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When more is less: Using multiple constraints to reduce tail risk

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ABSTRACT

Financial institutions suffered large trading losses during the 2007–2009 global financial crisis. These losses cast doubt on the effectiveness of regulations and risk management systems based on a *single* Value-at-Risk (VaR) constraint. While some researchers have recommended using Conditional Value-at-Risk (CVaR) to control tail risk, VaR remains popular among practitioners and regulators. Accordingly, our paper examines the effectiveness of *multiple* VaR constraints in controlling CVaR. Under certain conditions, we theoretically show that they are more effective than a single VaR constraint. Furthermore, we numerically find that the maximum CVaR permitted by the constraints is notably smaller than with a single constraint. These results suggest that regulations and risk management systems based on multiple VaR constraints are more effective in reducing tail risk than those based on a single VaR constraint.

1. Introduction

Financial institutions suffered large trading losses during the 2007–2009 global financial crisis (hereafter, 'recent crisis'). These losses cast doubt on the effectiveness of their regulation and risk management practices. At the heart of these practices is the use of Value-at-Risk (VaR) to measure and control tail risk. Indeed, financial institutions use VaR in determining minimum capital requirements and setting risk exposure limits for their trading books (see Basel Committee on Banking Supervision, 2006, 2009a; Scholes, 2000).

Even before the recent crisis, researchers were criticizing the use of VaR as a measure of tail risk since it does not consider the size of losses beyond VaR (see, e.g., Basak and Shapiro, 2001; Alexander and Baptista, 2006).[1] Hence, VaR-based risk management systems can lead to the selection of portfolios with substantial tail risk. Accordingly, these researchers recommend replacing VaR with Conditional Value-at-Risk (CVaR) since it considers the size of losses beyond VaR.[2]

Lately, practitioners and regulators have also recognized the ineffectiveness of VaR-based risk management systems in controlling tail risk. For example, nearly half of the respondents in a global survey of money managers are dissatisfied with the performance of VaR during the meltdown (see bfinance, 2009). Additionally, the Basel Committee on Banking Supervision (2009b) notes that a number of major banks suffered large trading losses during the recent crisis that were not captured by their VaRs. Nevertheless, VaR remains quite popular among practitioners and regulators. More than 60% of the money managers who participated in the aforementioned survey report the use of VaR (see bfinance, 2009). Furthermore, recent revisions of the Basel framework for trading books are based on the use of VaR (see Basel Committee on Banking Supervision, 2009a). Because practitioners and regulators still rely on VaR, the question of whether there are more effective VaR-based risk management systems is of particular interest.

Previous work recognizes the ineffectiveness of VaR-based risk management systems by assuming that a *single* VaR constraint is used in attempting to control CVaR. In this paper, we examine the effectiveness of risk management systems based on *multiple* VaR constraints. Since VaR does not capture the size of losses beyond VaR, we examine the case where the constraints use confidence levels equal to or higher than the confidence level used to estimate CVaR.[3] Our motivation is threefold. First, consistent with current practice and regulation, our paper takes as given that VaR is utilized in controlling tail risk. Second, it seems straightforward

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[1] Artzner et al. (1999) show that VaR fails to possess the subadditivity property (i.e., two assets in combination can have a VaR greater than the sum of their individual VaRs). However, Daníelsson et al. (2005) and Garcia et al. (2007) argue that the cases when VaR is not subadditive are rare.

[2] Note that CVaR possesses the subadditivity property. Empirical support for using CVaR as a measure of tail risk for hedge funds is provided by Agarwal and Naik (2004) and Liang and Park (2007, 2010).

[3] Nevertheless, we also examine the case where the former confidence levels are lower than or equal to the latter.

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to implement a system based on multiple VaR constraints if a system based on a single VaR constraint is already in place. Third, the Basel Committee on Banking Supervision (2009a) recently adopted minimum capital requirements for trading books that involve the use of two different confidence levels.[4]

Christoffersen (2003, p. 101) and Pritsker (2006) point out that historical simulation is widely used in practice to estimate VaR. For example, a survey by Pérignon and Smith (2010) indicates that 73% of the banks that disclose their VaR estimation methodology report the use of historical simulation. Accordingly, we use historical simulation in our analysis.[5]

We begin by theoretically examining the effectiveness of VaR constraints in controlling CVaR in a simple setting.[6] Feasible portfolios are assumed to have asset weights that are bounded from below and above. First, we examine the effectiveness of a single VaR constraint in controlling CVaR. We show that when the confidence level used by the constraint is equal to the one used to compute CVaR, the constraint does not preclude the selection of portfolios with large CVaRs. When the former confidence level is higher than the latter, the extent to which the constraint precludes the selection of portfolios with large CVaRs depends on its confidence level and bound. This extent is larger when the confidence level is higher or the bound is smaller, but it is nevertheless limited.

Second, we examine the effectiveness of multiple VaR constraints in controlling CVaR. We show that the extent to which the constraints preclude the selection of portfolios with large CVaRs depends on their confidence levels and bounds. This extent is larger when the confidence levels are higher or the bounds are smaller. Importantly, the use of appropriately chosen multiple VaR constraints more effectively precludes the selection of portfolios with larger CVaRs than the use of a single VaR constraint.

Third, we assess the effect of loosening portfolio weight restrictions (e.g., considering the case of allowing short selling relative to the case when it is disallowed) on the effectiveness of VaR constraints in controlling CVaR. We show that loosening them can reduce the extent to which portfolios with large CVaRs are precluded by single or multiple VaR constraints. Hence, the effectiveness of the constraints in controlling CVaR possibly depends on the tightness of portfolio weight restrictions.

Next, we examine whether VaR constraints are effective in controlling CVaR when an expected return constraint is also in place. Specifically, we investigate whether VaR constraints preclude the selection of *all* portfolios with substantial efficiency losses relative to the mean-CVaR frontier. Here, a portfolio belongs to the *mean-CVaR frontier* if there is no portfolio with the same expected return and a smaller CVaR. Also, a portfolio's *efficiency loss* is given by the difference between: (1) its CVaR and (2) the CVaR of the portfolio on the mean-CVaR frontier with the same expected return. Hence, this loss represents the increase in tail risk arising from selecting it instead of the portfolio with the same expected return that has minimum tail risk. Accordingly, if a set of constraints precludes all portfolios with substantial efficiency losses from being selected, then it is effective in controlling CVaR since any portfolio, no matter how selected (e.g., using a mean-variance model), will have tail risk that is similar in magnitude to that of the portfolio with the same expected return that minimizes CVaR. However, if the set of constraints allows the selection of portfolios with substantial efficiency losses, then it is not effectively controlling CVaR.

Since there are no closed-form solutions for the portfolios on the mean-CVaR frontier in the presence of portfolio weight restrictions, we examine the size of efficiency losses numerically. In doing so, we solve a plausible problem of wealth allocation among Treasury bills, government bonds, corporate bonds, and the six size/book-to-market Fama–French portfolios.[7] We begin by assuming that short selling is disallowed (motivation for this assumption can be found in, e.g., Jagannathan and Ma, 2003; Almazan et al., 2004). First, we examine the effectiveness of risk management systems based on a single VaR constraint. We find that when the confidence levels used to estimate VaR and CVaR coincide, the constraint allows the selection of portfolios with substantial efficiency losses. Moreover, we find that such portfolios can be selected even when the confidence level used by the VaR constraint exceeds the one used to estimate CVaR. These results suggest that risk management systems based on a single VaR constraint are ineffective in controlling CVaR.

Second, we examine the effectiveness of risk management systems based on multiple VaR constraints. We find that efficiency losses with multiple VaR constraints are smaller than with a single VaR constraint. Furthermore, if the number of constraints is sufficiently large, then losses are close to zero. These results suggest that risk management systems based on multiple VaR constraints are effective in controlling CVaR when short selling is disallowed.

Next, we assume that short selling is allowed (motivation for this assumption can be found in, e.g., Jackson et al., 1997). Again, we find that the use of a single VaR constraint allows the selection of portfolios with substantial efficiency losses, regardless of whether the confidence level used to estimate VaR is equal to or higher than the one used to estimate CVaR. However, we find that losses with multiple VaR constraints are notably smaller than those with a single VaR constraint (but larger than in the no short selling case). These results suggest that when short selling is allowed, the use of risk management systems based on multiple VaR constraints substantively reduces tail risk relative to the use of systems based on a single VaR constraint.

There are many reasons for why risk management systems based on VaR (or on any other risk measure) might fail. For example, such systems might not capture some risks (e.g., time-varying liquidity risks). Furthermore, the use of a short window for estimation purposes may result in a severely understated VaR. Our paper solely focuses on the drawback that VaR does not take into consideration losses beyond VaR. Regardless of whether this drawback is viewed as a significant driver of the sizeable trading losses that financial institutions suffered during the recent crisis, it should be recognized that regulations and risk management systems

[4] These capital requirements are based on: (a) VaR at the 99% confidence level (under current and stressed conditions) as described in Basel Committee on Banking Supervision (2009a); and (b) an incremental risk charge that represents an estimate of default and migration risks of unsecuritized products at the 99.90% confidence level as described in Basel Committee on Banking Supervision (2009b). Here, migration risk refers to the possibility of a credit rating downgrade or upgrade.

[5] Christoffersen (2003, pp. 101–103) discusses the pros and cons of this methodology. An assessment of the effectiveness of VaR constraints in controlling CVaR when using alternative estimation methodologies (e.g., Monte Carlo simulation) is beyond the scope of our paper, and is left for future research.

[6] Note that there are conditions under which the effect of a VaR constraint on the optimal portfolio is equivalent to that of a CVaR constraint. Using a single-period mean-variance model where asset returns have a normal distribution, Alexander and Baptista (2006) note that for any confidence level used for a VaR constraint, there is a lower confidence level for a CVaR constraint such that the constraints lead to the same optimal portfolio when the VaR and CVaR bounds coincide. Using a continuous-time expected utility model where asset prices have a lognormal distribution, Cuoco et al. (2008) similarly find that for any VaR constraint, there is a CVaR constraint with the same effect on the optimal portfolio. The equivalence between VaR and CVaR constraints does not occur in our paper since we use historical simulation and assume that asset returns have a discrete distribution with finitely many jumps.

[7] The choice of these asset classes can be motivated in the context of, for example, the trading books of large banks. Indeed, the Basel Committee on Banking Supervision (2006, p. 157) notes that market risk exposures subject to VaR-based minimum capital requirements include "the risks pertaining to interest rate related instruments and equities in the trading book."

based on multiple VaR constraints are more effective in reducing tail risk than those based on a single VaR constraint.[8]

Several papers explore the portfolio selection implications arising from attempting to control tail risk with either: (1) a single VaR constraint; or (2) a single or multiple stress testing constraints. Some of these papers use a continuous-time expected utility model. For example, Basak and Shapiro (2001) find that when an agent faces a VaR constraint at the initial date, he or she may select a larger exposure to risky assets, but Yiu (2004) shows that a dynamic VaR constraint leads an agent to reduce this exposure. Other papers use discrete-time models. For example, Alexander and Baptista (2004, 2006) show that, under certain conditions, adding a VaR constraint to the mean-variance model leads to the selection of portfolios with larger standard deviations than those of the portfolios selected without the constraint. However, they also show that there are conditions under which doing so leads to the selection of portfolios with smaller standard deviations (also see Sentana, 2003). Alexander and Baptista (2009) show that adding stress testing constraints to the mean-variance model tends to lead to an increase in the optimal holding of the risk-free asset, thereby decreasing the exposure to risky assets. Our work differs from these papers in that we examine the effectiveness of multiple VaR constraints in controlling CVaR.[9]

We proceed as follows. Section 2 describes the model. Section 3 theoretically examines the effectiveness of VaR constraints in controlling CVaR. Sections 4 and 5 numerically examine their effectiveness when short selling is disallowed and allowed, respectively. Section 6 assesses the robustness of our numerical results, and Section 7 concludes. Appendix A contains proofs of our theoretical results. Appendix B considers the case when estimation risk is present.

2. The model

Suppose that uncertainty is described by S states ($s = 1,\ldots,S$). Let $p_s > 0$ be the probability of state s. There are J risky assets ($j = 1,\ldots,J$) and a risk-free asset ($j = J + 1$). Asset returns are given by a $(J + 1) \times S$ matrix $\boldsymbol{R}$. The return of asset j in state s is R_{js}.

Let $\mathbf{1}$ denote the $(J + 1) \times 1$ vector $[1 \cdots 1]^\top$. A portfolio is a $(J + 1) \times 1$ vector $\boldsymbol{w} = [w_1 \cdots w_{J+1}]^\top$ with $\boldsymbol{w}^\top \mathbf{1} = 1$, where w_j represents the weight of asset j. Note that a positive (negative) weight on a given asset represents a long (short) position in the asset.

2.1. VaR

In defining VaR, we follow Rockafellar and Uryasev (2002, Proposition 8). Fix a confidence level $\alpha \in (1/2,1)$. Let $\widetilde{R}_{\boldsymbol{w}}$ denote the random return of portfolio $\boldsymbol{w}$. Let $z_{1,\boldsymbol{w}} < z_{2,\boldsymbol{w}} < \cdots < z_{N_{\boldsymbol{w}},\boldsymbol{w}}$ denote the ordered values that $\tilde{z}_{\boldsymbol{w}} \equiv -\widetilde{R}_{\boldsymbol{w}}$ can take where $N_{\boldsymbol{w}} \leqslant S$ is the number of these values. Define n_α as the unique index number with:

$$\sum_{n=1}^{n_\alpha} p_{n,\boldsymbol{w}} \geqslant \alpha > \sum_{n=1}^{n_\alpha - 1} p_{n,\boldsymbol{w}}, \tag{1}$$

where $p_{n,\boldsymbol{w}} \equiv P[\tilde{z}_{\boldsymbol{w}} = z_{n,\boldsymbol{w}}]$. Note that while n_α depends on $\boldsymbol{w}$, we write 'n_α' instead of '$n_{\alpha,\boldsymbol{w}}$.' Portfolio $\boldsymbol{w}$'s *VaR at the 100α% confidence level* is given by:

$$V_{\alpha,\boldsymbol{w}} \equiv z_{n_\alpha,\boldsymbol{w}}. \tag{2}$$

Eqs. (1) and (2) imply that:

$$P[\widetilde{R}_{\boldsymbol{w}} \geqslant -V_{\alpha,\boldsymbol{w}}] = P[\tilde{z}_{\boldsymbol{w}} \leqslant z_{n_\alpha,\boldsymbol{w}}] \geqslant \alpha, \tag{3}$$

$$P[\widetilde{R}_{\boldsymbol{w}} > -V_{\alpha,\boldsymbol{w}}] = P[\tilde{z}_{\boldsymbol{w}} < z_{n_\alpha,\boldsymbol{w}}] < \alpha. \tag{4}$$

As Eqs. (3) and (4) show, this definition of VaR is based on the upper quantile (see, for example, Acerbi and Tasche, 2002).

2.2. CVaR

In defining CVaR, we again follow Rockafellar and Uryasev (2002, Proposition 8). Portfolio $\boldsymbol{w}$'s *CVaR at the 100α% confidence level* is given by:

$$C_{\alpha,\boldsymbol{w}} \equiv \frac{1}{1-\alpha}\left[\left(\sum_{n=1}^{n_\alpha} p_{n,\boldsymbol{w}} - \alpha\right) z_{n_\alpha,\boldsymbol{w}} + \sum_{n=n_\alpha+1}^{N_{\boldsymbol{w}}} p_{n,\boldsymbol{w}} z_{n,\boldsymbol{w}}\right]. \tag{5}$$

Eqs. (2) and (5) imply that: (a) $C_{\alpha,\boldsymbol{w}} \geqslant V_{\alpha,\boldsymbol{w}}$; and (b) $C_{\alpha,\boldsymbol{w}} > V_{\alpha,\boldsymbol{w}}$ if $P[\widetilde{R}_{\boldsymbol{w}} < -V_{\alpha,\boldsymbol{w}}] > 0$.

2.3. VaR constraint

Given a confidence level α, consider the VaR constraint:

$$V_{\alpha,\boldsymbol{w}} \leqslant V, \tag{6}$$

where V is the VaR bound. The constraint can be thought of as being 'tightened' when either α increases or V decreases. As noted earlier, banks often use VaR constraints in attempting to control tail risk.

2.4. Estimating VaR and CVaR

Historical simulation is widely used in practice to estimate VaR as discussed earlier (see, e.g., Christoffersen, 2003, p. 101; Pritsker, 2006; Pérignon and Smith, 2010). Accordingly, we utilize historical simulation to estimate VaR and CVaR.

3. Theoretical results in a simple setting

This section theoretically examines the effectiveness of VaR constraints in controlling CVaR in a simple setting. Specifically, assume that all states are equally likely.[10] Also, assume that one or two VaR constraints are used in controlling CVaR at confidence level $\alpha = 1 - 2/S$.[11] While CVaR reflects the distribution of losses beyond VaR, VaR does not. Hence, assume that the confidence level used by each VaR constraint is equal to or higher than the one used to

[8] For excellent discussions of the recent financial market turmoil, see, e.g., Brunnermeier (2009), Gorton (2009), and Levine (2010a). Claessens et al. (2009), Caprio et al. (2010), Dewatripont et al. (2010, Chapters 2–4), Levine (2010b), and Moshirian (2011) offer recommendations on how the regulation of the financial system should be restructured in light of the recent crisis. More generally, there is an extensive literature examining the regulation of financial institutions; see, e.g., Koehn and Santomero (1980), Kim and Santomero (1988), Rochet (1992), Hellmann et al. (2000), John et al. (2000), Kane (2007), and Barth et al. (2008). Papers that criticize bank capital regulation based on VaR include Alexander and Baptista (2006) and Kane (2006). Bhattacharya et al. (1998) and Freixas and Rochet (2008, Chapter 9) provide reviews of the literature. Importantly, this literature does not consider the use of VaRs at two or more different confidence levels.

[9] Also related to our work are papers that investigate the portfolio selection implications arising from the presence of a floor (i.e., a restriction involving the minimum level of wealth); see, e.g., Black and Perold (1992), Grossman and Vila (1992), and Grossman and Zhou (1993). Our work differs from these papers in two respects. First, we examine whether VaR-based risk management systems are effective in controlling CVaR, whereas they focus on implications of a floor. Second, while we use a single-period model, they use continuous-time models.

[10] This assumption is consistent with the use of historical simulation to estimate VaR and CVaR. Nevertheless, our results extend to the case where different states possibly have different probabilities. Our results also extend to the case where asset returns have a continuous distribution. In this case, we have $C_{\alpha,\boldsymbol{w}} = \frac{1}{1-\alpha}\int_\alpha^1 V_{\alpha',\boldsymbol{w}} d\alpha'$ for every portfolio $\boldsymbol{w}$. While the use of finitely many VaR constraints in such a case will only approximate the use of CVaR, the approximation will improve as the number of VaR constraints increases.

[11] Since $\alpha = 1 - 2/S$, the use of three or more VaR constraints in controlling CVaR does not improve upon using only two VaR constraints with appropriately chosen confidence levels and bounds. Section 3.5 discusses the case when $\alpha < 1 - 2/S$.

compute CVaR. Lastly, assume that the weight of each asset is restricted to range from w_l to w_u, where $w_l < w_u$. Since the set of states is assumed to be finite, portfolio losses are bounded from above by some value $\bar{z}$. Eq. (5) implies that $\bar{z}$ is an upper bound on the CVaRs of feasible portfolios in the absence of VaR constraints.

3.1. One VaR constraint

Suppose that a VaR constraint is imposed. The following result provides an upper bound on the CVaRs of portfolios that meet the constraint.

