CFA2: A CONTEXT-FREE APPROACH
TO CONTROL-FLOW ANALYSIS

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ABSTRACT. In a functional language, the dominant control-flow mechanism is function call and return. Most higher-order flow analyses, including k-CFA, do not handle call and return well: they remember only a bounded number of pending calls because they approximate programs with control-flow graphs. Call/return mismatch introduces precision-degrading spurious control-flow paths and increases the analysis time.

We describe CFA2, the first flow analysis with precise call/return matching in the presence of higher-order functions and tail calls. We formulate CFA2 as an abstract interpretation of programs in continuation-passing style and describe a sound and complete summarization algorithm for our abstract semantics. A preliminary evaluation shows that CFA2 gives more accurate data-flow information than 0CFA and 1CFA.

INTRODUCTION

Higher-order functional programs can be analyzed using analyses such as the k-CFA family [26]. These algorithms approximate the valid control-flow paths through the program as the set of all paths through a finite graph of abstract machine states, where each state represents a program point plus some amount of abstracted environment and control context.

In fact, this is not a particularly tight approximation. The set of paths through a finite graph is a regular language. However, the execution traces produced by recursive function calls are strings in a *context-free language*. Approximating this control flow with regular-language techniques permits execution paths that do not properly match calls with returns. This is particularly harmful when analyzing higher-order languages, since flowing functional values down these spurious paths can give rise to further "phantom" control-flow structure, along which functional values can then flow, and so forth, in a destructive spiral that not only degrades precision but drives up the cost of the analysis.

Pushdown models of programs can match an unbounded number of calls and returns, tightening up the set of possible executions to strings in a context-free language. Such models have long been used for first-order languages. The functional approach of Sharir and Pnueli [25] computes transfer-functions for whole procedures by composing transfer-functions of their basic blocks. Then, at a call-node these functions are used to compute the

data-flow value of the corresponding return-node directly. This "summary-based" technique has seen widespread use [5, 23]. Other pushdown models include Recursive State Machines [2] and Pushdown Systems [3, 10].

In this paper, we propose CFA2, a pushdown model of higher-order programs.[1] Our contributions can be summarized as follows:

- CFA2 is a flow analysis with precise call/return matching that can be used in the compilation of both typed and untyped languages. No existing analysis for functional languages enjoys all of these properties. k-CFA and its variants support limited call/return matching, bounded by the size of k (section 3.1). Type-based flow analysis with polymorphic subtyping [21, 22] also supports limited call/return matching, and applies to typed languages only (section 7).
- CFA2 uses a stack and a heap for variable binding. Variable references are looked up in one or the other, depending on where they appear in the source code. Most references in typical programs are read from the stack, which results in significant precision gains. Also, CFA2 can filter certain bindings off the stack to sharpen precision (section 4). k-CFA with abstract garbage collection [20] cannot infer that it is safe to remove these bindings. Last, the stack makes CFA2 resilient to syntax changes like η-expansion (section 4.1). It is well known that k-CFA is sensitive to such changes [30, 31].
- We formulate CFA2 as an abstract interpretation of programs in continuation-passing style (CPS). The abstract semantics uses a stack of unbounded height. Hence, the abstract state space is infinite, unlike k-CFA. To analyze the state space, we extend the functional approach of Sharir and Pnueli [25]. The resulting algorithm is a search-based variant of summarization that can handle higher-order functions and tail recursion. Currently, CFA2 does not handle first-class-control operators such as `call/cc` (section 5).
- We have implemented 0CFA, 1CFA and CFA2 in the Twobit Scheme compiler [6]. Our experimental results show that CFA2 is more precise than 0CFA and 1CFA. Also, CFA2 usually visits a smaller state space (section 6).

1. PRELIMINARY DEFINITIONS AND NOTATIONAL CONVENTIONS

In flow analysis of λ-calculus-based languages, a program is usually turned to an intermediate form where all subexpressions are named before it is analyzed. This form can be CPS, administrative normal form [11], or ordinary direct-style λ-calculus where each expression has a unique label. Selecting among these is mostly a matter of taste, and an analysis using one form can be changed to use another form without much effort.

This work uses CPS. We opted for CPS because it makes contexts explicit, as continuation-lambda terms. Moreover, `call/cc`, which we wish to support in the future, is directly expressible in CPS without the need for a special primitive operator.

In this section we describe our CPS language. For brevity, we develop the theory of CFA2 in the untyped λ-calculus. Primitive data, explicit recursion and side-effects can be added using standard techniques [26, ch. 3] [19, ch. 9]. Compilers that use CPS [16, 29]

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[1]CFA2 stands for "a Context-Free Approach to Control-Flow Analysis". We use "context-free" with its usual meaning from language theory, to indicate that CFA2 approximates valid executions as strings in a context-free language. Unfortunately, "context-free" means something else in program analysis. To avoid confusion, we use "monovariant" and "polyvariant" when we refer to the abstraction of calling context in program analysis. CFA2 is polyvariant (*aka* context-sensitive), because it analyzes different calls to the same function in different environments.

$$
\begin{array}{rcll}
v \in \mathit{Var} & = & \mathit{UVar} + \mathit{CVar} \\
u \in \mathit{UVar} & = & \text{a set of identifiers} \\
k \in \mathit{CVar} & = & \text{a set of identifiers} \\
\psi \in \mathit{Lab} & = & \mathit{ULab} + \mathit{CLab} \\
l \in \mathit{ULab} & = & \text{a set of labels} \\
\gamma \in \mathit{CLab} & = & \text{a set of labels} \\
lam \in \mathit{Lam} & = & \mathit{ULam} + \mathit{CLam} \\
ulam \in \mathit{ULam} & ::= & [\![(\lambda_l\,(u\,k)\;call)]\!]
\end{array}
\qquad
\begin{array}{rcll}
clam \in \mathit{CLam} & ::= & [\![(\lambda_\gamma\,(u)\;call)]\!] \\
call \in \mathit{Call} & = & \mathit{UCall} + \mathit{CCall} \\
\mathit{UCall} & ::= & [\![(f\,e\,q)^l]\!] \\
\mathit{CCall} & ::= & [\![(q\,e)^\gamma]\!] \\
g \in \mathit{Exp} & = & \mathit{UExp} + \mathit{CExp} \\
f,e \in \mathit{UExp} & = & \mathit{ULam} + \mathit{UVar} \\
q \in \mathit{CExp} & = & \mathit{CLam} + \mathit{CVar} \\
pr \in \mathit{Program} & ::= & \mathit{ULam}
\end{array}
$$

Figure 1: Partitioned CPS

usually partition the terms in a program in two disjoint sets, the user and the continuation set, and treat user terms differently from continuation terms.

We adopt this partitioning for our language (Fig. 1). Variables, lambdas and calls get labels from ULab or CLab. Labels are pairwise distinct. User lambdas take a user argument and the current continuation; continuation lambdas take only a user argument. We apply an additional syntactic constraint: the only continuation variable that can appear free in the body of a user lambda $(\lambda_l\,(u\,k)\;call)$ is k. This simple constraint forbids first-class control [24]. Intuitively, we get such a program by CPS-converting a direct-style program without `call/cc`.

We assume that all variables in a program have distinct names. Concrete syntax enclosed in $[\![\cdot]\!]$ denotes an item of abstract syntax. Functions with a '?' subscript are predicates, $e.g.$, $\mathit{Var}_?(e)$ returns true if e is a variable and false otherwise.

We use two notations for tuples, $(e_1,\ldots,e_n)$ and $\langle e_1,\ldots,e_n\rangle$, to avoid confusion when tuples are deeply nested. We use the latter for lists as well; ambiguities will be resolved by the context. Lists are also described by a head-tail notation, $e.g.$, $3::\langle 1,3,-47\rangle$.

CFA2 treats references to the same variable differently in different contexts. We split references in two categories: stack and heap references. In direct-style, if a reference appears at the same nesting level as its binder, then it is a stack reference, otherwise it is a heap reference. For example, the program $(\lambda_1(\mathtt{x})(\lambda_2(\mathtt{y})(\mathtt{x}\ (\mathtt{x}\ \mathtt{y}))))$ has a stack reference to $\mathtt{y}$ and two heap references to $\mathtt{x}$. Intuitively, only heap references may escape. When a program p is CPS-converted to a program p', stack ($\mathit{resp.}$ heap) references in p remain stack ($\mathit{resp.}$ heap) references in p'. All references added by the transform are stack references.

We can give an equivalent definition of stack and heap references directly in CPS, without referring to the original direct-style program. Labels can be split into disjoint sets according to the innermost user lambda that contains them. In the program $(\lambda_1(\mathtt{x}\ \mathtt{k1})\ (\mathtt{k1}\ (\lambda_2(\mathtt{y}\ \mathtt{k2})\ (\mathtt{x}\ \mathtt{y}\ (\lambda_3(\mathtt{u})\ (\mathtt{x}\ \mathtt{u}\ \mathtt{k2})^4))^5))^6)$, which is the CPS translation of the previous program, these sets are $\{1,6\}$ and $\{2,3,4,5\}$. The "label to variable" map $LV(\psi)$ returns all the variables bound by any lambdas that belong in the same set as ψ, $e.g.$, $LV(4)=\{\mathtt{y},\mathtt{k2},\mathtt{u}\}$ and $LV(6)=\{\mathtt{x},\mathtt{k1}\}$. We use this map to model stack behavior, because all continuation lambdas that "belong" to a given user lambda λ_l get closed by extending λ_l's stack frame ($\mathit{cf.}$ section 4). Notice that, for any ψ, $LV(\psi)$ contains exactly one continuation variable. Using LV, we give the following definition.

