaluate computers in ancient prices, would give us sensible answers.

What is needed is a derivative which correctly modernises or antiquates the components of the basket of goods while holding its value constant. Thus we seek a derivative which will answer the question 'what would the basket of frankincense and myrrh be if it were given as a gift in modern times', or put more crudely, 'how many power rangers and smurfs would the wise men have given to the infant Jesus?'

It is possible for us to directly translate the basket for one important reason: at the point of time any new good is introduced, old ones are still in existence. This enables us at any time to evaluate the price at which one will be traded for the other, i.e. it gives us an ability to evaluate one basket in terms of another. An approximation to this can be found through the process of chaining indexes, but because of time gaps in chained baskets, the ability to translate any particular basket is unavailable. For instance, the U.S. CPI is a prominent representative example of a chained index whose links are approximately 10 years in length. Rapid changes in fields such as computers and the automotive industry mean that several generations of changes may occur between recalibrations of the reference basket. Consider, for example, the category referred to by the Bureau of Labor Statistics as "Information Processing Equipment"; there has been only one recalibration (in 1982-84) between the last slide rules and the modern lap tops of the 1990s. Obviously, this kind of rough taxonomy is insufficient to deal with rapidly changing components of the basket.

What the index should require is the ability to allow the composition of the basket to change, while maintaining constant some notion of value. We can maintain a "constant value" basket while allowing the composition of the basket to change by allowing trade-offs in the basket at any time based on the current market value of the trade-offs.

This can be accomplished mathematically using an adapted or covariant derivative. The tools we use to develop this are:

1. Projections onto the space of goods $[q(t)]$ and barters $\beta_{p(t)}$ i.e. $v = \lambda(t)q(t) + b(t)$

2. Total differentials of vector valued functions

We will now develop the covariant derivative which will be uniquely adapted to give us the desired price and quantity indexes. The following discussion represents a translation of various differential geometric concepts into the language of linear algebra and differential calculus. For a more formal differential geometric exposition see Appendix A.

Let us start with some definitions:

Definition 1 *We define $[w]$ to be the subspace of a vector space V given by all multiples of a vector $w \in V$.*

Definition 2 *Define C to be all points in $V \times V^*$ such that $p \cdot q = 0$.*

Definition 3 *We define Ξ to be $V \times V^* - C$.*

For most realistic purposes, the baskets of goods under consideration will have non-zero value. Thus confining ourselves to consideration of only those baskets and pricing system represented by elements of $\Xi \subset \mathbb{R}^{2n}$ will not present a serious restriction.

Given this motivation we give the following definition.

Definition 4 *An* **economic history** *is defined to be a curve $\alpha : I \longrightarrow \Xi$ where $I \subset \mathbb{R}$ is a connected subset of the real line (such as $[0,1]$).*

Such a path is intended to represent the collection of points $(p,q) \in V^* \times V$ which are traced out by an economy, enterprise or individual over the interval I.

Given a point $(q(t), p(t)) \in \Xi$, we determine the following projection maps $\Pi : V \longrightarrow V$ and $\Pi : V^* \longrightarrow V^*$

$$\Pi_{[q(t)]}(v) = (\frac{p(t) \cdot v}{p(t) \cdot q(t)})q(t) \tag{1.8}$$

$$\Pi_{\beta_{p(t)}}(v) = v - \Pi_{[q(t)]}(v) \tag{1.9}$$

$$\Pi_{[p(t)]}(\omega) = \frac{\omega \cdot q(t)}{p(t) \cdot q(t)} \cdot p(t) \tag{1.10}$$

$$\Pi_{\beta_{[v]}}(\omega) = \omega - \Pi_{[p(t)]}(\omega) \tag{1.11}$$

where $v \in V$ and $\omega \in V^*$. So, for example, it is immediate that

$$v = \Pi_{[q(t)]}(v) + \Pi_{\beta_{p(t)}}(v). \tag{1.12}$$

In order to intuitively understand these projections and their function, we will give an economic interpretation to the linear algebra. For the purposes of visualisation and ease of exposition, we examine the case where $n = 2$.

Let q represent the original basket containing goods x and y, within the two dimensional vector space of goods and services V. (see Figure 1) In the dual vector space, V^* we can picture corresponding price vector p. (see Figure 1) Once a price vector is specified we can define β_p to be the subset of V given by $\beta_p = \{v \in V : p \cdot v = 0\}$. This represents all baskets of x and y corresponding to 'barters' in the pricing system p; these are, as the name suggests, composites of debts and assets whose total value is identically zero. The corresponding subspace β_q can be defined within V^*.

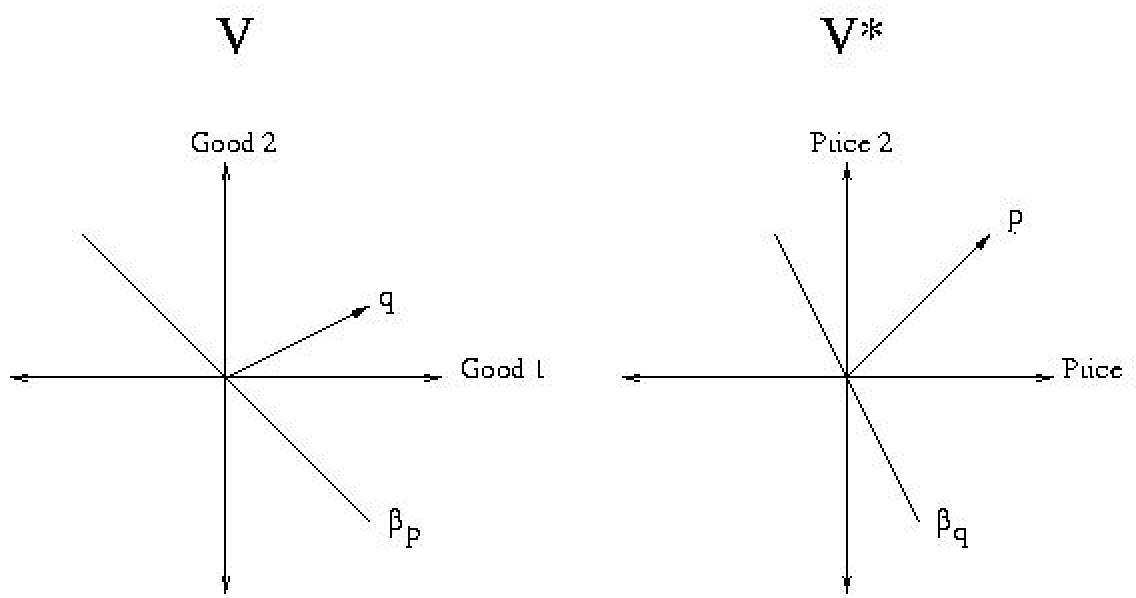

Figure 1.1: Vector subspaces in a two good economy.

Let us now look at the subspace $[q]$ spanned by the vector q in V. The dimension of $[q]$ will clearly be 1 so long as $q \neq 0$; likewise the dimension of the subspace β_p will be $n-1$ so long as $p \neq 0$. Therefore we can, in the generic situation, decompose any other basket v into its projections onto $[q]$ and β_p (see Figure 2). The $v_{[q]}$ component represents the multiple of q whose value is equal to that of v in the pricing system p. Given the equivalence in value between v and $v_{[q]}$, there must exist a barter in β_p which transforms v into $v_{[q]}$; this is the economic interpretation of v_{β_p}.

The subspaces q and β_p will, however, not be linearly independent on those occasions when $p \cdot q = 0$, in which case we will not be able to span the space V in this way. In order to avoid this degeneracy, we define C to be all points in $V \times V^*$ such that $p \cdot q = 0$, and remove these points from consideration.

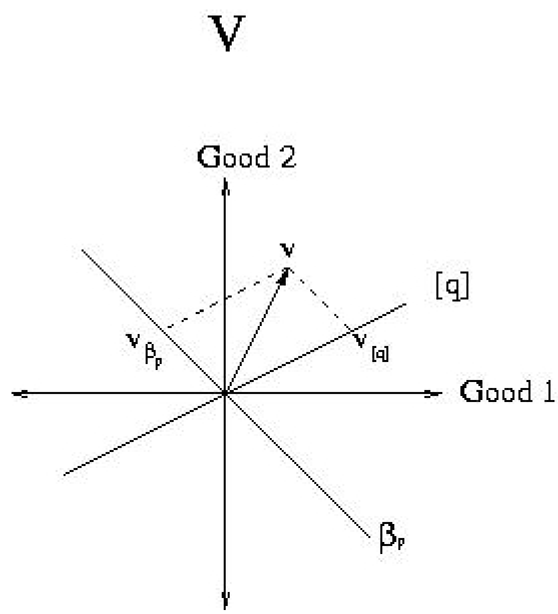

Figure 1.2: Decomposition of a vector v into components.

Let us assume that we have n real valued functions

$$f_i : \Xi \longrightarrow \mathbb{R}. \tag{1.13}$$

We can then assemble these real valued functions into a single vector valued function

$$\sigma : \Xi \longrightarrow V \tag{1.14}$$

given by

$$\sigma(q_1, ..., q_n, p_1, ..., p_n) = \sum_{i=1}^{n} f_i(q, p)e_i. \tag{1.15}$$

Such a vector valued function is called a **section** in differential Geometry (see appendix A). This mathematical construct will enable us to keep track of the best analog of our reference basket as we antiquate or modernise its constituents. The real-valued functions f_i which determine the vector-valued function σ are given by the contributions of the various basket components in their various units of measurement (kilograms, gallons, etc.).

The total differential, or Jacobian, of σ is defined to be

$$\nabla^o \sigma = \sum_{i=1}^{n} df_i(q,p) e_i \tag{1.16}$$

$$= \sum_{i=1}^{n} (\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} dq_j + \sum_{k=1}^{n} \frac{\partial f_i}{\partial p_k} dp_k) e_i \tag{1.17}$$

Let $\alpha : \mathbb{R} \longrightarrow \Xi$ be given by 2n functions $(q_1(t), ..., q_n(t), p_1(t), ..., p_n(t))$ with tangent vector $\dot{\alpha} = (\frac{d\alpha}{dt})$ then

$$\nabla^o_{\dot{\alpha}} \sigma = \sum_{i=1}^{n} (\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} \frac{dq_j}{dt} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial p_k} \frac{dp_k}{dt}) e_i \tag{1.18}$$

This is simply the Jacobian evaluated in the direction of the tangent field $\dot{\alpha} = \frac{d\alpha}{dt}$ along α.

Let us now define the derivative uniquely adapted to achieve the appropriate index number. The non trivial 'economic' covariant derivative of $\sigma : \Xi \longrightarrow V$ along α is:

$$\nabla^a_{\dot{\alpha}} \sigma = \Pi_{[q_t]} \nabla^o_{\dot{\alpha}} (\Pi_{[q_t]} \sigma) + \Pi_{\beta_{p_t}} \nabla^o_{\dot{\alpha}} (\Pi_{\beta_{p_t}} \sigma) \tag{1.19}$$

The interested reader may check that ∇^a satisfies the axioms for a covariant derivative found in Appendix A. [4]

If $\nabla^a_{\dot{\alpha}} \sigma = 0$, we will say that σ is **covariantly constant** along α with respect to the covariant derivative ∇^a. If σ is covariantly constant along α with respect to a given covariant derivative ∇ and $\sigma(\alpha(s)) = v$ we will refer to $\sigma(\alpha(d))$ as the parallel translation of v along α from the 'source' time $t = s$ to the 'destination' time $t = d$. We will denote this parallel translate by $\tau_s(d)$; that is $\tau_s(d)$ is determined as a particular value of a vector valued function σ, covariantly constant along a curve. It should be noted that it is possible to parallel translate using *any* covariant derivative. If the parallel

[4] It should be noted that when the vector valued function $\sigma(x)$ is a multiple of the quantity vector comprising $x \in \Xi$, the second summand is identically zero; this simplifies matters considerably and is the major case of interest in this paper. For example, if we are attempting to track the growth of a basket $q(0)$ from time $t = 0$ to time $t = 1$, the vector $\sigma(0)$ at the source time $t = 0$ will lie entirely in the subspace cut out by $q(0)$. The projection of σ onto the barter space, $\Pi_{\beta_{p_t}} \sigma$, will therefore be zero. In fact if $\nabla^a_{\dot{\alpha}} \sigma = 0$ then it will be the case that along α we will have $\Pi_{\beta_{p_t}} \sigma = 0$ for all t.

translation is done using the ordinary derivative, it will simply give us the original basket. This corresponds to the traditional notion of constancy. We will denote the parallel translate of $q(s)$ (or $p(s)$) along α with respect to the ordinary derivative ∇^o by τ^o; likewise, the parallel translate of $q(s)$ (or $p(s)$) along α using the adapted covariant derivative ∇^a will be written τ^a.

At this point one might ask 'what is the economic interpretation of the equation $\nabla^a_{\dot\alpha}\sigma = 0$?' In order to address this question, it will help to recall our earlier interpretation of the various projection maps. Let us begin by making the assumption that $\sigma(\alpha(0)) = q(0)$; while this is certainly not a generic assumption, all σ's considered in this paper will satisfy similar conditions. In order to induce modernisation or antiquation of the $\sigma(t)$ basket components while disallowing 'growth', we would like to require that $\sigma(t)$ be changing at any given time strictly by barters. This allows the time dependent basket $\sigma(t)$ to maintain the 'value' of the original basket $q(0)$ while changing its composition as the composition of $q(t)$ changes. This goal is achieved by selecting the covariant derivative $\nabla^a_{\dot\alpha}\sigma$, for by construction, the parallel translation equation requires that $\nabla^a_{\dot\alpha}\sigma = \Pi_{[q]}\nabla^o_{\dot\alpha}(\Pi_{[q]}\sigma) = 0$. In prose, this means that the projection of the "change in $\sigma(t)$" (namely $\nabla^o_{\dot\alpha}$) onto 'basket space' (namely $[q]$), must be zero, thereby confining all change to lie in barter space. This parallel translation of the basket $q(0)$ from time $t = 0$ to time $t = 1$ using $\nabla^a_{\dot\alpha}\sigma$ will thereby ensure that at time $t = 1$, $\sigma(t)$ will be some multiple of the basket $q(t)$.

We are therefore going to seek a vector valued function $\sigma : \Xi \longrightarrow V$ such that for a source time s and a destination time d we have:

1. $\sigma(\alpha(s)) = q(s)$
2. $\nabla^a_{\dot\alpha}\sigma = 0$

The first condition requires that our vector valued function σ agree with the source-time basket $q(s)$ at the point from which we are parallel translating. The second condition requires that we parallel translate this basket using the adapted derivative.

We note that $\sigma(\alpha(t))$ is uniquely determined by a theorem in differential geometry.[5]

It is clear that depending on the covariant derivative used, such a solution will have quite different properties. For the ordinary derivative ∇^o, a solution

[5] See Bishop and Goldberg pg.225-226.

will be of the form

$$\sigma(\alpha(t)) = q(s). \tag{1.20}$$

For the adapted derivative ∇^a, however, we will have,

$$\sigma(\alpha(t)) = \Lambda(t)q(t) \tag{1.21}$$

As discussed above, the parallel translate of the basket from the source time to the destination time using the adapted derivative is a multiple of the basket at the destination time t. This multiple, Λ expresses the "size" of the source time basket relative to the destination-time basket $q(t)$. **Therefore the geometric index is simply the multiple Λ.**

We should note at this point that there is a complementary version of this picture where we consider vector valued functions

$$\tilde{\sigma} : \Xi \longrightarrow V^* \tag{1.22}$$

and an adapted derivative built from ∇^0 and projections

$$\Pi : V^* \longrightarrow V^*. \tag{1.23}$$

which give us the corresponding price index.

1.4 An Extended Example

In order to more clearly understand how the differential geometric approach works, we will actually calculate this index for a simplified economy. The example is set up to show a discrepancy between the standard Paasche and Laspeyres indexes. In the following section we will proceed to show how the differential geometric index resolves this discrepancy.

Imagine an island economy where there are only two goods: coconuts (good 1) and lumber (good 2) (measured in kilotons, priced in dollars). Let us assume that the islanders produce lumber with a saw mill which is gradually rusting into disrepair, but that in partial compensation the islanders become increasingly inventive in their ability to produce coconuts. We further assume that the consumers on the island are developing a mild preference for coconuts over lumber. Mild in this context means that the increased demand is not strong enough to keep the coconut price from decreasing relative to lumber. The toy economy given below will be seen to exhibit the index number problem for the Laspeyres and Paasche quantity indices. While the example has obviously been abstracted, it does exhibit the full complexity of the paradox that we are trying to resolve.