Theorem 1. *Consider a VaR constraint using confidence level α_1 and bound V_1 where $\alpha_1 \geqslant \alpha$ and $V_1 < \bar{z}$. The CVaR of any portfolio that meets this constraint does not exceed: (i) $\bar{z}$ if $\alpha_1 = \alpha$; (ii) $(\bar{z} + V_1)/2$ if $\alpha < \alpha_1 \leqslant \alpha + 1/S$; and (iii) V_1 if $\alpha_1 > \alpha + 1/S$.*

Using Theorem 1, the size of the upper bound on the CVaRs of portfolios that meet the VaR constraint depends on confidence level α_1 and bound V_1.[12] First, assume that $\alpha_1 = \alpha$. Then, the upper bound is $\bar{z}$. Since $\bar{z}$ is also an upper bound on the CVaRs of feasible portfolios in the absence of the constraint, its presence does not reduce the size of the upper bound. This result follows from the fact that the constraint may not restrict the size of losses in the states that are used to find CVaR.[13]

Second, assume that $\alpha < \alpha_1 \leqslant \alpha + 1/S$. Then, the upper bound is $(\bar{z} + V_1)/2$. Since $V_1 < \bar{z}$, we have $(\bar{z} + V_1)/2 < \bar{z}$. Hence, if α_1 increases from α to any value in the interval $(\alpha, \alpha + 1/S]$, then the size of the upper bound decreases.[14] If V_1 decreases, then the size of the upper bound also decreases. These results can be understood by noting that the VaR constraint is tightened when either α_1 increases or V_1 decreases.

Third, assume that $\alpha_1 > \alpha + 1/S$. Then, the upper bound is V_1. Since $V_1 < \bar{z}$, we have $V_1 < (\bar{z} + V_1)/2$. Hence, if α_1 increases from any value in the interval $(\alpha, \alpha + 1/S]$ to any value in the interval $(\alpha + 1/S, 1)$, then the size of the upper bound decreases. If V_1 decreases, then the size of the upper bound also decreases. Again, these results can be understood by noting that the VaR constraint is tightened when either α_1 increases or V_1 decreases.

Generally, a VaR constraint restricts the size of losses in at most one of the states that is used to find CVaR. Hence, portfolios that meet the constraint may suffer notable losses and thus have large CVaRs. Therefore, the extent to which a VaR constraint precludes the selection of portfolios with large CVaRs is limited.

In sum, when the confidence level used by a VaR constraint is equal to the one used to compute CVaR, the constraint does not preclude the selection of portfolios with large CVaRs. When the former confidence level is higher than the latter, the extent to which the constraint precludes their selection depends on its confidence level and bound. This extent is larger when the confidence level is higher or the bound is smaller. However, such an extent is limited by the fact that the constraint restricts losses in at most one of the states that is used to find CVaR. Accordingly, we next examine the case when two VaR constraints are imposed.

3.2. Two VaR constraints

Suppose that two VaR constraints are imposed. The following result provides an upper bound on the CVaRs of portfolios that meet the constraints.

Theorem 2. *Consider two VaR constraints using confidence levels $\{\alpha'_k\}_{k=1}^2$ and bounds $\{V'_k\}_{k=1}^2$ where $\alpha \leqslant \alpha'_1 < \alpha'_2$ and $V'_1 < V'_2 < \bar{z}$. The CVaR of any portfolio that meets these constraints does not exceed: (i) $(\bar{z} + V'_2)/2$ if $\alpha'_1 = \alpha < \alpha'_2 \leqslant \alpha + 1/S$; (ii) V'_2 if $\alpha'_1 = \alpha < \alpha + 1/S < \alpha'_2$; and (iii) $(V'_1 + V'_2)/2$ if $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2$.*

Using Theorem 2, the size of the upper bound on the CVaRs of portfolios that meet the two VaR constraints depends on confidence levels $\{\alpha'_k\}_{k=1}^2$ and bounds $\{V'_k\}_{k=1}^2$.[15] First, assume that $\alpha'_1 = \alpha < \alpha'_2 \leqslant \alpha + 1/S$. Then, the upper bound is $(\bar{z} + V'_2)/2$. Since $V'_2 < \bar{z}$, we have $(\bar{z} + V'_2)/2 < \bar{z}$. Hence, the imposition of the constraints reduces the size of the upper bound (relative to the case where no constraint is imposed). While decreasing V'_1 does not reduce the size of the upper bound, decreasing V'_2 reduces it. This result can be seen by noting that the constraint using α'_1 may not restrict the size of losses in the states that are used to find CVaR, whereas the constraint using α'_2 restricts the size of losses in one of them.

Second, assume that $\alpha'_1 = \alpha < \alpha + 1/S < \alpha'_2$. Then, the upper bound is V'_2. Since $V'_2 < \bar{z}$, we have $V'_2 < (\bar{z} + V'_2)/2$. Hence, if α'_2 increases from any value in the interval $(\alpha, \alpha + 1/S]$ to any value in the interval $(\alpha + 1/S, 1)$, then the size of the upper bound decreases. While decreasing V'_1 does not reduce the size of the upper bound, decreasing V'_2 reduces it. Again, this result can be seen by noting that the constraint using α'_1 may not restrict the size of losses in the states that are used to find CVaR, whereas the constraint using α'_2 restricts the size of losses in one of them.

Third, assume that $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2$. Then, the upper bound is $(V'_1 + V'_2)/2$. Since $V'_1 < V'_2$, we have $(V'_1 + V'_2)/2 < V'_2$. Hence, if α'_1 increases from α to any value in the interval $(\alpha, \alpha + 1/S]$, then the size of the upper bound decreases. Note that decreasing either V'_1 or V'_2 reduces the size of the upper bound. This result can be seen by noting that each VaR constraint restricts the size of losses in one of the two states that are used to find CVaR.

In sum, the extent to which two VaR constraints preclude the selection of portfolios with large CVaRs depends on their confidence levels and bounds. This extent is larger when the confidence levels are higher or the bounds are smaller. Importantly, when the confidence levels and bounds are appropriately chosen, the joint use of two VaR constraints restricts losses in the two states that are used to find CVaR.

3.3. Comparing the effectiveness of one and two VaR constraints in controlling CVaR

Consider: (1) a VaR constraint with confidence level α_1 and bound V_1; and (2) two VaR constraints with confidence levels $\{\alpha'_k\}_{k=1}^2$ and bounds $\{V'_k\}_{k=1}^2$. Of particular interest is the question of whether the latter constraints are more effective in controlling CVaR than the former. In general, as Theorems 1 and 2 indicate, the answer to this question depends on the values of (α_1, V_1) and $\{(\alpha'_k, V'_k)\}_{k=1}^2$. For example, suppose that: (a) $\alpha_1 > \alpha + 1/S$ and $V_1 < \bar{z}$ as in Theorem 1(iii); and (b) $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2$ and $V'_1 < V'_2 < \bar{z}$ as in Theorem 2(iii). Then, the upper bound on CVaR with one VaR constraint is V_1, whereas that with two VaR

[12] Note that Theorem 1 provides the least upper bound on the CVaRs of the portfolios that meet the constraint. That is, there are cases where portfolios that meet the constraint have a CVaR equal to this upper bound. However, there are also cases where the upper bound on the CVaRs of such portfolios is smaller than the one presented in Theorem 1.

[13] For example, consider portfolios with different returns in different states. Since $\alpha_1 = \alpha = 1 - 2/S$, the VaR constraint restricts the size of the third largest loss that a portfolio suffers, whereas a portfolio's CVaR equals the average of its largest and second largest losses; see Eqs. (1), (2), and (5).

[14] In assessing the effect of a change in the value of some parameter (e.g., α_1) on the size of the upper bound, the values of other parameters (e.g., V_1) are assumed to remain unchanged.

[15] Like Theorem 1, Theorem 2 provides the least upper bound on the CVaRs of the portfolios that meet the constraints. That is, there are cases where portfolios that meet the constraints have a CVaR equal to this upper bound. However, there are also cases where the upper bound on the CVaRs of such portfolios is smaller than the one presented in Theorem 2.

constraints is $(V'_1 + V'_2)/2$. It follows that three cases are possible when comparing the effectiveness of one and two constraints in controlling CVaR. First, if $V_1 < (V'_1 + V'_2)/2$, then the upper bound on CVaR with one constraint is smaller than that with two constraints. Second, if $V_1 = (V'_1 + V'_2)/2$, then the upper bound on CVaR with one constraint is equal to that with two constraints. Third, if $V_1 > (V'_1 + V'_2)/2$, then the upper bound on CVaR with one constraint is larger than that with two constraints.

While there are conditions under which two constraints are less effective in controlling CVaR than one VaR constraint, there are also conditions under which they are more effective. Importantly, the latter conditions are not overly restrictive. For example, in the third case above, the condition that $V_1 > (V'_1 + V'_2)/2$ holds even in situations where the VaR constraint with confidence level α_1 and bound V_1 is not particularly loose. Specifically, suppose that $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2 = \alpha_1$ and $V_1 = (V'_1 + 2V'_2)/3$. Since $V'_1 < V'_2$, we have $V_1 > (V'_1 + V'_2)/2$ and $V'_1 < V_1 < V'_2$. Observe that the constraint with confidence level α_1 and bound V_1 is: (a) not looser than the constraint with confidence level α'_1 and bound V'_1 (since $\alpha_1 > \alpha'_1$)[16]; and (b) tighter than the constraint with confidence level α'_2 and bound V'_2 (since $\alpha_1 = \alpha'_2$ and $V_1 < V'_2$).

More generally, a VaR constraint restricts the loss in at most one of the states that is used to compute CVaR, whereas two appropriately chosen VaR constraints restrict the losses in the two states that are used to compute CVaR.[17] Hence, the use of two appropriately chosen VaR constraints is more effective in controlling CVaR than the use of one VaR constraint.

3.4. Loosening portfolio weight restrictions

We now assess the effect of loosening portfolio weight restrictions (e.g., considering the case of allowing short selling relative to the case when it is disallowed) on the extent to which VaR constraints preclude the selection of portfolios with large CVaRs. Specifically, suppose that the weight of each asset is now restricted to range from w'_l to w'_u, where $w'_l \leqslant w_l < w_u \leqslant w'_u$ and at least one of the two inequalities '$\leqslant$' is strict. Let $\bar{z}'$ denote the upper bound on portfolio losses when the weight of each asset is restricted to range from w'_l to w'_u. By construction, we have $\bar{z}' \geqslant \bar{z}$. First, assume that $\bar{z}' = \bar{z}$. Using Theorem 1, the upper bound on the CVaRs of portfolios that meet a VaR constraint remains unchanged. Similarly, using Theorem 2, the upper bound on the CVaRs of portfolios that meet two VaR constraints remains unchanged.

Second, assume that $\bar{z}' > \bar{z}$. Using Theorem 1, the effect of loosening portfolio weight restrictions on the size of the upper bound on the CVaRs of portfolios that meet a VaR constraint depends on confidence level α_1. Its size decreases if $\alpha_1 \leqslant \alpha + 1/S$ (see (i) and (ii) in Theorem 1), but it remains unchanged if $\alpha_1 > \alpha + 1/S$ (see (iii)). Similarly, using Theorem 2, the effect of loosening portfolio weight restrictions on the size of the upper bound on the CVaRs of portfolios that meet two VaR constraints depends on confidence levels $\{\alpha'_k\}_{k=1}^2$. Its size decreases if $\alpha'_1 = \alpha < \alpha'_2 \leqslant \alpha + 1/S$ (see (i) in Theorem 2), but it remains unchanged if $\alpha'_1 = \alpha < \alpha + 1/S < \alpha'_2$ or $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2$ (see (ii) and (iii)).

In sum, loosening portfolio weight restrictions can reduce the extent to which portfolios with large CVaRs are precluded by either one or two VaR constraints. Hence, the effectiveness of VaR constraints in controlling CVaR possibly depends on the tightness of such restrictions.

3.5. Using a lower confidence level in determining CVaR

Theorems 1 and 2 extend in a natural way to the case when the confidence level used to find CVaR is lower than $1 - 2/S$. However, in order to achieve a relatively small upper bound on CVaR, the number of required VaR constraints is possibly larger than two. Using Eq. (5), a portfolio's CVaR is approximately equal to the average of an appropriate set of its VaRs at various confidence levels. When the number of VaR constraints is smaller than the cardinality of this set, the constraints might not be effective in controlling CVaR, but their effectiveness increases as the number of VaR constraints increases. Note that the cardinality of the aforementioned set is larger for lower values of the confidence level used to find CVaR. Hence, the number of VaR constraints required to effectively control CVaR is larger for lower values of the confidence level used to find CVaR.

3.6. Adding an expected return constraint

Let $\overline{\boldsymbol{R}}$ denote the $(J+1) \times 1$ vector $[\overline{R}_1 \cdots \overline{R}_{J+1}]^\top$, where $\overline{R}_j$ is asset j's expected return. Consider the expected return constraint:

$$E_{\boldsymbol{w}} = E, \tag{7}$$

where $E_{\boldsymbol{w}} \equiv \overline{\boldsymbol{R}}'\boldsymbol{w}$ denotes portfolio $\boldsymbol{w}$'s expected return and E is the required expected return. Of particular interest is the question of whether VaR constraints are effective in controlling CVaR when an expected return constraint is also in place. Previous work suggests the use of a mean-CVaR model to control tail risk (see, e.g., Agarwal and Naik, 2004; Alexander and Baptista, 2004; Bertsimas et al., 2004). Accordingly, we examine whether VaR constraints lead to the selection of portfolios with small efficiency losses relative to the mean-CVaR frontier.

Since there are no closed-form solutions for portfolios on the mean-CVaR frontier in the presence of portfolio weight restrictions, we examine the size of efficiency losses numerically. We first consider the case where short selling is disallowed (i.e., $w_l = 0$), and then consider the case where it is allowed (i.e., $w_l < 0$).

4. Risk management when short selling is disallowed

This section explores the effectiveness of VaR-based risk management systems in controlling CVaR when short selling is disallowed.

4.1. Optimization inputs

We consider a problem of wealth allocation among: (1) Treasury bills, (2) government bonds, (3) corporate bonds, and (4) the six size/book-to-market Fama–French portfolios.[18] Monthly returns on Treasury bills and Fama–French portfolios are obtained from Kenneth French's website; the Merrill Lynch government and corporate bond master indices are extracted from Bloomberg to measure the monthly returns on, respectively, government and corporate bonds. Table 1 shows summary statistics on the asset class returns during 1978–2008.[19] Following the historical simulation approach, our

[16] Note that the constraint with confidence level α_1 and bound V_1 is also not tighter than the constraint with confidence level α'_1 and bound V'_1 (since $V_1 > V'_1$).

[17] Here, we use the terms 'appropriately chosen' to emphasize that the confidence levels and bounds for the two VaR constraints need to be carefully selected so that they are indeed more effective in controlling CVaR than one VaR constraint.

[18] Similar results (available upon request) are obtained if (1) Treasury bills, or (2) bonds, or (3) both Treasury bills and bonds are removed from consideration. Section 6 confirms these results by considering a larger set of asset classes.

[19] The standard deviation of the return on Treasury bills is reported as zero since Treasury bills are assumed to be risk-free.

Table 1
Summary statistics on asset class returns. This table presents summary statistics on the monthly returns on nine asset classes during the period 1978–2008: (i) Treasury bills (assumed to be risk-free), (ii) government bonds as measured by the Merrill Lynch government bond master index, (iii) corporate bonds as measured by the Merrill Lynch corporate bond master index, and (iv) the six size/book-to-market Fama–French portfolios. These portfolios result from sorting stocks along the dimensions of: (a) market capitalization (small and big) and (b) book-to-market ratio (low, intermediate, and high). Returns on Treasury bills and Fama–French portfolios are obtained from Kenneth French's website. Returns on bond indices are obtained from Bloomberg. All numbers are reported in percentage points per month.

	Treas. bills	Govt. bonds	Corp. bonds	Fama–French portfolios					
				Small			Big		
				Low	Inter.	High	Low	Inter.	High
Mean	0.48	0.71	0.68	0.84	1.35	1.44	0.95	1.01	1.08
Std. dev.	0.00	1.67	2.10	7.00	5.11	5.05	4.82	4.38	4.38
$VaR_{95\%}$	−0.48	1.85	2.50	9.98	6.73	6.64	7.23	5.88	5.90
$CVaR_{95\%}$	−0.48	2.76	4.27	15.96	12.21	12.54	10.40	9.76	10.57
$VaR_{99\%}$	−0.48	3.15	6.99	21.62	18.49	17.36	11.66	10.79	13.08
$CVaR_{99\%}$	−0.48	4.41	7.29	25.66	21.31	21.33	16.44	16.90	17.53

optimization inputs are based on these sample statistics.[20] Not surprisingly, average returns, standard deviations, VaRs, and CVaRs of asset classes involving stocks are larger than those involving bonds. Also, for any given risky asset class and confidence level, CVaRs are larger than VaRs. Lastly, for any given risky asset class, higher confidence levels are associated with larger VaRs and CVaRs.

4.2. Methodology

Fig. 1 summarizes the methodology that is used to assess the effectiveness of K VaR constraints in controlling CVaR, where $K = 1, 2, 3$.[21] Initially short selling is disallowed and then it is allowed, resulting in six basic cases. For each case we take six steps.

While the minimum required expected return $\underline{E}$ is assumed to be the risk-free rate, the maximum feasible expected return $\overline{E}$ is found in Step 1. Step 2 uses the values of $\underline{E}$ and $\overline{E}$ to calculate $\delta \equiv (\overline{E} - \underline{E})/100$. Using the value of δ, Step 3 creates a grid of 101 expected returns $\{E_i\}_{i=0}^{100}$ that range from $E_0 = \underline{E}$ to $E_{100} = \overline{E}$ in return increments of δ. Hence, E_i is $(i/100)$% of the way between $\underline{E}$ and $\overline{E}$ for $i = 1, \ldots, 99$.