[UEA] $(\llbracket (f\,e\,q)^l \rrbracket, \beta, ve, t) \rightarrow (proc, d, c, ve, l :: t)$
       $proc = \mathcal{A}(f, \beta, ve)$
       $d = \mathcal{A}(e, \beta, ve)$
       $c = \mathcal{A}(q, \beta, ve)$

$$\mathcal{A}(g, \beta, ve) \triangleq \begin{cases} (g, \beta) & Lam_?(g) \\ ve(g, \beta(g)) & Var_?(g) \end{cases}$$

Concrete domains:

[UAE] $(proc, d, c, ve, t) \rightarrow (call, \beta', ve', t)$
       $proc \equiv \langle \llbracket (\lambda_l\,(u\,k)\,call) \rrbracket, \beta \rangle$
       $\beta' = \beta[u \mapsto t][k \mapsto t]$
       $ve' = ve[(u, t) \mapsto d][(k, t) \mapsto c]$

$$\begin{aligned}
\varsigma \in State &= Eval + Apply \\
Eval &= UEval + CEval \\
UEval &= UCall \times BEnv \times VEnv \times Time \\
CEval &= CCall \times BEnv \times VEnv \times Time \\
Apply &= UApply + CApply \\
UApply &= UClos \times UClos \times CClos \times VEnv \times Time \\
CApply &= CClos \times UClos \times VEnv \times Time \\
Clos &= UClos + CClos
\end{aligned}$$

[CEA] $(\llbracket (q\,e)^\gamma \rrbracket, \beta, ve, t) \rightarrow (proc, d, ve, \gamma :: t)$
       $proc = \mathcal{A}(q, \beta, ve)$
       $d = \mathcal{A}(e, \beta, ve)$

$$\begin{aligned}
d \in UClos &= ULam \times BEnv \\
c \in CClos &= (CLam \times BEnv) + halt \\
\beta \in BEnv &= Var \rightharpoonup Time \\
ve \in VEnv &= Var \times Time \rightharpoonup Clos \\
t \in Time &= Lab^*
\end{aligned}$$

[CAE] $(proc, d, ve, t) \rightarrow (call, \beta', ve', t)$
       $proc = \langle \llbracket (\lambda_\gamma\,(u)\,call) \rrbracket, \beta \rangle$
       $\beta' = \beta[u \mapsto t]$
       $ve' = ve[(u, t) \mapsto d]$

Figure 2: Concrete semantics and domains for Partitioned CPS

Definition 1.1 (Stack and heap references)**.**
- Let ψ be a call site that refers to a variable v. The predicate $S_?(\psi, v)$ holds iff $v \in LV(\psi)$. We call v a **stack reference**.
- Let ψ be a call site that refers to a variable v. The predicate $H_?(\psi, v)$ holds iff $v \notin LV(\psi)$. We call v a **heap reference**.
- v is a **stack variable**, written $S_?(v)$, iff all its references satisfy $S_?$.
- v is a **heap variable**, written $H_?(v)$, iff some of its references satisfy $H_?$.

Then, $S_?(5, \mathtt{y})$ holds because $\mathtt{y} \in \{\mathtt{y}, \mathtt{k2}, \mathtt{u}\}$ and $H_?(5, \mathtt{x})$ holds because $\mathtt{x} \notin \{\mathtt{y}, \mathtt{k2}, \mathtt{u}\}$.

2. Concrete Semantics

Execution in Partitioned CPS is guided by the semantics of Fig. 2. In the terminology of abstract interpretation, this semantics is called the *concrete* semantics. In order to find properties of a program at compile time, one needs to derive a computable approximation of the concrete semantics, called the *abstract* semantics. CFA2 and k-CFA are such approximations.

Execution traces alternate between *Eval* and *Apply* states. At an *Eval* state, we evaluate the subexpressions of a call site before performing a call. At an *Apply*, we perform the call.

The last component of each state is a *time*, which is a sequence of call sites. *Eval* to *Apply* transitions increment the time by recording the label of the corresponding call site. *Apply* to *Eval* transitions leave the time unchanged. Thus, the time t of a state reveals the call sites along the execution path to that state.

Times indicate points in the execution when variables are bound. The binding environment β is a partial function that maps variables to their binding times. The variable environment *ve* maps variable-time pairs to values. To find the value of a variable v, we look up the time v was put in β, and use that to search for the actual value in *ve*.

Let's look at the transitions more closely. At a *UEval* state with call site $(f\,e\,q)^l$, we evaluate f, e and q using the function $\mathcal{A}$. Lambdas are paired up with β to become closures,

while variables are looked up in *ve* using β. We add the label l in front of the current time and transition to a *UApply* state (rule [UEA]).

From *UApply* to *Eval*, we bind the formals of a procedure $\langle [\![(\lambda_l\,(u\,k)\,call)]\!], \beta \rangle$ to the arguments and jump to its body. The new binding environment β' extends the procedure's environment, with u and k mapped to the current time. The new variable environment ve' maps (u, t) to the user argument d, and (k, t) to the continuation c (rule [UAE]).

The remaining two transitions are similar. We use *halt* to denote the top-level continuation of a program *pr*. The initial state $\mathcal{I}(pr)$ is $((pr, \emptyset), input, halt, \emptyset, \langle\rangle)$, where *input* is a closure of the form $\langle [\![(\lambda_l\,(u\,k)\,call)]\!], \emptyset \rangle$. The initial time is the empty sequence of call sites.

CPS-based compilers may or may not use a stack for the final code. Steele's view, illustrated in the Rabbit compiler [29], is that argument evaluation pushes stack and function calls are GOTOs. Since arguments in CPS are not calls, argument evaluation is trivial and Rabbit never needs to push stack. By this approach, every call in CPS is a tail call.

An alternative style was used in the Orbit compiler [16]. At every function call, Orbit pushes a frame for the arguments. By this approach, tail calls are only the calls where the continuation argument is a variable. These CPS call sites were in tail position in the initial direct-style program. *CEval* states where the operator is a variable are calls to the current continuation with a return value. Orbit pops the stack at tail calls and before calling the current continuation.

We will see later that the abstract semantics of CFA2 uses a stack, like Orbit. However, CFA2 computes safe flow information which can be used by both aforementioned approaches. The workings of the abstract interpretation are independent of what style an implementor chooses for the final code.

3. Limitations of k-CFA

In this section, we discuss the main causes of imprecision and inefficiency in k-CFA. Our motivation in developing CFA2 is to create an analysis that overcomes these limitations.

We assume some familiarity with k-CFA, and abstract interpretation in general. Detailed descriptions on these topics can be found in [19, 26]. We use Scheme syntax for our example programs.

3.1. k-CFA does not properly match calls and returns.
In order to make the state space of k-CFA finite, Shivers chose a mechanism similar to the call-strings of Sharir and Pnueli [25]. Thus, recursive programs introduce approximation by folding an unbounded number of recursive calls down to a fixed-size call-string. In effect, by applying k-CFA to a higher-order program, we turn it into a finite-state machine. Taken to the extreme, when k is zero, a function can return to any of its callers, not just to the last one.

For example, consider the function `len` that computes the length of a list. Fig. 3 shows the code for `len`, its CPS translation and the associated control-flow graph. In the graph, the top level of the program is presented as a function called `main`. Function entry and exit nodes are rectangles with sharp corners. Inner nodes are rectangles with rounded corners. Each call site is represented by a call node and a corresponding return node, which contains the variable to which the result of the call is assigned. Each function uses a local variable `ret` for its return value. Solid arrows are intraprocedural steps. Dashed arrows go from call sites to function entries and from function exits to return points. There is no edge between call and return nodes; a call reaches its corresponding return only if the callee
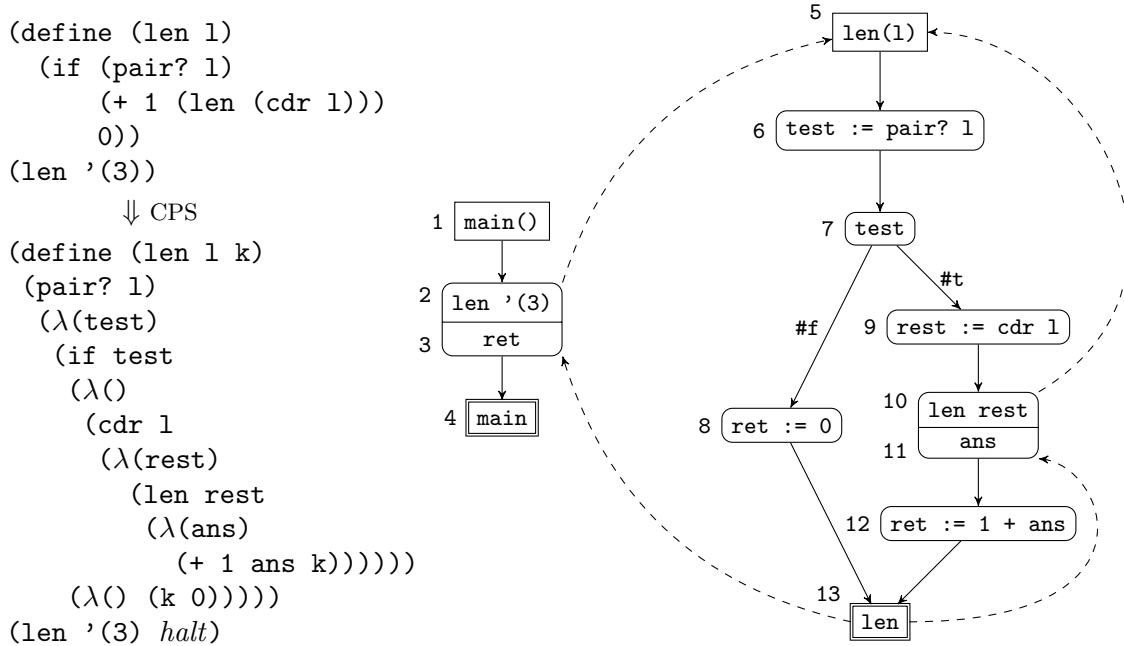
```
(define (len l)
  (if (pair? l)
      (+ 1 (len (cdr l)))
      0))
(len '(3))
         ⇓ CPS
(define (len l k)
 (pair? l)
  (λ(test)
   (if test
    (λ()
     (cdr l
      (λ(rest)
        (len rest
         (λ(ans)
           (+ 1 ans k))))))
    (λ() (k 0)))))
(len '(3) halt)
```


Figure 3: 0CFA on len

terminates. A monovariant analysis, such as 0CFA, considers every path from 1 to 4 to be a valid execution. In particular, it cannot exclude the path 1, 2, 5, 6, 7, 9, 10, 5, 6, 7, 8, 13, 3, 4. By following such a path, the program will terminate with a *non-empty* stack. It is clear that k-CFA cannot help much with optimizations that require accurate calculation of the stack change between program states, such as stack allocation of closure environments.

Spurious flows caused by call/return mismatch affect traditional data-flow information as well. For instance, 0CFA-constant-propagation for the program below cannot spot that n2 is the constant 2, because 1 also flows to x and is mistakenly returned by the second call to app. 1CFA also fails, because both calls to id happen in the body of app. 2CFA helps in this example, but repeated η-expansion of id can trick k-CFA for any k.

```
(let* ((app (λ(f e) (f e)))
       (id (λ(x) x))
       (n1 (app id 1))
       (n2 (app id 2)))
  (+ n1 n2))
```

In a non-recursive program, a large enough k can provide accurate call/return matching, but this is not desirable because the analysis becomes intractably slow even when k is 1 [30]. Moreover, the ubiquity of recursion in functional programs calls for a static analysis that can match an unbounded number of calls and returns. This can be done if we approximate programs using pushdown models instead of finite-state machines.