Example 1 *Consider an economy with two goods $g_1,\ g_2$ examined over a one year period $0 \leq t \leq 1$. Let $q : [0,1] \longrightarrow V \cong \mathbb{R}^2$ given by $q(t) = (t+1)e_1 + (-t+2)e_2$ be the output function of the economy at time t. That is $q(t)$ gives the production of goods (g_1, g_2) at time t. Let $p : [0,1] \longrightarrow V^* \cong \mathbb{R}^2$ given by $p(t) = (-10t + 100)e_1^* + (9t + 92)e_2^*$ be the price function of the economy. Likewise $p(t)$ gives the prices per unit quantity of the goods (g_1, g_2) at time t.*

Let us first parallel translate the production $q(0)$ at time $t = 0$ to time $t = 1$. That is we want to find a function $\Lambda : [0,1] \longrightarrow \mathbb{R}$ determining $\sigma : [0,1] \longrightarrow V$ with

1. $\sigma(t) = \Lambda(t)q(t)$
2. $\Lambda(0) = 1$
3. $\nabla^a_{\dot\alpha}(\sigma(t)) = \Pi_{[q(t)]}(\nabla^0_{\dot\alpha}\Pi_{[q(t)]}\sigma) = 0$

Inside of

$$[0,1] \times V \tag{1.24}$$

(referred to as a trivial vector bundle in Appendix A) the section

$$q : [0,1] \longrightarrow V \cong \mathbb{R}^2 \tag{1.25}$$

given by

$$q(t) = ((t+1)e_1 + (-t+2)e_2) \tag{1.26}$$

cuts out a changing 1-dimensional vector sub-space from V (a non-trivial sub-bundle with 1-dimensional fiber in the geometric language of Appendix A). Conversely, the price history

$$p(t) = ((-10t + 100)e_1^* + (9t + 92)e_2^*) \tag{1.27}$$

determines a complementary non-constant vector sub-space of ‘barters’ (also with 1-dimensional fiber). What we want now is to find

$$\Lambda(t) \text{ such that } \nabla^a_{\dot\alpha}\Lambda(t)q(t) = 0. \tag{1.28}$$

Thus we calculate

$$\nabla^a_{\dot{\alpha}}\Lambda(t)q(t) = \Pi_{[q(t)]}(\frac{d(\Lambda(t)q(t))}{dt}) \tag{1.29}$$

$$= \Pi_{[q(t)]}(\frac{d\Lambda}{dt}(t)q(t) + \Lambda(t)(e_1 - e_2)) \tag{1.30}$$

$$= \Pi_{[q(t)]}(\frac{d\Lambda}{dt}(t)((t+1)e_1 + (-t+2)e_2) + \Lambda(t)(e_1 - e_2)) \tag{1.31}$$

$$= \Pi_{[q(t)]}(\frac{d\Lambda}{dt}(t)(t+1) + \Lambda(t))e_1 + (\frac{d\Lambda}{dt}(t)(-t+2) - \Lambda(t))e_2) \tag{1.32}$$

Recalling that

$$\Pi_{[q(t)]}(v) = (\frac{p(t)\cdot v}{p(t)\cdot q(t)})q(t) \tag{1.33}$$

we have

$$\nabla^a_{\dot{\alpha}}\Lambda(t)q(t) = \frac{(\frac{d\Lambda}{dt}(t)(t+1) + \Lambda(t))(-10t+100) + (\frac{d\Lambda}{dt}(t)(-t+2) - \Lambda(t))(9t+92)}{(t+1)(-10t+100) + (-t+2)(9t+92)}q(t) \tag{1.34}$$

$$= \frac{(-19t^2+16t+284)\frac{d\Lambda}{dt}(t) + (-19t+8)\Lambda(t)}{(t+1)(-10t+100) + (-t+2)(9t+92)}q(t) \tag{1.35}$$

so we need to solve the equation

$$(-19t^2+16t+284)\frac{d\Lambda}{dt}(t) + (-19t+8)\Lambda(t) = 0 \tag{1.36}$$

with initial condition $\Lambda(0) = 1$ (which is an ordinary linear differential equation). We solve it as follows. Since

$$\frac{d\Lambda}{dt}(t) = \frac{-(-19t+8)}{(-19t^2+16t+284)}\Lambda(t) \tag{1.37}$$

we should expect that $\Lambda(t)$ will have the form $\Lambda(t) = e^{g(t)}$ with

$$\frac{dg}{dt}(t) = \frac{-(-19t+8)}{(-19t^2+16t+284)}. \tag{1.38}$$

We next note that

$$\frac{d(-19t^2+16t+284)}{dt} = -38t+16 = 2\cdot(-19t+8) \tag{1.39}$$

so we are looking for a function $g(t)$ with

$$\frac{dg}{dt}(t) = -\frac{1}{2}\frac{(-19t^2 + 16t + 284)'}{(-19t^2 + 16t + 284)} \tag{1.40}$$

which indicates that $g(t)$ is of the form $g(t) = -\frac{1}{2} \cdot ln(-19t^2 + 16t + 284) + c$ where c is a constant to be determined from initial conditions. Thus we have

$$\Lambda(t) = e^{-\frac{1}{2} \cdot ln(-19t^2+16t+284)+c} = \frac{c}{\sqrt{-19t^2 + 16t + 284}} \tag{1.41}$$

which when when supplemented by our requirement that $\Lambda(0) = 1$ fixes the value $c = ln(\sqrt{284})$ giving our final solution

$$\Lambda(t) = \frac{\sqrt{284}}{\sqrt{(-19t^2 + 16t + 284)}}. \tag{1.42}$$

Conversely, if we had wanted to parallel translate *backwards* from time $t = 1$ to $t = 0$ we would have sought $\tilde{\Lambda} : [0, 1] \longrightarrow \mathbb{R}$ determining $\tilde{\sigma} : [0, 1] \longrightarrow V$ with

1. $\tilde{\sigma}(t) = \tilde{\Lambda}(t)q(t)$
2. $\tilde{\Lambda}(1) = 1$
3. $\nabla^a_{\dot{\alpha}}(\tilde{\sigma}(t)) = \Pi_{[q(t)]}\nabla^o_{\dot{\alpha}}\Pi_{[q(t)]}\tilde{\sigma} = 0$

It is easy to see that this function is given by

$$\tilde{\Lambda}(t) = \frac{\sqrt{281}}{\sqrt{(-19t^2 + 16t + 284)}}. \tag{1.43}$$

1.5 Resolution Of the Index Number Problem

As noted previously, the two most commonly used indexes, the Paasche and Laspeyres, have an inherent discrepancy. Let us start by observing the effects of updating our notion of constancy for these two indexes. We will later extend these results to all bilateral indexes.

The Paasche and Laspeyres quantity indexes have the formulas

$$Q_L^{old} = \frac{p(0) \cdot q(1)}{p(0) \cdot q(0)} \qquad Q_P^{old} = \frac{p(1) \cdot q(1)}{p(1) \cdot q(0)} \tag{1.44}$$

In both of these indices, three of the four factors which appear are from a common time period (the period used to make the comparison) and the odd factor is imported unchanged across the time separating $t = 0$ from $t = 1$. It has been held <u>constant</u> in the usual sense of the term. This leads us to suspect that the problem is arising from the use of a derivative, un-adapted to our situation. We thus rewrite (1.44) in the form

$$Q_L^{new} = \frac{p(0) \cdot \tau_1(0)}{p(0) \cdot q(0)} \qquad Q_P^{new} = \frac{p(1) \cdot q(1)}{p(1) \cdot \tau_0(1)} \tag{1.45}$$

where Q^{new} refers to an index whose previously implicit choice of derivative (used in parallel translation) has been made explicit. $\tau_1(0)$ refers to the parallel translate of the basket $q(0)$ from time $t = 0$ to time $t = 1$, with an as yet unspecified choice of derivative being used for parallel translation; likewise, $\tau_0(1)$ refers to the parallel translate backwards of the basket $q(1)$ from time $t = 1$ to time $t = 0$.

Note that nothing has been lost in this reformulation as

$$Q_L^{old} = Q_L^{new} \qquad Q_P^{old} = Q_P^{new} \tag{1.46}$$

when parallel translation is performed using the ordinary derivative ∇^o (i.e. $\tau_1(0) = \tau_1^o(0)$ and $\tau_0(1) = \tau_0^o(1)$.) However, when we use the covariant derivative ∇^a for parallel translation to import the data we find that the adapted Paasche and Laspeyres (and hence the Fisher Ideal index) are in fact *identically equal*

$$Q_L^{new} = \frac{p(0) \cdot \tau_1^a(0)}{p(0) \cdot q(0)} = \frac{p(1) \cdot q(1)}{p(1) \cdot \tau_0^a(1)} = Q_P^{new} \tag{1.47}$$

thus resolving the so called index number problem.

Let us verify this for our example from the previous section:

In our example we see that the Laspeyres quantity index

$$\frac{p(0)q(1)}{p(0)q(0)} = \frac{292}{284} \tag{1.48}$$

indicates positive growth, whereas the Paasche quantity index

$$\frac{p(1)q(1)}{p(1)q(0)} = \frac{281}{292} \tag{1.49}$$

indicates negative growth.

Let us begin with the Paasche quantity index. We first observe that since the quantity $q(0)$ in the expression $p(t)q(0)$ is being evaluated at $p(t)$ prices, it is really the parallel translate of $q(0)$ from $\alpha(0)$ to $\alpha(t)$ using the *ordinary* derivative. That is

$$\frac{p(1)q(1)}{p(1)q(0)} = \frac{p(1)q(1)}{p(1)\tau_0^o(1)} \tag{1.50}$$

where $\tau_0^o(1)$ denotes the parallel translation from time $t = 0$ to time $t = 1$ using the *ordinary* derivative.

Likewise for the Laspeyres quantity index we have

$$\frac{p(0)q(1)}{p(0)q(0)} = \frac{p(0)\tau_1^o(0)}{p(0)q(0)} \tag{1.51}$$

Let us first parallel translate the production $q(0)$ at time $t = 0$ to time $t = 1$. As in the previous example we want to find a function $\Lambda : [0, 1] \longrightarrow \mathbb{R}$ determining $\sigma : [0, 1] \longrightarrow V$ with

1. $\sigma(t) = \Lambda(t)q(t)$
2. $\Lambda(0) = 1$
3. $\nabla^a_{\dot\alpha}(\sigma(t)) = \Pi_{[q(t)]}(\nabla^0_{\dot\alpha}\Pi_{[q(t)]}\sigma) = 0$

From our calculations in the last section we see that this function is given by

$$\Lambda(t) = \frac{\sqrt{284}}{\sqrt{(-19t^2 + 16t + 284)}}. \tag{1.52}$$

enabling us to calculate the corrected Paasche index:

$$\frac{\mathbf{p(1)q(1)}}{\mathbf{p(1)}\tau_{\mathbf{0}}^{\mathbf{a}}\mathbf{(1)}} = \frac{\sqrt{\mathbf{281}}}{\sqrt{\mathbf{284}}} \tag{1.53}$$

Similarly, to calculate the Laspeyres index we need to parallel translate back from $t = 1$ to time $t = 0$, which we do by finding the function $\tilde{\Lambda} : [0,1] \longrightarrow \mathbb{R}$ determining $\tilde{\sigma} : [0,1] \longrightarrow V$ with

1. $\tilde{\sigma}(t) = \tilde{\Lambda}(t)q(t)$
2. $\tilde{\Lambda}(0) = 1$
3. $\nabla^a_{\dot{\alpha}}(\tilde{\sigma}(t)) = \Pi_{[q(t)]}(\nabla^0_{\dot{\alpha}}\Pi_{[q(t)]}\tilde{\sigma}) = 0$

which we see from (equation 43) gives us

$$\tilde{\Lambda}(t) = \frac{\sqrt{281}}{\sqrt{(-19t^2 + 16t + 284)}}. \tag{1.54}$$

enabling us to calculate the corrected Laspeyres index:

$$\frac{\mathbf{p(0)}\tau_{\mathbf{1}}^{\mathbf{a}}\mathbf{(0)}}{\mathbf{p(0)q(0)}} = \frac{\sqrt{\mathbf{281}}}{\sqrt{\mathbf{284}}} \tag{1.55}$$

We thus see that:

$$\frac{\mathbf{p(0)}\tau_{\mathbf{1}}^{\mathbf{a}}\mathbf{(0)}}{\mathbf{p(0)q(0)}} = \frac{\mathbf{p(1)q(1)}}{\mathbf{p(1)}\tau_{\mathbf{0}}^{\mathbf{a}}\mathbf{(1)}} = \frac{\sqrt{\mathbf{281}}}{\sqrt{\mathbf{284}}} \cong \mathbf{.9947} \tag{1.56}$$

showing unambiguously that the economy actually *shrank*.

The Fisher 'ideal' Index, is defined as the geometric mean of the Paasche and Laspeyres indexes:

$$\sqrt{\frac{p(t)q(0)}{p(0)q(0)} \cdot \frac{p(t)q(t)}{p(0)q(t)}} \qquad \sqrt{\frac{p(0)q(t)}{p(0)q(0)} \cdot \frac{p(t)q(t)}{p(t)q(0)}} \tag{1.57}$$

As the Laspeyres and Paasche are actually equal for all t, the agreement with the Fisher index is immediate.

This result is not restricted to the Paasche, Laspeyres and Fisher indexes. What follows is a formal proof of the equivalence of *all* bilateral index numbers when the correct adapted derivative is used.

1.5.1 Technical Proof of Equivalence of Bilateral Index Numbers

We take the opportunity below to set notation before going into the proof of the equivalence of all bilateral adapted indices. Readers should consult appendix A for terminology. We follow the proof of equivalence with explicit formulas for the major indices and their differential geometric adaptations.

Preliminary Definitions

We begin by motivating the concept of bundle and sub-bundle. Intuitively the idea is as follows.

As we have seen, the evolution of the economy we wish to investigate is given by a path α in the space of all possible economies Ξ. It is convenient in such situations to introduce a second copy of the space of possible prices and quantities for use in tracking the 'best analogues' of the initial and final states of the economy during the period under study. Thus we will work with the space $\Xi \times V \times V^*$. The first factor Ξ will be referred to as the "base space". For every point $x \in \Xi$ there exists a (different) copy of the vector space $V \times V^*$; each copy $(V \times V^*)_x = x \times V \times V^*$ is referred to as the "fiber" above the point x. The totality of these spaces is referred to as a "vector bundle". The base space Ξ will function as home to the history α. The space of fibers $V \times V^*$ will be used to keep track of the parallel translates of the price and quantity vectors.

Let us now give a brief intuitive explanation of the following proof. We first observe that all of the *algebraic formulas* used by the common indexes satisfy the so called proportionality test; that is, if the original basket of goods $q(0)$ is a multiple $\frac{1}{\lambda}$ of the current basket $q(1) = \lambda \cdot q(0)$, then the quantity index is guaranteed to be λ.

As discussed in Section 2, all of the common bilateral index numbers tacitly require a notion of parallel translation. To take but one example, the Laspeyres quantity index purports to transport today's goods back to yesterday's prices. Further, all such indices make an implicit choice of the ordinary derivative for the purpose of this translation. By contrast, we will insist that the choice of derivative be made *explicit*. In addition, we will require that the data be expressed in a form which distinguishes the usual price and quantity vectors (at the source and destination times) from their parallel translates.

When one chooses the ordinary derivative, the parallel translation of the source-time vector $q(s)$ to the destination-time is merely the original vector $q(s)$. However, when the adapted derivative is selected, we are ensured that the parallel translate of the vector $q(s)$ will be a multiple of the vector $q(d)$ at the destination time. Therefore, when this data is input into any index number with an algebraic formula passing the proportionality test, it is guaranteed to give us the same answer: namely λ.

Definition 5 *We define sub-bundles of $\Xi \times V \times V^*$ by*

$$\Sigma_0 = \Xi \times V \times V^* \tag{1.58}$$

$$\Sigma_1 = \{(x, 0, v^*) \in \Sigma_0 \ s.t. \ v^* \in [p_x] \subset V^*\} \tag{1.59}$$

$$\Sigma_2 = \{(x, 0, v^*) \in \Sigma_0 \ s.t. \ \ v^* \in \beta_{q_x} \subset V^*\} \tag{1.60}$$

$$\Sigma_3 = \{(x, v, 0) \in \Sigma_0 \ s.t. \ \ v \in [q_x] \subset V\} \tag{1.61}$$

$$\Sigma_4 = \{(x, v, 0) \in \Sigma_0 \ s.t. \ \ v \in \beta_{p_x} \subset V\} \tag{1.62}$$

Definition 6

$$P(\Xi) = \{\alpha | \alpha : [0, 1] \longrightarrow \Xi \ (\textit{with } \alpha \textit{ piecewise differentiable})\} \tag{1.63}$$

Definition 7 *Given an element $\sigma(s)$ in the fiber $E_{\alpha(s)}$ of a vector bundle E over Ξ with connection ∇, we will let $\tau_\alpha^\nabla(\sigma(s), d)$ denote the parallel translate of $\sigma(s)$ at $\alpha(s)$ to $\alpha(d)$ along α relative to ∇.*

Definition 8 *Let ∇^u be a(n) (unadapted) connection on $\Xi \times V$. Then we define the covariant derivative adapted from ∇^u to be ∇^a by*

$$\nabla^a \sigma = \Pi_{[q]} \nabla^u (\Pi_{[q]} \sigma) + \Pi_{\beta_p} \nabla^u (\Pi_{\beta_p} \sigma). \tag{1.64}$$

Conversely, if ∇^u is a connection on $\Xi \times V^*$, then the covariant derivative adapted from ∇^u is defined to be ∇^a given by

$$\nabla^a \sigma = \Pi_{[p]} \nabla^u (\Pi_{[p]} \sigma) + \Pi_{\beta_{[q]}} \nabla^u (\Pi_{\beta_{[q]}} \sigma). \tag{1.65}$$

Likewise, if ∇^u is a connection on Σ_0, then the covariant derivative adapted from ∇^u is defined to be ∇^a given by

$$\nabla^a \sigma = \Pi_{[q]} \nabla^u (\Pi_{[q]} \sigma) + \Pi_{\beta_p} \nabla^u (\Pi_{\beta_p} \sigma) + \Pi_{[p]} \nabla^u (\Pi_{[p]} \sigma) + \Pi_{\beta_q} \nabla^u (\Pi_{\beta_q} \sigma). \tag{1.66}$$

We will say that a covariant derivative ∇ is adapted if $\nabla^a = \nabla$.

Definition 9 *Let us define* $\Theta := \bigoplus_{i=1}^{4} V \bigoplus_{i=1}^{4} V^*$ *and* $\kappa \subset \Theta$ *by the rule*

$$\kappa = \{\theta \in \Theta | \ s.t. \ \Theta = (A, \ y \cdot A, \ B, \ \frac{1}{y} \cdot B, \ C, \ z \cdot C, \ D, \ \frac{1}{z} \cdot D \ )\} \quad (1.67)$$

with $A, B \in V$, $C, D \in V^*$ *and* $y, z \in \mathbb{R}$

Definition 10 *Let* $\mathcal{A}$ *be the space of all smooth connections on* Σ_0.

Definition 11 *Let*

$$T : P(\Xi) \times \mathcal{A} \longrightarrow \Theta \quad (1.68)$$

be given by

$$T(\alpha, \nabla) = \quad (1.69)$$

$$(\ p(0), \ \tau_\alpha^\nabla(p(1), 0), \ p(1), \ \tau_\alpha^\nabla(p(0), 1),$$
$$q(0), \ \tau_\alpha^\nabla(q(1), 0), \ q(1), \ \tau_\alpha^\nabla(q(0), 1) \ )$$

Definition 12 *We will say that a price (or quantity) index formula*

$$F : \Theta \longrightarrow \mathbb{R} \quad (1.70)$$

satisfies the proportionality hypothesis if when restricted to κ *we have*

$$F(\ (A, \ y \cdot A, \ B, \ \frac{1}{y} \cdot B, \quad (1.71)$$

$$C, \ z \cdot C, \ D, \ \frac{1}{z} \cdot D \ ) \ ) = y \quad (or \ z).$$

for all non-degenerate choices of A,B,C and D with $y, z \in \mathbb{R}$.

In actuality, we do not need the function F to be defined on all of Θ but rather we require it to be well defined on a dense subset of non-degenerate data (which is always the case in practice). Also, for simplicity we need only consider the case of quantity indexes as the case of price indexes goes through *mutatis mutandis.*

Proof of Equivalence

Theorem 2 *Let $I_{1,2} : P(\Xi) \times \mathcal{A} \longrightarrow \mathbb{R}$ be a pair of quantity index formulas where $I_{1,2} = F_{1,2} \circ T$,*

$$T : P(\Xi) \times \mathcal{A} \longrightarrow \Theta \tag{1.72}$$

is as before and

$$F_{1,2} : \Theta \longrightarrow \mathbb{R} \tag{1.73}$$

are two formulae which satisfy the proportionality test. Then if $\nabla_1^a = \nabla_2^a = \nabla^a$, we have $I_1(\cdot, \nabla^a) = I_2(\cdot, \nabla^a)$ and $I_i(\alpha(t), \nabla^a) = \frac{1}{I_i(\alpha(1-t), \nabla^a)}$ (time reversal).