In Step 4, the confidence level used to compute CVaR, α, as well as the ones used by the VaR constraints, $\{\alpha_k\}_{k=1}^{K}$, are chosen. As before, we assume that $\alpha_k \geqslant \alpha$ for $k = 1, \ldots, K$.[22] In Step 4, we also determine the bounds for the VaR constraints. When confidence level α is used to compute CVaR, let $\boldsymbol{w}_{\alpha,E}$ denote the portfolio on the mean-CVaR frontier with an expected return of E. Note that the VaR of this portfolio at confidence level α_k is given by $V_{\alpha_k,\boldsymbol{w}_{\alpha,E}}$. Hence, the following bound $V_{\alpha_k,\alpha,E}$ is used for a VaR constraint with confidence level α_k when the confidence level to compute CVaR is α and the required expected return is E:

$$V_{\alpha_k,\alpha,E} \equiv V_{\alpha_k,\boldsymbol{w}_{\alpha,E}}. \tag{8}$$

This constraint: (i) allows (but typically does not force) the selection of portfolio $\boldsymbol{w}_{\alpha,E}$, which has by construction an expected return of E and a zero efficiency loss; and (ii) precludes (as much as possible) portfolios with an expected return of E that have the greatest efficiency losses, thereby leading to the smallest maximum efficiency loss. Generally, when a bound $V \neq V_{\alpha_k,\alpha,E}$ is used, either (i) or (ii) do not hold. First, the use of a bound $V < V_{\alpha_k,\alpha,E}$ precludes the selection of portfolio $\boldsymbol{w}_{\alpha,E}$. Second, the use of a bound $V > V_{\alpha_k,\alpha,E}$ typically results in a larger maximum efficiency loss. The choice of bound $V_{\alpha_k,\alpha,E}$ is thus appealing.[23]

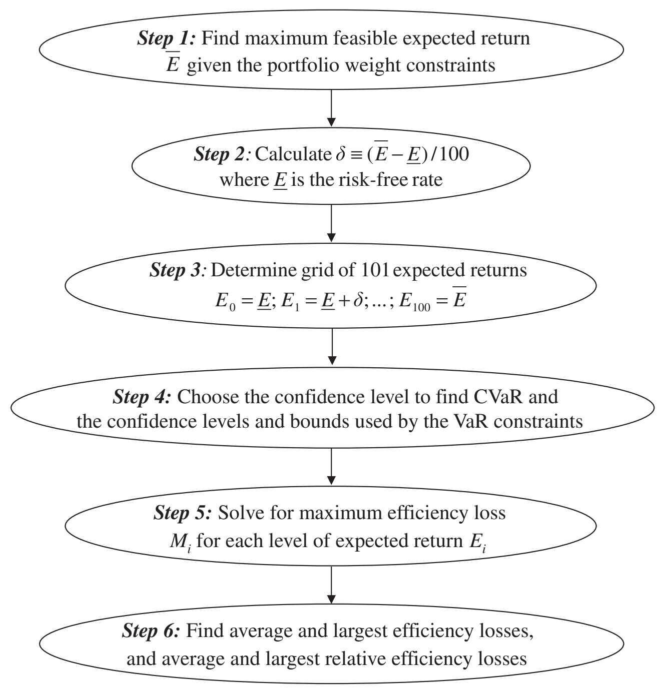

Fig. 1. Methodology. We assume that the number of VaR constraints, K, is either 1, 2, or 3. Initially short selling is disallowed and then it is allowed, resulting in six basic cases. Historical simulation is used to estimate VaR and CVaR for all portfolios.

In Step 5, we find *maximum efficiency loss* M_i for each level of required expected return E_i. Fig. 2 illustrates how M_i is determined. The curve represents portfolios on the mean-CVaR frontier. Point p_i^{max} represents the portfolio that has an expected return of E_i, satisfies the VaR constraints, and has maximum CVaR, denoted by C_i^{max}.[24] Point p_i^{min} represents the portfolio that has the same expected

[20] Since the average return on corporate bonds is smaller than that on government bonds, we consider two additional cases where the expected return on corporate bonds is adjusted upwards. The first case involves multiplying all sample returns on corporate bonds by 1.1. The second case involves adding seven basis points to the sample returns on corporate bonds. In both cases, the expected return on corporate bonds is 0.75%, which exceeds that on government bonds, 0.71%. The results in these two cases (available upon request) are similar to those presented when using the unadjusted sample average return on corporate bonds.

[21] Compared with the results with $K = 3$, the results with $K = 4$ and $K = 5$ (available upon request) indicate increased effectiveness of the VaR constraints in controlling CVaR.

[22] This assumption is motivated by our finding that the use of confidence levels with $\alpha_k \geqslant \alpha$ for $k = 1, \ldots, K$ is typically more effective in controlling CVaR than the use of confidence levels with $\alpha_k \leqslant \alpha$ for $k = 1, \ldots, K$. Nevertheless, Section 6 examines the case when the latter confidence levels are used.

[23] Note that this bound is typically increasing in the required expected return. Recognizing that a higher expected return is generally associated with more risk, the choice of such a bound is also motivated by the practical plausibility of setting larger bounds when required expected returns are higher since failure to do so could result in the non-existence of a feasible portfolio.

[24] Our approach does not suggest that money managers or banks maximize CVaR subject to an expected return constraint and one or more VaR constraints. The sole purpose of this approach is to assess whether the use of VaR constraints allows the selection of portfolios with large CVaRs.

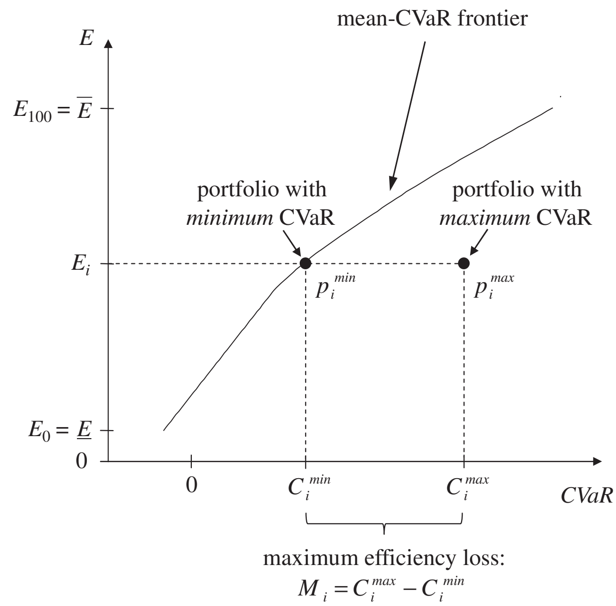

Fig. 2. Determining maximum efficiency losses. The curve represents portfolios on the mean-CVaR frontier for various levels of expected return. The minimum required expected return $E_0 = \underline{E}$ is assumed to be the risk-free rate. The maximum required expected return $E_{100} = \overline{E}$ depends on whether short selling is disallowed or allowed. Point p_i^{max} represents the portfolio that has an expected return of E_i, satisfies the VaR constraints, and has maximum CVaR, denoted by C_i^{max}. Point p_i^{min} represents the portfolio that has the same expected return and minimum CVaR, denoted by C_i^{min}. When the required expected return is E_i, the maximum efficiency loss is given by $M_i = C_i^{max} - C_i^{min}$.

return but with minimum CVaR (i.e., the portfolio on the mean-CVaR frontier with an expected return of E_i), denoted by C_i^{min}. Since $M_i = C_i^{max} - C_i^{min}$, M_i is the maximum increase in CVaR allowed by the VaR constraints given a required expected return of E_i.

In Step 6, the average of $\{M_i\}_{i=0}^{100}$, referred to as the *average efficiency loss*, is determined. The single largest maximum efficiency loss, referred to as the *largest efficiency loss*, is also determined. Next, *relative efficiency loss* $REL_i \equiv M_i/C_i^{min}$ is determined for each E_i. Note that REL_i is the ratio between efficiency loss M_i and the CVaR of the minimum CVaR portfolio with an expected return of E_i. Lastly, we compute average and largest relative efficiency losses from $\{REL_i\}_{i=0}^{100}$.[25] Subsequent analysis is based on the values of $\{M_i\}_{i=0}^{100}$ and $\{REL_i\}_{i=0}^{100}$, and the average and largest (relative) efficiency losses for each one of the six cases.[26]

An examination of maximum efficiency losses captures the idea of being agnostic regarding the portfolio selection model that is used in the presence of VaR constraints.[27] The motivation for this idea is twofold. First, we are interested in exploring the effectiveness of these constraints in controlling CVaR without making any assumption on the portfolio selection model that is used in their presence. If losses are relatively small for a given set of constraints, then this set of constraints is effective in controlling CVaR. Thus, any portfolio that meets the constraints, no matter how selected, will have a CVaR that is close to that of the portfolio with the same expected return that minimizes CVaR. However, if losses are substantial, then the set of constraints is not effectively controlling CVaR as it allows the selection of portfolios with relatively large CVaRs.

Second, while the use of VaR constraints by money managers and large banks is apparent, we do not know the exact models that they utilize for portfolio selection. For example, Lucas (2001) and Hull (2007, p. 198) note that trading desks might seek to take on as much risk as possible (subject to existing risk constraints) in attempting to: (a) recuperate losses, or (b) exploit options in compensation contracts. Consistent with the existence of incentives to take risks, Berkowitz and O'Brien (2002) find that large banks sometimes suffer losses in their trading books that are surprisingly larger than their VaRs.

4.3. One VaR constraint

Suppose that a single VaR constraint is imposed. The first two columns of Panel A of Table 2 show the values of α and α_1 that are used. The next four columns show that efficiency losses can be sizeable. For example, when $\alpha = \alpha_1 = 95\%$ (see the first row of the panel), average and largest efficiency losses are, respectively, 0.51% and 1.21%, whereas average and largest relative efficiency losses are, respectively, 15.69% and 39.74%. Furthermore, losses with $\alpha = \alpha_1$ can be either smaller or larger than with $\alpha < \alpha_1$. For example, the average efficiency loss is 0.51% if $\alpha = \alpha_1 = 95\%$ (see the first row) but it is: (a) smaller at 0.38% if $95\% = \alpha < \alpha_1 = 96.25\%$ (see the second row); and (b) larger at 0.73% if $95\% = \alpha < \alpha_1 = 97.50\%$ (see the third row). It follows that the constraint that uses a confidence level of 96.25% is more effective in controlling CVaR at the 95% confidence level than the constraint that uses a confidence level of 95%. However, the constraint that uses a confidence level of 97.50% is less effective in controlling CVaR at the 95% confidence level than the constraint that uses a confidence level of 95%. Thus, a constraint that uses a higher confidence level might be less effective in controlling CVaR if it also uses a larger bound as we assume.[28]

The average CVaR of portfolios with maximum efficiency losses is the sum of their average efficiency loss plus the average CVaR of portfolios on the mean-CVaR frontier. Since the latter average is 5.07% if $\alpha = 95\%$, the third column of Panel A of Table 2 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 5.37% [=5.07% + 0.30%] when $\alpha_1 = 98.25\%$ to 5.80% [=5.07% + 0.73%] when $\alpha_1 = 97.50\%$. Similar results are obtained if $\alpha = 99\%$. Hence, the use of one VaR constraint is not fully effective in controlling CVaR.[29]

Rows (1)–(4) of Fig. 3 provide box plots of maximum efficiency losses. In the top (bottom) panel, α and α_1 take the values used in first (last) four rows of Panel A of Table 2. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the

[25] There are two difficulties in using this measure of relative efficiency loss. First, portfolios on the mean-CVaR frontier with expected returns close to the risk-free return have negative CVaRs and thus their relative efficiency losses are negative. Second, even when the CVaRs of certain portfolios on the mean-CVaR frontier are positive, they could be arbitrarily close to zero, resulting in arbitrarily large relative efficiency losses. In order to circumvent these two difficulties, we compute the average and largest relative efficiency losses by solely using levels of expected return for which the correspondent portfolios on the mean-CVaR frontier have CVaRs greater than 1%. Note that such losses are larger if we include levels of expected return for which the correspondent portfolios on the mean-CVaR frontier have smaller (but positive) CVaRs.

[26] As a robustness check, we also compute the losses using only the interquartile range of expected returns $\{E_i\}_{i=25}^{75}$. The results (available upon request) are similar to those reported when using their entire range.

[27] However, we are not agnostic regarding the model that is used to control tail risk in the absence of VaR constraints. As noted earlier, we use the mean-CVaR model to find the efficiency losses.

[28] As Eq. (8) indicates, each VaR constraint uses a bound equal to the VaR of the portfolio with minimum CVaR that has the required level of expected return. Moreover, the constraint that uses a higher confidence level also uses this higher confidence level to determine the VaR of such a portfolio. Accordingly, this constraint uses a larger bound.

[29] Similarly, we find that the median CVaR of portfolios with maximum efficiency losses is larger than the median CVaR of portfolios on the mean-CVaR frontier. However, the former median is closer to the latter when more than one VaR constraint is imposed. These results are available upon request.

Table 2
Efficiency losses with one VaR constraint. This table examines whether CVaR at confidence level α can be controlled by utilizing a VaR constraint with confidence level α_1 and bound $V_{\alpha_1,\alpha,E}$ as defined in Eq. (8). The first two columns show the values of α and α_1 that are used. The third and fourth columns report, respectively, average and largest efficiency losses. The fifth and sixth columns report, respectively, average and largest relative efficiency losses. The last two columns report average and largest distances. Short selling is disallowed in Panel A, whereas it is allowed in Panel B. All numbers are reported in percentage points. Efficiency losses are per month.

Confidence levels		Efficiency loss		Relative efficiency loss		Distance	
CVaR	VaR constraint						
α	α_1	Average	Largest	Average	Largest	Average	Largest
Panel A: Short selling disallowed							
95.00	95.00	0.51	1.21	15.69	39.74	7.98	24.47
95.00	96.25	0.38	1.26	13.24	32.18	7.48	19.98
95.00	97.50	0.73	2.11	14.92	40.48	13.30	40.76
95.00	98.25	0.30	0.98	13.18	45.45	4.28	15.30
99.00	99.00	0.65	2.34	19.25	62.68	5.64	18.19
99.00	99.25	0.20	0.94	7.66	34.12	3.23	14.81
99.00	99.50	0.07	0.27	2.28	8.06	1.62	7.34
99.00	99.75	0.17	0.71	5.58	18.14	2.72	11.71
Panel B: Short selling allowed							
95.00	95.00	2.27	5.91	63.48	79.25	41.36	85.53
95.00	96.25	1.47	5.00	40.74	70.09	32.14	81.00
95.00	97.50	1.36	3.85	37.14	58.47	36.65	80.78
95.00	98.25	1.10	2.61	34.23	59.40	36.98	75.95
99.00	99.00	1.38	2.98	50.20	155.45	28.56	63.26
99.00	99.25	1.64	3.79	50.69	113.71	33.59	69.65
99.00	99.50	1.55	3.04	40.70	80.96	37.05	68.36
99.00	99.75	0.62	1.81	11.66	23.94	15.67	59.45

box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss.[30] Note that the median of losses with $\alpha = \alpha_1$ is larger than that with $\alpha < \alpha_1$ in all cases (in each panel compare row (1) with rows (2)–(4)). Also, the range of losses indicates that their size depends on the required expected return.

Panel A of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1 = 95\%$.[31] Three results are worth noting. First, this relation is non-monotonic. Note that the size of losses depends on the size of the set of portfolios that satisfy VaR and expected return constraints. Specifically, losses tend to be larger when such a set is also larger. Hence, the non-monotonic relation between maximum efficiency losses and required expected returns can be understood by noting that there is also a non-monotonic relation between the size of this set of portfolios and the required expected return.[32]

Second, losses are either zero or close to zero for both small levels of expected return (i.e., levels of expected return equal to or slightly larger than $\underline{E} = 0.48\%$) and large levels of expected return (i.e., levels of expected return slightly smaller than or equal to $\overline{E} = 1.44\%$). This result can be understood by noting the relatively small size of the set of portfolios that satisfy VaR and expected return constraints and do not involve short selling when such levels of expected return are required.

Third, losses exceed 0.5% for a relatively large fraction of the moderate levels of expected return (i.e., levels of expected return sufficiently larger than $\underline{E} = 0.48\%$ but sufficiently smaller than $\overline{E} = 1.44\%$). This result can be understood by noting the relatively large size of the set of portfolios that satisfy VaR and expected return constraints and do not involve short selling when such levels of expected return are required.

We use $|\boldsymbol{w}^1 - \boldsymbol{w}^2|/\sqrt{J+1}$ in assessing the *distance* between any two given portfolios $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ (here, $|\cdot|$ denotes the Euclidean norm).[33] Let D_i denote the distance between the portfolio with maximum efficiency loss and the portfolio on the mean-CVaR frontier when the required expected return is E_i. The average (largest value) of $\{D_i\}_{i=0}^{100}$ is referred to as *average* (*largest*) *distance*. The last two columns of Panel A of Table 2 show that average distances range from 1.62% to 13.30%, whereas largest distances range from 7.34% to 40.76%.

Since efficiency losses are possibly large, the effectiveness of VaR and CVaR constraints in controlling CVaR can notably differ in discrete-time settings where asset returns are assumed to have a realistic discrete distribution and historical simulation is used to compute VaR and CVaR. The result contrasts with Alexander and Baptista (2004). Indeed, they show that when the confidence level used to compute VaR is appropriately higher than the one used to compute CVaR, VaR and CVaR constraints have the same effect on portfolio selection within a discrete-time mean-variance model where asset returns are assumed to have a normal distribution and the variance-covariance approach is used to compute VaR and CVaR.

In sum, when the confidence levels used to estimate VaR and CVaR coincide, a VaR constraint allows the selection of portfolios with substantial losses. When the confidence level used by the constraint exceeds the one used to estimate CVaR, the extent to which the constraint allows the selection of portfolios with substantial losses might be smaller. However, the use of a relatively high confidence level for the constraint is possibly impractical if the confidence level used to find CVaR is also relatively high and the sample size is relatively small (e.g., VaR estimates might be imprecise due to the relatively small number of observations in the tail of the distribution).[34] Furthermore, portfolios with substantial losses might still be selected even when the confidence level used by the constraint exceeds the one used to estimate CVaR. Thus, the use of a single VaR constraint is either ineffective or impractical

[30] In rows (1)–(4) of Panel B, the left vertical and horizontal lines are not present because the lowest value of the loss equals either the lower quartile (in row (1)) or the median (in rows (2)–(4)).

[31] Similar results (available upon request) are obtained when either $\alpha = 95\% < \alpha_1$ or $\alpha = 99\% \leqslant \alpha_1$.

[32] For example, since short selling is disallowed and different assets are assumed to have different expected returns, this set contains a single portfolio if the required expected return is either $\underline{E} = 0.48\%$ or $\overline{E} = 1.44\%$. However, the set is possibly larger if the required expected return is strictly between $\underline{E}$ and $\overline{E}$.

[33] Note that $|\boldsymbol{w}^1 - \boldsymbol{w}^2|/\sqrt{J+1} = \sqrt{\Sigma_{j=1}^{J+1}\left(w_j^1 - w_j^2\right)^2}/\sqrt{J+1}$.

[34] For example, consider the case where a confidence level of 99% is used to estimate CVaR. Suppose that the confidence level used to find VaR is sufficiently high (e.g., 99.99%). Since we use historical simulation and 372 monthly observations, an estimate of the VaR of a given portfolio is the worst monthly loss that the portfolio suffers over the sample period.