3.2. **The environment problem and fake rebinding.** In higher-order languages, many bindings of the same variable can be simultaneously live. Determining at compile time whether two references to some variable will be bound in the same run-time environment is referred to as the *environment problem* [26]. Consider the following program:

```
(let ((f (λ(x thunk) (if (number? x) (thunk) (λ₁() x)))))
  (f 0 (f "foo" "bar")))
```

In the inner call to `f`, `x` is bound to `"foo"` and λ_1 is returned. We call `f` again; this time, `x` is 0, so we jump through `(thunk)` to λ_1, and reference `x`, which, despite the just-completed test, is *not* a number: it is the string `"foo"`. Thus, during abstract interpretation, it is generally *unsafe* to assume that a reference has some property just because an earlier reference had that property. This has an unfortunate consequence: sometimes an earlier reference provides *safe* information about the reference at hand and k-CFA does not spot it:

$$(\text{define (compose-same f x) (f (f x)}^1)^2)$$

In `compose-same`, both references to `f` are always bound at the same time. However, if multiple closures flow to `f`, k-CFA may call one closure at call site 1 and a different closure at call site 2. This flow never happens at run time.

Imprecise binding information also makes it difficult to infer the types of variable references. In `len`, the `cdr` primitive must perform a run-time check and signal an error if `l` is not bound to a pair. This check is redundant since we checked for `pair?` earlier, and both references to `l` are bound in the same environment. If `len` is called with both pair and non-pair arguments, k-CFA cannot eliminate the run-time check.

CFA2 tackles this problem by distinguishing stack from heap references. If a reference v appears in a static context where we know the current stack frame is its environment record, we can be precise. If v appears free in some possibly escaping lambda, we cannot predict its extent so we fall back to a conservative approximation.

3.3. **Imprecision increases the running time of the analysis.** k-CFA for $k > 0$ is not a cheap analysis, both in theory [30] and in practice [27]. Counterintuitively, imprecision in higher-order flow analyses can increase their running time: imprecision induces spurious control paths, along which the analysis must flow data, thus creating further spurious paths, and so on, in a vicious cycle which creates extra work whose only function is to degrade precision. This is why techniques that aggressively prune the search space, such as ΓCFA [20], not only increase precision, but can also improve the speed of the analysis.

In the previous subsections, we saw examples of information known at compile time that k-CFA cannot exploit. CFA2 uses this information. The enhanced precision of CFA2 has a positive effect on its running time (*cf.* section 6).

4. The CFA2 semantics

In this section we define the abstract semantics of CFA2. The abstract semantics approximates the concrete semantics. This means that each concrete state has a corresponding abstract state. Therefore, each concrete execution, *i.e.*, sequence of states related by $\rightarrow$, has a corresponding abstract execution that computes an approximate answer.

Each abstract state has a stack. Analyzing recursive programs requires states with stacks of unbounded size. Thus, the abstract state space is infinite and the standard algorithms for k-CFA [19,26] will diverge because they work by enumerating all states. We show how to solve the stack-size problem in section 5. Here, we describe the abstract semantics (section 4.1), show how to map concrete to abstract states and prove the correctness of the abstract semantics (section 4.2).

$[\widehat{\text{UEA}}]$ $([\![(f\,e\,q)^l]\!], st, h) \rightsquigarrow (ulam, \hat{\text{d}}, \hat{c}, st', h)$
$\quad ulam \in \hat{\mathcal{A}}_u(f, l, st, h)$
$\quad \hat{\text{d}} = \hat{\mathcal{A}}_u(e, l, st, h)$
$\quad \hat{c} = \hat{\mathcal{A}}_k(q, st)$
$\quad st' = \begin{cases} pop(st) & Var_?(q) \\ st & Lam_?(q) \wedge (H_?(l, f) \vee Lam_?(f)) \\ st[f \mapsto \{ulam\}] & Lam_?(q) \wedge S_?(l, f) \end{cases}$

$[\widehat{\text{UAE}}]$ $([\![(\lambda_l\,(u\,k)\,call)]\!], \hat{\text{d}}, \hat{c}, st, h) \rightsquigarrow (call, st', h')$
$\quad st' = push([u \mapsto \hat{\text{d}}][k \mapsto \hat{c}], st)$
$\quad h' = \begin{cases} h \sqcup [u \mapsto \hat{\text{d}}] & H_?(u) \\ h & S_?(u) \end{cases}$

$[\widehat{\text{CEA}}]$ $([\![(q\,e)^\gamma]\!], st, h) \rightsquigarrow (clam, \hat{\text{d}}, st', h)$
$\quad clam = \hat{\mathcal{A}}_k(q, st)$
$\quad \hat{\text{d}} = \hat{\mathcal{A}}_u(e, \gamma, st, h)$
$\quad st' = \begin{cases} pop(st) & Var_?(q) \\ st & Lam_?(q) \end{cases}$

$[\widehat{\text{CAE}}]$ $([\![(\lambda_\gamma\,(u)\,call)]\!], \hat{\text{d}}, st, h) \rightsquigarrow (call, st', h')$
$\quad st' = st[u \mapsto \hat{\text{d}}]$
$\quad h' = \begin{cases} h \sqcup [u \mapsto \hat{\text{d}}] & H_?(u) \\ h & S_?(u) \end{cases}$

$$\hat{\mathcal{A}}_u(e, \psi, st, h) \triangleq \begin{cases} \{e\} & Lam_?(e) \\ st(e) & S_?(\psi, e) \\ h(e) & H_?(\psi, e) \end{cases}$$

$$\hat{\mathcal{A}}_k(q, st) \triangleq \begin{cases} q & Lam_?(q) \\ st(q) & Var_?(q) \end{cases}$$

Abstract domains:

$\hat{\varsigma} \in \widehat{UEval} = UCall \times Stack \times Heap$
$\hat{\varsigma} \in \widehat{UApply} = ULam \times \widehat{UClos} \times \widehat{CClos} \times Stack \times Heap$
$\hat{\varsigma} \in \widehat{CEval} = CCall \times Stack \times Heap$
$\hat{\varsigma} \in \widehat{CApply} = \widehat{CClos} \times \widehat{UClos} \times Stack \times Heap$
$\hat{\text{d}} \in \widehat{UClos} = Pow(ULam)$
$\hat{c} \in \widehat{CClos} = CLam + halt$
$fr, tf \in Frame = (UVar \rightharpoonup \widehat{UClos}) \cup (CVar \rightharpoonup \widehat{CClos})$
$st \in Stack = Frame^*$
$h \in Heap = UVar \rightharpoonup \widehat{UClos}$

Stack operations:

$pop(tf :: st) \triangleq st$
$push(fr, st) \triangleq fr :: st$
$(tf :: st)(v) \triangleq tf(v)$
$(tf :: st)[u \mapsto \hat{\text{d}}] \triangleq tf[u \mapsto \hat{\text{d}}] :: st$

Figure 4: Abstract semantics and relevant definitions

4.1. **Abstract semantics.** The CFA2 semantics is an abstract interpreter that executes a CPS program, using a stack for variable binding and return-point information.

We describe the stack-management policy with an example. Assume that we run the `len` program of section 3. When calling (`len '(3) halt`) we push a frame $[l \mapsto (3)][k \mapsto halt]$ on the stack. The test (`pair? l`) is true, so we add the binding $[test \mapsto true]$ to the top frame and jump to the true branch. We take the `cdr` of `l` and add the binding $[rest \mapsto ()]$ to the top frame. We call `len` again, push a new frame for its arguments and jump to its body. This time the test is false, so we extend the top frame with $[test \mapsto false]$ and jump to the false branch. The call to `k` is a function return, so we pop a frame and pass `0` to (λ(`ans`)(`+ 1 ans k`)). Call site (`+ 1 ans k`) is also a function return, so we pop the remaining frame and pass `1` to the top-level continuation $halt$.

In general, we push a frame at function entries and pop at tail calls and at function returns. Results of intermediate computations are stored in the top frame. This policy enforces two invariants about the abstract interpreter. First, when executing inside a user function ($\lambda_l\,(u\,k)\,call$), the domain of the top frame is a subset of $LV(l)$. Second, the frame below the top frame is the environment of the current continuation.

Each variable v in our example was looked up in the top frame, because each lookup happened while executing inside the lambda that binds v. This is not always the case; in the first snippet of section 3.2 there is a heap reference to `x` in λ_1. When control reaches that reference, the top frame does not belong to the lambda that binds `x`. In CFA2, we

look up stack references in the top frame, and heap references in the heap. Stack lookups below the top frame never happen.

The CFA2 semantics appears in Fig. 4. An abstract value is either an abstract user closure (member of the set $\widehat{UClos}$) or an abstract continuation closure (member of $\widehat{CClos}$). An abstract user closure is a set of user lambdas. An abstract continuation closure is either a continuation lambda or *halt*. A frame is a map from variables to abstract values, and a stack is a sequence of frames. All stack operations except *push* are defined for non-empty stacks only. A heap is a map from variables to abstract values. It contains only user bindings because, without first-class control, every continuation variable is a stack variable.

On transition from a $\widehat{UEval}$ state $\hat{\varsigma}$ to a $\widehat{UApply}$ state $\hat{\varsigma}'$ (rule [$\widehat{UEA}$]), we first evaluate f, e and q. We evaluate user terms using $\hat{\mathcal{A}}_u$ and continuation terms using $\hat{\mathcal{A}}_k$. We non-deterministically choose one of the lambdas that flow to f as the operator in $\hat{\varsigma}'$.[2] The change to the stack depends on q and f. If q is a variable, the call is a tail call so we pop the stack (case 1). If q is a lambda, it evaluates to a new closure whose environment is the top frame, hence we do not pop the stack (cases 2, 3). Moreover, if f is a lambda or a heap reference then we leave the stack unchanged. However, if f is a stack reference, we set f's value in the top frame to $\{ulam\}$, possibly forgetting other lambdas that flow to f. This "stack filtering" prevents fake rebinding (*cf.* section 3.2): when we return to $\hat{c}$, we may reach more stack references of f. These references and the current one are bound at the same time. Since we are committing to $ulam$ in this transition, these references must also be bound to $ulam$.