Proof: The proof proceeds in two parts. First we we will show that any adapted connection on Σ_0 is "reducible" in that it is the sum of 4 connections on the Σ_i sub-bundles of Σ_0. The implication of reducibility is that if we parallel translate a vector $v(\alpha(s))$ belonging to a sub-bundle Σ_i using a reducible connection, then the parallel translate at the destination time d will remain in the same sub-bundle as the original vector at source time s. As both the sub-bundle of pricing systems Σ_1, and the sub-bundle of baskets Σ_3 have 1-dimensional fibers, we are guaranteed to have our parallel translates of $p(s)$ and $q(s)$ expressible as non-zero multiples of the price and quantity data at the destination time d.

The second part of the proof will be to show that for any such reducible connection ∇^a, the 8 pieces of data comprising $T(\alpha, \nabla^a)$ are guaranteed to fit the pattern found in the proportionality hypothesis. Since all index numbers can be represented by formulas satisfying the proportionality hypothesis above, the conclusion follows.

To begin, we must demonstrate that if ∇ is an adapted connection and σ is a section of one of the four sub-bundles of $\Sigma_i \subset \Sigma_0$, then $\nabla_X^a \sigma$ is again in Σ_i for any $X \in T\Xi$. For concreteness, let us look at $\sigma \subset \Sigma_3$. Now by hypothesis we have

$$\nabla_X \sigma = \nabla_X^a \sigma = \Pi_{[q]} \nabla_X (\Pi_{[q]} \sigma) = \Pi_{[q]} \nabla_X \sigma \tag{1.74}$$

proving our claim. The claim that given any connection ∇ on Σ_0, $\Pi_{\Sigma_i} \nabla (\Pi_{\Sigma_i} \cdot)$ is a connection on Σ_i can be verified by checking the axioms for a connection in Appendix A, and is straightforward.

Because the 4 summands of an adapted connection ∇^a are individually connections on the sub-bundles Σ_i, we are assured that the parallel translation of a vector in one of the summands Σ_i will remain in that summand. In particular, for the one dimensional $[q]$ sub-bundle we have

$$\tau_{\alpha}^{\nabla^a}(q(s), d) = \lambda_{q(d)} \cdot q(d) \tag{1.75}$$

for $\lambda_{q(d)} \in \mathbb{R}$. Now the equation for a section along α to be a parallel translate of our source data

$$\nabla^a_{\dot{\alpha}} \sigma(t) = 0 \tag{1.76}$$

is a first order linear ordinary differential equation with initial data $q(s)$. This implies that there exists a unique solution to the initial value problem which depends linearly on the initial conditions. Thus if σ satisfies (1.76) with

$$\sigma(\alpha(s)) = q(s) \tag{1.77}$$

then by (1.75) we have

$$\sigma(\alpha(d)) = \lambda_{q(d)} q(d). \tag{1.78}$$

Conversely, the section

$$\tilde{\sigma} = \frac{1}{\lambda_{q(d)}} \sigma \tag{1.79}$$

will also satisfy (1.76) but with the boundary condition

$$\tilde{\sigma}(d) = \frac{1}{\lambda_{q(d)}} \cdot \lambda_{q(d)} \cdot q(d) = q(d). \tag{1.80}$$

This proves:

Proposition 3 *For an adapted derivative ∇^a on Σ_0 there exist numbers $y, z \in \mathbb{R}$ such that*

$$T(\alpha, \nabla^a) = \tag{1.81}$$

$$(p(0),\ y \cdot p(0),\ p(1),\ \frac{1}{y} \cdot p(1),$$

$$q(0),\ z \cdot q(0),\ q(1),\ \frac{1}{z} \cdot q(1)\ ).$$

This means that in the case of an *adapted* covariant derivative ∇^a, we are guaranteed that $T(\alpha, \nabla^a) \in \kappa$. Thus by the hypothesis that our index formulas satisfy the proportionality test we can see that

$$I_1(\alpha(t), \tilde{A}) = I_2(\alpha(t), \tilde{A}) = \frac{1}{I_1(\alpha(1-t), \tilde{A})} = \frac{1}{I_2(\alpha(1-t), \tilde{A})} \tag{1.82}$$

for all piecewise smooth α. **QED**.

Adapted Index Numbers

Let us now see how this adjustment in the notion of constancy corrects some common bilateral index numbers. In the following price index number formulas the σ's refer to the parallel translates of prices:

<u>Laspeyres</u>

Original:

$$P_L(p^0, p^1, q^0, q^1) \equiv \frac{p^1 \cdot q^0}{p^0 \cdot q^0} \tag{1.83}$$

Adapted:

$$P_L(p^0, p^1, q^0, q^1) \equiv \frac{\sigma_1(0) \cdot q^0}{p^0 \cdot q^0} \tag{1.84}$$

<u>Paasche</u>

Original:

$$P_P(p^0, p^1, q^0, q^1) \equiv \frac{p^1 \cdot q^1}{p^0 \cdot q^1} \tag{1.85}$$

Adapted:

$$P_P(p^0, p^1, q^0, q^1) \equiv \frac{p^1 \cdot q^1}{\sigma_0(1) \cdot q^1} \tag{1.86}$$

<u>Fisher</u>

Original:

$$P_F(p^0, p^1, q^0, q^1) \equiv \sqrt{\frac{p^1 \cdot q^0}{p^0 \cdot q^0} \cdot \frac{p^1 \cdot q^1}{p^0 \cdot q^1}} \tag{1.87}$$

Adapted:

$$P_F(p^0, p^1, q^0, q^1) \equiv \sqrt{\frac{\sigma_1(0) \cdot q^0}{p^0 \cdot q^0} \cdot \frac{p^1 \cdot q^1}{\sigma_0(1) \cdot q^1}} \tag{1.88}$$

<u>Tornqvist</u>

Original:

$$P_T(p^0, p^1, q^0, q^1) \equiv \prod_{n=1}^{N} \sqrt{\left(\frac{p_n^1}{p_n^0}\right)^{(p_n^0 q_n^0 / p^0 \cdot q^0)} \cdot \left(\frac{p_n^1}{p_n^0}\right)^{(p_n^1 q_n^1 / p^1 \cdot q^1)}} \tag{1.89}$$

Adapted:

$$P_T(p^0, p^1, q^0, q^1) \equiv \prod_{n=1}^{N} \sqrt{\left(\frac{\sigma_1(0)}{p_n^0}\right)^{(p_n^0 q_n^0 / p^0 0 \cdot q^0)} \cdot \left(\frac{p_n^1}{\sigma_0(1)}\right)^{(p_n^1 q_n^1 / p^1 \cdot q^1)}} \tag{1.90}$$

<u>Walsh</u>

Original:

$$P_W(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{N} (q_i^0 q_i^1)^{1/2} p_i^1}{\sum_{j=1}^{N} (q_j^0 q_j^1)^{1/2} p_j^0} \tag{1.91}$$

Adapted:

$$P_W(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{N} (q_i^0 q_i^1)^{1/2} p_i^1}{\sum_{j=1}^{N} (q_j^0 q_j^1)^{1/2} \sigma_0(1)_j} = \frac{\sum_{i=1}^{N} (q_i^0 q_i^1)^{1/2} \sigma_1(0)_i}{\sum_{j=1}^{N} (q_j^0 q_j^1)^{1/2} p_j^0} \tag{1.92}$$

<u>Jevons</u>

Original:

$$P_J(p^0, p^1) \equiv \prod_{i=1}^{N} (\frac{p^1}{p^0})^{1/N} \tag{1.93}$$

Adapted:

$$P_J(p^0, p^1) \equiv \prod_{i=1}^{N} (\frac{p^1}{\sigma_0(1)})^{1/N} = \prod_{i=1}^{N} (\frac{\sigma_1(0)}{p^0})^{1/N} \tag{1.94}$$

1.6 The Divisia Index

What is this new index? In this section we will prove that this index is the same as the Divisia index. Divisia developed this index in the 1920's in a somewhat ad hoc fashion. In fact, Bennet [1920] writing a few years before Divisia using essentially the same argument, came up with almost the same index which differs from Divisia's by its absence of Log functions. The following is a description of the traditional development of the Divisia index[6]:

1.6.1 Traditional Derivation of the Divisia

We seek a pair of functions

$$Q : \mathbb{R} \longrightarrow \mathbb{R} \qquad P : \mathbb{R} \longrightarrow \mathbb{R} \tag{1.95}$$

such that

$$p(t) \cdot q(t) = \sum_{i=1}^{n} p_i(t) q_i(t) = P(t)Q(t) \quad \forall t. \tag{1.96}$$

Of course, the only constraint this implies is

$$P(t) = \frac{V(t)}{Q(t)} \tag{1.97}$$

with Q(t) arbitrary. Divisia's idea was to take the derivatives of the logarithms

$$\frac{dln(p(t) \cdot q(t))}{dt} = \frac{dln(P(t)Q(t))}{dt} \tag{1.98}$$

expand using the chain and product rules

$$\frac{p'(t) \cdot q(t)}{v(t)} + \frac{p(t) \cdot q'(t)}{v(t)} = \frac{P'(t)}{P(t)} + \frac{Q'(t)}{Q(t)} \tag{1.99}$$

and set the summands

$$\frac{p'(t) \cdot q(t)}{v(t)} = \frac{P'(t)}{P(t)} \tag{1.100}$$

$$\frac{p(t) \cdot q'(t)}{v(t)} = \frac{Q'(t)}{Q(t)} \tag{1.101}$$

[6] See also Diewert and Nakamura , pp. 42

to be equal. We can then see that this *does* specify unique function $P(t), Q(t)$ by the formulas

$$P(t) = exp(\int_0^1 \sum_{i=1}^{n} \frac{p_i(t)q_i(t)}{\sum_{j=1}^{n} p_j(t)q_j(t)} \frac{p_i'(t)}{p_i(t)}) \tag{1.102}$$

$$Q(t) = exp(\int_0^1 \sum_{i=1}^{n} \frac{p_i(t)q_i(t)}{\sum_{j=1}^{n} p_j(t)q_j(t)} \frac{q_i'(t)}{q_i(t)}). \tag{1.103}$$

Theorem 4 *The differential geometric index is equal to the Bennett-Divisia index .*

Proof of Theorem: The Bennett-Divisia Price index can be expressed concisely using the 'differential form' notation for line integrals as

$$= e^{\int_\alpha \sum_{i=1}^{n} w_i(\alpha) d\ log(p_i)} \tag{1.104}$$

where $\frac{p_i \cdot q_i}{p \cdot q} = w_i$ is the budget share of the i^{th} commodity [7] .

The differential geometric price index will be given in the general situation by $\frac{p(1)}{\sigma_0^p(1)} = \frac{1}{\Lambda(t)}$ where Λ is determined by

$$\nabla^a_{\dot\alpha}\sigma = 0 \qquad \sigma(t) = \Lambda(t)p(t) \qquad \sigma(0) = p(0). \tag{1.105}$$

Now if

$$0 = \nabla^a_{\dot\alpha}\sigma = \Pi_{[p]}(\nabla^o_{\dot\alpha}\Pi_{[p]}(\sigma)) + \Pi_{[\beta]}(\nabla^o_{\dot\alpha}\Pi_{[\beta]}(\sigma)) \tag{1.106}$$

$$= \Pi_{[p]}(\nabla^o_{\dot\alpha}\Pi_{[p]}(\sigma)) = \Pi_p(\frac{d\Lambda}{dt}(t) \cdot p(t) + \Lambda(t)\frac{dp}{dt}(t)) \tag{1.107}$$

$$= \frac{((\frac{d\Lambda}{dt}(t) \cdot p(t) + \Lambda(t)\frac{dp}{dt}(t)) \cdot q(t))}{p(t) \cdot q(t)} p(t) \tag{1.108}$$

$$= (\frac{d\Lambda}{dt}(t)p(t) + \Lambda(t)\frac{dp}{dt}(t)) \cdot q(t) \tag{1.109}$$

so

$$\frac{d\Lambda}{dt}(t) = -(\frac{\frac{dp}{dt}(t) \cdot q(t)}{p(t) \cdot q(t)})\Lambda(t) \tag{1.110}$$

[7] Deaton and Muellbauer pp. 174-175

hence

$$\Lambda(t) = exp(-\int_0^t (\frac{\frac{dp}{dr}(r)\cdot q(r)}{p(r)\cdot q(r)})dr) \tag{1.111}$$

$$= exp(-\int_0^t \sum_{i=1}^{n} (\frac{\frac{dp_i}{dr}(r)\cdot q_i(r)}{p(r)\cdot q(r)})dr) \tag{1.112}$$

$$= exp(-\int_0^t \sum_{i=1}^{n} (\frac{p_i(r)\cdot q_i(r)}{p(r)\cdot q(r)})\frac{1}{p_i(r)}\frac{dp_i(r)}{dr}dr) \tag{1.113}$$

$$= exp(-\int_\alpha \sum_{i=1}^{n} (\frac{p_i\cdot q_i}{p\cdot q})\frac{dp_i}{p_i}) \tag{1.114}$$

Thus

$$\text{Diff. Geom. Index} = \frac{1}{\Lambda(t)} = e^{\sum_{i=1}^{n}\int_\alpha w_i(\alpha)d\ log(p_i)} = \text{Divisia Index} \tag{1.115}$$

QED.

1.7 Results

As we have shown in this paper, much of index number theory can be viewed as a special topic in differential geometry. When the traditional index number problem is formulated in the context of differential geometry, a new feature emerges. Unlike the ordinary calculus, differential geometry points out that there is not a unique derivative as is often assumed. There is however a unique choice of derivative adapted from the familiar one to the problems of index theory. Once that choice is made successfully, all index numbers currently in favor become equivalent and are equal to the Divisia index. The shift in emphasis from the algebraic formulas of various index numbers, and towards the choice of derivative thereby enables us to resolve the so called Index Number Problem. Thus by using these mathematical techniques, we have a natural answer to the question "What is the correct index number?" Casting the question in the right framework not only allows us to unify and recover previous results, it also enables us to understand better the nature of the question.

One of the primary debates one sees in the current literature on index number theory centers around whether or not to chain bilateral indexes. With the sophistication of the differential geometric framework, we can cast this question in an entirely new light. As we have seen above, *all* index numbers make a choice of derivative. The choice has till now been made implicitly. When we view the issue differential geometrically we are forced to make the choice explicit. What then becomes clear is that there are two different derivatives being used. The ordinary derivative, which has *no* justification whatsoever economically, and the adapted derivative, whose economic justification has been presented in this paper. Once one is forced to justify the choice of derivative, one must realise the arbitrary nature of all bilateral indexes using the ordinary derivative. The Divisia is the only index which uses an economically justified derivative and is therefore superior to all bilateral indexes using the ordinary derivative. *Any* further approximation to the Divisia is therefore an improvement. Chaining is simply one such approximation. The issue should no longer be whether or not to chain bilateral indexes, but rather to find *the best* way of approximating the Divisia.

The one characteristic of the Divisia index most troubling to economists is that it is given by a path dependent line integral. What we would claim is that even with this path dependence (or perhaps because of it), the Divisia is the fair and correct way to calculate cost of living adjustments in our real world filled with changing preferences. Let us attempt to make an argument for this in a particular setting in order to show that path dependence can be more of a virtue than a vice. Let us assume that we have two individuals starting and finishing by consuming exactly the same basket of goods, and look at a situation where the path dependent Divisia might give us a different index for each of these individuals. Let us say that the first, as a connoisseur of the finer things in life, starts cultivating his taste in red wine, while this might not be a particularly popular habit. As he is functioning under a budget constraint he thus shifts consumption away from other goods, for instance soda pop. If we assume there is then a faddish move toward the consumption of red wine we should expect that wine prices will start rising. The second consumer, following the fad, starts shifting his consumption only after the prices have already risen. The Divisia would require compensating the first consumer, but not the second, precisely *because* their paths were different. The first started consuming wine while prices were low, and then got hit with the price increase. The second *chose* to shift his consumption only after prices had increased. What we would claim is that it is only the path dependent Divisia index that can give a *fair* cost of living adjustment.

What we see in the mathematics is that a Divisia index C.O.L.A. is a guarantee that that the recipient will be able to maintain his previous life-style with the freedom to change to an equivalently priced life-style at any time. Unlike the Konus C.O.L.A. discussed in Sen [1979], the person will not be compensated if they become increasingly depressed nor penalized for greater happiness. Unlike the Paasche and Laspeyres C.O.L.A., her adjustment is based on a lifestyle which is allowed to change at any time. But there is an important feature of Divisia compensation not found amongst the other indices: the element of risk. Under a Divisia C.O.L.A., your *lifestyle is an investment*. If you change your spending patterns, there is no guarantee that you will be able to get back to your previous level of consumption. Far from being exotic, these risks are well known to all those who must decide how much currency to change when visiting nations with volatile exchange rates. What the differential geometry emphasizes is that path dependence is *preferable* and that the path dependence of the Divisia is no more counter-intuitive than the path dependence underlying the theory of investment.

As previously noted, the differential geometric framework allows us to see that this index is the natural one to emerge from both the mathematics and the economics, and that it enables us to resolve consistency issues raised by the plethora of existing indexes. What it also enables us to do is recover previous results about the Divisia in a unified framework and to extend these results in ways that can practically facilitate our use of the index.

For economists concerned with issues of path dependence, an important result is path independence of the Divisia under unchanging homothetic preferences [see Hulten 1973]. What does the mathematics have to say about this?

Differential geometry allows us to calculate something called a curvature tensor. Recall that with the ordinary notion of a derivative, mixed partials commute. This is no longer true of general covariant derivatives. The curvature tensor measures the failure of the covariant derivative to commute. i.e. $\nabla_X \nabla_Y \neq \nabla_Y \nabla_X$. What differential geometry tells us is that when the curvature tensor is equal to zero on a subspace, our index will be path independent. Homothetic preferences are a special case of a zero curvature tensor.

Another important result is Diewert's 1980 demonstration that when certain approximations are valid, the Paasche, Laspeyres and Tornqvist indexes can be regarded as discrete approximations to the Divisia. In addition, it has been noted that chaining usually improves the approximation and leads to less variation among index numbers. As discussed previously, the differential geometry shows us that chaining is a crude way of approximating our adapted derivative. Given that the adapted derivative resolves the consistency issue, greater agreement under chaining is therefore explained.

Thus we see that not only does our approach recover the above results, but it also gives a unifying framework in which to understand them.

1.7.1 Extensions

Besides recovering previous results in a consistent framework, differential geometry enables us to extend our knowledge about the Divisia in practically applicable ways:

I.) Using the theory of connections, it is possible to do for the space of barters what has been done for the space of baskets. In this paper we have shown how to modernize or antiquate the components of the usual 'fixed basket' to reflect changes in production and/or consumption.