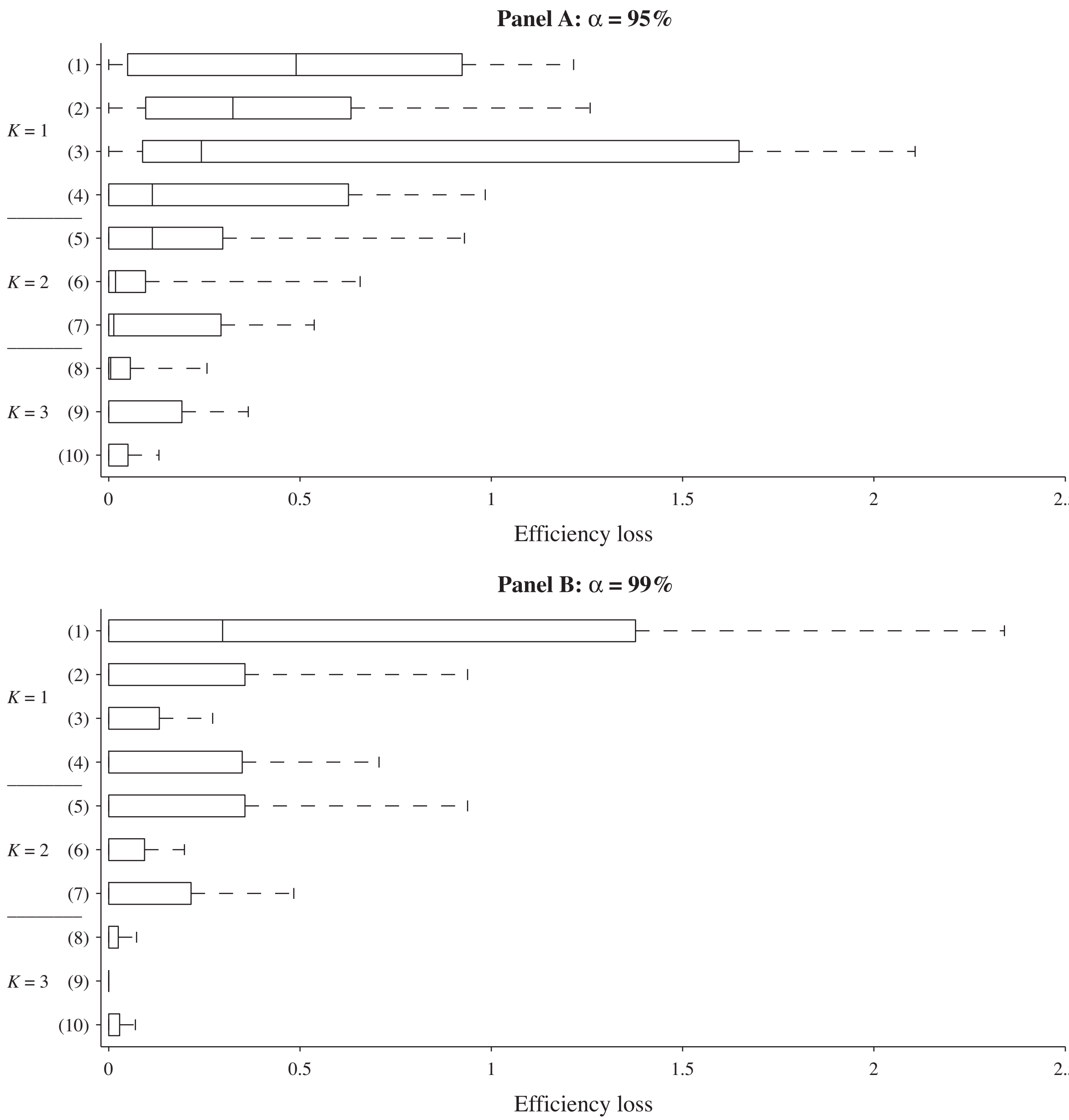

Fig. 3. Box plots of maximum efficiency losses with short selling disallowed. This figure shows box plots of maximum efficiency losses when short selling is disallowed. These losses arise when seeking to control CVaR at confidence level α with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{K}$ as defined in Eq. (8). While α = 95% in Panel A, α = 99% in Panel B. In rows (1)–(4), $K = 1$ and α_1 takes the values used in Table 2. In rows (5)–(7), $K = 2$ and $\{\alpha_k\}_{k=1}^{2}$ take the values used in Table 3. In rows (8)–(10), $K = 3$ and $\{\alpha_k\}_{k=1}^{3}$ take the values used in Table 6. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

in controlling CVaR. Accordingly, we next examine whether the use of multiple VaR constraints is effective in controlling CVaR.

4.4. Two VaR constraints

Suppose that two VaR constraints are imposed. The first three columns of Panel A of Table 3 show the values of α and $\{\alpha_k\}_{k=1}^{2}$ that are used.[35] The next four columns show that efficiency losses are generally smaller than when a single VaR constraint is imposed. For example, compare the cases of: (1) one constraint and $\alpha = \alpha_1$ = 95% (see the first row of Panel A of Table 2); and (2) two constraints, $\alpha = \alpha_1$ = 95%, and α_2 = 96.25% (see the first row of Panel A of Table 3). Average and largest efficiency losses in the first case are, respectively, 0.51% and 1.21%, whereas they are, respectively, 0.18% and 0.93% in the second one. Similarly, average and largest relative efficiency losses in the former case are, respectively, 15.69% and 39.74%, whereas they are, respectively, 6.47% and 18.03% in the latter.

As noted earlier, the use of a relatively high confidence level in estimating VaR is possibly impractical if the confidence level used to find CVaR is also relatively high and the sample size is relatively small. Hence, it is pertinent to assess the effectiveness of two VaR

[35] In our paper, a portfolio's VaR at the 99.90% confidence level equals its VaR at the 99.75% confidence level since we use a sample with 372 observations. Hence, the last row of Panels A and B of Table 3 covers the confidence levels required by the new Basel framework for trading books (i.e., 99% and 99.90%); see Basel Committee on Banking Supervision (2009a,b).

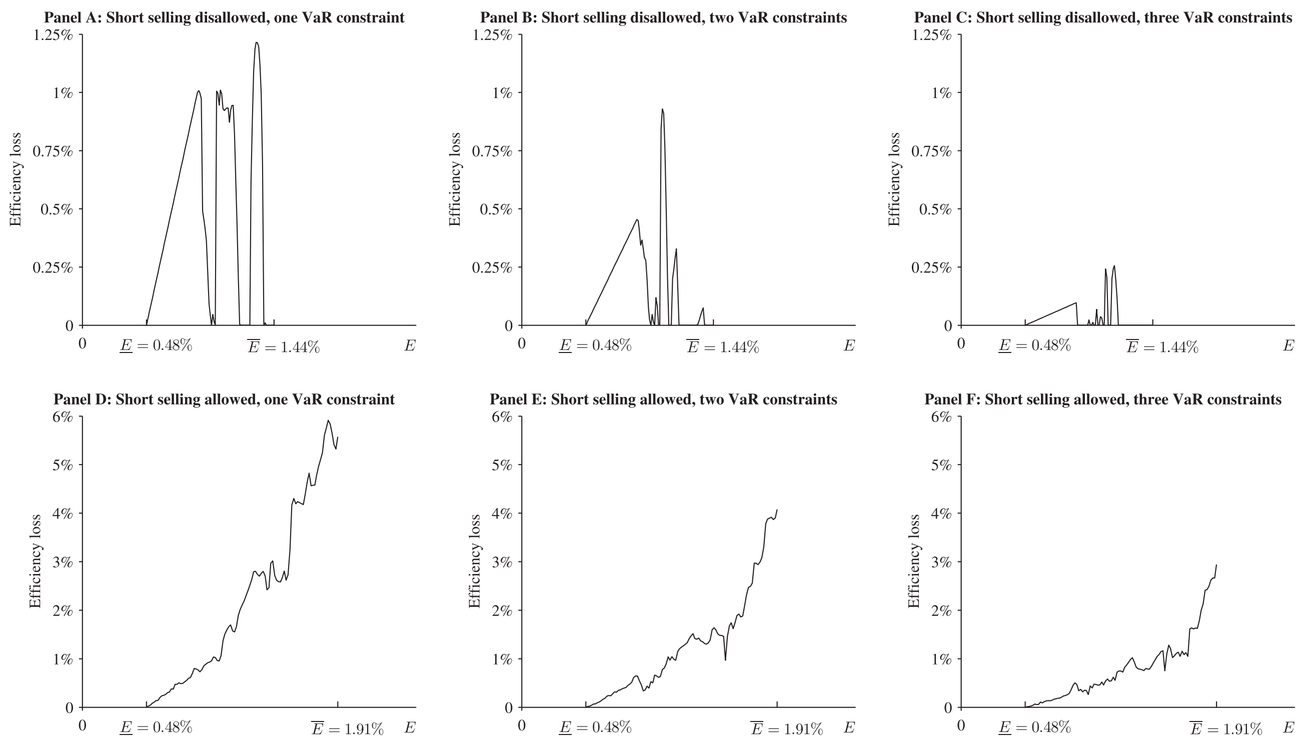

Fig. 4. Maximum efficiency losses and required expected returns. This figure shows the relation between maximum efficiency losses and required expected returns (denoted by E). These losses arise when seeking to control CVaR at confidence level $\alpha = 95\%$ with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{K}$ as defined in Eq. (8). While short selling is disallowed in Panels A, B, and C, it is allowed in Panels D, E, and F. In Panels A and D, $K = 1$ and $\alpha_1 = 95\%$. In Panels B and E, $K = 2$, $\alpha_1 = 95\%$, and $\alpha_2 = 96.25\%$. In Panels C and F, $K = 3$, $\alpha_1 = 95\%$, $\alpha_2 = 96.25\%$, and $\alpha_3 = 97.50\%$.

Table 3
Efficiency losses with two VaR constraints. This table examines whether CVaR at confidence level α can be controlled by utilizing two VaR constraints with confidence levels $\{\alpha_k\}_{k=1}^{2}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{2}$ as defined in Eq. (8). The first three columns show the values of α and $\{\alpha_k\}_{k=1}^{2}$ that are used. The fourth and fifth columns report, respectively, average and largest efficiency losses. The sixth and seventh columns report, respectively, average and largest relative efficiency losses. The last two columns report average and largest distances. Short selling is disallowed in Panel A, whereas it is allowed in Panel B. All numbers are reported in percentage points. Efficiency losses are per month.

Confidence levels			Efficiency loss		Relative efficiency loss		Distance	
CVaR	VaR constraints							
α	α_1	α_2	Average	Largest	Average	Largest	Average	Largest
Panel A: Short selling disallowed								
95.00	95.00	96.25	0.18	0.93	6.47	18.03	4.27	13.43
95.00	95.00	97.50	0.07	0.66	2.09	9.29	1.81	18.38
95.00	95.00	98.25	0.14	0.54	6.34	20.94	2.17	8.70
99.00	99.00	99.25	0.19	0.94	7.57	34.12	3.19	14.81
99.00	99.00	99.50	0.05	0.20	1.63	6.12	1.94	9.51
99.00	99.00	99.75	0.11	0.48	3.67	12.42	0.83	3.63
Panel B: Short selling allowed								
95.00	95.00	96.25	1.29	4.08	35.67	62.77	33.33	79.06
95.00	95.00	97.50	1.15	3.37	31.24	53.67	35.41	91.09
95.00	95.00	98.25	0.84	1.99	25.73	43.65	30.44	65.03
99.00	99.00	99.25	0.80	2.52	34.75	113.72	21.47	62.78
99.00	99.00	99.50	0.61	1.84	23.59	80.95	20.61	51.96
99.00	99.00	99.75	0.12	0.66	3.35	17.60	4.65	18.95

constraints in controlling CVaR at confidence level $\alpha = 99\%$ when the constraints use confidence levels that are not too high (e.g., when they use confidence levels lower than or equal to 99.50%). Note that the average efficiency loss is only 0.05% if the constraints use confidence levels of $\alpha_1 = 99\%$ and $\alpha_2 = 99.50\%$ (see the fifth row). In comparison, the average efficiency loss when a single VaR constraint is imposed ranges from 0.07% to 0.65% (see the last four rows of Panel A of Table 2). Hence, the use of two constraints is beneficial (relative to the use of a single constraint) even if the former constraints use confidence levels that are not too high.

Recall that the average CVaR of portfolios on the mean-CVaR frontier is 5.07% if $\alpha = 95\%$. Hence, the fourth column of Panel A of Table 3 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 5.14% [=5.07% + 0.07%] when $\alpha_1 = 95\%$ and $\alpha_2 = 97.50\%$ to 5.25% [=5.07% + 0.18%] when $\alpha_1 = 95\%$

Table 4
Comparing distributions and medians of efficiency losses with one and two VaR constraints. We assess the statistical significance of differences between the distributions of maximum efficiency losses and of differences between their medians with one and two VaR constraints using, respectively, the two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests. These losses arise when seeking to control CVaR at confidence level α by using either: (a) one VaR constraint with confidence level α_1 and bound $V_{\alpha_1,\alpha,E}$ as defined in Eq. (8); or (b) two VaR constraints with confidence levels $\{\alpha'_k\}_{k=1}^2$ and bounds $\{V_{\alpha'_k,\alpha,E}\}_{k=1}^2$ as defined in Eq. (8). The first four columns show the values of α, α_1, and $\{\alpha'_k\}_{k=1}^2$ that are used. The Kolmogorov-Smirnov test statistic corresponds to the null hypothesis that the cdf of losses with one VaR constraint equals the cdf of losses with two VaR constraints. The alternative hypothesis is that the two cdfs differ. The Wilcoxon rank sum test statistic corresponds to the null hypothesis that the median of losses with one VaR constraint equals the median of losses with two VaR constraints. The alternative hypothesis is that the two medians differ. "–" indicates that the test is not conducted since losses with one constraint are, by construction, equal to or larger than losses with two constraints. Short selling is disallowed in Panel A, whereas it is allowed in Panel B.

Confidence levels				Kolmogorov-Smirnov test	Wilcoxon rank sum test	Kolmogorov-Smirnov test	Wilcoxon rank sum test
CVaR	VaR constraints						
α	α_1	α'_1	α'_2				
				Panel A: Short selling disallowed		*Panel B: Short selling allowed*	
95.00	95.00	95.00	96.25	–	–	–	–
95.00	95.00	95.00	97.50	–	–	–	–
95.00	95.00	95.00	98.25	–	–	–	–
95.00	96.25	95.00	96.25	–	–	–	–
95.00	96.25	95.00	97.50	0.60***	7.82***	0.23***	1.91
95.00	96.25	95.00	98.25	0.36***	6.01***	0.30***	3.71***
95.00	97.50	95.00	96.25	0.38***	4.88***	0.13	0.09
95.00	97.50	95.00	97.50	–	–	–	–
95.00	97.50	95.00	98.25	0.40***	6.27***	0.26***	2.67***
95.00	98.25	95.00	96.25	0.28***	1.47	0.14	-0.73
95.00	98.25	95.00	97.50	0.39***	3.99***	0.20**	0.49
95.00	98.25	95.00	98.25	–	–	–	–
99.00	99.00	99.00	99.25	–	–	–	–
99.00	99.00	99.00	99.50	–	–	–	–
99.00	99.00	99.00	99.75	–	–	–	–
99.00	99.25	99.00	99.25	–	–	–	–
99.00	99.25	99.00	99.50	0.50***	5.25***	0.53***	7.60***
99.00	99.25	99.00	99.75	0.49***	4.24***	0.79***	10.34***
99.00	99.50	99.00	99.25	0.48***	2.34**	0.46***	6.13***
99.00	99.50	99.00	99.50	–	–	–	–
99.00	99.50	99.00	99.75	0.45***	2.16**	0.82***	11.30***
99.00	99.75	99.00	99.25	0.51***	3.57***	0.18*	−0.41
99.00	99.75	99.00	99.50	0.50***	5.61***	0.12	0.32
99.00	99.75	99.00	99.75	–	–	–	–

* The null hypothesis is rejected at the 10% level.
** The null hypothesis is rejected at the 5% level.
*** The null hypothesis is rejected at the 1% level.

and α_2 = 96.25%. Noting that this average ranges from 5.37% to 5.80% with one VaR constraint, there are benefits from adding a second VaR constraint (similar results are obtained if α = 99%).

Rows (5)–(7) of Fig. 3 provide box plots of maximum efficiency losses. In the top (bottom) panel, α and $\{\alpha_k\}_{k=1}^2$ take the values used in the first (last) three rows of Panel A of Table 3. Observe that the median of losses is: (i) close to zero in Panel A of Fig. 3; and (ii) zero in Panel B of Fig. 3. The upper quartile of losses is less than 0.5% in both panels.

Panel B of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1$ = 95% and α_2 = 96.25%. Like the case where a single VaR constraint is imposed, the relation is non-monotonic and losses are either zero or close to zero for small and large levels of expected return. Unlike the case where a single VaR constraint is imposed, losses exceed 0.5% for a relatively small fraction of the moderate levels of expected return.

The last two columns of Panel A of Table 3 show that average distances range from 0.83% to 4.27%, whereas largest distances range from 3.63% to 18.38%. Note that the average distance is smaller than when a single constraint is imposed if α = 95%, whereas it can be smaller or larger if α = 99% (compare the next to last column of Panel A of Tables 3 and 2). For both values of α, the largest distance can be smaller or larger than when a single constraint is imposed (compare the last column of Panel A of Tables 3 and 2).

Panel A of Table 4 assesses the statistical significance of differences between the distributions of losses with one and two VaR constraints.[36] Using the two-sample Kolmogorov-Smirnov test, we consider the null hypothesis that the cumulative distribution function (cdf) of losses with one constraint equals the cdf of losses with two constraints against the alternative hypothesis that the two cdfs differ. Similarly, using the Wilcoxon rank sum test, we consider the null hypothesis that the median of the distribution of losses with one constraint equals the median with two constraints against the alternative hypothesis that the two medians differ.

By construction, the loss with one VaR constraint is equal to or larger than the loss with two VaR constraints if α_1 equals either α'_1 or α'_2. Hence, we do not perform tests in such cases (the table uses "–" to identify these cases). The null hypotheses are rejected for both tests at the 1% level in nine of the twelve remaining cases. Hence, efficiency losses are significantly smaller with two constraints.

Panel A of Table 5 reports the reduction in efficiency losses due to an increase in the number of VaR constraints from one to two. This reduction represents the marginal benefit arising from imposing an additional VaR constraint relative to the case when a single

[36] While we report test results when outliers are included, similar results (available upon request) are obtained when outliers are excluded. Following the default option of Matlab for box plots, we define an outlier as a value of the loss that is smaller (larger) than the lower (upper) quartile by an amount that exceeds 1.5 times the interquartile range (i.e., the difference between the upper and lower quartiles).

Table 5
Reduction in efficiency losses arising from increasing the number of VaR constraints from one to two. Suppose that one seeks to control CVaR at confidence level α by using K VaR constraints with confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ as defined in Eq. (8). This table reports the reduction in average and largest efficiency losses due to an increase in K from one to two. This reduction represents the marginal benefit arising from imposing an additional VaR constraint relative to the case when a single VaR constraint is imposed. The first three columns show the values of α and $\{\alpha_k\}_{k=1}^2$ that are used when two VaR constraints are imposed. The fourth and fifth columns report the marginal benefits of adding the VaR constraint using confidence level α_1. The next two columns report the marginal benefits of adding the VaR constraint using confidence level α_2. Short selling is disallowed in Panel A, whereas it is allowed in Panel B. All numbers are reported in percentage points. Reductions in efficiency losses are per month.

Confidence levels			Marginal benefit of VaR constraint using confidence level			
CVaR	VaR constraints		α_1		α_2	
α	α_1	α_2	Average	Largest	Average	Largest
Panel A: Short selling disallowed						
95.00	95.00	96.25	0.20	0.33	0.33	0.29
95.00	95.00	97.50	0.67	1.45	0.44	0.56
95.00	95.00	98.25	0.16	0.45	0.36	0.68
99.00	99.00	99.25	0.01	0.00	0.45	1.40
99.00	99.00	99.50	0.02	0.07	0.60	2.14
99.00	99.00	99.75	0.06	0.22	0.54	1.86
Panel B: Short selling allowed						
95.00	95.00	96.25	0.18	0.92	0.98	1.83
95.00	95.00	97.50	0.21	0.48	1.12	2.54
95.00	95.00	98.25	0.26	0.63	1.43	3.93
99.00	99.00	99.25	0.84	1.27	0.58	0.45
99.00	99.00	99.50	0.94	1.21	0.77	1.14
99.00	99.00	99.75	0.49	1.15	1.25	2.32

VaR constraint is imposed. The first three columns show the values of α and $\{\alpha_k\}_{k=1}^2$ that are used when two constraints are imposed. The next (last) two columns report marginal benefits of the constraint using confidence level α_1 (α_2). In nearly all cases where two constraints are imposed, both of them reduce losses. For example, assume that $\alpha = \alpha_1 = 95\%$ and $\alpha_2 = 96.25\%$ (see the first row). The fourth (fifth) column shows that the constraint using $\alpha_1 = 95\%$ reduces the average (largest) efficiency loss by 0.20% (0.33%).[37] Similarly, the sixth (last) column shows that the constraint using $\alpha_2 = 96.25\%$ reduces the average (largest) efficiency loss by 0.33% (0.29%). Note that adding a constraint using a higher confidence level (α_2) tends to result in a greater reduction in the average efficiency loss than adding a constraint using a lower confidence level (α_1). Nevertheless, the latter constraint still reduces the average efficiency loss in all cases.