In the $\widehat{UApply}$-to-$\widehat{Eval}$ transition (rule [$\widehat{UAE}$]), we push a frame for the procedure's arguments. In addition, if u is a heap variable we must update its binding in the heap. The join operation $\sqcup$ is defined as:

$$(h \sqcup [u \mapsto \hat{d}])(v) \triangleq \begin{cases} h(v) & v \not\equiv u \\ h(v) \cup \hat{d} & v \equiv u \end{cases}$$

In a $\widehat{CEval}$-to-$\widehat{CApply}$ transition (rule [$\widehat{CEA}$]), we are preparing for a call to a continuation so we must reset the stack to the stack of its birth. When q is a variable, the $\widehat{CEval}$ state is a function return and the continuation's environment is the second stack frame. Therefore, we pop a frame before calling *clam*. When q is a lambda, it is a newly created closure thus the stack does not change. Note that the transition is deterministic, unlike [$\widehat{UEA}$]. Since we always know which continuation we are about to call, call/return mismatch *never* happens. For instance, the function `len` may be called from many places in a program, so multiple continuations may flow to `k`. But, by retrieving `k`'s value from the stack, we always return to the correct continuation.

In the $\widehat{CApply}$-to-$\widehat{Eval}$ transition (rule [$\widehat{CAE}$]), our stack policy dictates that we extend the top frame with the binding for the continuation's parameter u. If u is a heap variable, we also update the heap.[3]

---

[2]An abstract execution explores one path, but the algorithm that searches the state space considers all possible executions (*cf.* section 5), as is the case in the operational formulation of k-CFA [19].

[3]All temporaries created by the CPS transform are stack variables; but a compiler optimization may rewrite a program to create heap references to temporaries.

Examples. When the analyzed program is not recursive, the stack size is bounded so we can enumerate all abstract states without diverging. Let's see how the abstract semantics works on a simple program that applies the identity function twice and returns the result of the second call. The initial state $\hat{\mathcal{I}}(pr)$ is a $\widehat{UApply}$.

$$([\![(\lambda(\text{id h})(\text{id 1 }(\lambda_1(\text{n1})(\text{id 2 }(\lambda_2(\text{n2})(\text{h n2}))))))]\!], \{[\![(\lambda_3(\text{x k})(\text{k x}))]\!]\}, halt, \langle\rangle, \emptyset)$$

All variables in this example are stack variables, so the heap will remain empty throughout the execution. In frames, we abbreviate lambdas by their labels. By rule $[\widehat{UAE}]$, we push a frame for id and h and transition to a $\widehat{UEval}$ state.

$$([\![(\text{id 1 }(\lambda_1(\text{n1})(\text{id 2 }(\lambda_2(\text{n2})(\text{h n2})))))]\!], \langle[\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We look up id in the top frame. Since the continuation argument is a lambda, we do not pop the stack. The next state is a $\widehat{UApply}$.

$$([\![(\lambda_3(\text{x k})(\text{k x}))]\!], \{1\}, \lambda_1, \langle[\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We push a frame for the arguments of λ_3 and jump to its body.

$$([\![(\text{k x})]\!], \langle[\text{x} \mapsto \{1\}][\text{k} \mapsto \lambda_1], [\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

This is a $\widehat{CEval}$ state where the operator is a variable, so we pop a frame.

$$([\![(\lambda_1(\text{n1})(\text{id 2 }(\lambda_2(\text{n2})(\text{h n2}))))]\!], \{1\}, \langle[\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We extend the top frame to bind n1 and jump to the body of λ_1.

$$([\![(\text{id 2 }(\lambda_2(\text{n2})(\text{h n2})))]\!], \langle[\text{n1} \mapsto \{1\}][\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

The new call to id is also not a tail call, so we do not pop.

$$([\![(\lambda_3(\text{x k})(\text{k x}))]\!], \{2\}, \lambda_2, \langle[\text{n1} \mapsto \{1\}][\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We push a frame and jump to the body of λ_3.

$$([\![(\text{k x})]\!], \langle[\text{x} \mapsto \{2\}][\text{k} \mapsto \lambda_2], [\text{n1} \mapsto \{1\}][\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We pop a frame and jump to λ_2.

$$([\![(\lambda_2(\text{n2})(\text{h n2}))]\!], \{2\}, \langle[\text{n1} \mapsto \{1\}][\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

We extend the top frame to bind n2 and jump to the body of λ_2.

$$([\![(\text{h n2})]\!], \langle[\text{n2} \mapsto \{2\}][\text{n1} \mapsto \{1\}][\text{id} \mapsto \{\lambda_3\}][\text{h} \mapsto halt]\rangle, \emptyset)$$

The operator is a variable, so we pop the stack. The next state is a final state, so the program terminates with value $\{2\}$.

$$(halt, \{2\}, \langle\rangle, \emptyset)$$

1CFA would also find the precise answer for this program. However, if we η-expand λ_3 to $(\lambda_3(\text{x k})((\lambda_4(\text{y k2})(\text{k2 y}))\text{ x k}))$, 1CFA will return $\{1, 2\}$ because both calls to λ_4 happen at the same call site. CFA2 is more resilient to η-expansion. It will return the precise answer in the modified program because the change did not create any heap references. However, if we change λ_3 to $(\lambda_3(\text{x k})((\lambda_4(\text{y k2})(\text{k2 x}))\text{ x k}))$, then both 1 and 2 flow to the heap reference to x and CFA2 will return $\{1, 2\}$.

$|(\llbracket(g_1 \ldots g_n)^\psi\rrbracket, \beta, ve, t)|_{ca} = (\llbracket(g_1 \ldots g_n)^\psi\rrbracket, \ toStack(LV(\psi), \beta, ve), \ |ve|_{ca})$

$|(\langle\llbracket(\lambda_l(u\,k)\ call)\rrbracket, \beta\rangle, d, c, ve, t)|_{ca} = (\llbracket(\lambda_l(u\,k)\ call)\rrbracket, |d|_{ca}, |c|_{ca}, st, |ve|_{ca})$

where $st = \begin{cases} \langle\rangle & c = halt \\ toStack(LV(\gamma), \beta', ve) & c = (\llbracket(\lambda_\gamma(u')call')\rrbracket, \beta') \end{cases}$

$|(\langle\llbracket(\lambda_\gamma(u)\ call)\rrbracket, \beta\rangle, d, ve, t)|_{ca} = (\llbracket(\lambda_\gamma(u)\ call)\rrbracket, |d|_{ca}, toStack(LV(\gamma), \beta, ve), |ve|_{ca})$

$|(halt, d, ve, t)|_{ca} = (halt, |d|_{ca}, \langle\rangle, |ve|_{ca})$

$|(\llbracket(\lambda_l(u\,k)\ call)\rrbracket, \beta)|_{ca} = \{\llbracket(\lambda_l(u\,k)\ call)\rrbracket\}$

$|(\llbracket(\lambda_\gamma(u)\ call)\rrbracket, \beta)|_{ca} = \llbracket(\lambda_\gamma(u)\ call)\rrbracket$

$|halt|_{ca} = halt$

$|ve|_{ca} = \{\,(u, \bigsqcup_t |ve(u,t)|_{ca}) \ : \ H_?(u)\}$

$toStack(\{u_1, \ldots, u_n, k\}, \beta, ve) \triangleq \begin{cases} \langle[\overline{u_i \mapsto \hat{\mathrm{d}}_i}][k \mapsto halt]\rangle & halt = ve(k, \beta(k)) \\ [\overline{u_i \mapsto \hat{\mathrm{d}}_i}][k \mapsto \llbracket(\lambda_\gamma(u)\ call)\rrbracket] :: st & (\llbracket(\lambda_\gamma(u)\ call)\rrbracket, \beta') = ve(k, \beta(k)) \end{cases}$

where $\hat{\mathrm{d}}_i = |ve(u_i, \beta(u_i))|_{ca}$ and $st = toStack(LV(\gamma), \beta', ve)$

Figure 5: From concrete states to abstract states

$(call, st_1, h_1) \sqsubseteq (call, st_2, h_2)$ iff $st_1 \sqsubseteq st_2 \ \wedge \ h_1 \sqsubseteq h_2$

$(ulam, \hat{\mathrm{d}}_1, \hat{c}, st_1, h_1) \sqsubseteq (ulam, \hat{\mathrm{d}}_2, \hat{c}, st_2, h_2)$ iff $\hat{\mathrm{d}}_1 \sqsubseteq \hat{\mathrm{d}}_2 \ \wedge \ st_1 \sqsubseteq st_2 \ \wedge \ h_1 \sqsubseteq h_2$

$(\hat{c}, \hat{\mathrm{d}}_1, st_1, h_1) \sqsubseteq (\hat{c}, \hat{\mathrm{d}}_2, st_2, h_2)$ iff $\hat{\mathrm{d}}_1 \sqsubseteq \hat{\mathrm{d}}_2 \ \wedge \ st_1 \sqsubseteq st_2 \ \wedge \ h_1 \sqsubseteq h_2$

$h_1 \sqsubseteq h_2$ iff $h_1(u) \sqsubseteq h_2(u)$ for each $u \in \mathrm{dom}(h_1)$

$tf_1 :: st_1 \sqsubseteq tf_2 :: st_2$ iff $tf_1 \sqsubseteq tf_2 \ \wedge \ st_1 \sqsubseteq st_2$

$\langle\rangle \sqsubseteq \langle\rangle$

$tf_1 \sqsubseteq tf_2$ iff $tf_1(v) \sqsubseteq tf_2(v)$ for each $v \in \mathrm{dom}(tf_1)$

$\hat{\mathrm{d}}_1 \sqsubseteq \hat{\mathrm{d}}_2$ iff $\hat{\mathrm{d}}_1 \subseteq \hat{\mathrm{d}}_2$

$\hat{c} \sqsubseteq \hat{c}$

Figure 6: The $\sqsubseteq$ relation on abstract states

4.2. **Correctness of the abstract semantics.** We proceed to show that the CFA2 semantics safely approximates the concrete semantics. First, we define a map $|\cdot|_{ca}$ from concrete to abstract states. Next, we show that if ς transitions to ς' in the concrete semantics, the abstract counterpart $|\varsigma|_{ca}$ of ς transitions to a state $\hat{\varsigma}'$ which approximates $|\varsigma'|_{ca}$. Hence, we ensure that the possible behaviors of the abstract interpreter include the actual run-time behavior of the program.