Understanding how barters change is essential to understanding the economy as a whole. For example, if our basket $q(t)$ is taken to be the U.S. GDP, an individual's basket will consist of a non-zero barter vector B added to a scaled basket vector $\lambda \cdot q(t)$ (here B is asserted to be non-zero because individuals never purchase multiples of the GDP). Further, the entire theory of option pricing rests on the fact that barters today will not in general be so tomorrow. It can be shown however that the adapted derivative gives us a natural identification between the spaces of barters at two different instants of time. This in turn results in an n-1 $\times$ n-1 matrix analog for barters of the usual Divisia index for baskets. The significance of this matrix index is not yet clear and and is the subject of work in progress with E. Weinstein.

II.) Differential geometry allows us to discuss path independence in the absence of knowledge of homothetic preferences. Let us assume that we are not given access to preferences, but that we are instead given some region $R \subset V \times V^*$ such that when we restrict the curvature tensor F^{∇^a} to R we get zero. Then if α_1, α_2 are two curves with the same end points and α_1 can be deformed to α_2 within R while keeping the endpoints fixed, then the Divisia of α_1 equals α_2.

III.) When the curvature tensor $F^{\nabla^a} \neq 0$ we can use the 'size' of the curvature tensor to estimate how far we are from path independence. In other words, we can use the size of the curvature tensor to estimate the local contribution to the path dependence of the Divisia index. Thus, this allows us to deviate from pure homothetic preferences and estimate data needs based on the curvature intensity of a region.

These issues will be further explored in following papers.

1.8 Appendix A: Differential Geometric Explanation: Notation, Conventions and Tools from Gauge Theory

1.8.1 Notation for Goods and Services

Let us assume that we are interested in a sector of the economy comprised of some n goods and services. We will assume that each of these goods and services come equipped with a natural price independent unit of measurement. Thus gasoline will be measured in gallons and labor in hours. We will also assume that all prices are linear and are therefore measured in $\frac{\text{dollars}}{\text{unit}}$. Let us order the goods and services together with their system of units and denote the collection by $\{g_i, u_i\}_{i=1}^n$. Given this initial data we will describe the evolution of our sector by the collection of data $\Upsilon = (V^n, \alpha, \nabla^0)$ given as a triple. We will postpone the definition of ∇^0 until section 1.8.3.

Let V^n be the vector space of all quantities of goods and services in the sector of the economy under consideration. That is we let $\{e_i\}_{i=1}^n \subset V^n$ be a basis for V indexed by the (ordered) set of n goods and services under consideration in their natural units of measurement (kilograms, units, etc...). We will interpret negative multiples of these goods as debts or obligations to provide.

We will let V^* be the dual space of linear functionals $\phi : V \longrightarrow \mathbb{R}$. This is a vector space of the same dimension as V. We note that given our basis $\{e_i\}_{i=1}^n \subset V^n$ there is a corresponding basis $\{e_i^*\}_{i=1}^n \subset V^*$ where e_i^* is defined by specifying that $e_i^*(e_j)$ is equal to 1 if $i = j$ and 0 otherwise.

We wish to comment here that the reader may choose to represent the elements of V as 'column' vectors and the elements of V^* as 'row' vectors. In this notation the act of evaluating an element μ^* of V^* on a vector v is obtained by matrix multiplying the $1 \times n$ row vector against the $n \times 1$ column vector to get a real number (i.e. a 1×1 matrix).

If we assume that we are interested in an initial state of the economy at time τ_0 contrasted against a later state of the economy at time τ_1 then we can define $\rho : [\tau_0, \tau_1] \longrightarrow [0, 1]$ by $\rho(x) = (\frac{x-\tau_0}{\tau_1-\tau_0})$ in order to restrict our formalism to the interval $[0, 1]$. From now on we will assume that the time interval of interest will always be $[0, 1]$.

Let $\Xi = \{(v, \phi) \;\; v \in V, \; \phi \in V^* \text{ s.t. } \phi(v) \neq 0\}$ be the collection of pairs of baskets of goods and possible pricing systems.

Then let $\alpha = (m, \pi)$ where $\alpha : [0, 1] \longrightarrow \Xi$ and $\pi : [0, 1] \longrightarrow V^*$ gives the prices of the goods and services under consideration. Conversely $m : [0, 1] \longrightarrow V$ denotes the (possibly) time dependent basket of goods and services with which we are attempting to calibrate the economy.

Given any non-zero vector w in a vector space W, we will denote by $[w]$ the linear subspace spanned by w.

With this defined we argue that if we are given any element $(v, \phi) \in \Xi$ we get a decomposition of our vector space V as follows. In the first place ϕ determines a subspace $\beta_{[\phi]}$ defined by $\beta_{[\phi]} = \{h \in V \; s.t. \; \phi(h) = 0\}$. Our bracket notation is justified by the fact that $\beta_{[\phi]}$ is independent of whether one chooses to work with ϕ or $r \cdot \phi$ $(r \neq 0)$. Now by our above definition of Ξ we are guaranteed that

Proposition 5 *For any point $(v, \phi) \in \Xi$ we get a canonical decomposition $V = [v] \oplus \beta_{[\phi]}$. That is, for any vector $w \in V$ there exist unique vectors*

$w_{[v]} \in [v]$ *and* $w_{\beta_{[\phi]}} \in \beta_{[\phi]}$ *such that* $w = w_{[v]} + w_{\beta_{[\phi]}}$.

Proof: By hypothesis we know that $\phi(v) \neq 0$ which implies in particular that $\phi \neq 0$ and thus that $\phi : V^n \longrightarrow \mathbb{R}$ is a surjective linear map. This means that the dimension of its kernel $\beta_{[\phi]}$ is $n-1$. Further, we know that $v \notin \beta_{[\phi]}$ and our conclusion follows from the dimension count (n-1)+1=n. QED

In the same vein, if we let $\beta_{[v]}$[8]$= \{\kappa \in V^* \; s.t. \; \kappa(v) = 0\}$ then we have

Proposition 6 *For any point* $(v, \phi) \in \Xi$ *we get a canonical decomposition* $V^* = [\phi] \oplus \beta_{[v]}$.

Proof: The argument above goes over word for word in this case with V^* and V interchanged. QED

The importance of the decompositions is that they determine projection maps onto the various subspaces. Given vectors $y \in V$, $\omega \in V^*$ and our point $(v, \phi) \in \Xi$ we get projection maps

$$\Pi_{[v]} : V \longrightarrow [v] \tag{1.116}$$

$$\Pi_{\beta_{[\phi]}} : V \longrightarrow \beta_{[\phi]} \tag{1.117}$$

$$\Pi_{[\phi]} : V^* \longrightarrow [\phi] \tag{1.118}$$

$$\Pi_{\beta_{[v]}} : V^* \longrightarrow \beta_{[v]} \tag{1.119}$$

defined as follows.

$$\Pi_{[v]}(y) = \frac{\phi(y)}{\phi(v)} \cdot v \tag{1.120}$$

$$\Pi_{\beta_{[\phi]}}(y) = y - \Pi_{[v]}(y) \tag{1.121}$$

$$\Pi_{[\phi]}(\omega) = \frac{\omega(v)}{\phi(v)} \cdot \phi \tag{1.122}$$

$$\Pi_{\beta_{[v]}}(\omega) = \omega - \Pi_{[\phi]}(\omega) \tag{1.123}$$

[8]One might ask whether there exists a word corresponding to 'barter' for currencies which view our basket $m(t)$ as worthless. Unfortunately the word exists; the *Oxford English Dictionary* defines Floccinaucinihilipilification as "The action or habit of estimating as worthless" so we rise to the challenge and suggest to the brave reader that the noun describing the elements of $\beta_{[v]}$ is probably 'Floccinaucinihilipilificators.'

1.8.2 Vector Bundles, Sections etc....

We now turn to a discussion of vector bundles. We wish to say at the outset that we will only be discussing a restricted class of vector bundles and will not need the full formalism. Given a vector space W^n of dimension n and a (topological) space X (set, manifold etc...) we refer to $X \times W$ as the **total space** of the **trivial vector bundle** with **base space** X and **fiber** W. For most all of our discussion, X will be an open dense subset of $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ so that $X^{2n} = \mathbb{R}^{2n} - C$ on which we will use the standard coordinates for $\mathbb{R}^{2n}$, agreeing at the outset simply to ignore the points of C. (Note: C is a $2n-1$ dimensional space with singularities).

Let $P_m(W^n)$ denote the space of (oriented) m-dimensional planes in an n-dimensional vector space W^n. That is, the points of the space $P_m(W^n)$ are in one to one correspondence with the oriented m-planes through the origin $0 \in W^n$. Denote by $\mathfrak{P}_p \subset W^n$ the plane corresponding to a point $p \in P_m(W^n)$.

Now if the points $x \in X$ parameterise a family of m-dimensional subspaces $\mathfrak{P}_{Z(x)} \subset W$ by a map $Z : X \longrightarrow P_m(W^n)$ we can define a new non-trivial bundle; we refer to the space $\Sigma = \{(x,u) : (x,u) \in X \times W^n \text{ and } u \in \mathfrak{P}_{Z(x)}\}$ as a (non-trivial) vector bundle with fiber $\mathbb{R}^m$ and base space X (the idea being that a vector bundle is trivial if it is given as a cartesian product). The bundle we have described is a **sub-bundle** of the trivial bundle $X \times W^n$. Note that a trivial bundle is always a sub-bundle of itself (ie. when $n = m$). In this paper m will usually be either $n-1$ or 1; we note that in these cases we have $P_m(W^n) = S^{n-1}$.

The surjective map $\pi : \Sigma \longrightarrow X$ given by $\pi((x,u)) = x$ (defined for a sub-bundle of a trivial bundle) is referred to as the '**projection map** onto the base space'. The collection of points $\mathfrak{P}_{Z(x)} = \pi^{-1}(x)$ are called the **fiber** of the vector bundle at x (or 'above x').

Now a map $\sigma : X \longrightarrow \Sigma$ is called a **section** of Σ if for all $x \in X$ we have $\sigma(x) = (x,u)$ for some $u \in \pi^{-1}(x)$. This is equivalent to requiring that $\pi \circ \sigma(x) = x$. The space of all sections is denoted by $\Gamma(X,\Sigma)$. The space of smooth (that is infinitely differentiable) sections is denoted by $\Gamma^\infty(X,\Sigma)$.

It is important for our paper to know that given a curve $\alpha : [0,1] \longrightarrow X$, then every vector bundle over our space X determines a vector bundle over $[0,1]$ as follows. If E is a vector bundle over X with projection map π, define the **pull-back** bundle $\alpha^*(E)$ to be the set of pairs $\alpha^*(E) = \{(t,u) \text{ s.t. } t \in [0,1],\ \ u \in \pi^{-1}(\alpha(t))\}$. Thus if $E = X \times W^n$ then $\alpha^*(E)$ is simply the trivial bundle $[0,1] \times W^n$. Likewise if we have a sub-bundle of $X \times W^n$ given by a map $Z : X \longrightarrow P_m(W^n)$ then we get a sub-bundle of $[0,1] \times W^n$ given by the map $\hat{Z} : [0,1] \longrightarrow P_m(W^n)$which sends t to the point $Z(\alpha(t)) \in P_m(W^n)$.

1.8.3 Discussion of Tangent vectors, Co-vectors and Covariant Derivatives

How does one define the (intuitive notion) of the space of tangent vectors at a point $x \in X$? If we are lucky enough to be in the situation where there exist natural coordinates on our space X (in our case Ξ) then there exists a straightforward description. This situation is considerably simpler than the general case and it is this case to which we address ourselves.

Let us first review the total differential df of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. In general economics usage the total differential df is defined by $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n)dx_i$ where the symbols dx_i are referred to as differentials and are somewhat abstract. In differential geometry the 'total differential' of a function is referred to as the 'exterior derivative' and the dx_i's are defined as 'co-vectors'. The idea is the following. We identify the intuitive concept of the 'unit tangent vector in the x_i direction' with the partial derivative $\frac{\partial}{\partial x_i}$. The span of the space of these symbols at a point $x \in X$ is denoted by T_xX and is referred to as the **tangent space** to X at the point x.

With this stated we return to the symbols dx_i. The idea here is that these symbols belong to the **dual** space T_x^*X which is referred to as the **cotangent** space at the point x. That is, these symbols eat vectors and spit out numbers in a linear fashion. For example $dx_i(\frac{\partial}{\partial x_j})$ is equal to 1 if i=j and 0 otherwise. With this in mind we see that the total differential is really a convenient bookkeeping device which keeps track of all directional derivatives.

A connection (or equivalently co-variant derivative) is really a kind of 'total differential' in the setting of vector bundles. The first thing to remark is that it is a total differential of 'vector valued functions' which are the sections we have met above.

Definition 13 *A covariant derivative ∇ on a vector bundle Σ over a manifold X is a map $\nabla : \Gamma^\infty(X, \Sigma) \longrightarrow \Gamma^\infty(X, T^*(X) \otimes \Sigma)$ such that we have*

1. *Pointwise linearity in the tangent vector argument:*
   $\nabla_{(fY+gZ)}\sigma = f\nabla_Y\sigma + g\nabla_Z\sigma$

2. *Additivity in the section argument:* $\nabla_Y(f\sigma + g\mu) = \nabla_Y f\sigma + \nabla_Y g\mu$

3. Product Rule: $\nabla_Y f\sigma = df(Y)\sigma + f\nabla_Y \sigma$

where $f, g \in C^\infty(X)$, $Y, Z \in \Gamma^\infty(X, TX)$ *and* $\sigma, \mu \in \Gamma^\infty(X, E)$.

The simplest connection is just the familiar total differential of ordinary real valued functions.

Example 7 *Usual derivative of ordinary vector valued functions of several variables,* ∇^0.

Consider a trivial vector bundle $\mathbb{R}^n \times W^m$ with base space $\mathbb{R}^n$ and $\{e_i\}_{i=1}^m$ a basis for W^m. Then if $s : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times W^n$ is a smooth section given by

$$s(x) = (x, \sum_{i=1}^{m} f_i(x)e_i) \qquad f_i(x) \in C^\infty(\mathbb{R}^n) \tag{1.124}$$

then we define the trivial connection ∇^0 by the formula

$$\nabla^0 s(x) = (x, \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f_i(x)}{\partial x_j} e_i \otimes dx_j). \tag{1.125}$$

If $V = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} \in T_x\mathbb{R}^n$ is a tangent vector at the point x we define the covariant derivative of s in the direction V to be given by

$$\nabla^0_V s(x) = (x, \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f_i(x)}{\partial x_j} a_j e_i) \in W_x. \tag{1.126}$$

If we have a vector bundle Σ over a base space $\mathbb{R}^1$ (or any connected subset thereof) and a (possibly non-trivial) connection ∇ on Σ, then we define the **parallel translations** $T(w_p, \nabla)$ of a vector w_p in the fiber above p to be given by the section $\sigma : \mathbb{R}^1 \longrightarrow \Sigma$ which satisfies

$$\nabla\sigma = 0 \text{ and } \sigma(p) = w_p. \tag{1.127}$$

This is guaranteed to exist and be unique by a theorem in ordinary linear differential equations. We solve an explicit problem of this kind in section 1.4

Chapter 2

Welfare Implications of Divisia Indices[1]

2.1 Introduction

In recent months the U.S. Senate has been actively debating the Consumer Price Index (CPI) and its relation to the "true" cost of living index. The importance of this question for the U.S. economy is undeniable. According to recent Senate hearings on the CPI, 30% of federal outlays and 45% of federal revenues are indexed to the CPI, and a 1% overstatement in the CPI costs the U.S. government $280,000,000,000 over 7 years.

The current CPI issued by the U.S. Bureau of Labor Statistics is based on a Laspeyres index $P_L = \frac{P(1)\cdot Q(0)}{P(0)\cdot Q(0)}$ for a representative consumer whose fixed basket is updated once every 10 years or so. It is widely accepted, however, that this is not a cost of living index. There are several reasons for this, both technical (eg. the way data on housing is handled by the BLS) and theoretical. We will focus here on that part of the failure of the CPI to evaluate the true cost of living adjustment that arises from the theoretical issues such as substitution effects, the new goods bias, and changes in quality.

The goal of a 'cost of living adjustment' $P_E(t)$ is that a utility maximising consumer purchasing a basket $q(0)$ with expenditure $E(0)$ at time 0, should receive the same utility from an expenditure of

$$E(t) = P_E(t)E(0) \tag{2.1}$$

[1]This chapter is co-authored with Eric Ross Weinstein in the MIT mathematics department.

at time t.[2]

The Konus index (often referred to as the 'true cost of living index' or simply as 'the economic index'), is believed by most economists to be a complete theoretical (if incalculable) solution to this welfare problem. Pollak (1983) defines the Konus as a ratio of expenditures in the following way:

$$P_K(P^a, P^b, s, R) = \frac{E(P^a, s, R)}{E(P^b, s, R)} \tag{2.2}$$

where P^a and P^b are prices at $t = a$ and $t = b$, R represents a preference ordering, and s represents the choice of a base indifference curve from that map.

While the Konus may seem the most natural and intuitively appealing solution for the index P_E given normal neoclassical assumptions, it does in fact make severely restrictive demands and is therefore not the most realistic measure of welfare in the real world. Perhaps the two most severe requirements are that the Konus requires access to an individual's preference map, and that no changes in taste occur during the period in question. In addition, using this index involves dealing with an "index number problem". Whereas the Paasche and Laspeyres indexes must fix a reference basket, the Paasche-Konus and Laspeyres-Konus are fixed utility indexes which involve the choice of a reference indifference curve, and thereby suffer from much of the arbitrariness plaguing data based bilateral indices.

In this paper we will examine the welfare implications of the Divisia price index

$$P_D = exp \int_0^1 \sum_{i=1}^{n} \frac{p_i q_i}{\sum_{j=1}^{n} p_j q_j} \frac{dp_i}{p_i} \tag{2.3}$$

defined for a consumer consuming a basket $q(t)$ under $p(t)$ prices. While this index is given by a path dependent line integral, as a data based index it suffers from none of the preference restrictions under which the Konus labours.

It is clear that the Laspeyres index, currently being used for the CPI is an overestimate of the Konus index, P_K. What is not clear, however, is by *how much*. The economists who have been asked by the government for

[2] This paper addresses itself to the benchmark case where marginal analysis is applicable. Accordingly, all functions of time should be assumed once continuously differentiable unless otherwise specified. If care is taken, this assumption can be weakened in some cases to continuous or continuous and piecewise differentiable.

their estimations of the overstatement have delivered a range of suggested adjustments to the CPI. This raises two important questions:

1. Without access to preference maps, how do we know if any of our guesses are correct?

2. In a society in which changes in preferences are far from negligible is there a more appropriate way to estimate the welfare of a representative consumer than the standard fixed preference Konus Index?