4.5. Three VaR constraints

Suppose now that three VaR constraints are imposed. The first four columns of Panel A of Table 6 show the values of α and $\{\alpha_k\}_{k=1}^3$ that are used. The next four columns show that efficiency losses are generally smaller than when two VaR constraints are imposed. For example, compare the cases of: (a) two constraints, $\alpha = \alpha_1 = 95\%$ and $\alpha_2 = 96.25\%$ (see the first row of Panel A of Table 3); and (b) three constraints, $\alpha = \alpha_1 = 95\%$, $\alpha_2 = 96.25\%$, and $\alpha_3 = 97.50\%$ (see the first row of Panel A of Table 6). Average and largest efficiency losses with two constraints are, respectively, 0.18% and 0.93%, whereas they are 0.04% and 0.26% with three constraints.[38] Similarly, average and largest relative efficiency losses with two constraints are, respectively, 6.47% and 18.03%, whereas they are 1.24% and 3.86% with three constraints.

Recall that the average CVaR of portfolios on the mean-CVaR frontier is 5.07% if $\alpha = 95\%$. Hence, the fifth column of Panel A of Table 6 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 5.10% [=5.07% + 0.03%] when $\alpha_1 = 95\%$, $\alpha_2 = 97.50\%$, and $\alpha_3 = 98.25\%$ to 5.16% [=5.07% + 0.09%] when $\alpha_1 = 95\%$, $\alpha_2 = 96.25\%$, and $\alpha_3 = 98.25\%$. Noting that this average ranges from 5.14% to 5.25% with two VaR constraints, there are benefits from adding a third VaR constraint (similar results are obtained if $\alpha = 99\%$).

Rows (8)–(10) of Fig. 3 provide box plots of maximum efficiency losses. In the top (bottom) panel, α and $\{\alpha_k\}_{k=1}^3$ take the values used in the first (last) three rows of Panel A of Table 6. Observe that the median of losses is either zero or very close to zero in both panels, whereas the upper quartile of losses is less than 0.25%.

Panel C of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1 = 95\%$, $\alpha_2 = 96.25\%$, and $\alpha_3 = 97.50\%$. Like the case where one or two VaR constraints are imposed, the relation is non-monotonic and losses are either zero or close to zero for small and large levels of expected return. Unlike the case where one or two VaR constraints are imposed, losses do not exceed roughly 0.25% for moderate levels of expected return.

The last two columns of Panel A of Table 6 show that average distances range from 0.00% to 3.03%, whereas largest distances range from 0.00% to 12.67%. Note that these distances can be either smaller or larger than when two VaR constraints are imposed.

Panel A of Table 7 assesses the statistical significance of differences between the distributions of losses with two and three VaR constraints. By construction, the loss with two VaR constraints is equal to or larger than the loss with three VaR constraints if α_2 equals either α'_2 or α'_3 since $\alpha_1 = \alpha'_1$. Hence, we do not perform tests in such cases (the table uses "–" to identify these cases). Note that

[37] Note that the reduction in the average efficiency loss might exceed the reduction in the largest efficiency loss. Specifically, consider the marginal benefit of using two VaR constraints with confidence levels of $\alpha_1 = 99\%$ and $\alpha_2 = 99.25\%$ relative to the case where a single VaR constraint with a confidence level of 99.25% is imposed; see the fourth row of Panel A of Table 5, fourth and fifth columns. When a VaR constraint with a confidence level of 99.25% is imposed, average and largest efficiency losses are 0.20% and 0.94%, respectively; see the sixth row of Panel A of Table 2. Similarly, when two VaR constraints with confidence levels of 99% and 99.25% are imposed, average and largest efficiency losses are 0.19% and 0.94%, respectively; see the fourth row of Panel A of Table 3. Hence, while the reduction in the average efficiency loss is 0.01% [=0.20% − 0.19%], the reduction in the largest efficiency loss is 0.00% [=0.94% − 0.94%]; see the fourth row of Panel A of Table 5, fourth and fifth columns. Intuitively, the addition of the constraint with a confidence level of 99% reduces the size of the maximum efficiency loss for some levels of expected return, but it does not affect the size of the maximum efficiency loss at the level of expected return where the largest efficiency loss occurs.

[38] Observe that efficiency losses (as well as relative efficiency losses and distances) are zero in the case when $\alpha = \alpha_1 = 99\%$, $\alpha_2 = 99.25\%$, and $\alpha_3 = 99.75\%$; see the fifth row of Panel A of Table 6. This result can be understood by noting that in such a case the portfolios with maximum efficiency losses are on (or very close to) the mean-CVaR frontier.

the null hypotheses are rejected for both tests at the 1% level in five of the six remaining cases. Hence, efficiency losses are significantly smaller with three constraints.

Panel A of Table 8 reports the reduction in efficiency losses due to an increase in the number of VaR constraints from two to three. This reduction represents the marginal benefit arising from imposing an additional VaR constraint relative to the case when two VaR constraints are imposed. The first four columns show the values of α and $\{\alpha_k\}_{k=1}^{3}$ that are used when three constraints are imposed. The fifth and sixth columns report marginal benefits of the constraint using confidence level α_1. The next (last) two columns report marginal benefits of the constraint using confidence level α_2 (α_3). When three constraints are imposed, all of them can reduce losses. For example, assume that $\alpha_1 = 95\%$, $\alpha_2 = 96.25\%$, and $\alpha_3 = 97.50\%$ (see the first row). The fifth (sixth) column shows that the constraint using $\alpha_1 = 95\%$ reduces the average (largest) efficiency loss by 0.13% (0.84%).[39] Similarly, the seventh (eighth) column shows that the constraint using $\alpha_2 = 96.25\%$ reduces the average (largest) efficiency loss by 0.03% (0.40%). Finally, the ninth (last) column shows that the constraint using $\alpha_3 = 97.50\%$ reduces the average (largest) efficiency loss by 0.14% (0.67%). Note that adding a constraint using a higher confidence level (α_3) tends to result in a greater reduction in the average efficiency loss than adding either of the constraints using lower confidence levels (α_1 and α_2). Nevertheless, adding the constraint using α_1 still reduces the average efficiency loss if α is 95%, whereas adding the constraint using α_2 still reduces the average efficiency loss if α is either 95% or 99%.

In sum, efficiency losses with three constraints are smaller than with two constraints, which in turn are smaller than with a single constraint. Also, with three constraints the losses are either zero or close to zero. Lastly, all of the constraints can be beneficial in reducing efficiency losses, particularly the ones using higher confidence levels. Thus, the use of multiple VaR constraints appears to be effective in controlling CVaR when short selling is disallowed.

5. Risk management when short selling is allowed

This section explores the effectiveness of VaR-based risk management systems in controlling CVaR when short selling is allowed. In doing so, the weight of each asset class is restricted to be between −50% and 150%.[40] Here, a portfolio's *leverage ratio* is defined as the sum of its positive weights. Note that our weight restrictions allow portfolios with a *maximum* leverage ratio of 400%.[41] The allowance of short selling and the range of leverage ratios permitted by our weight restrictions are realistic in the context of the trading books of large banks. Consider US depository institutions with total assets of $100 billion or more and positive trading assets as of December 31, 2009, which consists of seventeen institutions (see Federal Deposit Insurance Corporation, 2010). First, the trading portfolios of sixteen out of these seventeen institutions involve short selling. Second, of the fourteen institutions for which leverage ratios are well-defined, twelve have leverage ratios less than 400%.[42] Third, the average leverage ratio across these fourteen institutions is 219%.

5.1. One VaR constraint

Suppose that a single VaR constraint is imposed.[43] The results when short selling is allowed mainly differ from those when it is disallowed in that losses are notably larger (compare Panels B and A of Table 2).[44] For example, consider the case when $\alpha = \alpha_1 = 95\%$ (see the first row of each panel). Average and largest efficiency losses are, respectively, 2.27% and 5.91% if short selling is allowed, whereas they are, respectively, 0.51% and 1.21% if it is disallowed. Similarly, average and largest relative efficiency losses are, respectively, 63.48% and 79.25% if short selling is allowed, whereas they are, respectively, 15.69% and 39.74% if it is disallowed.

Since the average CVaR of portfolios on the mean-CVaR frontier is 3.62% if $\alpha = 95\%$, the third column of Panel B of Table 2 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 4.72% [=3.62% + 1.10%] when $\alpha_1 = 98.25\%$ to 5.89% [=3.62% + 2.27%] when $\alpha_1 = 95\%$. Hence, the use of one VaR constraint is ineffective in controlling CVaR (similar results are obtained if $\alpha = 99\%$).

Fig. 5 provides box plots of maximum efficiency losses. Importantly, the lower quartile, median, and upper quartile of losses are noticeably larger than when short selling is disallowed (noting that the horizontal axes use different scales, compare rows (1)–(4) of Figs. 3 and 5).

Fig. 6 illustrates the difference between losses with short selling disallowed and allowed. The solid line represents the portfolios on the mean-CVaR frontier. In Panel A, short selling is disallowed. The minimum and maximum required expected returns are $\underline{E} = 0.48\%$ and $\overline{E} = 1.44\%$, respectively. Point *A* represents the portfolio on the mean-CVaR frontier with an expected return of 1%. Point *B* represents the portfolio with maximum efficiency loss when the required expected return is 1% and a VaR constraint with a confidence level of 99% and a bound given by Eq. (8) is imposed. The efficiency loss of this portfolio is 1.59% [=9.64% − 8.05%]. In Panel B, short selling is allowed. In this case, the minimum and maximum required expected returns are $\underline{E} = 0.48\%$ and $\overline{E} = 1.91\%$, respectively. Point *C* represents the portfolio on the mean-CVaR frontier with an expected return of 1%. Point *D* represents the portfolio with maximum efficiency loss when the required expected return is 1% and a VaR constraint with a confidence level

[39] Note that the reduction in the average efficiency loss might exceed the reduction in the largest efficiency loss; see the second row. The intuition for this result is similar to that presented in footnote 37.

[40] Qualitatively similar results (available upon request) are obtained when the weight of each asset class is restricted to be between −100% and 200%. Quantitatively, losses are generally larger than when it is restricted to be between −50% and 150%.

[41] Since portfolio weights must sum to one, a leverage ratio of 400% is achieved by any portfolio with the following weights: (a) 150% in each of two asset classes; (b) 100% in one asset class; and (c) −50% in each of the remaining six asset classes.

[42] The difference between trading assets and trading liabilities is negative for three institutions. Accordingly, the leverage ratios of such institutions are not well-defined.

[43] In order to restrict our attention to a plausible range of required expected returns when short selling is allowed, we proceed as follows. First, as noted earlier, we let $\underline{E}$ equal the risk-free return, 0.48%. While there exist portfolios with expected returns smaller than $\underline{E}$ when short selling is allowed, it is not sensible for risk-averse investors to select such portfolios. Second, we let $\overline{E} = 1.91\%$ instead of setting it equal to the highest feasible expected return, 2.92%. Nevertheless, results with $\overline{E} = 2.92\%$ (available upon request) are similar to those with $\overline{E} = 1.91\%$. Note that the portfolio with a weight of: (a) 150% in small cap/high book-to-market stocks (the asset class with the highest expected return); (b) −50% in T-bills (the asset class with the lowest expected return); and (c) zero in the seven remaining asset classes has an expected return of 1.91% [=1.5 × 1.44% − 0.5 × 0.48%]. Observe that the calculation within square brackets is off by one basis point due to rounding. Similarly, the portfolio with a weight of: (a) 150% in small cap/intermediate book-to-market stocks and in small cap/high book-to-market stocks (the two asset classes with the highest expected returns); (b) 100% in large cap/high book-to-market stocks (the asset class with the third highest expected return); and (c) −50% in the six remaining asset classes has an expected return of 2.92% [=1.5 × (1.35% + 1.44%) + 1 × 1.08% − 5 × (0.48% + 0.71% + 0.68% + 0.84% + 0.95% + 1.01%)]. Again, the calculation within square brackets is off by one basis point due to rounding.

[44] Intuitively, allowing short selling is equivalent to adding a set of nine asset classes with risk profiles that are notably different from those of the existing ones. By construction, for fixed levels of the expected return and bound for which there are portfolios satisfying a VaR constraint: (1) the CVaR of the minimum CVaR portfolio in the absence of the constraint is non-increasing in the number of asset classes; and (2) the CVaR of the maximum CVaR portfolio in the presence of the constraint is non-decreasing in the number of asset classes. Thus, the larger the number of asset classes that are available, the less likely the VaR constraints will be effective in controlling CVaR. Hence, VaR constraints are less likely to be effective in controlling CVaR when short selling is allowed. Using similar arguments, efficiency losses are possibly larger if derivative securities were considered in our paper.

Table 6

Efficiency losses with three VaR constraints. This table examines whether CVaR at confidence level α can be controlled by utilizing three VaR constraints with confidence levels $\{\alpha_k\}_{k=1}^3$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^3$ as defined in Eq. (8). The first four columns show the values of α and $\{\alpha_k\}_{k=1}^3$ that are used. The fifth and sixth columns report, respectively, average and largest efficiency losses. The seventh and eighth columns report, respectively, average and largest relative efficiency losses. The last two columns report average and largest distances. Short selling is disallowed in Panel A, whereas it is allowed in Panel B. All numbers are reported in percentage points. Efficiency losses are per month.

Confidence levels				Efficiency loss		Relative efficiency loss		Distance	
CVaR	VaR constraints								
α	α_1	α_2	α_3	Average	Largest	Average	Largest	Average	Largest
Panel A: Short selling disallowed									
95.00	95.00	96.25	97.50	0.04	0.26	1.24	3.86	1.26	9.30
95.00	95.00	96.25	98.25	0.09	0.36	4.15	14.30	3.03	12.67
95.00	95.00	97.50	98.25	0.03	0.13	1.30	5.20	1.26	6.20
99.00	99.00	99.25	99.50	0.01	0.07	0.52	2.12	1.26	6.56
99.00	99.00	99.25	99.75	0.00	0.00	0.00	0.00	0.00	0.00
99.00	99.00	99.50	99.75	0.01	0.07	0.51	1.83	0.42	1.98
Panel B: Short selling allowed									
95.00	95.00	96.25	97.50	0.81	2.94	23.09	48.98	28.04	74.73
95.00	95.00	96.25	98.25	0.59	1.49	17.83	30.61	25.11	58.59
95.00	95.00	97.50	98.25	0.57	1.45	17.31	28.36	25.47	59.37
99.00	99.00	99.25	99.50	0.43	1.84	21.62	80.95	13.41	48.21
99.00	99.00	99.25	99.75	0.03	0.13	1.27	8.80	2.02	14.08
99.00	99.00	99.50	99.75	0.05	0.51	0.47	3.93	1.86	19.77

Table 7

Comparing distributions and medians of efficiency losses with two and three VaR constraints. We assess the statistical significance of differences between the distributions of maximum efficiency losses and of differences between their medians with two and three VaR constraints using, respectively, the two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests. These losses arise when seeking to control CVaR at confidence level α by using either: (a) two VaR constraints with confidence levels $\{\alpha_k\}_{k=1}^2$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^2$ as defined in Eq. (8); or (b) three VaR constraints with confidence levels $\{\alpha'_k\}_{k=1}^3$ and bounds $\{V_{\alpha'_k,\alpha,E}\}_{k=1}^3$ as defined in Eq. (8). The first six columns show the values of α, $\{\alpha_k\}_{k=1}^2$, and $\{\alpha'_k\}_{k=1}^3$ that are used. The Kolmogorov-Smirnov test statistic corresponds to the null hypothesis that the cdf of losses with two VaR constraints equals the cdf of losses with three VaR constraints. The alternative hypothesis is that the two cdfs differ. The Wilcoxon rank sum test statistic corresponds to the null hypothesis that the median of losses with two VaR constraints equals the median of losses with three VaR constraints. The alternative hypothesis is that the two medians differ. "–" indicates that the test is not conducted since losses with two constraints are, by construction, equal to or larger than losses with three constraints. Short selling is disallowed in Panel A, whereas it is allowed in Panel B.

Confidence levels						Kolmogorov-Smirnov test	Wilcoxon rank sum test	Kolmogorov-Smirnov test	Wilcoxon rank sum test
CVaR	VaR constraints								
α	α_1	α_2	α'_1	α'_2	α'_3				
						Panel A: Short selling disallowed		*Panel B: Short selling allowed*	
95.00	95.00	96.25	95.00	96.25	97.50	–	–	–	–
95.00	95.00	96.25	95.00	96.25	98.25	–	–	–	–
95.00	95.00	96.25	95.00	97.50	98.25	0.48***	5.79***	0.44***	5.33***
95.00	95.00	97.50	95.00	96.25	97.50	–	–	–	–
95.00	95.00	97.50	95.00	96.25	98.25	0.21**	0.17	0.32***	4.04***
95.00	95.00	97.50	95.00	97.50	98.25	–	–	–	–
95.00	95.00	98.25	95.00	96.25	97.50	0.34***	2.77***	0.16	0.86
95.00	95.00	98.25	95.00	96.25	98.25	–	–	–	–
95.00	95.00	98.25	95.00	97.50	98.25	–	–	–	–
99.00	99.00	99.25	99.00	99.25	99.50	–	–	–	–
99.00	99.00	99.25	99.00	99.25	99.75	–	–	–	–
99.00	99.00	99.25	99.00	99.50	99.75	0.35***	3.63***	0.61***	9.19***
99.00	99.00	99.50	99.00	99.25	99.50	–	–	–	–
99.00	99.00	99.50	99.00	99.25	99.75	0.41***	5.17***	0.77***	8.92***
99.00	99.00	99.50	99.00	99.50	99.75	–	–	–	–
99.00	99.00	99.75	99.00	99.25	99.50	0.38***	3.05***	0.30***	-1.89*
99.00	99.00	99.75	99.00	99.25	99.75	–	–	–	–
99.00	99.00	99.75	99.00	99.50	99.75	–	–	–	–

* The null hypothesis is rejected at the 10% level.
** The null hypothesis is rejected at the 5%, level.
*** The null hypothesis is rejected at the 1% level.

of 99% and a bound given by Eq. (8) is imposed. The efficiency loss of this portfolio is 2.98% [=5.38% − 2.40%]. Hence, the allowance of short selling leads to an increase in the maximum efficiency loss of 1.39% [=2.98% − 1.59%].

By construction, the set of portfolios that meet the portfolio weight restrictions is larger if short selling is allowed. Hence, the allowance of short selling affects the size of maximum efficiency losses through three channels. First, it permits the selection of portfolios with smaller CVaRs. As Fig. 6 shows, the allowance of short selling shifts the mean-CVaR frontier leftward for nearly all levels of expected return.[45] Since efficiency losses are determined

[45] The only exception to this shift occurs when the required expected return is equal to the risk-free rate. The portfolio on the mean-CVaR frontier with this expected return is the risk-free portfolio regardless of whether short selling is allowed.