The map $|\cdot|_{ca}$ appears in Fig. 5. The abstraction of an *Eval* state ς of the form $(\llbracket(g_1 \ldots g_n)^\psi\rrbracket, \beta, ve, t)$ is an $\widehat{Eval}$ state $\hat{\varsigma}$ with the same call site. Since ς does not have a stack, we must expose stack-related information hidden in β and ve. Assume that λ_l is

the innermost user lambda that contains ψ. To reach ψ, control passed from a $\widehat{UApply}$ state $\hat{\varsigma}'$ over λ_l. According to our stack policy, the top frame contains bindings for the formals of λ_l and any temporaries added along the path from $\hat{\varsigma}'$ to $\hat{\varsigma}$. Therefore, the domain of the top frame is a subset of $LV(l)$, *i.e.*, a subset of $LV(\psi)$. For each user variable $u_i \in (LV(\psi) \cap \text{dom}(\beta))$, the top frame contains $[u_i \mapsto |ve(u_i, \beta(u_i))|_{ca}]$. Let k be the sole continuation variable in $LV(\psi)$. If $ve(k, \beta(k))$ is *halt* (the return continuation is the top-level continuation), the rest of the stack is empty. If $ve(k, \beta(k))$ is $([\![(\lambda_\gamma(u)\ call)]\!], \beta')$, the second frame is for the user lambda in which λ_γ was born, and so forth: proceeding through the stack, we add a frame for each live activation of a user lambda until we reach *halt*.

The abstraction of a *UApply* state over $\langle [\![(\lambda_l(u\,k)\ call)]\!], \beta \rangle$ is a $\widehat{UApply}$ state $\hat{\varsigma}$ whose operator is $[\![(\lambda_l(u\,k)\ call)]\!]$. The stack of $\hat{\varsigma}$ is the stack in which the continuation argument was created, and we compute it using *toStack* as above.

Abstracting a *CApply* is similar to the *UApply* case, only now the top frame is the environment of the continuation operator. Note that the abstraction maps drop the time of the concrete states, since the abstract states do not use times.

The abstraction of a user closure is the singleton set with the corresponding lambda. The abstraction of a continuation closure is the corresponding lambda. When abstracting a variable environment *ve*, we only keep heap variables.

We can now state our simulation theorem. The proof proceeds by case analysis on the concrete transition relation. The relation $\hat{\varsigma}_1 \sqsubseteq \hat{\varsigma}_2$ is a partial order on abstract states and can be read as "$\hat{\varsigma}_1$ is more precise than $\hat{\varsigma}_2$" (Fig. 6). The proof can be found in the appendix.

Theorem 4.1 (Simulation). *If* $\varsigma \to \varsigma'$ *and* $|\varsigma|_{ca} \sqsubseteq \hat{\varsigma}$, *then there exists* $\hat{\varsigma}'$ *such that* $\hat{\varsigma} \rightsquigarrow \hat{\varsigma}'$ *and* $|\varsigma'|_{ca} \sqsubseteq \hat{\varsigma}'$.

5. Computing CFA2

5.1. **Pushdown models and summarization.** In section 3, we saw that a monovariant analysis like 0CFA treats the control-flow graph of `len` as a finite-state machine (FSM), where all paths are valid executions. For $k > 0$, k-CFA still approximates `len` as a FSM, albeit a larger one that has several copies of each procedure, caused by different call strings.

But in reality, calls and returns match; the call from 2 returns to 3 and each call from 10 returns to 11. Thus, by thinking of executions as strings in a context-free language, we can do more precise flow analysis. We can achieve this by approximating `len` as a pushdown system (PDS) [3, 10]. A PDS is similar to a pushdown automaton, except it does not read input from a tape. For illustration purposes, we take the (slightly simplified) view that the state of a PDS is a pair of a program point and a stack. The transition rules for call nodes push the return point on the stack:

$$(2, s) \hookrightarrow (5, 3 :: s), \qquad (10, s) \hookrightarrow (5, 11 :: s)$$

Function exits pop the node at the top of the stack and jump to it:

$$(13, n :: s) \hookrightarrow (n, s)$$

All other nodes transition to their successor(s) and leave the stack unchanged, *e.g.*

$$(3, s) \hookrightarrow (4, s), \qquad (7, s) \hookrightarrow (8, s), \qquad (7, s) \hookrightarrow (9, s)$$

Suppose we want to find all nodes reachable from 1. Obviously, we cannot do it by enumerating all states. Thus, algorithms for pushdown reachability use a dynamic programming technique called *summarization*. The intuition behind summarization is to flow facts from a program point n with an *empty* stack to a point n' in the same procedure. We say that n' is *same-context reachable* from n. These facts are then suitably combined to get flow facts for the whole program.

We use summarization to explore the state space in CFA2. Our algorithm is based on Sharir and Pnueli's functional approach [25, pg. 207], adapted to the more modern terminology of Reps *et al.* [23]. Summarization requires that we know all call sites of a function. Therefore, it does not apply directly to higher-order languages, because we cannot find the call sites of a function by looking at a program's source code. We need a *search-based* variant of summarization, which records callers as it discovers them.

We illustrate our variant on `len`. We find reachable nodes by recording *path edges*, *i.e.*, edges whose source is the entry of a procedure and target is some program point in the same procedure. Path edges should not be confused with the edges already present in `len`'s control-flow graph. They are artificial edges used by the analysis to represent intraprocedural paths, hence the name. From 1 we can go to 2, so we record $\langle 1, 1 \rangle$ and $\langle 1, 2 \rangle$. Then 2 calls `len`, so we record the call $\langle 2, 5 \rangle$ and jump to 5. In `len`, we reach 6 and 7 and record $\langle 5, 5 \rangle$, $\langle 5, 6 \rangle$ and $\langle 5, 7 \rangle$. We do not assume anything about the result of the test, so we must follow both branches. By following the false branch, we discover $\langle 5, 8 \rangle$ and $\langle 5, 13 \rangle$. Node 13 is an exit, so each caller of `len` can reach its corresponding return point. We keep track of this fact by recording the *summary* edge $\langle 5, 13 \rangle$. We have only seen a call from 2, so we return to 3 and record $\langle 1, 3 \rangle$. Finally, we record $\langle 1, 4 \rangle$, which is the end of the program. By analyzing the true branch, we discover edges $\langle 5, 9 \rangle$ and $\langle 5, 10 \rangle$, and record the new call $\langle 10, 5 \rangle$. Reachability inside `len` does not depend on its calling context, so from the summary edge $\langle 5, 13 \rangle$ we infer that 10 can reach 11 and we record $\langle 5, 11 \rangle$ and subsequently $\langle 5, 12 \rangle$. At this point, we have discovered all possible path edges.

Summarization works because we can temporarily forget the caller while analyzing inside a procedure, and remember it when we are about to return. A consequence is that if from node n with an empty stack we can reach n' with stack s, then n with s' can go to n' with $append(s, s')$.

5.2. **Local semantics.** Summarization-based algorithms operate on a finite set of program points. Hence, we cannot use (an infinite number of) abstract states as program points. For this reason, we introduce *local states* and define a map $|\cdot|_{al}$ from abstract to local states (Fig. 7). Intuitively, a local state is like an abstract state but with a single frame instead of a stack. Discarding the rest of the stack makes the local state space finite; keeping the top frame allows precise lookups for stack references.

The local semantics describes executions that do not touch the rest of the stack (in other words, executions where functions do not return). Thus, a $\widetilde{CEval}$ state with call site $(k\, e)^\gamma$ has no successor in this semantics. Since functions do not call their continuations, the frames of local states contain only user bindings. Local steps are otherwise similar to abstract steps. The metavariable $\tilde{\varsigma}$ ranges over local states. We define the map $|\cdot|_{cl}$ from concrete to local states to be $|\cdot|_{al} \circ |\cdot|_{ca}$.

We can now see how the local semantics fits in a summarization algorithm for CFA2. Essentially, CFA2 approximates a higher-order program as a PDS. The local semantics

$$\tilde{\mathcal{A}}_u(e, \psi, tf, h) \triangleq \begin{cases} \{e\} & Lam_?(e) \\ tf(e) & S_?(\psi, e) \\ h(e) & H_?(\psi, e) \end{cases}$$

$[\widetilde{UEA}]$ $(\llbracket (f\, e\, q)^l \rrbracket, tf, h) \approx (ulam, \hat{\mathrm{d}}, h)$
$\quad ulam \in \tilde{\mathcal{A}}_u(f, l, tf, h)$
$\quad \hat{\mathrm{d}} = \tilde{\mathcal{A}}_u(e, l, tf, h)$

$[\widetilde{UAE}]$ $(\llbracket (\lambda_l(u\, k)\, call) \rrbracket, \hat{\mathrm{d}}, h) \approx (call, [u \mapsto \hat{\mathrm{d}}], h')$
$\quad h' = \begin{cases} h \sqcup [u \mapsto \hat{\mathrm{d}}] & H_?(u) \\ h & S_?(u) \end{cases}$

$[\widetilde{CEA}]$ $(\llbracket (clam\, e)^\gamma \rrbracket, tf, h) \approx (clam, \hat{\mathrm{d}}, tf, h)$
$\quad \hat{\mathrm{d}} = \tilde{\mathcal{A}}_u(e, \gamma, tf, h)$

$[\widetilde{CAE}]$ $(\llbracket (\lambda_\gamma(u)\, call) \rrbracket, \hat{\mathrm{d}}, tf, h) \approx (call, tf', h')$
$\quad tf' = tf[u \mapsto \hat{\mathrm{d}}]$
$\quad h' = \begin{cases} h \sqcup [u \mapsto \hat{\mathrm{d}}] & H_?(u) \\ h & S_?(u) \end{cases}$

Local domains:

$\widetilde{Eval} = Call \times \widetilde{Stack} \times Heap$
$\widetilde{UApply} = ULam \times \widetilde{UClos} \times Heap$
$\widetilde{CApply} = \widetilde{CClos} \times \widetilde{UClos} \times \widetilde{Stack} \times Heap$
$\widetilde{Frame} = UVar \rightharpoonup \widetilde{UClos}$
$\widetilde{Stack} = \widetilde{Frame}$

Abstract to local maps:

$|(call, st, h)|_{al} = (call, |st|_{al}, h)$
$|(ulam, \hat{\mathrm{d}}, \hat{c}, st, h)|_{al} = (ulam, \hat{\mathrm{d}}, h)$
$|(\hat{c}, \hat{\mathrm{d}}, st, h)|_{al} = (\hat{c}, \hat{\mathrm{d}}, |st|_{al}, h)$
$|tf :: st'|_{al} = \{ (u, tf(u)) : UVar_?(u) \}$
$|\langle\rangle|_{al} = \emptyset$

Figure 7: Local semantics

describes the PDS transitions that do not return (intraprocedural steps and function calls). We discover return points by recording callers and summary edges.