It is primarily towards answering these two questions that this paper is directed: i.e. which experimental index P_E in Equation 2.1 ensures that the consumer's utility stays constant. We show that under the right assumptions, the Divisia index provides a complete solution to these questions.

This paper should be viewed as a companion paper to [Malaney I]. In [Malaney I] a new relationship between the traditional bilateral indexes (eg. Laspeyres, Fisher) and the Divisia was explored. This relationship involved no use of the chain principle but instead relied on the use of the calculus of covariant derivatives used in differential geometry.

Unlike chaining, the differential geometric method immediately illuminated welfare aspects of the Divisia. In that paper it was shown that there exists a special "economic" derivative uniquely capable of separating income and substitution effects. It was then shown that all bilateral index formulas made implicit and unjustified use of a non-economic derivative which confuses the two effects; once a systematic replacement was carried out all formulas were shown to produce indexes exactly equivalent to the Divisia.

Somewhat surprisingly, path dependence was seen to emerge from this analysis as being advantageous with important welfare implications. As path dependence has long been considered the major drawback of the the Divisia index, the novelty of the assertion prompted the question of how it was possible to derive welfare implications from an index which resulted from a change in derivatives. As measurements of welfare and changes in standard of living are at the heart of index number theory, the importance of this question was clear. When viewed from a differential geometric standpoint, the answer to the question also becomes apparent. In this paper we attempt to translate these insights into a framework familiar to economists using advanced calculus. The geometric details will appear elsewhere.

2.2 Welfare and Path Dependence

Let us start by looking at why the issue of changing preferences is of fundamental importance to a welfare index.

Standard neoclassical economic analysis generally makes the simplifying assumption that preferences are static; as mentioned before, the definition of the Konus index makes this assumption implicitly. In (Malaney I), however, we discuss the example of a "fad" based upon the changes in taste one is likely to see in any realistic model of an economy. The situation described was that of two persons starting and ending with the same basket of goods comprised of red wine and soda pop.

Let us assume at time t_0 that Consumer A and Consumer B share not only preferences but also a budget constraint $I(0)$. For the sake of simplicity, let us consider a toy economy in which the only goods are wine and cola. Assume that at t_0 the preferences of both consumers cause them each to purchase 70% cola and 30% wine at $p(0)$ prices. Assume that between t_0 and t_a demand, supply and prices remain nearly constant but that consumer A begins to cultivate a taste in red wine causing her to shift her consumption towards 30% cola and 70% wine; consumer B's preferences and purchases remain static.

If one then assumes that between time t_a and t_1 a fad for red wine increases society's demand for wine, we should expect to see an increase in price as well. If consumer B is carried along with the fad, she will increase her consumption of wine despite the fact that the price of wine is increasing. We note that this common behavior associated with fads and trends, is associated only with the exotic Giffen goods under static preferences.

We may then assume that at time t_1 both consumers are purchasing 30% cola and 70% wine. If we compute their Divisia Indexes we will find that Consumer A will have undergone considerable inflation while Consumer B will not have. This despite the fact that their initial and final baskets are the same.

The point here is that A was hit with a price increase in the basket she was already consuming, while B increased her consumption into a good whose price had already undergone much of the increase.

A continuous C.O.L.A. based on the Divisia price index would protect the rights of both consumers to keep their life styles intact in the face of a price increase but it would compensate both consumers differently. This led us in [Malaney 1] to make the point that the path dependence of the Divisia index

C.O.L.A represents something directly analogous to the path dependence of currency exchange or stock portfolio composition; with a Divisia C.O.L.A. *your lifestyle is an investment*. You can always retain your way of life but you cannot necessarily return to it if you change your preferences.

2.3 Implicit Assumptions Underlying a Cost-of-Living Index.

Before considering any proposed experimental income adjustment $P_E(t)$ meant to neutralise the effects of inflation, it is important to consider the implicit assumptions underlying our notions of welfare. Let us look once again at Pollak's definition of the Konus index:

> "The cost of living index is the ratio of the minimum expenditures required to attain a particular indifference curve under two price regimes. We denote the cost of living index by $I(P^a, P^b, s, R)$:
>
> $$I(P^a, P^b, s, R) = \frac{E(P^a, s, R)}{E(P^b, s, R)} \tag{2.4}$$
>
> The notation emphasizes that the index depends not only on the two sets of prices, P^a and P^b, but also on an initial choice of an indifference map or preference ordering, R and the choice of a base indifference curve s from that map. One set of prices is called "reference prices" and the other, "comparison prices'. If the comparison prices are twice the reference prices, the index is 2; if they are one-half, the index is one-half." -The Theory of the Cost of Living Index Robert Pollak Pg. 94

It is plain from the above definition that 'the cost of living index' (otherwise known as the Konus price index P_K) is only defined under the hypothesis of unchanging ordinal preferences. The assertion commonly made is that if a consumer with a given cardinal utility function U maintains such a static indifference map R, then $P_K = I(P^a, P^b, s, R)$ by its definition insures 'constant utility'. We do not take issue with this statement in this paper except to point out that this assertion appeals to an implicit condition on U which we wish to make explicit.

Let us assume for the moment that we were interested in a consumer whose preference map R_t was derived from a dynamic utility function $U : \mathbb{R}_{\text{Time}} \times V \longrightarrow \mathbb{R}$. It is quite possible that our consumer's "efficiency as a [cardinal] pleasure machine" might change even if her indifference map R_t was constant. For example, consider a static cardinal utility function $U_0 : V \longrightarrow \mathbb{R}$ with an indifference map R_0. Then

$$U_a(t, v) = U_0(v) \tag{2.5}$$

and

$$U_b(t, v) = (\frac{1}{2}Sin(2\pi t) + 1)U_0(v) \tag{2.6}$$

are both time dependent utility functions with a common indifference map: R_0.

Let us then assume that our utility maximising consumer possesses just such a static indifference map R derived from the cardinal utility function $U(s, v)$ and at time t expends an adjusted income of $I(t) = P_K(t)I(0)$. In this case, the assumption that her Konus compensation yields constant cardinal utility u_c.

$$U(t, q(t)) = u_c \quad \forall t \tag{2.7}$$

implies that

$$\frac{dU}{dt} = \frac{\partial U}{\partial s}\frac{ds}{dt} + \frac{\partial U}{\partial q}\frac{dq}{dt} = 0. \tag{2.8}$$

We know however from the definition of the Konus that $\frac{dq}{dt}$ lies tangent to the indifference curve while $\frac{\partial U}{\partial q} = \nabla U$ is orthogonal to the tangent planes of U's level sets implying

$$\frac{\partial U}{\partial q}\frac{dq}{dt} = 0. \tag{2.9}$$

Therefore, for $U(t, q(t))$ to be constant under Konus compensation, we must make the single assumption

$$\frac{\partial U(s, q(t))}{\partial s} = 0 \quad \text{whenever t=s.} \tag{2.10}$$

Relative to the basket $q(t)$, equation (2.10) refers to the constancy of what what Sen calls the 'change in man's efficiency as a pleasure machine' and what Balk refers to as the 'cost of living effect of pure preference change'.

2.3.1 Changing Preferences and Psychological Neutrality

The Konus index in its standard form is of no use in evaluating welfare considerations in the context of variable preference maps. However, if we restrict our attention to those utility functions which:

1. Possess a static indifference map R_0.
2. Satisfy equation (2.10).

then the Konus guarantees constant utility. This raises the question of whether there is any way to guarantee constant cardinal utility when the first assumption is lifted.

Happily this question can be answered in the affirmative; we will show in this paper that the Divisia index P_D can be seen to be an extension of the Konus index into the realm of changing preferences where only the above "condition 2" is retained. To this end we make the following precise definition based on equation (2.10).

Definition 14 *Let $U : \mathbb{R} \times V \longrightarrow \mathbb{R}$ be a (time dependent) utility function on V. That is that at any instant of time s_0, the function $U(s_0, \cdot) : V \longrightarrow \mathbb{R}$ has a system of differentiable convex indifference curves. Let us assume that we are given a time dependent basket of goods $q : \mathbb{R} \longrightarrow V$ with $q(t)$ representing the goods purchased at time t.*

We will say that U is **psychologically neutral**[3] *for the basket q if*

$$\frac{dU(s,v)}{ds}|_{s=t,v=q(t)} = 0 \tag{2.11}$$

for all t. Given a dynamic utility function $\hat{U}$, we will refer to any U which shares the same indifference curves as $\hat{U}$ while satisfying (2.11) as a psychological neutralisation of $\hat{U}$.

While a given $\hat{U}$ will not in general satisfy (2.11), it is always possible to replace such a function with a function U which is psychologically neutral and shares the same indifference curves as the original $\hat{U}$.

[3] The terminology is based on Sen's discussion of the welfare considerations surrounding a manic depressive whose utility function is akin to that of equation (2.6).

With this in mind, we can introduce the function

$$\tilde{U}_q(s,v) = \hat{U}(s,v) - \int_0^s \frac{\partial \hat{U}(t,q(s))}{\partial t} dt \tag{2.12}$$

which we will refer to as the canonical neutralisation of $\hat{U}$ relative to q. This function has the same indifference curves as $\hat{U}$ at every instant of time but is psychologically neutral by its construction. Of course this is but one among many satisfying the P.N. hypothesis.

Theorem 8 *Consider a representative consumer with budget constraint $I(0)$ at time $t = 0$ and a differentiable time-dependent convex utility function $\hat{U} : \mathbb{R} \times V \longrightarrow \mathbb{R}$. Let us further assume the consumer's expenditure at time t is given by an experimental price index $P_E(t)$ according to the formula*

$$I(t) = P_E(t)I(0) \tag{2.13}$$

Let $p(t)$ be a differentiable 1-parameter family of price vectors with $q(t)$ be the basket of goods which maximises $U(t, \cdot)$ subject to the income constraint $I(t)$. Then if U is **any** *psychological neutralisation of $\hat{U}$ and P_K, P_D, Q_D are the Konus price, Divisia price and Divisia quantity indices we have:*

1. *Utility is constant if and only if the Divisia quantity index is 1:*
   $U(t, q(t)) = U(0, q(0))\ \forall t$ *if and only if* $Q_D(t) = 1\ \forall t$.
2. *Within the realm of marginal analysis, the Divisia Index is a perfect "cardinal indicator of ordinal utility"*[4]*:*
   (a) $\frac{dU(t,q(t))}{dt} < 0$ *if and only if* $\frac{dQ_D(t)}{dt} < 0$.
   (b) $\frac{dU(t,q(t))}{dt} > 0$ *if and only if* $\frac{dQ_D(t)}{dt} > 0$.
   (c) $\frac{dU(t,q(t))}{dt} = 0$ *if and only if* $\frac{dQ_D(t)}{dt} = 0$.
3. *If the hypotheses needed for the definition of the Konus price index (i.e. preferences and utility are fixed) are placed on the Divisia, then the Divisia is equal to the Konus:*
   $Q_D(t) = 1\ \forall t$ *and* $\frac{\partial U(s,v)}{\partial s} = 0\ \forall s, v \ \Rightarrow P_K(t) = P_D(t)\ \forall t$.

Proof: For part 1, we first show that if the consumer's PN utility $U(t, q(t))$ is constant for all time t then his Divisia quantity index $Q_D(t)$ is equal to unity.

$$0 = \frac{dU(s,q)}{dt} = \frac{\partial U(s,q)}{\partial s}\frac{ds}{dt} + \frac{\partial U(s,q)}{\partial q}\frac{dq}{dt} \tag{2.14}$$

but by the assumption of psychological neutrality the first summand vanishes and we have

$$\frac{\partial U(s,q)}{\partial s}\frac{ds}{dt} = 0 \text{ at } q(t) \tag{2.15}$$

[4]Samuelson & Swamy 1974, Page 568 and Sen 1979, Page 404

Now define β_t to be the time dependent space of *barters*

$$\beta_t = \{v \in V \text{ s.t. } p_t \cdot v = 0\} \tag{2.16}$$

i.e. all baskets which our pricing system at time t evaluates as representing a 'fair trade'.

Then by the assumption that the basket $q(t)$ represents the purchases of a utility maximiser, we know that any instant of time t there exists a strictly positive constant λ_t such that $(\nabla U)_{q(t)} = \lambda_t p_{q(t)}$

$$\therefore \frac{\partial U(s,q)}{\partial q}\frac{dq}{dt} = \lambda_t p_t(\frac{dq}{dt}) = 0 \Rightarrow p_t(\frac{dq}{dt}) = 0 \Rightarrow \frac{dq}{dt} \in \beta_t \Rightarrow Q_D = 1 \tag{2.17}$$

To establish the converse, assume that the Divisia quantity index is equal to unity for all t.

$$Q_D = 1 \Rightarrow \frac{dq}{dt} \in \beta_t \ \forall t \ \Rightarrow p_t(\frac{dq}{dt}) = 0 \tag{2.18}$$

$$\beta_p(t) = TU^{-1}(U(q)) \perp \nabla U \tag{2.19}$$

$$\frac{dq}{dt} \cdot \nabla U = 0 \tag{2.20}$$

$$\frac{\partial U}{\partial q}\frac{dq}{dt} = 0 \tag{2.21}$$

By PN:

$$\frac{\partial U}{\partial s}\frac{ds}{dt} = 0 \text{ at } s = t, \ v = q(t) \tag{2.22}$$

$$\therefore \frac{\partial U}{\partial s}\frac{ds}{dt} + \frac{\partial U}{\partial q}\frac{dq}{dt} = \frac{dU}{dt} \tag{2.23}$$

The second point can be deduced from the above argument.

$$Q_D(t) = exp(\int_0^t \sum_{i=1}^n \frac{p_i \cdot q_i}{\sum_{j=1}^n p_j \cdot q_j}\frac{dq_i}{q_i}) = exp(\int_0^t \sum_{i=1}^n \frac{p_i(s) \cdot \frac{dq_i(s)}{ds}}{\sum_{j=1}^n p_j(s) \cdot q_j(s)} ds) \tag{2.24}$$

but by P.N., maximisation and line (2.17) we have

$$\frac{1}{\lambda_t}\frac{\partial U(s,q)}{\partial q}\frac{dq}{dt} = p_t(\frac{dq}{dt}) \tag{2.25}$$

$$exp(\int_0^t \sum_{i=1} n \frac{p_i(s) \cdot \frac{dq_i(s)}{ds}}{\sum_{j=1}^n p_j(s) \cdot q_j(s)} ds) = exp(\int_0^t \frac{1}{\lambda_s} \frac{\nabla U(s,\cdot) \cdot \frac{dq_i(s)}{ds}}{\sum_{j=1}^n p_j(s) \cdot q_j(s)} ds) \tag{2.26}$$

$$= exp(\int_0^t \mu(s)(\nabla U(s,\cdot) \cdot \frac{dq_i(s)}{ds})\ ds) \tag{2.27}$$

The point here is that the function μ is strictly positive by the positivity of λ_s and $I(s) = p(s) \cdot q(s)$. Thus the sign of the integrand is negative, positive or zero according to whether utility is decreasing, increasing or stagnant. Hence

$$\frac{dQ_D(t)}{dt} = \frac{d}{dt} exp(\int_0^t \mu(s)(\nabla U(s,\cdot) \cdot \frac{dq_i(s)}{ds})\ ds \tag{2.28}$$

$$= exp(\int_0^t \mu(s)(\nabla U(s,\cdot) \cdot \frac{dq_i(s)}{ds})\ ds)\mu(t)(\nabla U(t,\cdot) \cdot \frac{dq_i(t)}{dt}) \tag{2.29}$$

$$= \nu(t)(\nabla U(t,\cdot) \cdot \frac{dq_i(t)}{dt}) \tag{2.30}$$

where $\nu(t)$ is strictly positive; this proves the assertion.

In order to establish the third point let us introduce the notation $v(t) = p(t) \cdot q(t) = I(t)$ for the value of a basket We then multiply the Divisia price and quantity indices to obtain:

$$P_D(t)Q_D(t) = exp(ln(P_D) + ln(Q_D)) = exp(\int \frac{dp \cdot q + p \cdot dq}{p \cdot q}) \tag{2.31}$$

$$= exp(\ \int \frac{dv}{v}\ ) = exp(\ \int dln(v)\ ) = exp(\ ln(\ v(t)\ ) + c\ ) = v(t)e^c \tag{2.32}$$

but since $P_D(0) = Q_D(0) = 1$ we must have $c = -ln(\ v(0)\ )$ yielding the expression

$$P_D(t)Q_D(t) = \frac{v(t)}{v(0)}. \tag{2.33}$$

Therefore, if $Q_D(t) = 1$ we must have

$$P_D(t) = \frac{v(t)}{v(0)} = \frac{p(t) \cdot q(t)}{p(0) \cdot q(0)} = P_K(t) \tag{2.34}$$

since $q(t)$ is the basket of minimal cost on the indifference curve C by the maximisation hypothesis.

QED.

Corollary 9 *Given the same assumptions and notation as in Theorem 2:*

1. *Utility is constant if and only if the experimental index is the Divisia. That is:*
$$\frac{dU(t,q(t))}{dt} = 0 \text{ if and only if } P_E(t) = P_D(t). \tag{2.35}$$

2. *There is no path dependence in the index $P_D(t)$ of a consumer with C.O.L.A. expenditure $I(t) = P_D(t)I(0)$ under constant preferences:*
$$P_D(t) \text{ Path dependent} \Rightarrow \ \exists \tilde{s} \in \mathbb{R}, \tilde{v} \in V \quad s.t. \ \frac{\partial U(s,v)}{\partial s}|_{\tilde{s},\tilde{v}} \neq 0. \tag{2.36}$$

Proof: If $\frac{dU(t,q(t))}{dt} = 0$ then by part 1 of the preceding theorem, we know that $Q_D(t) = 1 \ \forall t$. But

$$P_D(t) = \frac{p(t) \cdot q(t)}{Q_D(t)I(0)} = \frac{I(t)}{I(0)} = \frac{P_E(t)I(0)}{I(0)} = P_E(t) \tag{2.37}$$

Conversely, if we know that $P_D(t) = P_E(t)$ we deduce that

$$Q_D(t) = \frac{p(t) \cdot q(t)}{P_D(t)I(0)} = \frac{P_E(t)I(0)}{P_D(t)I(0)} = 1 \tag{2.38}$$

and the conclusion follows from part 1 of Theorem 2.

To see that path dependence implies changing preferences we reason as follows. Assuming the consumer with expenditure $P_D(t)I(0)$ at time t has constant preferences, we know from Part 3 of Theorem 2 that the $P_D(t)$ must equal $P_K(t)$. However it is clear that $P_K(t)$ is path independent when it is defined. This implies that if P_D is path dependent, the only possibility is that the consumer must have experienced an alteration of her preference map.

QED.