Table 8
Reduction in efficiency losses arising from increasing the number of VaR constraints from two to three. Suppose that one seeks to control CVaR at confidence level α by using K VaR constraints with confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{K}$ as defined in Eq. (8). This table reports the reduction in average and largest efficiency losses due to an increase in K from two to three. This reduction represents the marginal benefit arising from imposing an additional VaR constraint relative to the case when two VaR constraints are imposed. The first four columns show the values of α and $\{\alpha_k\}_{k=1}^{3}$ that are used when three VaR constraints are imposed. The fifth and sixth columns report the marginal benefits of adding the VaR constraint using confidence level α_1. The next two columns report the marginal benefits of adding the VaR constraint using confidence level α_2. The last two columns report the marginal benefits of adding the VaR constraint using confidence level α_3. Short selling is disallowed in Panel A, whereas it is allowed in Panel B. All numbers are reported in percentage points. Reductions in efficiency losses are per month.

Confidence levels				Marginal benefit of VaR constraint using confidence level					
CVaR	VaR constraints			α_1		α_2		α_3	
α	α_1	α_2	α_3	Average	Largest	Average	Largest	Average	Largest
Panel A: Short selling disallowed									
95.00	95.00	96.25	97.50	0.13	0.84	0.03	0.40	0.14	0.67
95.00	95.00	96.25	98.25	0.01	0.00	0.05	0.17	0.09	0.57
95.00	95.00	97.50	98.25	0.03	0.22	0.12	0.41	0.04	0.53
99.00	99.00	99.25	99.50	0.00	0.00	0.03	0.13	0.18	0.87
99.00	99.00	99.25	99.75	0.00	0.00	0.11	0.48	0.19	0.94
99.00	99.00	99.50	99.75	0.00	0.00	0.09	0.41	0.03	0.13
Panel B: Short selling allowed									
95.00	95.00	96.25	97.50	0.12	0.58	0.33	0.43	0.48	1.14
95.00	95.00	96.25	98.25	0.05	0.28	0.25	0.49	0.71	2.59
95.00	95.00	97.50	98.25	0.10	0.02	0.27	0.54	0.58	1.92
99.00	99.00	99.25	99.50	0.39	0.00	0.18	0.00	0.36	0.69
99.00	99.00	99.25	99.75	0.16	0.46	0.09	0.53	0.76	2.40
99.00	99.00	99.50	99.75	0.30	0.48	0.08	0.15	0.56	1.32

relative to the mean-CVaR frontier, this shift generally increases the size of maximum efficiency losses.

Second, the value of the VaR bound when short selling is allowed may be smaller than that when it is disallowed, assuming a level of expected return that is also feasible when short selling is disallowed. Since the allowance of short selling causes the mean-CVaR frontier to shift leftward, the VaRs of the portfolios on this frontier are possibly smaller when short selling is allowed. For example, when the required expected return is 1%, the value of the bound is 7.13% if short selling is disallowed, and 2.40% if it is allowed.[46] The use of smaller bounds generally decreases the size of maximum efficiency losses.

Third, the range of required expected returns is larger when short selling is allowed. As Fig. 6 shows, the maximum required expected return $\overline{E}$ is 1.44% if short selling is disallowed, and 1.91% if it is allowed. The effect of using a larger range of required expected returns on the size of maximum efficiency losses is generally unclear.

Panel D of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1 = 95\%$. This relation differs from the one when short selling is disallowed in two respects. First, losses tend to increase in the required expected return. Second, losses notably exceed 0.5% except for small levels of expected return (i.e., levels of expected return equal to or slightly larger than $\underline{E} = 0.48\%$). These results can be understood by noting that: (1) the size of the set of portfolios that satisfy VaR and expected return constraints and possibly involve short selling tends to increase in the required expected return; and (2) losses tend to be larger when such a set is also larger.

In sum, the use of a single VaR constraint is ineffective in controlling CVaR when short selling is allowed. Accordingly, we next examine whether the use of multiple VaR constraints is effective in controlling CVaR.

[46] Due to rounding, the VaR bound when short selling is allowed appears to coincide with the CVaR of the minimum CVaR portfolio. The reason why this occurs is that such a portfolio has losses very close to 2.40% in the states used to determine VaR and CVaR.

5.2. Two VaR constraints

Panel B of Table 3 shows the results with two VaR constraints. Two results can be seen. First, as when short selling is disallowed, losses are smaller than when a single constraint is imposed. For example, compare the cases of: (1) one constraint, $\alpha = \alpha_1 = 95\%$ (see the first row of Panel B of Table 2); and (2) two constraints, $\alpha = \alpha_1 = 95\%$, and $\alpha_2 = 96.25\%$ (see the first row of Panel B of Table 3). Average and largest efficiency losses in the first case are, respectively, 2.27% and 5.91%, whereas they are, respectively, 1.29% and 4.08% in the second one. Similarly, average and largest relative efficiency losses in the former case are, respectively, 63.48% and 79.25%, whereas they are, respectively, 35.67% and 62.77% in the latter.

Second, as when a single constraint is imposed, losses are notably larger than when short selling is disallowed (compare Panels A and B of Table 3). For example, consider the case when $\alpha = \alpha_1 = 95\%$ and $\alpha_2 = 96.25\%$ (see the first row of each panel). Average and largest efficiency losses are, respectively, 1.29% and 4.08% if short selling is allowed, whereas they are, respectively, 0.18% and 0.93% if it is disallowed. Similarly, average and largest relative efficiency losses are, respectively, 35.67% and 62.77% if short selling is allowed, whereas they are, respectively, 6.47% and 18.03% if it is disallowed.

Recall that the average CVaR of portfolios on the mean-CVaR frontier is 3.62% if $\alpha = 95\%$. Hence, the fourth column of Panel B of Table 3 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 4.46% [=3.62% + 0.84%] when $\alpha_1 = 95\%$ and $\alpha_2 = 98.25\%$ to 4.91% [=3.62% + 1.29%] when $\alpha_1 = 95\%$ and $\alpha_2 = 96.25\%$. Since this average ranges from 4.72% to 5.89% with one VaR constraint, there are benefits from adding a second VaR constraint (similar results are obtained if $\alpha = 99\%$).

Comparing rows (5)–(7) of Figs. 3 and 5, it can be seen that in nearly all cases the lower quartile, median, and upper quartile of losses are much larger than when short selling is disallowed (again noting the different scales on the horizontal axes). Also, the average (largest) distance is roughly five (four) times or more larger than when short selling is disallowed (compare the last two columns of Panels A and B of Table 3).

Panel E of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1 = 95\%$

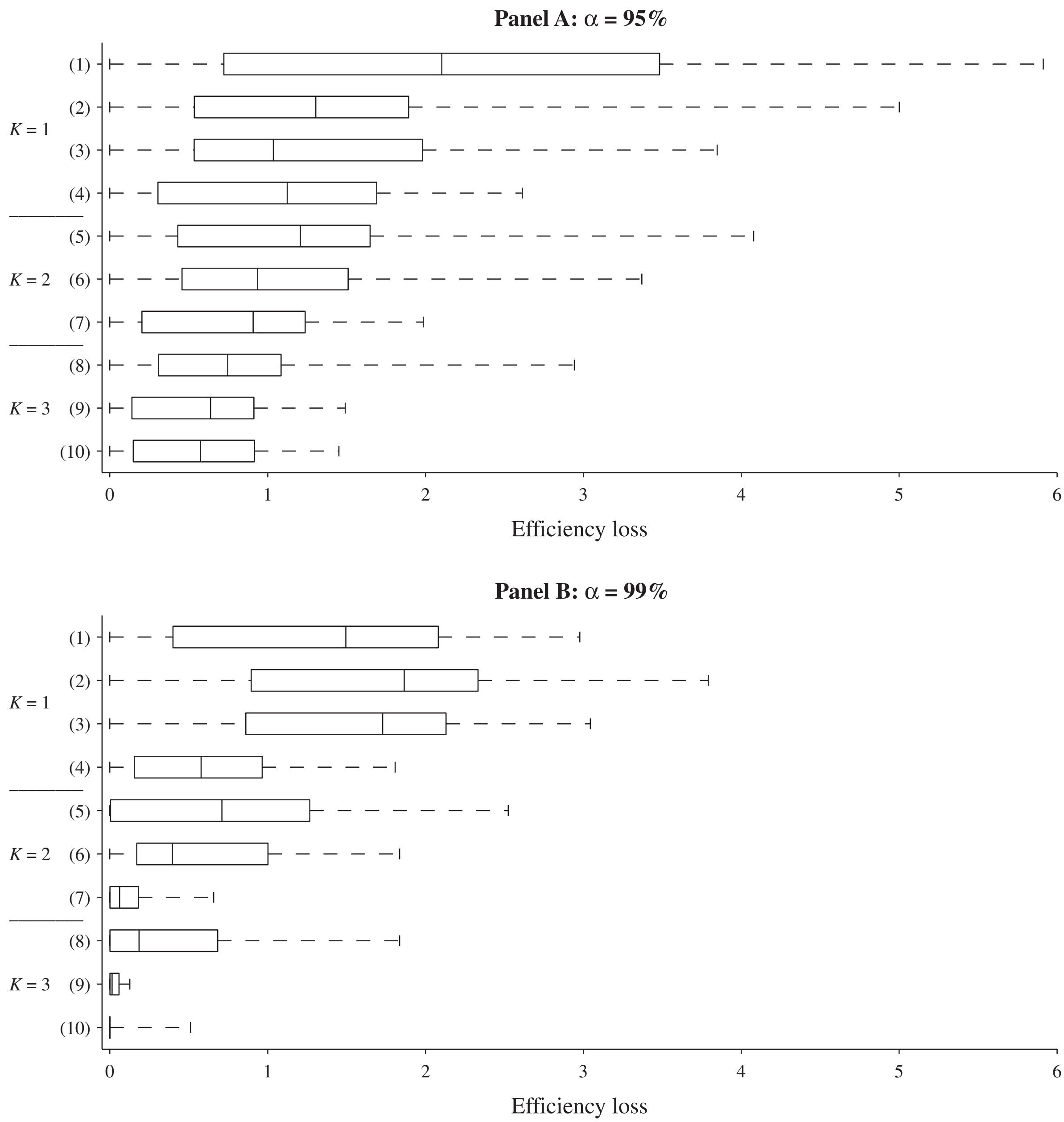

Fig. 5. Box plots of maximum efficiency losses with short selling allowed. This figure shows box plots of maximum efficiency losses when short selling is allowed. These losses arise when seeking to control CVaR at confidence level α with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ as defined in Eq. (8). While $\alpha = 95\%$ in Panel A, $\alpha = 99\%$ in Panel B. In rows (1)–(4), $K = 1$ and α_1 takes the values used in Table 2. In rows (5)–(7), $K = 2$ and $\{\alpha_k\}_{k=1}^2$ take the values used in Table 3. In rows (8)–(10), $K = 3$ and $\{\alpha_k\}_{k=1}^3$ take the values used in Table 6. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

and $\alpha_2 = 96.25\%$. The results are similar to those with one VaR constraint (see Panel D of Fig. 4), except that losses are smaller for nearly all levels of expected return.

Panel B of Table 4 assesses the statistical significance of differences between the distributions of losses with one and two VaR constraints. In comparison to the case when short selling is disallowed, there is less statistical evidence that efficiency losses with two constraints are smaller than those with one constraint. Nevertheless, we still obtain statistical significance at the 1% level for: (a) seven out of the twelve cases of the Kolmogorov-Smirnov test; and (b) six out of the twelve cases of the Wilcoxon rank sum test. Hence, there is still statistical evidence that efficiency losses are significantly smaller with two constraints.

Panel B of Table 5 reports the reduction in efficiency losses due to an increase in the number of VaR constraints from one to two. In all cases where two VaR constraints are imposed, both of them reduce losses. For example, assume that $\alpha = \alpha_1 = 95\%$ and $\alpha_2 = 96.25\%$ (see the first row of Panel B). The fourth (fifth) column shows that the constraint using $\alpha_1 = 95\%$ reduces the average (largest) efficiency loss by 0.18% (0.92%). Similarly, the sixth (last) column shows that the constraint using $\alpha_2 = 96.25\%$ reduces the average (largest) efficiency loss by 0.98% (1.83%). Note that adding a

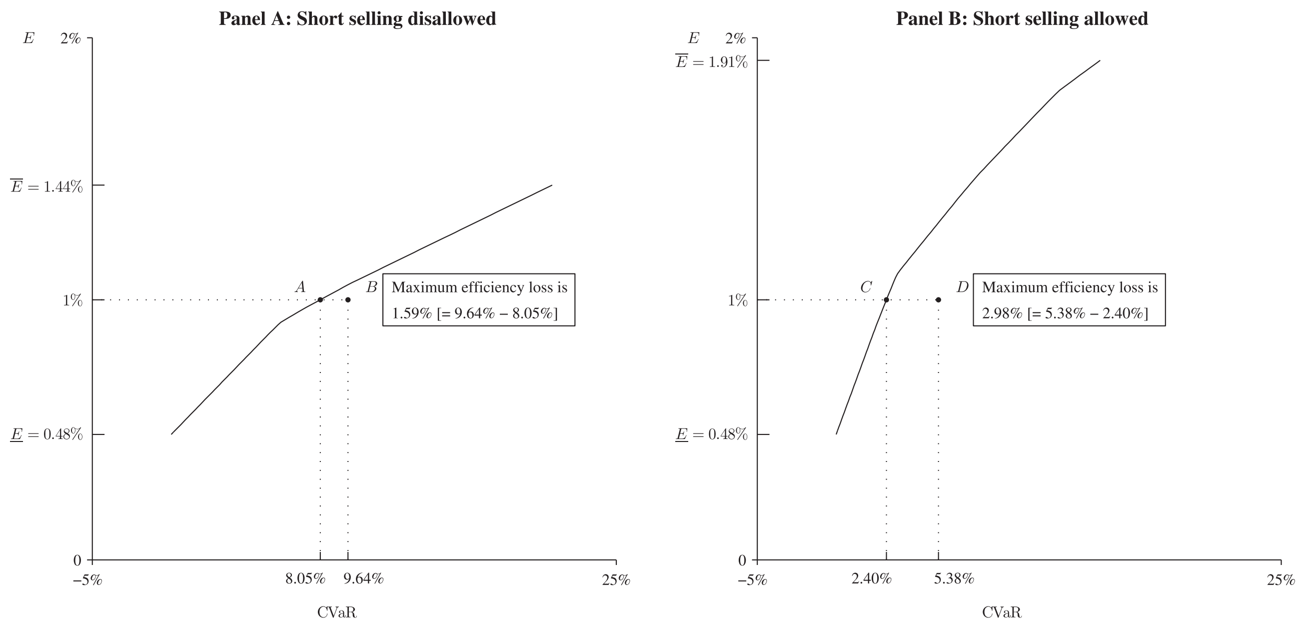

Fig. 6. Maximum efficiency losses with short selling disallowed and allowed. This figure illustrates the size of maximum efficiency losses. The solid line represents the portfolios on the mean-CVaR frontier. Panel A considers the case when short selling is disallowed. In this case, the minimum and maximum required expected returns are given by $\underline{E}$ = 0.48% and $\overline{E} = 1.44\%$, respectively. Point *A* represents the portfolio on the mean-CVaR frontier with an expected return of 1%. Point *B* represents the portfolio with maximum efficiency loss when the required expected return is 1% and a VaR constraint with a confidence level of 99% and a bound given by Eq. (8) is imposed. The efficiency loss of this portfolio is 1.59% [=9.64% − 8.05%]. Panel B considers the case when short selling is allowed. In this case, the minimum and maximum required expected returns are given by $\underline{E}$ = 0.48% and $\overline{E} = 1.91\%$, respectively. Point *C* represents the portfolio on the mean-CVaR frontier with an expected return of 1%. Point *D* represents the portfolio with maximum efficiency loss when the required expected return is 1% and a VaR constraint with a confidence level of 99% and a bound given by Eq. (8) is imposed. The efficiency loss of this portfolio is 2.98% [=5.38% − 2.40%].

constraint using a higher confidence level (α_2) tends to result in a greater reduction in the average efficiency loss than adding a constraint using a lower confidence level (α_1). Nevertheless, adding the latter constraint still reduces the average efficiency loss in all cases.

5.3. Three VaR constraints

Panel B of Table 6 shows the results with three VaR constraints. Two results can be seen. First, as when short selling is disallowed, losses are smaller than when two VaR constraints are imposed (compare Panel B of Tables 6 and 3). Second, as when two VaR constraints are imposed, losses are notably larger than when short selling is disallowed (compare Panels B and A of Table 6).

Recall that the average CVaR of portfolios on the mean-CVaR frontier is 3.62% if α = 95%. Hence, the fifth column of Panel B of Table 6 implies that the average CVaR of portfolios with maximum efficiency losses ranges from 4.19% [=3.62% + 0.57%] when α_1 = 95%, α_2 = 97.50%, and α_3 = 98.25% to 4.43% [=3.62% + 0.81%] when α_1 = 95%, α_2 = 96.25%, and α_3 = 97.50%. Since this average ranges from 4.46% to 4.91% with two VaR constraints, there are benefits from adding a third VaR constraint (similar results are obtained if α = 99%).

Comparing rows (8)–(10) of Figs. 3 and 5, it can be seen that in nearly all cases the median and upper quartile of losses are noticeably larger than when short selling is disallowed. Furthermore, average and largest distances are considerably larger than when short selling is disallowed (compare the last two columns of Panels A and B of Table 6).

Panel F of Fig. 4 shows the relation between maximum efficiency losses and required expected returns when $\alpha = \alpha_1$ = 95%, α_2 = 96.25%, and α_3 = 97.50%. The results are similar to those with two VaR constraints (see Panel E of Fig. 4), except that losses are smaller for nearly all levels of expected return.

Panel B of Table 7 assesses the statistical significance of differences between the distributions of losses with two and three VaR constraints. The results are similar to those when short selling is disallowed in that efficiency losses are significantly smaller with three constraints.

Panel B of Table 8 reports the reduction in efficiency losses due to an increase in the number of VaR constraints from two to three. Note that adding a constraint using a higher confidence level (α_3) tends to result in a greater reduction in the average efficiency loss than adding either of the constraints using the lower confidence levels (α_1 and α_2). Nevertheless, adding either of the latter constraints still reduces the average efficiency loss in all cases.

5.4. Significance of differences with short selling disallowed and allowed

Next, we assess the statistical significance of the difference between the distributions of losses with short selling disallowed and allowed. Using the two-sample Kolmogorov-Smirnov test, we consider the null hypothesis that the cdf of losses when short selling is disallowed coincides with the cdf of losses when it is allowed against the alternative hypothesis that the two cdfs differ. Similarly, using the Wilcoxon rank sum test, we consider the null hypothesis that the median of the distribution of losses when short selling is disallowed equals the median when it is allowed against the alternative hypothesis that the two medians differ. Table 9 shows that both null hypotheses are rejected at the 1% level in all cases. Hence, efficiency losses are significantly larger when short selling is allowed.

In sum, our results suggest that the use of multiple VaR constraints is not fully effective when short selling is allowed. Nevertheless, losses with multiple VaR constraints are smaller than with a single VaR constraint. Also, all of these constraints can be beneficial in reducing efficiency losses, particularly the ones using

Table 9
Comparing distributions and medians of efficiency losses with short selling disallowed and allowed. We assess the statistical significance of differences between the distributions of maximum efficiency losses and of differences between their medians with short selling disallowed and allowed by using, respectively, the two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests. These losses arise when seeking to control CVaR at confidence level α with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ as defined in Eq. (8). The first five columns show the values of K, α, and $\{\alpha_k\}_{k=1}^K$ that are used. The sixth column reports the Kolmogorov-Smirnov test statistic corresponding to the null hypothesis that the cdf of losses when short selling is disallowed coincides with the cdf of losses when it is allowed. The alternative hypothesis is that the two cdfs differ. Similarly, the last column reports the Wilcoxon rank sum test statistic corresponding to the null hypothesis that the median of losses when short selling is disallowed coincides with the median of losses when it is allowed. The alternative hypothesis is that the two medians differ.