Summarization distinguishes between different kinds of states: entries, exits, calls, returns and inner states. CPS lends itself naturally to such a categorization:

- A $\widetilde{UApply}$ state corresponds to an **entry** node—control is about to enter the body of a function.
- A $\widetilde{CEval}$ state where the operator is a variable is an **exit** node—a function is about to pass its result to its context.
- A $\widetilde{CEval}$ state where the operator is a lambda is an **inner** state.
- A $\widetilde{UEval}$ state where the continuation argument is a variable is an **exit**—at tail calls control does not return to the caller.
- A $\widetilde{UEval}$ state where the continuation argument is a lambda is a **call**.
- A $\widetilde{CApply}$ state is a **return** if its predecessor is an exit, or an **inner** state if its predecessor is also an inner state. Our algorithm will not need to distinguish between the two kinds of $\widetilde{CApply}$s; the difference is just conceptual.

Last, we generalize the notion of summary edges to handle tail recursion. Consider an earlier example, written in CPS.

```
((λ(app id k)
    (app id 1 (λ₁(n1) (app id 2 (λ₂(n2) (+ n1 n2 k)))))))
 (λ(f e k) (f e k))
 (λ(x k) (k x))
 halt)
```

```
01      Summary, Callers, TCallers, Final ⟵ ∅
02      Seen, W ⟵ {(Ĩ(pr), Ĩ(pr))}
03      while W ≠ ∅
04        remove (ς̃₁, ς̃₂) from W
05        switch ς̃₂
06          case ς̃₂ of Entry, CApply, Inner-CEval
07            for each ς̃₃ in succ(ς̃₂) Propagate(ς̃₁, ς̃₃)
08          case ς̃₂ of Call
09            for each ς̃₃ in succ(ς̃₂)
10                Propagate(ς̃₃, ς̃₃)
11                insert (ς̃₁, ς̃₂, ς̃₃) in Callers
12                for each (ς̃₃, ς̃₄) in Summary  Update(ς̃₁, ς̃₂, ς̃₃, ς̃₄)
13          case ς̃₂ of Exit-CEval
14            if ς̃₁ = Ĩ(pr) then
15                Final(ς̃₂)
16            else
17                insert (ς̃₁, ς̃₂) in Summary
18                for each (ς̃₃, ς̃₄, ς̃₁) in Callers  Update(ς̃₃, ς̃₄, ς̃₁, ς̃₂)
19                for each (ς̃₃, ς̃₄, ς̃₁) in TCallers Propagate(ς̃₃, ς̃₂)
20          case ς̃₂ of Exit-TC
21            for each ς̃₃ in succ(ς̃₂)
22                Propagate(ς̃₃, ς̃₃)
23                insert (ς̃₁, ς̃₂, ς̃₃) in TCallers
24                for each (ς̃₃, ς̃₄) in Summary Propagate(ς̃₁, ς̃₄)

        Propagate(ς̃₁, ς̃₂) ≜
25        if (ς̃₁, ς̃₂) not in Seen then insert (ς̃₁, ς̃₂) in Seen and W

        Update(ς̃₁, ς̃₂, ς̃₃, ς̃₄) ≜
26        ς̃₁ of the form  ([[(λₗ₁(u₁ k₁) call₁)]] , d̂₁, h₁)
27        ς̃₂ of the form  ([[(f e₂ (λ_{γ₂} (u₂) call₂))^{l₂}]], tf₂, h₂)
28        ς̃₃ of the form  ([[(λₗ₃(u₃ k₃) call₃)]] , d̂₃, h₂)
29        ς̃₄ of the form  ([[(k₄ e₄)^{γ₄}]], tf₄, h₄)
30        d̂ ⟵  Ãᵤ(e₄, γ₄, tf₄, h₄)
31        tf ⟵  { tf₂[f ↦ {[[(λₗ₃(u₃ k₃) call₃)]]}]   S?(l₂,f)
                 { tf₂                                  H?(l₂,f) ∨ Lam?(f)
32        ς̃ ⟵  ([[(λ_{γ₂}(u₂) call₂)]], d̂, tf, h₄)
33        Propagate(ς̃₁, ς̃)

        Final(ς̃) ≜
34        ς̃ of the form  ([[(k e)^γ]], tf, h)
35        insert  (halt, Ãᵤ(e, γ, tf, h), ∅, h) in Final
```

Figure 8: CFA2 workset algorithm

The call (f e k) in the body of app is a tail call, so no continuation is born there. Upon return from the first call to id, we must be careful to pass the result to λ_1. Also, we must restore the environment of the first call to app, *not* the environment of the tail call. Similarly, the second call to id must return to λ_2 and restore the correct environment. We achieve these by recording a "cross-procedure" summary from the entry of app to call site (k x), which is the exit of id. This transitive nature of summaries is essential for tail recursion.

5.3. **Summarization for CFA2.** The algorithm for CFA2 is shown in Fig. 8. It is a search-based summarization for higher-order programs with tail calls. Its goal is to compute which

local states are reachable from the initial state of a program through paths that respect call/return matching.

Overview of the algorithm's structure. The algorithm uses a workset W, which contains path edges and summaries to be examined. An edge $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ is an ordered pair of local states. We call $\tilde{\varsigma}_1$ the *source* and $\tilde{\varsigma}_2$ the *target* of the edge. At every iteration, we remove an edge from W and process it, potentially adding new edges in W. We stop when W is empty.

The algorithm maintains several sets. The results of the analysis are stored in the set *Seen*. It contains path edges (from a procedure entry to a state in the same procedure) and summary edges (from an entry to a $\widetilde{CEval}$ exit, not necessarily in the same procedure). The target of an edge in *Seen* is reachable from the source and from the initial state (*cf.* theorem 5.3). Summaries are also stored in *Summary*. *Final* records final states, *i.e.*, $\widetilde{CApply}$s that call *halt* with a return value for the whole program. *Callers* contains triples $\langle \tilde{\varsigma}_1, \tilde{\varsigma}_2, \tilde{\varsigma}_3 \rangle$, where $\tilde{\varsigma}_1$ is an entry, $\tilde{\varsigma}_2$ is a call in the same procedure and $\tilde{\varsigma}_3$ is the entry of the callee. *TCallers* contains triples $\langle \tilde{\varsigma}_1, \tilde{\varsigma}_2, \tilde{\varsigma}_3 \rangle$, where $\tilde{\varsigma}_1$ is an entry, $\tilde{\varsigma}_2$ is a tail call in the same procedure and $\tilde{\varsigma}_3$ is the entry of the callee. The initial state $\tilde{\mathcal{I}}(pr)$ is defined as $|\mathcal{I}(pr)|_{cl}$. The helper function $succ(\tilde{\varsigma})$ returns the successor(s) of $\tilde{\varsigma}$ according to the local semantics.

Edge processing. Each edge $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ is processed in one of four ways, depending on $\tilde{\varsigma}_2$. If $\tilde{\varsigma}_2$ is an entry, a return or an inner state (line 6), then its successor $\tilde{\varsigma}_3$ is a state in the same procedure. Since $\tilde{\varsigma}_2$ is reachable from $\tilde{\varsigma}_1$, $\tilde{\varsigma}_3$ is also reachable from $\tilde{\varsigma}_1$. If we have not already recorded the edge $(\tilde{\varsigma}_1, \tilde{\varsigma}_3)$, we do it now (line 25).

If $\tilde{\varsigma}_2$ is a call (line 8) then $\tilde{\varsigma}_3$ is the entry of the callee, so we propagate $(\tilde{\varsigma}_3, \tilde{\varsigma}_3)$ instead of $(\tilde{\varsigma}_1, \tilde{\varsigma}_3)$ (line 10). Also, we record the call in *Callers*. If an exit $\tilde{\varsigma}_4$ is reachable from $\tilde{\varsigma}_3$, it should return to the continuation born at $\tilde{\varsigma}_2$ (line 12). The function Update is responsible for computing the return state. We find the return value $\hat{d}$ by evaluating the expression e_4 passed to the continuation (lines 29-30). Since we are returning to λ_{γ_2}, we must restore the environment of its creation which is tf_2 (possibly with stack filtering, line 31). The new state $\tilde{\varsigma}$ is the corresponding return of $\tilde{\varsigma}_2$, so we propagate $(\tilde{\varsigma}_1, \tilde{\varsigma})$ (lines 32-33).

If $\tilde{\varsigma}_2$ is a $\widetilde{CEval}$ exit and $\tilde{\varsigma}_1$ is the initial state (lines 14-15), then $\tilde{\varsigma}_2$'s successor is a final state (lines 34-35). If $\tilde{\varsigma}_1$ is some other entry, we record the edge in *Summary* and pass the result of $\tilde{\varsigma}_2$ to the callers of $\tilde{\varsigma}_1$ (lines 17-18). Last, consider the case of a tail call $\tilde{\varsigma}_4$ to $\tilde{\varsigma}_1$ (line 19). No continuation is born at $\tilde{\varsigma}_4$. Thus, we must find where $\tilde{\varsigma}_3$ (the entry that led to the tail call) was called from. Then again, all calls to $\tilde{\varsigma}_3$ may be tail calls, in which case we keep searching further back in the call chain to find a return point. We do the backward search by transitively adding a cross-procedure summary from $\tilde{\varsigma}_3$ to $\tilde{\varsigma}_2$ (line 25).

If $\tilde{\varsigma}_2$ is a tail call (line 20), we find its successors and record the call in *TCallers* (lines 21-23). If a successor of $\tilde{\varsigma}_2$ goes to an exit, we propagate a cross-procedure summary transitively (line 24). Figure 9 shows a complete run of the algorithm for a small program.

5.4. **Correctness of the workset algorithm.** The local state space is finite, so there is a finite number of path and summary edges. We record edges as seen when we insert them in W, which ensures that no edge is inserted in W twice. Therefore, the algorithm terminates.