Part 1 of Theorem 2 together with the definition of psychological neutrality, indicates that constant utility in the context of changing preferences is a meaningful and observable concept. Part 3 of Theorem 2 together with the first part of Corollary 3 shows that in this respect the Divisia index outperforms the more brittle Konus. That is, it agrees with the Konus when both are defined under constant preferences and utility but continues to guarantee constant utility in the absence of constant preferences. This suggests the idea that the definition of the Konus fixed utility price index be extended by defining it to equal P_D when $Q_D = 1$ $\forall t$. In the next section we back up this assertion by showing that even more is possible. First we define four variable-preference Konus indexes and then show through an advanced calculus argument (translated from differential geometry) utilising a traditional chaining approach, that they are all directly equal to the Divisia even under changing utility.

2.4 The Konus Index under Changing Preferences

As discussed before, the Konus Price Index

$$P_K = \frac{E(P^a, s, R)}{E(P^b, s, R)} \tag{2.39}$$

measures the change in the minimal cost of getting the consumer to the level of utility u^r associated with a reference basket. Depending on whether we take the reference basket to be the base time basket or the current time basket, therefore, we get different indexes. The Laspeyres Konus (P_{LK}) uses the base time basket as the reference basket, whereas the Paasche Konus (P_{PK}) uses the current time basket. In general these are not the same, except in the case of homothetic preferences.

Let us define $Q(U_a(q(b)), p(c)) \in V$ to be the basket of minimal cost in the pricing at time c with utility equivalent to the consumer's basket at time b as measured by the consumer's utility function at time a. With the above notation, the traditional Laspeyres Konus and Paasche Konus price indices are given by the formulas:

$$P_{LK}(p(1), p(0); U(q_0)) = \frac{p(1) \cdot Q(U(q(0)), p(1))}{p(0) \cdot Q(U(q(0)), p(0))} \tag{2.40}$$

$$P_{PK}(p(1), p(0); U(q_1)) = \frac{p(1) \cdot Q(U(q(1)), p(1))}{p(0) \cdot Q(U(q(1)), p(0))} \tag{2.41}$$

Here, the subscript on the utility function is suppressed because both these indexes implicitly assume that the preferences of the consumer, i.e. the utility function with which he is maximising, remain absolutely constant over time. As pointed out before, the Divisia index does not make this unrealistic assumption. If we relax this assumption for the Konus Price index, allowing preferences to change, we find that as well as specifying which *basket* is the reference basket, we need to specify which utility function we will use as the reference function, base time preferences or current time preferences. We thus define *four* Konus indexes:

Definition 15 *The dynamic utility Konus price indexes are defined by the formulas:*

$$P_{LKL}(p_1, p_0; U_0(q_0)) = \frac{p(1) \cdot Q(U_0(q(0)), p(1))}{p(0) \cdot Q(U_0(q(0)), p(0))} \tag{2.42}$$

$$P_{LKP}(p_1, p_0; U_1(q_0)) = \frac{p(1) \cdot Q(U_1(q(0)), p(1))}{p(0) \cdot Q(U_1(q(0)), p(0))} \quad (2.43)$$

$$P_{PKL}(p_1, p_0; U_0(q_1)) = \frac{p(1) \cdot Q(U_0(q(1)), p(1))}{p(0) \cdot Q(U_0(q(1)), p(0))} \quad (2.44)$$

$$P_{PKP}(p_1, p_0; U_1(q_1)) = \frac{p(1) \cdot Q(U_1(q(1)), p(1))}{p(0) \cdot Q(U_1(q(1)), p(0))} \quad (2.45)$$

P_{LKL} gives the change in the cost of achieving the utility associated with the base time basket under base time preferences, whereas P_{LKP} gives the change in the cost of achieving the utility associated with the base time basket under current preferences. Similarly, P_{PKL} represents the change in the cost of achieving the utility associated with the current basket under base time preferences, while P_{PKP} gives the change in the cost of achieving the utility associated with the current basket under current preferences.

In this section we will prove the equality between the infinitely chained versions of these indexes and the Divisia.

Theorem 10 *Assume that as vector valued functions, $q : [0,1] \longrightarrow V$ and $p : [0,1] \longrightarrow V^*$ are respectively continuous and once differentiable on the closed unit interval. Assume further that U_t is a smooth 1-parameter (time dependent) family of utility functions $U_t : V \longrightarrow \mathbb{R}$, each possessing a system of convex indifference curves. Then the infinitely chained Laspeyres-Konus-Laspeyres price index P^{∞}_{LKL} exists and is equal to the Divisia price index P_D.*

The infinitely chained Laspeyres-Konus-Laspeyres price Index over the interval $[t_0, t_1] =$

$$P^{\infty}_{LKL}(t_0, t_1) = \lim_{s \to \infty} \prod_{a=0}^{s-1} P_{LKL}(\frac{(a)(t_1 - t_0)}{s} + t_0, \frac{(a+1)(t_1 - t_0)}{s} + t_0) \quad (2.46)$$

For the sake of simplicity we shall assume that the intervals under consideration have been shifted and scaled to coincide with $[0,1]$.

Our strategy to demonstrate the equivalence

$$P^{\infty}_{LKL}(0,1) = P_D(0,1)$$

will be to

1. Re-express the infinite limit of finite products as an infinite limit of finite sums.

2. Construct a sequence of functions whose integrals over $[0,1]$ are equal to these (finite) sums.

3. Show that these functions converge uniformly to the Divisia integrand. This shows

Divisia integral

=Integral of Limit=Limit of Integrals

=Limit of Sums=Limit of Products of Laspeyres

= Infinitly Chained Laspeyres-Konus-Laspeyres

Let us look at the infinitely chained Laspeyres-Konus-Laspeyres price index over the interval $[0,1]$.

$$P_{LKL}^{\infty}[0,1] = \lim_{s\to\infty}\prod_{a=0}^{s-1} P_{LKL}^{\frac{a}{s},\frac{a+1}{s}} \tag{2.47}$$

This limit of products can be re-expressed as a limit of sums:

$$\lim_{s\to\infty}\prod_{a=0}^{s-1} P_{LKL}^{\frac{a}{s},\frac{a+1}{s}} = \lim_{s\to\infty}\prod_{a=0}^{s-1} e^{ln(P_{LKL}^{\frac{a}{s},\frac{a+1}{s}})} = \lim_{s\to\infty} exp(\sum_{a=0}^{s-1} ln(P_{LKL}^{\frac{a}{s},\frac{a+1}{s}})) \tag{2.48}$$

$$= exp(\lim_{s\to\infty}\sum_{a=0}^{s-1} ln(P_{LKL}^{\frac{a}{s},\frac{a+1}{s}})) \tag{2.49}$$

Consider the sequence of functions on the interval $[0,1]$

$$\phi_s^{LKL}(x) = s\ ln(P_{LKL}(\frac{a}{s},\frac{a+1}{s})) \qquad \frac{a}{s} < x \leq \frac{(a+1)}{s} \tag{2.50}$$

where s represents the number of 'links' in our chain and a is a non-negative integer less than s. These functions have been specifically constructed so that their integrals

$$\int_0^1 \phi_s^{LKL}(x)dx = \sum_{a=0}^{s-1} ln(P_{LKL}(\frac{a}{s},\frac{a+1}{s})) \tag{2.51}$$

are exactly equal to the natural logarithms of the chained P_{LKL}-indices.

Let us look at the limit as $s \to \infty$ of values of these functions $\phi_s^{LKL}(x_0)$ for a fixed point $x_0 \in [0, 1]$.

Let $a_s(x_0)$ be the sequence of integers such that

$$\frac{a_s(x_0)}{s} < x_0 \leq \frac{(a_s(x_0) + 1)}{s} \tag{2.52}$$

for all s. Obviously from our definition,

$$\lim_{s\to\infty} \frac{a_s(x_0)}{s} = x_0. \tag{2.53}$$

We now examine the infinite limit:

$$\lim_{s\to\infty} \phi_s^{LKL}(x_0) = \lim_{s\to\infty} s \; ln(P_{LKL}(\frac{a_s(x_0)}{s}, \frac{a_s(x_0) + 1}{s})) \tag{2.54}$$

Setting

$$r = \frac{1}{s} \tag{2.55}$$

we get

$$\lim_{s\to\infty} \phi_s^{LKL}(x_0) = \lim_{r\to 0} \frac{ln(P_{LKL}(\frac{a_s(x_0)}{s}, \frac{a_s(x_0)}{s} + r))}{r} \tag{2.56}$$

$$= \lim_{r\to 0} \frac{ln(P_{LKL}(\frac{a_s(x_0)}{s}, \frac{a_s(x_0)}{s} + r)) - ln(P_{LKL}(\frac{a_s(x_0)}{s}, \frac{a_s(x_0)}{s}))}{r} \tag{2.57}$$

$$= \lim_{r\to 0} \frac{ln(\frac{p(\frac{a_s(x_0)}{s}+r)\cdot Q(U_{\frac{a_s(x_0)}{s}}(q(\frac{a_s(x_0)}{s})), p(\frac{a_s(x_0)}{s}+r))}{p(\frac{a_s(x_0)}{s})\cdot q(\frac{a_s(x_0)}{s})}) - ln(\frac{p(\frac{a_s(x_0)}{s})\cdot Q(U_{\frac{a_s(x_0)}{s}}(q(\frac{a_s(x_0)}{s})), p(\frac{a_s(x_0)}{s}))}{p(\frac{a_s(x_0)}{s})\cdot Q(U_{\frac{a_s(x_0)}{s}}(q(\frac{a_s(x_0)}{s})), p(\frac{a_s(x_0)}{s}))})}{r} \tag{2.58}$$

$$= \frac{\partial ln(P_{LKL}(t_0, t_1))}{\partial t_1} \quad |_{t_0=t_1=x_0} \tag{2.59}$$

$$= \frac{1}{P_{LKL}(t_0, t_1)} \cdot \frac{1}{p(t_0)\cdot q(t_0)} \cdot (\frac{dp(t_1)}{dt_1} \cdot q(t_0) + p(t_1) \cdot \frac{\partial Q(U_{t_0}(q(t_0)), p_{t_1})}{\partial t_1}) \;\; |_{t_0=t_1=x_0} \tag{2.60}$$

$$= \frac{1}{p(x_0) \cdot q(x_0)} \cdot (\frac{dp(t_1)}{dt_1} \cdot q(x_0) + p(t_1) \cdot \frac{\partial Q(U_{x_0}(q(x_0)), p_{t_1})}{\partial t_1})|_{t_1 = x_0} \quad (2.61)$$

Since we have a utility maximising consumer at an optimum where

$$\lambda \frac{\partial U_{t_0}}{\partial Q} = p(t_1) \quad (2.62)$$

when evaluated at $t_0 = t_1 = x_0$ we must have

$$p(t_1) \cdot \frac{\partial Q(U_{t_0}(q(t_0)), p_{t_1})}{\partial t_1} \quad (2.63)$$

$$= \lambda \frac{\partial U_{t_0}(Q(U_{t_0}(q(t_0)), p_{t_1})))}{\partial Q} \cdot \frac{\partial Q(U_{t_0}(q(t_0)), p_{t_1})}{\partial t_1} \quad (2.64)$$

$$= \lambda \frac{\partial U_{t_0}(Q(U_{t_0}(q(t_0)), p_{t_1})))}{\partial t_1} \quad (2.65)$$

$$= \lambda \frac{\partial U(q(t_0))}{\partial t_1} \quad (2.66)$$

$$= 0 \quad (2.67)$$

by the definition of $Q(U_{t_0}(q(t_0)), p_{t_1})$ implying

$$\frac{1}{p(x_0) \cdot q(x_0)} \cdot (\frac{dp(t_1)}{dt_1} \cdot q(x_0) + p(t_1) \cdot \frac{\partial Q(U_{x_0}(q(x_0)), p_{t_1})}{\partial t_1})|_{t_1 = x_0} \quad (2.68)$$

$$= \frac{\dot{p}(x_0) \cdot q(x_0)}{p(x_0) \cdot q(x_0)} \quad (2.69)$$

$$= \text{Divisia Integrand} \quad (2.70)$$

Now, by the assumption that $q(t)$ and $p(t)$ are continuous on the **closed** unit interval, we know that they must be uniformly continuous. This in turn guarantees that the ϕ_s^{LKL} converge uniformly to the Divisia integrand. Finally this implies that the limit of the integrals is integral of the limit by a theorem in advanced calculus. Thus, with the definitions and arguments above we have:

$$P_{LKL}^{\infty}[0,1] = \lim_{s\to\infty} \prod_{a=0}^{s-1} P_{LKL}^{\frac{a}{s},\frac{a+1}{s}} \tag{2.71}$$

$$= exp(\lim_{s\to\infty} \sum_{a=0}^{s-1} ln(P_{LKL}^{\frac{a}{s},\frac{a+1}{s}})) \tag{2.72}$$

$$= exp(\lim_{s\to\infty} \int_0^1 \phi_s^{LKL}(t)\ dt) = exp(\int_0^1 \lim_{s\to\infty} \phi_s^{LKL}(t)\ dt) \tag{2.73}$$

$$= exp(\int_0^1 \phi_\infty^{LKL}(t)\ dt) \tag{2.74}$$

$$= exp(\int_0^1 \frac{\dot{p}(t)\cdot q(t)}{p(t)\cdot q(t)}\ dt) \tag{2.75}$$

$$= exp(\int_0^1 \sum_{i=1}^{n} \frac{p_i \cdot q_i}{\sum_{j=1}^{n} p_j \cdot q_j} \frac{dp_i}{p_i}) = P_D[0,1] \tag{2.76}$$

QED

2.5 Conclusions

A comparatively small amount of work seems to have been done on the analysis of the welfare implications of the "little understood"[5] Divisia index. This is especially curious as in this paper we have seen that for an idealised representative consumer with static or dynamic preferences, the Divisia price index is peerless in its ability to maintain constant utility.

In retrospect this is perhaps not so surprising for two main reasons:

1. The traditional chaining argument (connecting the Divisia to the more common bilateral indexes) obscures the advantages and function of the path dependent index. The differential geometric approach from which this paper sprang is the first vantage point which views the path dependence in its natural mathematical setting. From this perspective the Divisia index is the unique index which correctly separates income from substitution effects.

2. The issue of changing preferences, while sometimes discussed in the literature, is too often treated as overly obscure or complicated. We are fortunate in this paper to be able to reach strong welfare conclusions without having to correct for more than 'psychological neutrality'. This assumption is no more than the implicit assumption needed to assure that the Konus index guarantees constant cardinal utility.

While common simplifying assumptions like static preferences may be non-distortionary in the case of short term economic models, index numbers exist primarily to make evaluations over long periods of time. The assumption that preferences do not change over these periods is clearly invalid. As the Divisia does not make this assumption, but the Konus index, which represents our intuitive notion of welfare does, it requires adjusting our basic assumptions in order to evaluate the welfare implications of the Divisia. We prove in this paper that when we restrict the Divisia to unchanging utility, we have equivalence between the Divisia and the Konus price indexes. Moreover, when we lift the simplifying assumption of unchanging preferences, and adjust the Konus to accommodate changes in taste, we prove that in this more realistic case the Divisia price index is once again equal to the Konus price index. The relationship between the two is also shown to provide a

[5]Samuelson and Swamy, pg. 578

means for evaluating estimates of the CPI, which is of particular relevance given the current Congressional debate on this issue.

Chapter 3

Household Migration Decisions Under Uncertainty

3.1 Introduction

The problems involved with the control of rural migration into overtaxed urban areas presents one of the great challenges in development economics. Thirty years of intensive study by economists have spawned numerous policy initiatives intended to combat the phenomenon. Nevertheless, despite scattered success, many LDCs continue to report an increasing demographic shift towards urban areas.

The pioneering work of Lewis (1954) and Todaro (1969) approached the question of rural-to-urban labor migration in the face of high urban unemployment and a high urban to rural wage ratio as an individual decision making problem. The basic behavioral equation[1] developed by Todaro can be expressed by

$$V(0) = \int_{t=0}^{\tau} [P(t)Y_u(t) - Y_r(t)]e^{-rt}dt - C(0) \tag{3.1}$$

where $V(0)$ represents the present discounted value of the move, P represents the probability of getting a job in the urban market, Y_u represents urban income, and Y_r represents rural income. C is the one time cost of moving. This model was then extended by Harris and Todaro (1970) to discuss productivity and policy implications.

[1] Cf. for example to Cole and Sanders 1985

While this straightforward approach has enabled economists to understand some of the decision making involved in migration, it has been pointed out that there are pervasive phenomena that are unexplained by this model. One such issue is that of remittances. It has been found that 10%-30% of a migrant's income may be transferred in remittances (Lucas and Stark 1988). There are various theories based on pure self interest or pure altruism to explain these remittances. Lucas and Stark also discuss migration as a contractual risk sharing proposition: as there are basically non-correlated risks involved in rural production and urban job search, the two parties can co-insure, thereby diversifying the risk. The extent to which such co-operative behavior is found however, suggests that it might be useful to think of the decision making process not at the individual level, but instead at the household level. As Lauby and Stark (1988) contend

> "the decision to migrate and the choice of migration patterns are often not made by the individual alone but rather are determined by family resources and needs."[2]

In fact there were already indications from the authors of the the original individual migrant models that the ultimate goal should be to move the level of analysis from laborer to household. According to Harris and Todaro:

> "...this notion that migrants retain their ties to the rural sector is quite common and manifested by the phenomenon of the extended family system and the flow of remittances to rural relatives of large proportions of urban earnings. However, the reverse flow, i.e. rural-urban monetary transfers is also quite common in cases where the migrant is temporarily unemployed and therefore, must be supported by rural relatives."[3]

In this paper we will model rural-to-urban migration as a household decision. Once this shift is made, it is possible to view a family's migration pattern as a portfolio allocation problem. The low wage rural market, where it has traditionally been assumed that employment is full, is a safe, low return asset. The high wage urban market is associated with uncertain employment, and represents a risky asset. The wealth to be 'invested' in these assets is taken to be the number of working family members.

[2] Lauby J. and Stark O. 1988.

[3] Harris and Todaro 1970 pg. 127.

Policy Considerations

In the face of high urban unemployment there has been an ongoing discussion of the best policies by which to control rural to urban migration. This discussion has focused mainly on whether the migration incentive is best viewed as a "push" from the low wage rural sector, or a "pull" from the high wage urban sector. Throughout the development literature it is often stated as self-evident that

> "...an increase in agricultural income will induce reverse [urban-to-rural] migration..."-Harris and Todaro [4]

without supporting argument. As recently pointed out by Stark:

> "[a] widely held view is that, since expected utility maximisation under substantial intersectoral income differentials induces rural-to-urban migration, **policy measures designed to** dampen urban incomes (a freeze on urban real wages) or to **increase rural incomes** (farm price supports) **are essential**... [Emphasis added]." [5]

When one views the increase in agricultural incomes simply in light of their direct effect, it is clear why one would arrive at the conclusion that an increase in the rural wage level would lead people to want to invest more workers in this asset that is now more valuable. This, however, does not take into account the possible 'income' effects on the investment decision associated with the increased renumeration the household will experience if it maintains at least some workers in the rural labour market. Given the focus of migration literature on the expected income rather than expected utility approach, it is perhaps not surprising that these effects have been overlooked:

> "Originating largely in Todaro's pioneering article (Todaro, 1969), the dominance of the expected income motive soon became virtually exclusive. This is somewhat surprising, especially since during the very same period both risk and (especially) risk avoidance have assumed major significance in mainstream economics.