Number of VaR constraints	Confidence levels				Kolmogorov-Smirnov test	Wilcoxon rank sum test
	CVaR	VaR constraints				
K	α	α_1	α_2	α_3		
1	95.00	95.00			0.60***	−7.75***
1	95.00	96.25			0.55***	−7.83***
1	95.00	97.50			0.42***	−4.54***
1	95.00	98.25			0.58***	−8.29***
1	99.00	99.00			0.44***	−6.32***
1	99.00	99.25			0.73***	−10.48***
1	99.00	99.50			0.93***	−11.68***
1	99.00	99.75			0.53***	−7.83***
2	95.00	95.00	96.25		0.68***	−9.80***
2	95.00	95.00	97.50		0.82***	−11.03***
2	95.00	95.00	98.25		0.63***	−9.10***
2	99.00	99.00	99.25		0.62***	−7.94***
2	99.00	99.00	99.50		0.72***	−9.88***
2	99.00	99.00	99.75		0.55***	−3.83***
3	95.00	95.00	96.25	97.50	0.84***	−11.28***
3	95.00	95.00	96.25	98.25	0.60***	−8.96***
3	95.00	95.00	97.50	98.25	0.82***	−11.16***
3	99.00	99.00	99.25	99.50	0.62***	−8.87***
3	99.00	99.00	99.25	99.75	0.91***	−11.76***
3	99.00	99.00	99.50	99.75	0.58***	−3.73***

*** The null hypothesis is rejected at the 1% level.

higher confidence levels. Hence, the use of multiple VaR constraints substantively reduces tail risk relative to the use of a single VaR constraint.

6. Additional robustness checks

Next, we further assess the robustness of the result that the use of multiple VaR constraints substantively reduces tail risk relative to the use of a single VaR constraint. For brevity, we focus on the case when a confidence level of 95% is used to estimate CVaR. Nevertheless, similar results (available upon request) are obtained when a confidence level of 99% is used.

6.1. Using an alternative set of confidence levels for the VaR constraints

We begin by considering VaR constraints that use confidence levels lower than or equal to the one used to estimate CVaR. First, suppose that short selling is disallowed. Panel A of Fig. 7 provides box plots of maximum efficiency losses with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ given by Eq. (8). In row (1), $K = 1$ and $\alpha_1 = 95\%$. In rows (2)–(4), $K = 2$ and $\alpha_1 = 95\%$. Also, α_2 is 93.75% in row (2), 92.50% in row (3), and 91.25% in row (4). In rows (5)–(7), $K = 3$ and $\alpha_1 = 95\%$. Also, (α_2, α_3) is (93.75%, 92.50%) in row (5), (93.75%, 91.25%) in row (6), and (92.50%, 91.25%) in row (7). Note that losses with either two or three constraints are smaller than those with a single constraint (compare rows (2)–(7) with row (1)). However, the benefits of using multiple constraints are possibly smaller than when the constraints use confidence levels equal to or higher than the one used to estimate CVaR (compare rows (2)–(7) of Panel A of Fig. 7 with rows (5)–(10) of Panel A of Fig. 3, noting that the horizontal axes of the figures use different scales).

Second, suppose that short selling is allowed. While the results in Panel B of Fig. 7 differ from the results in Panel A of Fig. 7 in that losses are larger, losses with multiple constraints are again smaller than those with a single constraint (compare rows (2)–(7) with row (1)). Also, the benefits of using multiple constraints are smaller than when the constraints use confidence levels equal to or higher than the one used to estimate CVaR (compare rows (2)–(7) of Panel B of Fig. 7 with rows (5)–(10) of Panel A of Fig. 5 noting that the horizontal axes of the figures use different scales). In sum, regardless of whether short selling is allowed, the use of multiple VaR constraints reduces tail risk relative to the use of a single VaR constraint. However, the benefits of using multiple constraints are smaller than when the constraints use confidence levels equal to or higher than the one used to estimate CVaR.

6.2. Using an alternative set of VaR bounds

Next, we consider a set of VaR bounds that no longer relies on the location of the mean-CVaR frontier. When confidence level α_k is used to compute VaR, let $\overline{\boldsymbol{w}}_{\alpha_k,E}$ denote the portfolio on the mean-VaR frontier with an expected return of E. Suppose that the VaR constraint with confidence level α_k uses a bound given by:

$$V_{\alpha_k,\alpha,E} \equiv V_{\alpha_k,\overline{\boldsymbol{w}}_{\alpha_k,E}} + \eta, \tag{9}$$

where $\eta = 1\%$.[47] Here, the use of a positive value of η is appealing. First, it allows us to be conservative in assessing the effectiveness of a single VaR constraint in controlling CVaR. By construction,

[47] We consider two additional cases where η is either 0.9% or 1.1%. Losses when η is 0.9% (1.1%) are generally smaller (larger) than those when η is 1%. Nevertheless, we still find that losses with either two or three VaR constraints are generally smaller than those with one VaR constraint. These results are available upon request.

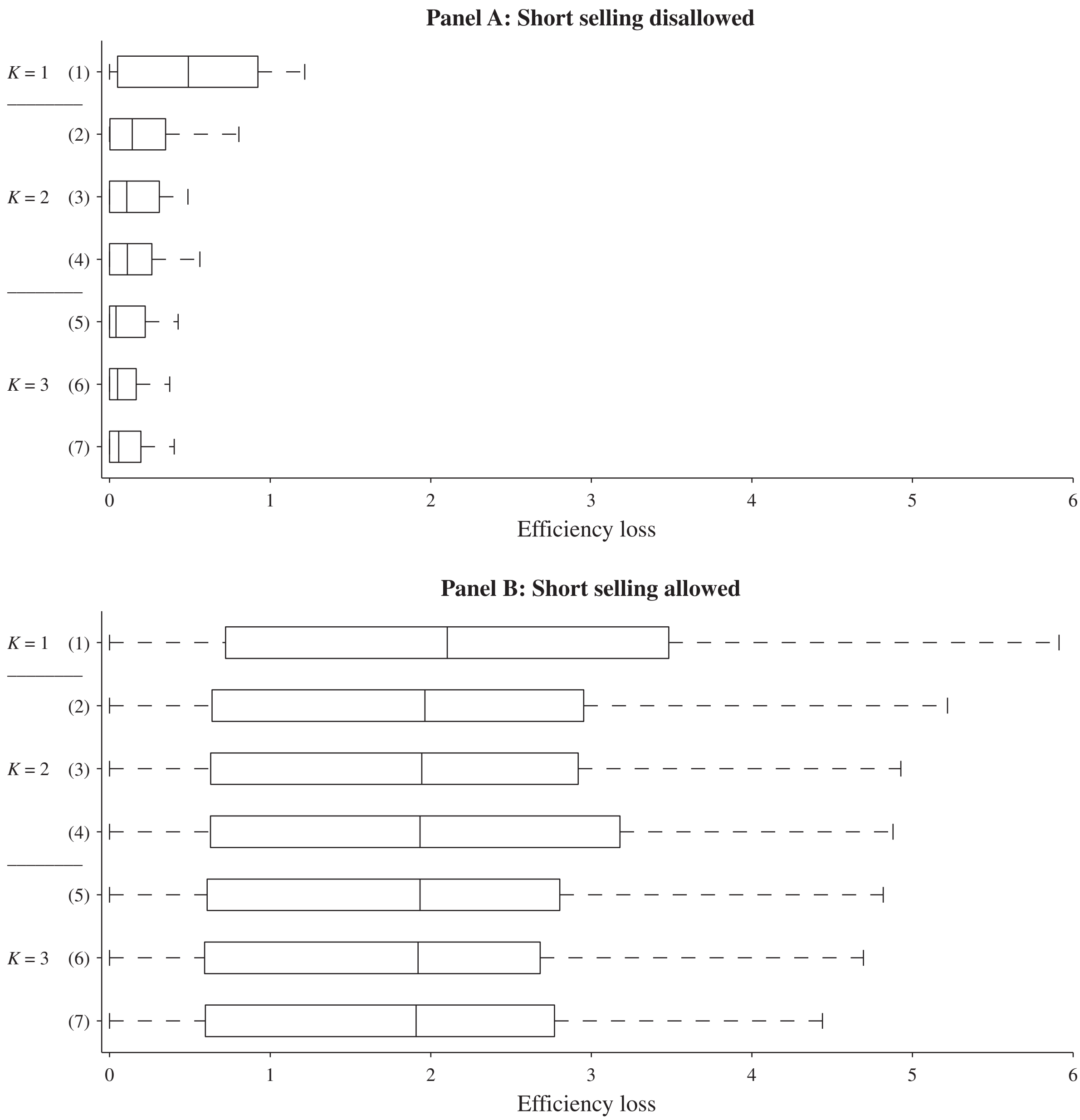

Fig. 7. Box plots of maximum efficiency losses when the VaR constraints use lower confidence levels. This figure shows box plots of maximum efficiency losses. These losses arise when seeking to control CVaR at confidence level α = 95% with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ as defined in Eq. (8). While short selling is disallowed in Panel A, it is allowed in Panel B. In row (1), K = 1 and α_1 = 95%. In rows (2)–(4), K = 2 and α_1 = 95%. Also, α_2 is 93.75% in row (2), 92.50% in row (3), and 91.25% in row (4). In rows (5)–(7), K = 3 and α_1 = 95%. Also, (α_2, α_3) is (93.75%, 92.50%) in row (5), (93.75%, 91.25%) in row (6), and (92.50%, 91.25%) in row (7). The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

efficiency losses when η = 0 are smaller than or equal to those when η > 0. Second, it allows us to examine the effectiveness of multiple VaR constraints in controlling CVaR. Note that the set of portfolios that satisfy such constraints is non-empty if η is sufficiently large, but is possibly empty if η is (sufficiently close to) zero.

First, suppose that short selling is disallowed. Panel A of Fig. 8 provides box plots of maximum efficiency losses with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^K$ given by Eq. (8). In rows (1)–(10), the number of VaR constraints and confidence levels take the values used in Fig. 3. The results in Panel A of Fig. 8 differ from the results in Panel A of Fig. 3 in that losses are generally larger (note that the horizontal axes of the figures use different scales). Nevertheless, losses with either two or three constraints are typically smaller than those with a single constraint (compare rows (5)–(10) with rows (1)–(4)).

Second, suppose that short selling is allowed. While the results in Panel B of Fig. 8 differ from the results in Panel A of Fig. 8 in that losses tend to be larger, losses with two or three constraints are again typically significantly smaller than those with a single constraint (compare rows (5)–(10) with rows (1)–(4)). In sum,

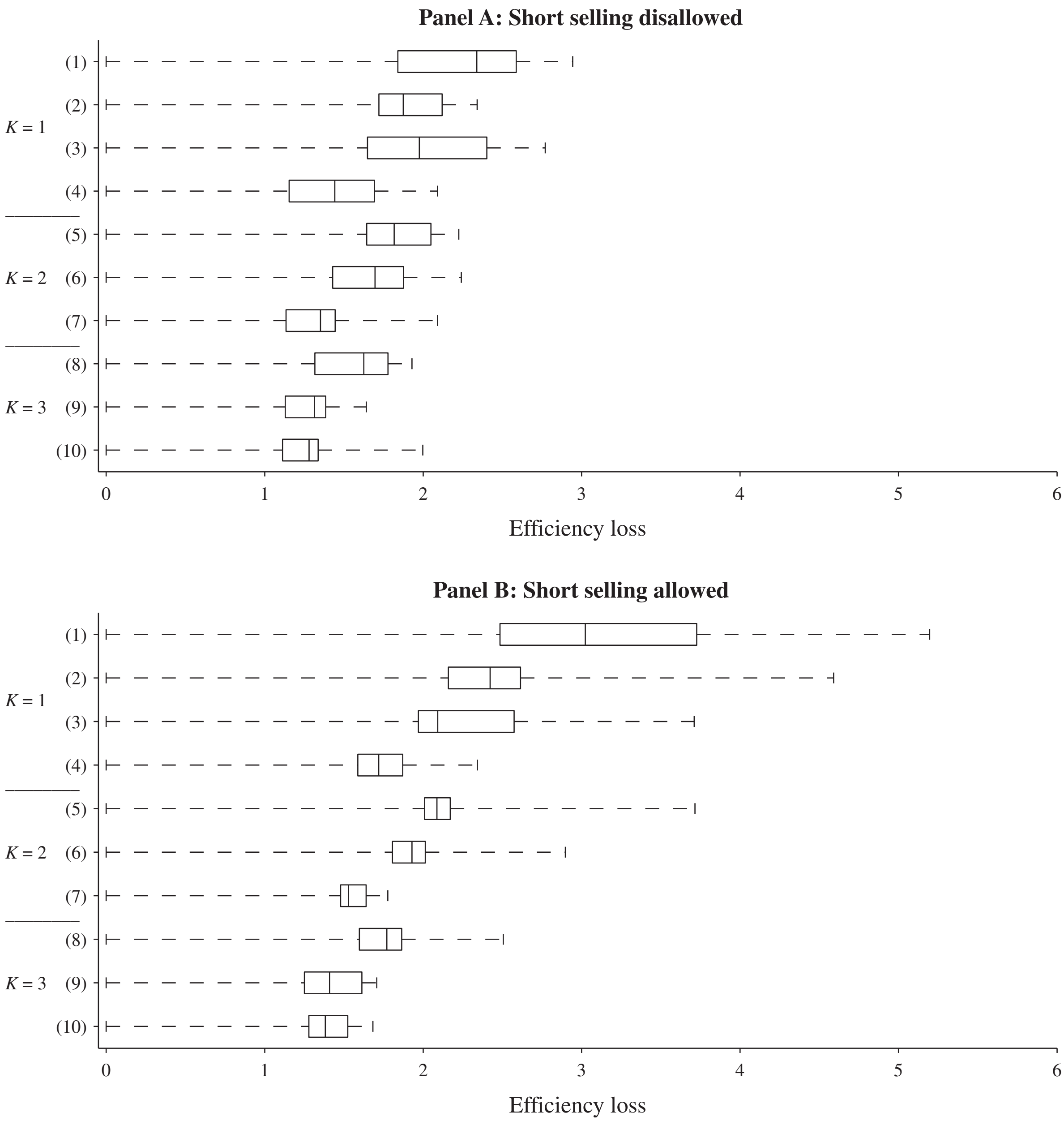

Fig. 8. Box plots of maximum efficiency losses when the VaR bounds are based on the mean-VaR frontier. This figure shows box plots of maximum efficiency losses. These losses arise when seeking to control CVaR at confidence level $\alpha = 95\%$ with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{K}$ as defined in Eq. (9). While short selling is disallowed in Panel A, it is allowed in Panel B. In rows (1)–(4), $K = 1$ and α_1 takes the values used in Table 2. In rows (5)–(7), $K = 2$ and $\{\alpha_k\}_{k=1}^{2}$ take the values used in Table 3. In rows (8)–(10), $K = 3$ and $\{\alpha_k\}_{k=1}^{3}$ take the values used in Table 6. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

regardless of whether short selling is allowed, the use of multiple VaR constraints substantively reduces tail risk relative to the use of a single VaR constraint.

6.3. Using an alternative set of asset classes

We now assess the case when a larger number of asset classes is available. Specifically, consider the following asset classes: (i) Treasury bills, (ii) government bonds with six maturity ranges that are used by the Merrill Lynch government bond indices (1–3, 3–5, 5–7, 7–10, 10–15, and 15+ years), (iii) corporate bonds with five maturity ranges that are used by the Merrill Lynch corporate bond indices (1–3, 3–5, 5–10, 10–15, and 15+ years), and (iv) the 25 size/book to-market Fama–French portfolios.[48] Due to a finer partition of bond and stock markets, there are now 37 asset classes (instead of the nine asset classes considered before).

First, suppose that short selling is disallowed. The results in Panel A of Fig. 9 mainly differ from the results in Panel A of Fig. 3 in that losses are generally larger (note that the horizontal axes of the figures use different scales). Nevertheless, losses with either two or three constraints are smaller than those with a single constraint (compare rows (5)–(10) with rows (1)–(4)).

[48] The reason why we use the maturity range of 5–10 years for corporate bonds is that index data for the maturity ranges of 5–7 and 7–10 years (that we use for government bonds) are only available for a period shorter than our sample period.

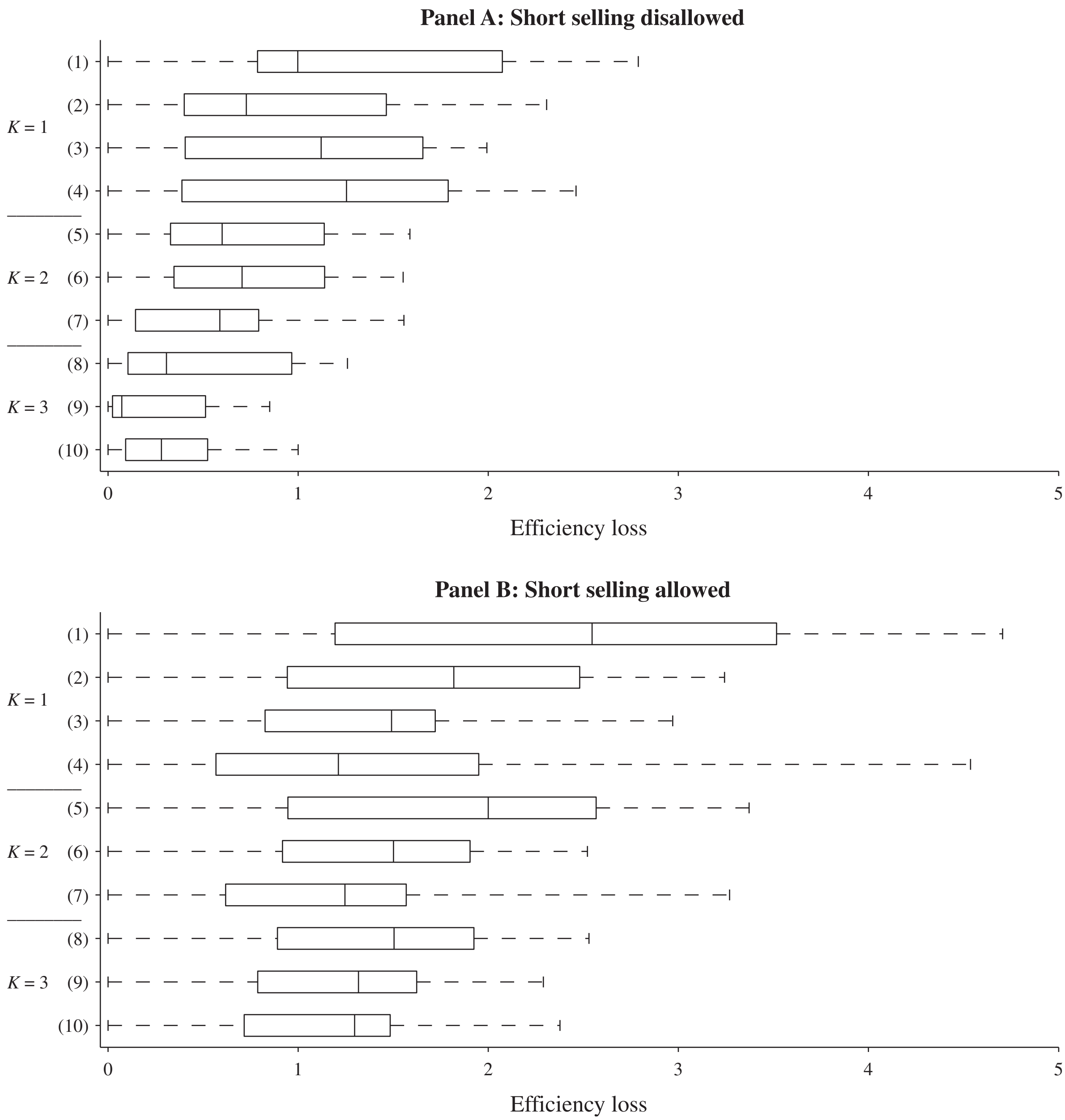

Fig. 9. Box plots of maximum efficiency losses when other asset classes are used. This figure shows box plots of maximum efficiency losses when the following asset classes are available: (i) Treasury bills (assumed to be risk-free), (ii) government bonds with six maturity ranges used by the Merrill Lynch government bond indices (1–3, 3–5, 5–7, 7–10, 10–15, and 15+ years), (iii) corporate bonds with five maturity ranges used by the Merrill Lynch corporate bond indices (1–3, 3–5, 5–10, 10–15, and 15+ years), and (iv) the 25 size/book-to-market Fama–French portfolios. These losses arise when seeking to control CVaR at confidence level $\alpha = 95\%$ with K VaR constraints using confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{V_{\alpha_k,\alpha,E}\}_{k=1}^{K}$ as defined in Eq. (8). While short selling is disallowed in Panel A, it is allowed in Panel B. The values of K and $\{\alpha_k\}_{k=1}^{K}$ are the same as those used in Fig. 8. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

Second, suppose that short selling is allowed. As before, we restrict the weight of each asset class to be between −50% and 150%. Also, we restrict leverage ratios to be less than or equal to 400%.[49] While the losses in Panel B of Fig. 9 are typically larger than the ones in Panel A of Fig. 9, losses with two or more constraints are still smaller than those with a single constraint (compare rows (5)–(10) with rows (1)–(4)). In sum, regardless of whether short selling is allowed, the use of multiple VaR constraints substantively reduces tail risk relative to the use of a single VaR constraint.[50]

[49] Without this restriction, portfolios with leverage ratios that notably exceed 400% (up to 13,500%) would be feasible. Such portfolios are unrealistic in the context of the trading books of large banks; see our discussion in Section 5.