We obviously cannot visit an infinite number of abstract states. To establish the soundness of our analysis, we show that if a state $\hat{\varsigma}$ is reachable from $\hat{\mathcal{I}}(pr)$, then the algorithm

Name	Kind	Value
$\tilde{\mathcal{I}}(pr)$	Entry	$([\![(\lambda_2(\text{id h})(\text{id 1 }(\lambda_3(\text{u})(\text{id 2 h}))))]\!], \{[\![(\lambda_1(\text{x k})(\text{k x}))]\!]\}, \emptyset)$
$\tilde{\varsigma}_1$	Call	$([\![(\text{id 1 }(\lambda_3(\text{u})(\text{id 2 h})))]\!], [\text{id} \mapsto \{\lambda_1\}], \emptyset)$
$\tilde{\varsigma}_2$	Entry	$(\lambda_1, \{1\}, \emptyset)$
$\tilde{\varsigma}_3$	Exit $\widetilde{CEval}$	$([\![(\text{k x})]\!], [\text{x} \mapsto \{1\}], \emptyset)$
$\tilde{\varsigma}_4$	$\widetilde{CApply}$	$(\lambda_3, \{1\}, [\text{id} \mapsto \{\lambda_1\}], \emptyset)$
$\tilde{\varsigma}_5$	Exit tail call	$([\![(\text{id 2 h})]\!], [\text{id} \mapsto \{\lambda_1\}][\text{u} \mapsto \{1\}], \emptyset)$
$\tilde{\varsigma}_6$	Entry	$(\lambda_1, \{2\}, \emptyset)$
$\tilde{\varsigma}_7$	Exit $\widetilde{CEval}$	$([\![(\text{k x})]\!], [\text{x} \mapsto \{2\}], \emptyset)$
$\tilde{\varsigma}_8$	$\widetilde{CApply}$ (final state)	$(halt, \{2\}, \emptyset, \emptyset)$

W	Summary	Callers	TCallers	Final
$(\tilde{\mathcal{I}}(pr), \tilde{\mathcal{I}}(pr))$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\tilde{\varsigma}_2, \tilde{\varsigma}_2)$	$\emptyset$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$\emptyset$	$\emptyset$
$(\tilde{\varsigma}_2, \tilde{\varsigma}_3)$	$\emptyset$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$\emptyset$	$\emptyset$
$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_4)$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$\emptyset$	$\emptyset$
$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_5)$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$\emptyset$	$\emptyset$
$(\tilde{\varsigma}_6, \tilde{\varsigma}_6)$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_5, \tilde{\varsigma}_6)$	$\emptyset$
$(\tilde{\varsigma}_6, \tilde{\varsigma}_7)$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_5, \tilde{\varsigma}_6)$	$\emptyset$
$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_7)$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3),\ (\tilde{\varsigma}_6, \tilde{\varsigma}_7)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_5, \tilde{\varsigma}_6)$	$\emptyset$
$\emptyset$	$(\tilde{\varsigma}_2, \tilde{\varsigma}_3),\ (\tilde{\varsigma}_6, \tilde{\varsigma}_7)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_1, \tilde{\varsigma}_2)$	$(\tilde{\mathcal{I}}(pr), \tilde{\varsigma}_5, \tilde{\varsigma}_6)$	$\tilde{\varsigma}_8$

Figure 9: A complete run of CFA2. Note that λ_1 is applied twice and returns to the correct context both times. The program evaluates to 2. For brevity, we first show all reachable states and then refer to them by their names. $\tilde{\mathcal{I}}(pr)$ shows the whole program; in the other states we abbreviate lambdas by their labels. All heaps are $\emptyset$ because there are no heap variables. The rows of the table show the contents of the sets at line 3 for each iteration. *Seen* contains all pairs entered in W.

visits $|\hat{\varsigma}|_{al}$ (*cf.* theorem 5.3). For instance, CFA2 on `len` tells us that we reach program point 5, *not* that we reach 5 with a stack of size 1, 2, 3, *etc.*

Soundness guarantees that CFA2 does not miss any flows, but it may also add flows that do not happen in the abstract semantics. For example, a sound but useless algorithm would add all pairs of local states in *Seen*. We establish the completeness of CFA2 by proving that every visited edge corresponds to an abstract flow (*cf.* theorem 5.4), which means that there is no loss in precision when going from abstract to local states.

The theorems use two definitions. The first associates a state $\hat{\varsigma}$ with its *corresponding entry*, *i.e.*, the entry of the procedure that contains $\hat{\varsigma}$. The second finds all entries that reach $CE_p(\hat{\varsigma})$ through tail calls. We include the proofs of the theorems in the appendix.

Definition 5.1. The Corresponding Entry $CE_p(\hat\varsigma)$ of a state $\hat\varsigma$ in a path p is:

- $\hat\varsigma$, if $\hat\varsigma$ is an Entry
- $\hat\varsigma_1$, if $\hat\varsigma$ is not an Entry, $\hat\varsigma_2$ is not an Exit-CEval, $p \equiv p_1 \rightsquigarrow \hat\varsigma_1 \rightsquigarrow^* \hat\varsigma_2 \rightsquigarrow \hat\varsigma \rightsquigarrow p_2$, and $CE_p(\hat\varsigma_2) = \hat\varsigma_1$
- $\hat\varsigma_1$, if $\hat\varsigma$ is not an Entry, $p \equiv p_1 \rightsquigarrow \hat\varsigma_1 \rightsquigarrow^+ \hat\varsigma_2 \rightsquigarrow \hat\varsigma_3 \rightsquigarrow^+ \hat\varsigma_4 \rightsquigarrow \hat\varsigma \rightsquigarrow p_2$, $\hat\varsigma_2$ is a Call and $\hat\varsigma_4$ is an Exit-CEval, $CE_p(\hat\varsigma_2) = \hat\varsigma_1$, and $\hat\varsigma_3 \in CE_p^*(\hat\varsigma_4)$

Definition 5.2. For a state $\hat\varsigma$ and a path p, $CE_p^*(\hat\varsigma)$ is the smallest set such that:

- $CE_p(\hat\varsigma) \in CE_p^*(\hat\varsigma)$
- $CE_p^*(\hat\varsigma_1) \subseteq CE_p^*(\hat\varsigma)$, when $p \equiv p_1 \rightsquigarrow \hat\varsigma_1 \rightsquigarrow \hat\varsigma_2 \rightsquigarrow^* \hat\varsigma \rightsquigarrow p_2$, $\hat\varsigma_1$ is a Tail Call, $\hat\varsigma_2$ is an Entry, and $\hat\varsigma_2 = CE_p(\hat\varsigma)$

Theorem 5.3 (Soundness)**.** *If $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat\varsigma$ then, after summarization:*

- *if $\hat\varsigma$ is not a final state then $(|CE_p(\hat\varsigma)|_{al}, |\hat\varsigma|_{al}) \in Seen$*
- *if $\hat\varsigma$ is a final state then $|\hat\varsigma|_{al} \in Final$*
- *if $\hat\varsigma$ is an Exit-CEval and $\hat\varsigma' \in CE_p^*(\hat\varsigma)$ then $(|\hat\varsigma'|_{al}, |\hat\varsigma|_{al}) \in Seen$*

Theorem 5.4 (Completeness)**.** *After summarization:*

- *For each $(\tilde\varsigma_1, \tilde\varsigma_2)$ in Seen, there exist $\hat\varsigma_1$, $\hat\varsigma_2$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat\varsigma_1 \rightsquigarrow^* \hat\varsigma_2$ and $\tilde\varsigma_1 = |\hat\varsigma_1|_{al}$ and $\tilde\varsigma_2 = |\hat\varsigma_2|_{al}$ and $\hat\varsigma_1 \in CE_p^*(\hat\varsigma_2)$*
- *For each $\tilde\varsigma$ in Final, there exist $\hat\varsigma$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^+ \hat\varsigma$ and $\tilde\varsigma = |\hat\varsigma|_{al}$ and $\hat\varsigma$ is a final state.*

5.5. **Complexity.** A simple calculation shows that CFA2 is in EXPTIME. The size of the domain of *Heap* is n and the size of the range is 2^n, so there are 2^{n^2} heaps. Similarly, there are 2^{n^2} frames. The size of $\widetilde{State}$ is dominated by the size of $\widetilde{CApply}$ which is $n \cdot 2^n \cdot 2^{n^2} \cdot 2^{n^2} = O(n \cdot 2^{2n^2+n})$. The size of *Seen* is the product of the sizes of $\widetilde{UApply}$ and $\widetilde{State}$, which is $(n \cdot 2^n \cdot 2^{n^2}) \cdot (n \cdot 2^{2n^2+n}) = O(n^2 \cdot 2^{3n^2+2n})$.

The running time of the algorithm is bounded by the number of edges in W times the cost of each iteration. W contains edges from *Seen* only, so its size is $O(n^2 \cdot 2^{3n^2+2n})$. The most expensive iteration happens when line 19 is executed. There are $O(n^3 \cdot 2^{4n^2+2n})$ *TCallers* and for each one we call `Propagate`, which involves searching *Seen*. Therefore, the loop costs $O(n^3 \cdot 2^{4n^2+2n}) \cdot O(n^2 \cdot 2^{3n^2+2n}) = O(n^5 \cdot 2^{7n^2+4n})$. Thus, the total cost of the algorithm is $O(n^2 \cdot 2^{3n^2+2n}) \cdot O(n^5 \cdot 2^{7n^2+4n}) = O(n^7 \cdot 2^{10n^2+6n})$.

Showing that CFA2 is in EXPTIME does not guarantee the existence of a program that, when analyzed, triggers the exponential behavior. Is there a such a program? The answer is yes. Consider the following program, suggested to us by Danny Dubé:

```
(let* ((merger   (λ₁(f) (λ₂(_) f)))
       (_        (merger (λ₃(x) x)))
       (clos     (merger (λ₄(y) y)))
       (f1       (clos _)₁)
       (_        (f1 _)₁′)
       (f2       (clos _)₂)
       (_        (f2 _)₂′)

                  ⋮

       (fn       (clos _)ₙ)
       (_        (fn _)ₙ′))
    _ )
```

The idea is to create an exponential number of frames by exploiting the strong updates CFA2 does on the top frame. The code is in direct style for brevity; the let-bound variables would be bound by continuation lambdas in the equivalent CPS program. The only heap reference appears in the body of λ_2. We use underscores for unimportant expressions.

The `merger` takes a function, binds `f` to it and returns a closure that ignores its argument and returns `f`. We call the `merger` twice so that `f` is bound to $\{\lambda_3, \lambda_4\}$ in the heap. Now `clos` is bound to λ_2 in the top frame and every call to `clos` returns $\{\lambda_3, \lambda_4\}$. Thus, after call site 1 the variable `f1` is bound to $\{\lambda_3, \lambda_4\}$. At $1'$, execution splits in two branches. One calls λ_3 and filters the binding of `f1` in the top frame to $\{\lambda_3\}$. The other calls λ_4 and filters the binding to $\{\lambda_4\}$. Each branch will split in two more branches at call $2'$, *etc.* By binding each `fi` to a set of two elements and applying it immediately, we force a strong update and create exponentially many frames.