[4] Harris and Todaro 1970, pp. 132.

[5] Stark, O., 1991, pg. 16.

> Yet the expected income hypothesis, even in its revised formulations, is devoid of any explicit decisional risk content (Todaro, 1976, 1980); the hypothesis does not incorporate a random variable (multiplicative or other), and the implied utility function is linear."[6]

It is only once one incorporates the effects of risk aversion into the family's decision making that one realises that there may be an income effect associated with the rise in rural incomes that may have adverse effects.

It is shown in this paper that if household utility can be characterised by Decreasing Absolute Risk Averse functions, then it is possible for increases in rural wages to in fact lead to 'perverse migration'; i.e. depending on which point of their utility function the family is at, an increase in rural wages may increase the incentive to send workers into the city.

3.2 'Income' vs 'Substitution' Effects

The standard policy recommendation of raising rural wages to stem the flow of rural-to-urban migration relies on what can be see as a substitution effect. In the case of a consumption good X, a decrease in the price of X makes the good more attractive, and the consumer will substitute away from other goods towards good X. Similarly, when the return to assets invested in the rural labour market increases, the family is expected to substitute its investment towards the more attractive rural labour market. However, if the family is risk averse, the change in income may have other effects.

It is has been empirically shown that the utility functions of economic agents with respect to risk generally display the property of Decreasing Absolute Risk Aversion (DARA). It can be shown (as it is below in Proposition 11) that when an agent with a DARA utility function is confronted by an investment decision between a safe and a risky asset, increases in wealth will lead the agent to invest more in the risky asset. This is suggestive of a possible analogy with the migration decision to send more family members to look for urban employment.

The following proposition is well known, and the notation and discussion here is based closely on the treatment in Green, Whinston and Mas-Collel.

[6] Stark O. 1991 Pg 18.

Proposition 11 *Assume that $u(x)$ is the Bernoulli utility function of an investor displaying decreasing absolute risk aversion (DARA). Assume further that the investor is given the opportunity to invest amounts of wealth $w = w_1, w_2$ in shares of a safe asset returning r dollars for every r dollars invested and a risky asset returning z dollars with cumulative distribution function F. Then if $w_1 < w_2$, the optimal investments in the risky asset will obey $\alpha_1^* < \alpha_2^*$.*

Proof:

We will show that if the utility function displays the property of Decreasing Absolute Risk Aversion, then at higher levels of wealth the individual will invest more in the risky asset.

Consider the two levels of initial wealth $w_1 < w_2$. Denote the increments or decrements to wealth by z. Then the individual evaluates risk at w_i by the Bernoulli utility functions $u_i(z) = u(w_i + z)$.

Because u is posited to exhibit DARA we are assured that whenever $w_1 < w_2$, $u_1(z) = u(w_1 + z)$ is a concave transformation of $u_2(z) = u(w_2 + z)$.

The expected utility maximising investor with wealth w_i seeks to maximise the expression

$$\int u(w_i - \alpha_i + \alpha_i z) dF(z) = \int u_i(-\alpha_i + \alpha_i z) dF(z) = 0. \qquad (3.2)$$

Assuming an interior solution, the first order conditions determining the optimal level of investment α_i^* in the risky asset is:

$$\phi_i(\alpha_i^*) = \int (z-1) u_i'(\alpha_i^*[z-1]) dF(z) = 0 \qquad (3.3)$$

As we know, the concavity of $u(\cdot)$ implies that the functions $\phi_i(\cdot)$ are decreasing. Therefore if we show $\phi_1(\alpha_2^*) < 0$, it must follow that $\alpha_1^* < \alpha_2^*$, which is the result we are seeking. Now by the hypothesis that u exhibits DARA there must exist an increasing concave function $\psi(x)$ such that the concave transformation, $u_1(x) = \psi(u_2(x))$ holds for all x. $\psi'(x)$ is therefore positive and decreasing. This allows us to assert that

$$\phi_1(\alpha_2^*) = \int (z-1) \psi'(u_2(\alpha_2^*[z-1])) u_2'(\alpha_2^*[z-1]) dF(z) < 0. \qquad (3.4)$$

The reasoning is that integral

$$\phi_2(\alpha_2^*) = \int (z-1)u_2'(\alpha_2^*[z-1])dF(z) \tag{3.5}$$

$$= \int_{-\infty}^{1} (z-1)u_2'(\alpha_2^*[z-1])dF(z) + \int_{1}^{\infty} (z-1)u_2'(\alpha_2^*[z-1])dF(z) = 0. \tag{3.6}$$

is closely related to (3.4) with the first summand in line (3.6) exhibiting a negative integrand and the second summand containing a positive one. The integral in line (3.5) differs only from the integral in equation (3.4) by the presence of the 'weighting function' $\psi'(u_2(\alpha_2^*[z-1]))$ which due to its decreasing nature will weight the negative integral more heavily. This gives the desired result.

QED

Thus we see that increases in the wealth variable w lead to greater investment in the risky asset. Under such a shift in the budget line we see that the risky asset behaves as a normal good.

The migration model can be regarded as just such a portfolio allocation decision but with an important twist: the 'wealth' of the household is represented not by an amount of money but by the labor force within the unit. In such a situation, direct application of the proposition above applies to an increase in the family size n leading to an increase in the number of 'assets' (namely workers) being sent to the urban market. While this would give rise to interesting population policy considerations, it is not the case being discussed here.

It is reasonably clear that an increase in the rural wage cannot be formulated to fit the structure of the above proposition. Were this possible, we would be left with the uncomfortable conclusion that rural wage increases could lead only to greater rural-to-urban migration and migration models would be universally perverse. That this cannot be the case can be seen by considering an increase in the rural wage beyond urban levels.

In the case under consideration, the potential for an increase in expected return on investment is coming from an increase in the return to the 'safe asset'. Therefore, while we would expect something like the income effect discussed above, there is also a substitution effect: the increase in the rural wage naturally increases the relative attractiveness of the rural market. In fact migration theory has focused on this substitution effect to the exclusion of the 'income' effect, thereby positing the sort of policy discussed above.

What we see here is that with DARA utility functions, and the fact that the increase in income is coming precisely from an increase in the return to the safe asset, income and substitution effects move in opposite directions.

In the case of consumer goods it is assumed that even when these effects conflict, it is highly unlikely that the income effect will outweigh the substitution effect. This is generally true because in order for the income effect of a price rise in any good to be large, the good must constitute a large proportion of the consumption basket. While this is unlikely to be true for any good in a consumption situation, it is almost certainly true in the case of employment. As there are only two assets between which a consumer must choose, it is very likely that income effects will be substantial. The omission of any discussion of this when forming policy can therefore lead to counter productive efforts.

3.3 A Risk Sensitive Household Migration Model.

Our aim in this section is to construct household migration models which incorporate risk sensitivity into labour allocation decisions. While this seems straightforward enough, a literature search indicates that such models are either quite obscure or have not been introduced at all.[7]

There is much debate in the current development literature about the appropriate methods of modeling household decision making. The problem is rich in intricacies which include principal-agent problems, gender specific labour[8], migrant network structures and various game theoretic considerations. Given the wide variety of phenomena which are being uncovered, it is unlikely that a 'universal' model will be settled upon any time soon. For the sake of simplicity we will thus work with a household whose decision making is well approximated by the maximisation of a household utility function that views the decisions as being made by a single entity. While such a choice does not pretend to universality, the situation is known to arise from a variety of situations (e.g. a benevolent dictatorship, a nuclear family whose interests are similar enough to work well in aggregation or an individual decision maker) and may lend itself to most easily to varied adaptation.

We define below the category of rural household models used in this paper.

Definition 16 *We define the data specifying a Rural Household Labour-Market Allocation Model H to be*

$$H = (\nu, n, w_r, w_u, \pi_r, \pi_u, c) \tag{3.7}$$

where

1. *The sub-utility function of the household is given by a twice continuously differentiable function $\nu : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\frac{d\nu(x)}{dx} > 0 \quad \forall x \geq 0 \tag{3.8}$$

guaranteeing that the function is monotonically increasing.

[7]In his 1991 book, Stark encountered a related lacuna writing "To the best of our knowledge, the argument that aversion to risk is a major cause of rural-to-urban migration has appeared only in Stark (1978,1981)".

[8]Gender issues in particular may play a role in migration decision making according to the suggestion of Lauby and Stark 1988.

2. *The number of workers in the household is given by a positive integer* $n \in \mathbb{Z}^+$.

3. *The rural wage is given as* $0 < w_r$.

4. *The urban wage* w_u *is given by a value satisfying* $w_r \leq w_u$.

5. *Rural employment is full, i.e.* $\pi_r = 1$.

6. *The probability of urban employment is* $0 < \pi_u \leq \pi_r$.

7. *A Utility function* $U_H(m)$ *giving the expected utility of allocating* m *of the household's* n *workers for rural labour while sending the remaining* $n - m$ *workers to search for urban employment.*

We can then calculate the expected utility of a family allocating m of its n workers for rural labour. In this paper we shall assume for simplicity that both labour markets make all employment offers at a single time (e.g. spot labour markets in which all work opportunities are offered before the beginning of the work day so that the chance of later employment is negligible.)

Proposition 12 *The expected utility of allocating* m *of the family's* n *workers to the rural labour market is given by*

$$U_{(\nu,n,w_r,w_u,\pi_r,\pi_u,c)}(m) = U^H(m) \tag{3.9}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n-m} \left(\begin{array}{c} m \\ i \end{array} \right) \left(\begin{array}{c} n-m \\ j \end{array} \right) (\pi_r)^i (1-\pi_r)^{m-i} (\pi_u)^j (1-\pi_u)^{n-m-j} \nu(i(w_r) + j(w_u) - (n-m)c).$$

Proof:

Let us assume that we have an ordered set of r identical though unrelated binary events $\{X_i\}_{i=1}^{r}$ (e.g. urban job searches) which all carry the probability π_A of outcome A (urban employment) and $1 - \pi_A$ of outcome B (urban unemployment). Then the probability π that only the first s events yield the outcome A is given by

$$\pi = \pi_A^s (1 - \pi_A)^{r-s}. \tag{3.10}$$

If one wishes to instead calculate the probability $\tilde{\pi}$ that exactly s of the possible outcomes result in A (without respect to which events return the result) the probability is given by the discrete binomial distribution:

$$\tilde{\pi} = \frac{r!}{s!(r-s)!} \pi_A^s (1 - \pi_A)^{r-s} = \left(\begin{array}{c} r \\ s \end{array} \right) \pi_A^s (1 - \pi_A)^{r-s} \tag{3.11}$$

where the binomial coefficient $\begin{pmatrix} r \\ s \end{pmatrix}$ is the number of distinct ways of getting the outcome A from exactly s of the r events $\{X_i\}_{i=1}^r$.

The above analysis tells us that if either all workers are allocated to a single labour market ($m = n$ or $m = 0$) or both rural and urban probabilities are identical ($\pi_r = \pi_u$), the resulting probability structure will be given by a discrete binomial distribution. In our situation however we have two markets which are governed by non-conditional probabilities which are in general not equal. Therefore, since such probabilities are multiplicative, the expected utility of a household which allocates m of its n workers for rural labour is given by the following 'hybrid' distribution:

$$U(m, n-m) = U^H(m) \tag{3.12}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n-m} \frac{m!}{i!(m-i)!} (\pi_r)^i (1-\pi_r)^{m-i} \frac{(n-m)!}{j!(n-m-j)!} (\pi_u)^j (1-\pi_u)^{n-m-j} \nu(iw_r + jw_u - (n-m)c) \tag{3.13}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n-m} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n-m \\ j \end{pmatrix} (\pi_r)^i (1-\pi_r)^{m-i} (\pi_u)^j (1-\pi_u)^{n-m-j} \nu(iw_r + jw_u - (n-m)c)$$

QED

3.4 Perverse Migration Incentives

We will show below that the migration model introduced in the previous section may behave in a fashion exactly opposite to that expected from the migration literature. In such situations, attempts to boost the attractiveness of rural labour will serve to increase the appeal of migration to the urban labour market.

While such phenomena can be shown to occur for families with any number of workers n exceeding 2, the higher cases do not contribute any qualitatively new phenomena; we will thus restrict our attentions to a 2 worker household for ease of illustration.

In order to clarify our exposition, it will be helpful to define a condition somewhat weaker than perverse migration which refers only to the effect on incentive structure. The following definition is given for the two worker household although the concept can be defined more generally:

Definition 17 *We will define a rural wage increase to have a perverse effect on incentives (perverse utility for short) if the marginal increase of* $U(1,1)$ *is greater than* $U(2,0)$*. Thus perverse migration implies perverse utility.*

In what follows it is also helpful to set concrete representatives for the functions whose coefficients of risk aversion are equal to unity:

Definition 18 *The functions representing CARA and CRRA equal to unity are taken to be* $-exp(-x)$ *and* $ln(x)$ *respectively.*

With these definitions set, let us look at the case of a two worker household. In this particular case we have the explicit formulas for the expected utility of the 3 possible labour allocations:

$$U_{w_r}(2,0) = \nu(2w_r) \tag{3.14}$$

$$U_{w_r}(1,1) = (1-\Pi_u)\nu(w_r) + (\Pi_u)\nu(w_r + w_u) \tag{3.15}$$

$$U_{w_r}(0,2) = (1-\Pi_u)^2\nu(0) + 2(\Pi_u)(1-\Pi_u)\nu(w_u) + (\Pi_u)^2\nu(2w_u) \tag{3.16}$$

so these utilities change at the marginal rates

$$\frac{\partial U_{w_r}(2,0)}{\partial w_r} = 2\dot{\nu}(2w_r) \tag{3.17}$$

$$\frac{\partial U_{w_r}(1,1)}{\partial w_r} = (1-\Pi_u)\dot{\nu}(w_r) + (\Pi_u)\dot{\nu}(w_r + w_u) \tag{3.18}$$

$$\frac{\partial U_{w_r}(0,2)}{\partial w_r} = 0 \tag{3.19}$$

relative to changes in the rural wage.

As can be seen from the formula in line (3.16), the strategy of sending both workers to the urban labour market carries a significant risk of a complete loss of income. It is thus important to make sure that whatever functional form is posited for $\nu(x)$ above w_r, the value for $\nu(0)$ must be low enough to prevent 'gambling' with the $(0,2)$ strategy in all but the most extreme circumstances as the migrants under consideration are taken to belong to risk averse subsistence level households.[9]

If we assume that the family was initially indifferent between having 1 or 2 workers in the rural market

$$U_{w_r}(2,0) - U_{w_r}(1,1) = 0 \tag{3.20}$$

then asking whether rural wage increases will lead to perverse migration is equivalent to asking whether

$$\frac{\partial U_{w_r}(2_r,0_u)}{\partial w_r} - \frac{\partial U_{w_r}(1_r,1_u)}{\partial w_r} = 2\dot{\nu}(2w_r) - (1-\Pi_u)\dot{\nu}(w_r) - (\Pi_u)\dot{\nu}(w_r + w_u) \tag{3.21}$$

is positive or negative.

In order for the increase in rural wages to cause an increase in the utility of sending workers to the urban labour market,

$$2\dot{\nu}(2w_r) < (1-\Pi_u)\dot{\nu}(w_r) + (\Pi_u)\dot{\nu}(w_r + w_u) \tag{3.22}$$

On the pages that follow it is shown that (for a two worker household) any unbounded DARA utility function ν which is a concave transformation of the CRRA function $ln(x)$ is capable of producing perverse migration at every rural wage level. Intuitively this indicates that functions which are 'intermediate' between CARA and CRRA will exhibit perversity for a two worker household.

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[9] See for example Sen (1984) Pg.260

> "In matters of 'life and death' such as these decisions involve, affecting one's entire economic existence, the assumption of expected utility maximization is recognized to be quite restrictive."

First, we establish that for any concave transformation of the CRRA function $ln(x)$, there are always urban employment probabilities leading to perverse incentives (without restriction on the urban wage).

Theorem 13 *Assume that a 2 worker household is governed by a DARA sub-utility function*

$$\nu(x) = \psi(ln(x)) \tag{3.23}$$

which can be represented by an increasing transformation ψ of the constant relative risk averse utility function $ln(x)$ such that

$$c > \frac{\psi'(ln(2w_r))}{\psi'(ln(w_r))} \qquad \forall w_r \in [a, b] \tag{3.24}$$

for some constant $c \in [0, 1]$. Then for any probability of urban employment Π_u satisfying

$$1 - c \geq \Pi_u \tag{3.25}$$

all rural wage increases will lead to perverse incentives by increasing the relative expected utility of the urban labour market.

Proof:

By hypothesis we have

$$1 - \Pi_u \geq c \tag{3.26}$$

implying the inequality

$$(1 - \Pi_u) > \frac{\psi'(ln(2w_r))}{\psi'(ln(w_r))} \tag{3.27}$$

for any choice $w_r \in [a, b]$. This may be rewritten successively as

$$\psi'(ln(2w_r)) < (1 - \Pi_u)\psi'(ln(w_r)) \tag{3.28}$$

$$2\psi'(ln(2w_r))\frac{1}{2w_r} < (1 - \Pi_u)\psi'(ln(w_r))\frac{1}{w_r} \tag{3.29}$$

$$2\frac{d\psi(ln(x)) \mid_{2w_r}}{dx} < (1 - \Pi_u)\frac{d\psi(ln(x)) \mid_{w_r}}{dx} \tag{3.30}$$

$$2\dot{\nu}(2w_r) < (1 - \Pi_u)\dot{\nu}(w_r) \tag{3.31}$$

implying

$$2\dot{\nu}(2w_r) < (1 - \Pi_u)\dot{\nu}(w_r) + (\Pi_u)\dot{\nu}(w_r + w_u) \tag{3.32}$$

$$\frac{\partial U_{w_r}(2,0)}{\partial w_r} < \frac{\partial U_{w_r}(1,1)}{\partial w_r} \tag{3.33}$$

This shows that for $\nu = \psi(ln(x))$, the possibility of sending 1 of the two workers to the urban labour market becomes comparatively more attractive for marginal increases in the rural wage.
QED

From this theorem we can see that while the standard CARA function $-exp(-x)$ is not everywhere a concave transformation of the CRRA $ln(x)$, it nevertheless leads to perverse incentives.