[50] For each of the three robustness checks (i.e., alternative confidence levels, alternative bounds, and alternative asset classes), we find that the distribution of losses with K constraints significantly differs from that with $K + 1$ constraints for $K = 1,2$ in most cases. A notable exception occurs in the case where alternative confidence levels are used and short selling is allowed. In this case, there is no statistical evidence that the use of three VaR constraints improves upon the use of two VaR constraints (there is still statistical evidence that the use of two VaR constraints improves upon the use of one VaR constraint). These findings are based on two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests (available upon request).

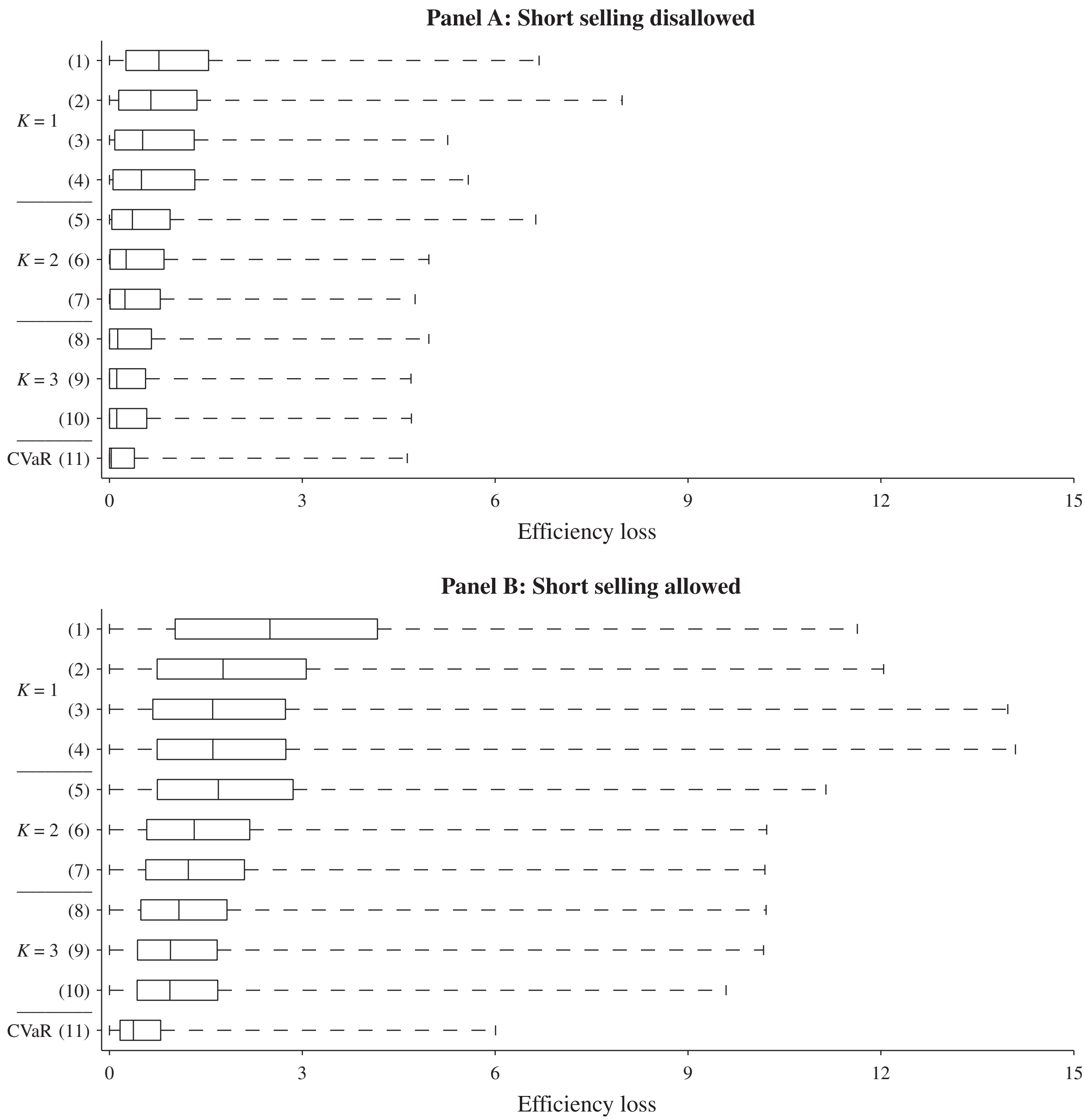

Fig. 10. Box plots of maximum efficiency losses with estimation risk. This figure shows box plots of maximum efficiency losses with estimation risk. These losses arise when seeking to control CVaR at confidence level α = 95% by using K constraints on estimated VaR with confidence levels $\{\alpha_k\}_{k=1}^{K}$ and bounds $\{\widehat{V}_{\alpha_k,\alpha,\widehat{E}}\}_{k=1}^{K}$ as defined in Eq. (10). We consider 100 simulated distributions of asset returns. Each of these distributions is based on 300 vectors of monthly asset returns that are drawn from their empirical distribution (with replacement). While short selling is disallowed in Panel A, it is allowed in Panel B. In rows (1)–(4), K = 1 and α_1 takes the values used in Table 2. In rows (5)–(7), K = 2 and $\{\alpha_k\}_{k=1}^{2}$ take the values used in Table 3. In rows (8)–(10), K = 3 and $\{\alpha_k\}_{k=1}^{3}$ take the values used in Table 6. Row (11) plots the efficiency losses of portfolios on the estimated mean-CVaR frontiers. The three vertical lines in the box represent the lower quartile, median, and upper quartile of losses. The dashed horizontal lines extending from each end of the box show the range of losses. Hence, the vertical line at the extreme left (right) shows the lowest (highest) value of the loss. Efficiency losses are reported in percentage points per month.

7. Conclusion

Financial institutions suffered large trading losses during the 2007–2009 global financial crisis. These losses cast doubt on the effectiveness of their regulation and risk management practices. At the heart of these practices is the use of Value-at-Risk (VaR). Some researchers have criticized the use of VaR as a measure of tail risk since it does not consider the size of losses beyond VaR. Accordingly, these researchers recommend replacing VaR with Conditional Value-at-Risk (CVaR) since it considers the size of losses beyond VaR. While practitioners and regulators have recognized that VaR-based risk management systems were ineffective in controlling tail risk during the recent crisis, VaR remains quite popular among them. Hence, the question of whether there exist more effective VaR-based risk management systems is of particular interest.

Previous work recognizes the ineffectiveness of VaR-based risk management systems by assuming that a *single* VaR constraint is used in attempting to control CVaR. Our paper examines the effectiveness of systems based on *multiple* VaR constraints in controlling tail risk as measured by CVaR. Under certain conditions, we theoretically show that the use of multiple VaR constraints is more effective in controlling CVaR than the use of a single VaR constraint. Furthermore, we numerically find that the use of multiple

VaR constraints leads to the selection of portfolios with relatively small CVaRs when short selling is disallowed. While the constraints are less effective in controlling tail risk when short selling is allowed, the maximum CVaR permitted by the constraints is notably smaller than with a single constraint. These results suggest that the use of risk management systems based on multiple VaR constraints substantively reduces tail risk relative to the use of systems based on a single VaR constraint.

There are many reasons for why risk management systems based on VaR (or on any other risk measure) might fail. For example, such systems might not capture some risks (e.g., time-varying liquidity risks). Furthermore, the use of a short window for estimation purposes may result in a severely understated VaR. Our paper solely focuses on the drawback that VaR does not take into consideration losses beyond VaR. Regardless of whether this drawback is viewed as a significant driver of the sizeable trading losses that financial institutions suffered during the recent crisis, it should be recognized that regulations and risk management systems based on multiple VaR constraints are more effective in reducing tail risk than those based on a single VaR constraint.

Acknowledgments

Our paper has benefited from the valuable comments and suggestions of Peter Christoffersen, Basile Maire, an anonymous referee, and session participants at the 2010 Australasian Finance and Banking Conference in Sydney, the 2011 Midwest Finance Association Meeting in Chicago, and the 2011 Financial Management Association Asian Conference in Queenstown. Baptista gratefully acknowledges research support from the School of Business at The George Washington University.

Appendix A. Proofs of theoretical results

Proof of Theorem 1. Recall that the confidence level used in determining CVaR is $\alpha = 1 - 2/S$. Consider a VaR constraint using confidence level α_1 and bound V_1 where $\alpha_1 \geqslant \alpha$ and $V_1 < \bar{z}$. First, assume that $\alpha_1 = \alpha$. As noted earlier, the CVaRs of feasible portfolios in the absence of the constraint do not exceed $\bar{z}$. Observe that the constraint may not restrict the losses in any of the states that is used to compute CVaR. Hence, $\bar{z}$ is also an upper bound on the CVaRs of feasible portfolios in the presence of the constraint. This completes the first part of the proof.

Second, assume that $\alpha < \alpha_1 \leqslant \alpha + 1/S$. It is enough to show that for any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (\bar{z} + V_1)/2$, we have $V_{\alpha_1,\boldsymbol{w}} > V_1$. Fix any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (\bar{z} + V_1)/2$. Since portfolio losses are bounded from above by $\bar{z}$, Eq. (5) implies that portfolio $\boldsymbol{w}$ suffers a loss larger than V_1 in at least two states. Using the fact that all states are equally likely, we have $P[\tilde{z}_{\boldsymbol{w}} \leqslant V_1] \leqslant 1 - 2/S = \alpha < \alpha_1$. It follows from Eqs. (1) and (2) that $V_{\alpha_1,\boldsymbol{w}} > V_1$. This completes the second part of the proof.

Third, assume that $\alpha_1 > \alpha + 1/S$. It is enough to show that for any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > V_1$, we have $V_{\alpha_1,\boldsymbol{w}} > V_1$. Fix any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > V_1$. Eq. (5) implies that portfolio $\boldsymbol{w}$ suffers a loss larger than V_1 in at least one state. Using the fact that all states are equally likely, we have $P[\tilde{z}_{\boldsymbol{w}} \leqslant V_1] \leqslant 1 - 1/S = \alpha + 1/S < \alpha_1$. It follows from Eqs. (1) and (2) that $V_{\alpha_1,\boldsymbol{w}} > V_1$. This completes the third part of the proof. □

Proof of Theorem 2. Recall that the confidence level used in determining CVaR is $\alpha = 1 - 2/S$. Consider two VaR constraints using confidence levels $\{\alpha'_k\}_{k=1}^2$ and bounds $\{V'_k\}_{k=1}^2$ where $\alpha \leqslant \alpha'_1 < \alpha'_2$ and $V'_1 < V'_2 < \bar{z}$. First, assume that $\alpha'_1 = \alpha < \alpha'_2 \leqslant \alpha + 1/S$. It is enough to show that for any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (\bar{z} + V'_2)/2$, we have $V_{\alpha'_2,\boldsymbol{w}} > V'_2$. Fix any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (\bar{z} + V'_2)/2$. Since portfolio losses are bounded from above by $\bar{z}$, Eq. (5) implies that portfolio $\boldsymbol{w}$ suffers a loss larger than V'_2 in at least two states. Using the fact that all states are equally likely, we have $P[\tilde{z}_{\boldsymbol{w}} \leqslant V'_2] \leqslant 1 - 2/S = \alpha < \alpha'_2$. It follows from Eqs. (1) and (2) that $V_{\alpha'_2,\boldsymbol{w}} > V'_2$. This completes the first part of the proof.

Second, assume that $\alpha'_1 = \alpha < \alpha + 1/S < \alpha'_2$. It is enough to show that for any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > V'_2$, we have $V_{\alpha'_2,\boldsymbol{w}} > V'_2$. Fix any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > V'_2$. Eq. (5) implies that portfolio $\boldsymbol{w}$ suffers a loss larger than V'_2 in at least one state. Using the fact that all states are equally likely, we have $P[\tilde{z}_{\boldsymbol{w}} \leqslant V'_2] \leqslant 1 - 1/S = \alpha + 1/S < \alpha'_2$. It follows from Eqs. (1) and (2) that $V_{\alpha'_2,\boldsymbol{w}} > V'_2$. This completes the second part of the proof.

Third, assume that $\alpha < \alpha'_1 \leqslant \alpha + 1/S < \alpha'_2$. It is enough to show that for any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (V'_1 + V'_2)/2$, we have $V_{\alpha'_1,\boldsymbol{w}} > V'_1$ or $V_{\alpha'_2,\boldsymbol{w}} > V'_2$. Fix any portfolio $\boldsymbol{w}$ with $C_{\alpha,\boldsymbol{w}} > (V'_1 + V'_2)/2$. Suppose by way of a contradiction that $V_{\alpha'_1,\boldsymbol{w}} \leqslant V'_1$ and $V_{\alpha'_2,\boldsymbol{w}} \leqslant V'_2$. Using Eqs. (1) and (2) and the fact that all states are equally likely, portfolio $\boldsymbol{w}$ suffers a loss: (1) smaller than or equal to V'_1 in at least $S - 1$ states; and (2) smaller than or equal to V'_2 in all states. Eq. (5) implies that $C_{\alpha,\boldsymbol{w}} \leqslant (V'_1 + V'_2)/2$, a contradiction. This completes the third part of the proof. □

Appendix B. Adding estimation risk

While a detailed examination of the impact of estimation risk on VaR and CVaR is beyond the scope of our paper, we next explore the effect of estimation risk on our results.[51] We use 100 simulated distributions of asset returns. Each of these distributions is based on 300 vectors of monthly asset returns that are drawn from their empirical distribution (with replacement).[52] Here we measure the error of a given estimate by the difference between the value of this estimate and the corresponding value based on the empirical distribution.

When confidence level α is used to estimate CVaR, let $\widehat{\boldsymbol{w}}_{\alpha,\widehat{E}}$ denote the portfolio on the estimated mean-CVaR frontier with an estimated expected return of $\widehat{E}$. Consider a constraint on estimated VaR with confidence level α_k and a bound given by:

$$\widehat{V}_{\alpha_k,\alpha,\widehat{E}} \equiv \widehat{V}_{\alpha_k,\widehat{\boldsymbol{w}}_{\alpha,\widehat{E}}}. \tag{10}$$

First, suppose that short selling is disallowed. Rows (1)–(10) of Panel A of Fig. 10 show box plots of maximum efficiency losses with K constraints on estimated VaR using confidence levels $\{\alpha_k\}_{k=1}^K$ and bounds $\{\widehat{V}_{\alpha_k,\alpha,\widehat{E}}\}_{k=1}^K$ given by Eq. (10) across the grids of estimated expected returns that are associated with the 100 simulated distributions.[53] Panel A of Fig. 10 differs from Panel A of Fig. 3 in that losses are larger (note that the horizontal axes of the figures use different scales). Nevertheless, losses with three constraints are

[51] For work that recognizes estimation risk, see, e.g., the November 2000 issue of the *Journal of Empirical Finance*, the July 2002 issue of the *Journal of Banking and Finance*, Christoffersen and Gonçalves (2005), Pritsker (2006), Gao and Song (2008), Alexander et al. (2009), and references therein.

[52] Similar results are obtained when using 200 vectors of monthly asset returns. Furthermore, similar results are obtained when using either 200 or 300 vectors of monthly asset returns that are drawn from their empirical distribution without replacement.

[53] Note that we consider a grid of 101 estimated expected returns that possibly depends on each of the 100 simulated distributions of asset returns. Specifically, while the lowest value in this grid is common across such distributions (the risk-free return), the highest value is not. For any given simulated distribution, this highest value is: (1) the highest feasible estimated expected return if short selling is disallowed; and (2) the estimated expected return on a portfolio solely involving a weight of 150% in the asset with the highest estimated expected return and −50% in the risk-free asset if short selling is allowed.

typically smaller than those with two constraints, which in turn are smaller than those with a single constraint. Row (11) shows a box plot of efficiency losses for portfolios on the estimated mean-CVaR frontier. These losses represent the increase in CVaR arising from using the estimated mean-CVaR frontier instead of the mean-CVaR frontier based on the empirical distribution. Comparing rows (8)–(10) with row (11), note that the efficiency losses with three constraints are relatively close to those when using the estimated mean-CVaR frontier.[54]

Second, suppose that short selling is allowed. While Panel B of Fig. 10 differs from Panel A of Fig. 10 in that losses are generally larger, losses with three constraints are again typically smaller than those with two constraints, which in turn are smaller than those with a single constraint. Also, losses with three constraints are relatively close to those when using the estimated mean-CVaR frontier (compare rows (8)–(10) with row (11)). In sum, regardless of whether short selling is allowed, the use of multiple VaR constraints substantively reduces tail risk relative to the use of a single VaR constraint even with estimation risk.[55]

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[54] While beyond the scope of our paper, the popularity of VaR relative to CVaR might be justified by the possibility of CVaR being more difficult to estimate than VaR in the presence of estimation risk. Consistent with this justification, Cont et al. (2010) find that CVaR is less "robust" than VaR in the sense that small changes in the distribution of returns (arising from, e.g., estimation errors) lead to changes in CVaR estimates that are larger than those in VaR estimates.

[55] When estimation risk is present, we also find that the distribution of losses with K constraints significantly differs from that with $K+1$ constraints for $K = 1,2$ in most cases. This finding is based on two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests (available upon request).