Corollary 14 *The function*

$$\nu(x) = -exp(-x) \tag{3.34}$$

with constant absolute risk aversion leads to perverse incentives. Specifically, let

$$\psi = -exp(-exp(x)) \tag{3.35}$$

Then if $w_r > 1$ and

$$\Pi_u < 1 - \frac{\psi'(ln(2w_r))}{\psi'(ln(w_r))} \tag{3.36}$$

then increases in the rural wage lead to perverse utility.

Proof: The function

$$\psi = -exp(-exp(x)) \tag{3.37}$$

is a smooth increasing function which for $x > 0$ is always concave. Therefore for $w_r > 1$ we have the function of constant absolute risk aversion as a concave transformation of the function of constant relative risk aversion

$$\nu(x) = -exp(-x) = -exp(-exp(ln(x))) = \psi(ln(x)) \tag{3.38}$$

as $ln(1) = 0$. By the fact that $\psi(x)$ is an increasing concave function above $x = ln(1) = 0$, we are assured that for w_r large

$$\frac{\psi'(ln(2w_r))}{\psi'(ln(w_r))} < 1 \tag{3.39}$$

and there will thus exist probabilities $1 > \Pi_u > 0$ such that

$$\frac{\psi'(ln(2w_r))}{\psi'(ln(w_r))} < 1 - \Pi_u \tag{3.40}$$

and the corollary follows.
QED

While perverse incentives indicate that raises in the rural wage are ineffective for combating rural to urban migration, they do not necessarily indicate that effort is counterproductive to the point of inducing migration. The following theorem shows that if the concave transformation ψ is unbounded, then for every choice of w_r, it will always be possible to find an urban wage w_u which will lead to perverse migration for a marginal increase in the rural wage.

Theorem 15 *Let the notation be as in the above theorem with ψ a concave transformation and Π_u a probability leading to perverse utility for $w_r \in [a, b]$. If ψ has*

$$\lim_{x \to +\infty} \psi(x) = \infty \tag{3.41}$$

then

$$\forall w_r \in [a, b] \qquad \exists w_u > w_r \tag{3.42}$$

such that the family is indifferent between the pure rural strategy and the mixed rural/urban strategy

$$U_{w_r}(2, 0) = U_{w_r}(1, 1) \tag{3.43}$$

with

$$\frac{\partial U_{w_r}(2, 0)}{\partial w_r} < \frac{\partial U_{w_r}(1, 1)}{\partial w_r} \tag{3.44}$$

leading to perverse migration.

Proof:

First, we recall that the expected utility for the 'mixed strategy' is given by:

$$U_{w_r}(1,1) = (1-\Pi_u)\nu(w_r) + (\Pi_u)\nu(w_r + w_u). \tag{3.45}$$

Now by hypothesis we know that

$$\nu(x) = \psi(ln(x)) \tag{3.46}$$

with $\psi(x)$ unbounded from above. However, since $ln(x)$ is also unbounded, we see from (3.45) that

$$\lim_{w_u \to +\infty} U_{w_r,w_u}(1,1) = \infty. \tag{3.47}$$

However, since the family is assumed to be risk averse we are assured that if w_u were set equal to w_r we would have

$$U_{w_r,w_u}(2,0) > U_{w_r,w_u}(1,1). \tag{3.48}$$

Thus by the intermediate value theorem we are assured that

$$\exists w_u > w_r \; s.t. \; U_{w_r,w_u}(2,0) = U_{w_r,w_u}(1,1) \tag{3.49}$$

as $U_{w_r,w_u}(2,0)$ has no dependence on w_u. Thus if w_u is so chosen for a particular rural wage $w_r \in [a,b]$, the household will exhibit perverse migration for marginal increases in the rural wage w_r.

QED

In the following corollary, the previous theorem is applied to a DARA function specified by its coefficient of absolute risk aversion. This produces an infinite number of models whose values for Π_u and w_u are given as functions of w_r and which lead to perverse migration.

Corollary 16 *Let $\kappa > 1$. Consider the DARA function $\nu(x)$ whose coefficient of absolute risk aversion for $x > \kappa$ is given by*

$$-\frac{v''(x)}{v'(x)} = \frac{1}{x}(1 + \frac{1}{ln(x)}) \tag{3.50}$$

Then if w_r is any value greater than κ, the choices

$$\Pi_u = 1 - \frac{ln(w_r)}{ln(2w_r)} \tag{3.51}$$

$$w_u = exp(exp(\frac{ln(ln(2w_r)) - \frac{ln(w_r)}{ln(2w_r)}ln(ln(w_r))}{1 - \frac{ln(w_r)}{ln(2w_r)}})) - w_r \tag{3.52}$$

lead to a 2 worker household being indifferent between the pure rural strategy and a mixed urban/rural strategy with marginal increases in the rural wage leading to perverse migration.

Proof: First, it can be readily checked that (3.50) is decreasing for $x > 1$ and can be solved by

$$\nu(x) = ln(ln(x)) + c \tag{3.53}$$

where for simplicity we take $c = 0$.

Next we see that if we take the difference in utilities between the two strategies under consideration we have

$$U_{w_r}(2,0) - U_{w_r}(1,1) \tag{3.54}$$

$$= \nu(2w_r) - (1 - \Pi_u)\nu(w_r) - (\Pi_u)\nu(w_r + w_u) \tag{3.55}$$

which for the choices above lead to:

$$= ln(ln(2w_r)) - \frac{ln(w_r)}{ln(2w_r)}ln(ln(w_r)) - \tag{3.56}$$

$$(1 - \frac{ln(w_r)}{ln(2w_r)})\frac{ln(ln(2w_r)) - \frac{ln(w_r)}{ln(2w_r)}ln(ln(w_r))}{1 - \frac{ln(w_r)}{ln(2w_r)}} = 0 \tag{3.57}$$

Lastly we see by design that

$$2\dot{\nu}(2w_r) = (1 - \Pi_u)\dot{\nu}(w_r) < (1 - \Pi_u)\dot{\nu}(w_r) + (\Pi_u)\dot{\nu}(w_r + w_u) \tag{3.58}$$

So perverse migration is assured.
QED

Relations to Income Stabilisation

While this paper has focused on raising a certain rural wage, the problem of increasing effective rural incomes may take a slightly different form. We discuss below a proposed method of relating the models of the current paper with models which deal with uncertainty in the agricultural incomes.

The policies proposed for raising the attractiveness of rural labour may attempt to do so by raising rural wages directly, or by attempting to reduce the fluctuations associated with agricultural incomes. In the latter case it is possible to model rural incomes as lotteries in which the certainty of rural employment is preserved but the possible incomes fluctuate with agricultural prices. The importance of this change in perspective is that it could be used to incorporate the effects of agricultural price stabilisation policy into the rural-to-urban migration decision.

In this situation we would imagine that we would have two cumulative distribution functions with $G_1(x)$ representing the natural variation in rural incomes and $G_2(x)$ giving the variation which would result from government stabilisation efforts. If the government is successful in doing no more that stabilising rural incomes we could expect $G_1(x)$ to determine a mean preserving spread of $G_2(x)$. In either case, if we call $Z_0^{1,2}$ the certainty equivalents

$$Z_0^i = \nu^{-1}(\int \nu(x)\ dG_i(x)) \tag{3.59}$$

of the two lotteries $G_1(x),\ G_2(x)$, we know that for a household governed by a risk averse utility function $\nu(x)$ that

$$Z_0^1 < Z_0^2 \tag{3.60}$$

as G_2 second-order stochastically dominates G_1

Thus if one is able to impose a policy so that the original rural distribution of incomes forms a mean preserving spread of the distribution resulting from the intervention, a situation will result which is in large part analogous to the raising of the rural wage (where w_r is analogous to Z_0) considered in the current paper.

3.5 Examples

Once one is made aware of the possibility of perverse migration, examples of perverse models appear to be ubiquitous. It is thus not too difficult to find examples of models where:

1. The utility function $\nu(x)$ exhibits decreasing absolute risk aversion.

2. The values for probabilities, and wages are not terribly different from those found in actual labour markets.

While such models are not intended to mirror any particular labour market, they do indicate that such models exist without recourse to unrealistic hypotheses.

Let us then look at examples in which we expect perverse migration. We will illustrate the phenomena by graphing the utility differences of the two main allocation strategies as a function of the rural wage.

It has been shown that any unbounded DARA utility function $\nu(x)$ which is a concave transformation of the CRRA function $ln(x)$ is capable of producing perverse migration. Here we look at a specific function and evaluate ranges of rural and urban wages and urban probabilities that cause perversity.

Let us assume the household's utility function is equal to

$$\nu(x) = ln(ln\frac{x}{10}) \tag{3.61}$$

in the range of interest (e.g. Rs. 11 and up).

We are able to plot a graph of the Rural Advantage Function (RAF) for the household with the utility function ν. This looks at how the difference between the utility of keeping two workers in the country and the utility of keeping one worker in the city and one in the country changes with rises in the rural income.

$$RAF(w_r) = U_{w_r}(2,0) - U_{w_r}(1,1) = \nu(2w_r) - (1-\Pi)\nu(w_r) - (\Pi)\nu(w_r + w_u) \tag{3.62}$$

Let us look at this for particular values of urban employment probabilities Π, and urban wage rates w_u. On the X axis we plot rural wage rates, and on the Y axis is measured the difference in utilities. When rural wages increase into the realm where the RAF becomes negative, the wage increase is seen to cause perverse migration as it becomes preferable for the family to send another worker into the city.

We will look at two specific cases; in the first case we set the urban probability equal to .5 and observe the range of rural wages w_r which will cause perverse migration with an urban wage of Rs. 200 (It will be noted in this example that a rural wage increase would have to more than double the rural earnings before it brought the migrant out of the range of perversity). In the second case we look at a situation with an urban probability of employment of .4 and urban wage rate set at Rs. 500.

We assume in these examples that the negative utility of a zero wage is taken to be large enough that the family will consider it prohibitive to gamble both workers in the city (as this strategy necessarily entails a high risk of earning nothing).

Case 1
We set:

$$\Pi_u = .5$$

$$w_u = Rs.200$$

Perverse migration is caused whenever rural wages w_r rise or fall to any wage within the range of Rs. 15 to Rs. 44. (See Graph 1)

Graph 1:

Probability of Urban Employment = .5
Urban Wage = Rs.200

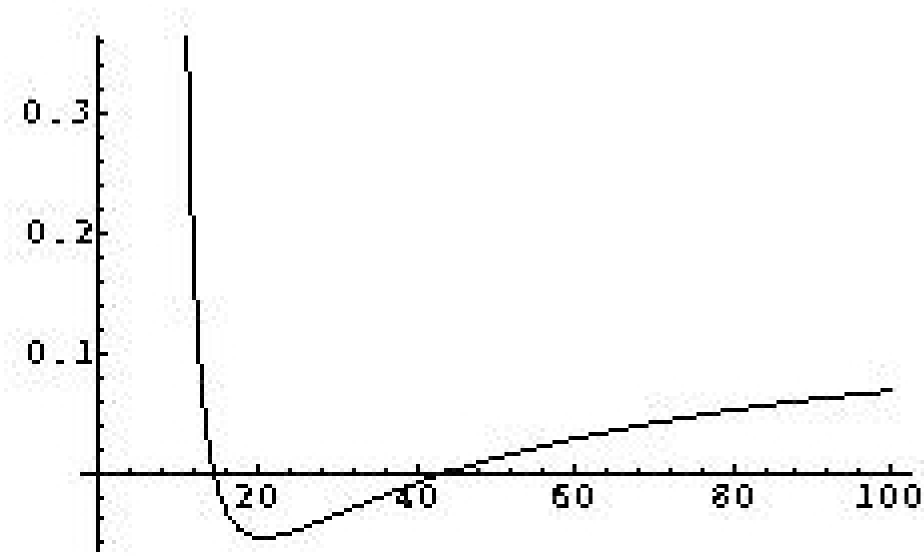

Figure 3.1: First Graphical Example of Perverse Migration

Case 2
We set:

$$\Pi_u = .4$$

$$w_u = Rs.500$$

Perverse migration is caused whenever rural wages w_r rise or fall to any wage within the range of Rs. 20 to Rs. 56. (See Graph 2)

Graph 2:

Probability of Urban Employment = .4
Urban Wage = Rs.500

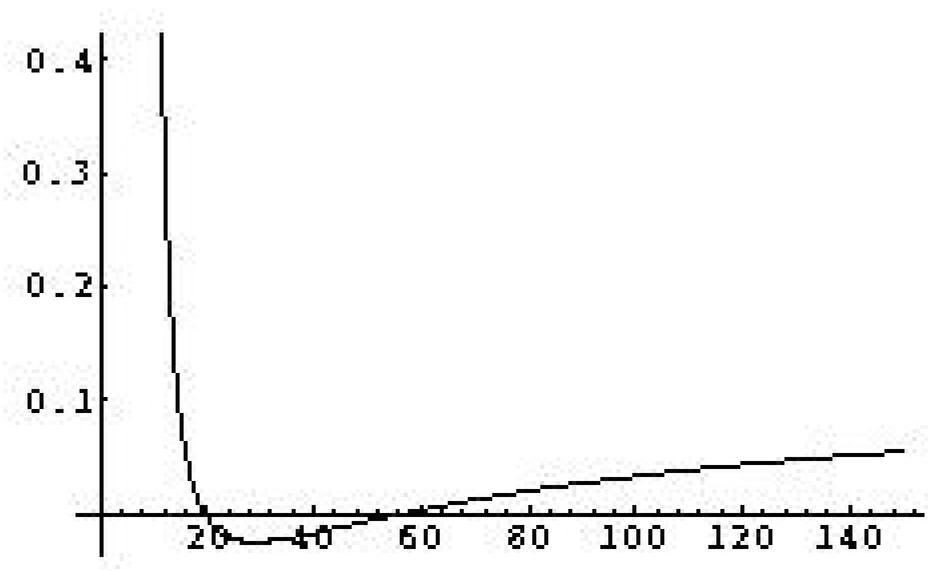

Figure 3.2: Second Graphical Example of Perverse Migration

3.6 Conclusion

When developing migration control policy, understanding the decision making processes behind rural to urban migration is essential. It has in general been assumed in the literature that there is a theoretical equivalence between raising rural wages to dampen the "push" factors of the low wage rural sector, and putting in place urban wage controls, to deal with the "pull" factor of a high wage urban sector. It is shown here, however, that once attitudes to risk are incorporated, it is is possible to find that raising rural wages may in fact have a counterproductive effect.

The migration literature has, until very recently, been based closely on the Todaro model. While this model captures the basic incentive structure for migration, it makes two simplifying assumptions that are generally acknowledged to be unrealistic, i.e.

1. The atomic hypothesis: migration decisions are made by the migrants alone (i.e. workers are unremitted and unremitting).

2. Pure risk neutrality: Expected income rather than expected utility is maximised.

Once these assumptions are lifted, it is found that one of the basic conclusions of the Todaro model can no longer be relied upon.

Perverse incentives have been shown in this paper to afflict the category of Decreasing Absolute Risk Averse utility functions. As this is precisely the utility structure one expects to find in the general situation we are left with the question of how to evaluate the likely migration effects of a generic rural wage support program. This paper would argue that without specific information on the particulars of the potential migrants being targeted (e.g. family sizes, objective functions, wage rates and probabilities) the usual conclusion drawn in the literature is not assured.

3.7 Appendix B: The Relationship Between the Rural Household Labour Allocation Model and the Todaro Model

The original Todaro equation serves something of a dual role in the development literature; while it is in some sense one of the simplest migration models, it also functions as the progenitor for the more sophisticated models which followed it. It may therefore be worthwhile to see how the data specifying a rural labour allocation model (RHLMAM) compares with the data specifying Todaro's equation.

In Todaro's model (3.1), the basic behavioral equation gives a single number $V(0)$ whose sign is the answer to the discrete optimisation problem; if and only if the number is positive will the rational individual choose to migrate. In the case of a household, the number of options is enlarged so that there are $\frac{n^2+n}{2}$ decisions to evaluate. We thus posit that the natural analog of Todaro's equation for a remitting household is given by the following 'behavioral matrix' M^H.

Definition 19 ***The*** $(n+1)\times(n+1)$ ***skew-symmetric behavioral matrix*** M^H ***associated to a RHLMAM is given by the entries*** $\{M^H_{ij}\}^n_{i,j=0}$ ***where*** $M^H_{ij} = U^H(i) - U^H(j)$

The number of rural workers m which will optimise the expected utility of the household H is then the number of the row whose entries are all non-negative. The greatest number row with this property will be called $O(H)$. If we denote the collection of household models by $\mathcal{H}$:

Definition 20 ***Let*** $O : \mathcal{H} \longrightarrow \mathbb{Z}$ ***be the integer valued function which gives the number of workers who should be allocated to rural labour in order to maximise the household's expected utility.***

With these conventions established we are now in a position to explain how these RHLMAMs relate to the established literature. In the following proposition it is shown that these models generalise the (static) Todaro model to families or networks of labourers governed by risk-sensitive household welfare functions.

Proposition 17 ***Let us assume that the following hypotheses are made:***

1. ***The household work force consists of an individual:*** $n = 1$***.***
2. ***The individual is risk neutral:*** $\nu(x) = x$***.***
3. ***The interest rate is negligible:*** $r = 0$***.***
4. ***The probabilities of urban and rural employment are given by constants as are the urban and rural wages:*** $\pi_u = P(t),\ \pi_r = 1,\ Y_u(t) = w_u$ ***and*** $Y_r(t) = w_r$***.***
5. ***The time period*** τ ***in Todaro's equation is set equal to unity:*** $\tau = 1$***.***

Then the RHLMAM is equivalent to the Todaro equation (3.1)

Proof: In order to evaluate the migration decision presented by a single worker RHLMAM, one need only examine the M_{01} component of the 2×2 behavioral matrix.

$$M_{01}^H = U_H(0) - U_H(1) \tag{3.63}$$

which when expanded from equation (3.13) yields

$$= \pi_u \nu(w_u - c) + (1 - \pi_u)\nu(0 - c) - \pi_r \nu(w_r) - (1 - \pi_r)\nu(0) \tag{3.64}$$

$$= \pi_u w_u - \pi_r w_r + (-c)(\pi_u + (1 - \pi_u)) \tag{3.65}$$

$$= \pi_u w_u - w_r - c \tag{3.66}$$

$$= \int_{t=0}^{\tau} [P(t)Y_u(t) - Y_r(t)]e^{-rt}dt - C(0) = V(0) \tag{3.67}$$

giving equation (3.1) (the Todaro behavioral equation).
QED.

This shows that in a well defined sense the category of RHLMAMs can be considered as an expansion of the seminal Todaro model generalised to include risk preferences and household objective functions. The following corollary is then immediate.

Corollary 18 ***Every RHLMAM*** H ***with*** $n = 1$ ***and*** $\nu_H(x) = x$ ***specifies a Todaro model.***

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