Even though strong update can be subverted, it can also speed up the analysis of some programs by avoiding spurious flows. In `compose-same` (*cf.* sec. 3.2), if two lambdas λ_1 and λ_2 flow to `f`, 0CFA will apply each lambda at each call site, resulting in four flows. CFA2 will only examine two flows, one that uses λ_1 in both call sites and one that uses λ_2.

We tried to keep the algorithm of Fig. 8 simple because it is meant to be a model. There are many parameters one can tune to improve the performance and/or asymptotic complexity of CFA2:

- *no stack filtering*: CFA2 is sound without stack filtering, but less precise. Permitting fake rebinding may not be too harmful in practice. Suppose that a set $\{\lambda_1, \lambda_2\}$ flows to a variable v with two stack references v_l and $v_{l'}$. Even with stack filtering, both lambdas will flow to both references. Stack filtering just prevents us from using λ_1 at v_l and λ_2 at $v_{l'}$ along the same execution path.
- *heap widening*: implementations of flow analyses rarely use one heap per state. They use a global heap instead and states carry timestamps [26, ch. 5]. *Heap* is a lattice of height $O(n^2)$. Since the global heap grows monotonically, it can change at most $O(n^2)$ times during the analysis.
- *summary reuse*: we can avoid some reanalyzing of procedures by creating general summaries that many callers can use. One option is to create more approximate summaries by widening. Another option is to include only relevant parts of the state in the summary [4].
- *representation of the sets*: in calculating the exponential upper bound, we pessimistically assumed that looking up an element in a set takes time linear in the size of the set. This need not be true if one uses efficient data structures to represent *Seen* and the other sets.

An in-depth study of the performance and complexity of the proposed variants would increase our understanding of their relative merits. Also, we do not know if CFA2 has an exponential lower bound. Our evaluation, presented in the next section, shows that CFA2 compares favorably to 0CFA, a cubic algorithm.

6. Evaluation

We implemented CFA2, 0CFA and 1CFA for the Twobit Scheme compiler [6] and used them to do constant propagation and folding. In this section we report on some initial measurements and comparisons.

0CFA and 1CFA use a standard workset algorithm. CFA2 uses the algorithm of section 5.3. To speed up the analyses, the variable environment and the heap are global.

	$S_?$	$H_?$	0CFA		1CFA		CFA2	
			visited	constants	visited	constants	visited	constants
`len`	9	0	81	0	126	0	55	2
`rev-iter`	17	0	121	0	198	0	82	4
`len-Y`	15	4	199	0	356	0	131	2
`tree-count`	33	0	293	2	2856	6	183	10
`ins-sort`	33	5	509	0	1597	0	600	4
`DFS`	94	11	1337	8	6890	8	1719	16
`flatten`	37	0	1520	0	6865	0	478	5
`sets`	90	3	3915	0	54414	0	4251	4
`church-nums`	46	23	19130	0	19411	0	22671	0

Figure 10: Benchmark results

We compared the effectiveness of the analyses on a small set of benchmarks (Fig. 10). We measured the number of stack and heap references in each program and the number of constants found by each analysis. We also recorded what goes in the workset in each analysis, *i.e.*, the number of abstract states visited by 0CFA and 1CFA, and the number of path and summary edges visited by CFA2. The running time of an abstract interpretation is proportional to the amount of things inserted in the workset.

We chose programs that exhibit a variety of control-flow patterns. `Len` computes the length of a list recursively. `Rev-iter` reverses a list tail-recursively. `Len-Y` computes the length of a list using the Y-combinator instead of explicit recursion. `Tree-count` counts the nodes in a binary tree. `Ins-sort` sorts a list of numbers using insertion-sort. `DFS` does depth-first search of a graph. `Flatten` turns arbitrarily nested lists into a flat list. `Sets` defines the basic set operations and tests De Morgan's laws on sets of numbers. `Church-nums` tests distributivity of multiplication over addition for a few Church numerals.

CFA2 finds the most constants, followed by 1CFA. 0CFA is the least precise. CFA2 is also more efficient at exploring its abstract state space. In five out of nine cases, it visits fewer paths than 0CFA does states. The visited set of CFA2 can be up to 3.2 times smaller (`flatten`), and up to 1.3 times larger (`DFS`) than the visited set of 0CFA. 1CFA is less efficient than both 0CFA (9/9 cases) and CFA2 (8/9 cases). The visited set of 1CFA can be significantly larger than that of CFA2 in some cases (15.6 times in `tree-count`, 14.4 times in `flatten`, 12.8 times in `sets`).

Naturally, the number of stack references in a program is much higher than the number of heap references; most of the time, a variable is referenced only by the lambda that binds it. Thus, CFA2 uses the precise stack lookups more often than the imprecise heap lookups.

7. RELATED WORK

We were particularly influenced by Chaudhuri's paper on subcubic algorithms for recursive state machines [5]. His clear and intuitive description of summarization helped us realize that we can use this technique to explore the state space of CFA2.

Our workset algorithm is based on Sharir and Pnueli's functional approach [25, pg. 207] and the tabulation algorithm of Reps *et al.* [23], extended for tail recursion and higher-order functions. In section 5.2, we mentioned that CFA2 essentially produces a pushdown system. Then, the reader may wonder why we designed a new algorithm instead of using an

existing one like $post^*$ [3, 10]. The reason is that callers cannot be identified syntactically in higher-order languages. Hence, algorithms that analyze higher-order programs must be based on search. The tabulation algorithm can be changed to use search fairly naturally. It is less clear to us how to do that for $post^*$. In a way, CFA2 creates a pushdown system and analyzes it *at the same time*, much like what k-CFA does with control-flow graphs.

Melski and Reps [17] reduced Heintze's set-constraints [13] to an instance of context-free-language (*abbrev.* CFL) reachability, which they solve using summarization. Therefore, their solution has the same precision as 0CFA.

CFL reachability has also been used for points-to analysis of imperative higher-order languages. For instance, Sridharan and Bodík's points-to analysis for Java [28] uses CFL reachability to match writes and reads to object fields. Precise call/return matching is achieved only for programs without recursive methods. Hind's survey [14] discusses many other variants of points-to analysis.

Debray and Proebsting [7] used ideas from parsing theory to design an interprocedural analysis for first-order programs with tail calls. They describe control-flow with a context-free grammar. Then, the FOLLOW set of a procedure represents its possible return points. Our approach is quite different on the surface, but similar in spirit; we handle tail calls by computing summaries transitively.

Mossin [21] created a type-based flow analysis for functional languages, which uses polymorphic subtyping for polyvariance. The input to the analysis is a program p in the simply-typed λ-calculus with recursion. First, the analysis annotates the types in p with labels. Then, it computes flow information by assigning labeled types to each expression in p. Thus, flow analysis is reduced to a type-inference problem. The annotated type system uses let-polymorphism. As a result, it can distinguish flows to different references of let- and letrec-bound variables. In the following program, it finds that n2 is a constant.

```
(let* ((id (λ(x) x))
       (n1 (id 1))
       (n2 (id 2)))
   (+ n1 n2))
```

However, the type system merges flows to different references of λ-bound variables. For instance, it cannot find that n2 is a constant in the app example of section 3.1. Mossin's algorithm runs in time $O(n^8)$.

Rehof and Fähndrich [9, 22] used CFL reachability in an analysis that runs in cubic time and has the same precision as Mossin's. They also extended the analysis to handle polymorphism in the target language. Around the same time, Gustavsson and Svenningsson [12] formulated a cubic version of Mossin's analysis without using CFL reachability. Their work does not deal with polymorphism in the target language.

Midtgaard and Jensen [18] created a flow analysis for direct-style higher-order programs that keeps track of "return flow". They point out that continuations make return-point information explicit in CPS and show how to recover this information in direct-style programs. Their work does not address the issue of unbounded call/return matching.

Earl *et al.* followed up on CFA2 with a pushdown analysis that does not use frames [8]. Rather, it allocates all bindings in the heap with context, in the style of k-CFA [26]. For $k = 0$, their analysis runs in time $O(n^6)$, where n is the size of the program. Like all pushdown-reachability algorithms, Earl *et al.*'s analysis records pairs of states $(\varsigma_1, \varsigma_2)$ where ς_2 is same-context reachable from ς_1. However, their algorithm does not classify states as entries, exits, calls, *etc.* This has two drawbacks compared to the tabulation algorithm.

First, they do not distinguish between path and summary edges. Thus, they have to search the whole set of edges when they look for return points, even though only summaries can contribute to the search. More importantly, path edges are only a small subset of the set S of all edges between same-context reachable states. By not classifying states, their algorithm maintains the whole set S, not just the path edges. In other words, it records edges whose source is not an entry. In the graph of `len`, some of these edges are $\langle 6, 8 \rangle$, $\langle 6, 13 \rangle$, $\langle 7, 11 \rangle$. Such edges slow down the analysis and do not contribute to call/return matching, because they cannot evolve into summary edges.

In CFA2, it is possible to disable the use of frames by classifying each reference as a heap reference. The resulting analysis has similar precision to Earl *et al.*'s analysis for $k = 0$. We conjecture that this variant is not a viable alternative in practice, because of the significant loss in precision.

Might and Shivers [20] proposed ΓCFA (abstract garbage collection) and μCFA (abstract counting) to increase the precision of k-CFA. ΓCFA removes unreachable bindings from the variable environment, and μCFA counts how many times a variable is bound during the analysis. The two techniques combined reduce the number of spurious flows and give precise environment information. Stack references in CFA2 have a similar effect, because different calls to the same function use different frames. However, we can utilize ΓCFA and μCFA to improve precision in the heap.

Recently, Kobayashi [15] proposed a way to statically verify properties of typed higher-order programs using model-checking. He models a program by a higher-order recursion scheme $\mathcal{G}$, expresses the property of interest in the modal μ-calculus and checks if the infinite tree generated by $\mathcal{G}$ satisfies the property. This technique can do flow analysis, since flow analysis can be encoded as a model-checking problem. The target language of this work is the simply-typed lambda calculus. Programs in a Turing-complete language must be approximated in the simply-typed lambda calculus in order to be model-checked.

8. Conclusions

In this paper we propose CFA2, a pushdown model of higher-order programs, and prove it correct. CFA2 provides precise call/return matching and has a better approach to variable binding than k-CFA. Our evaluation shows that CFA2 gives more accurate data-flow information than 0CFA and 1CFA.

Stack lookups make CFA2 polyvariant because different calls to the same function are analyzed in different environments. We did not add polyvariance in the heap to keep the presentation simple. Heap polyvariance is *orthogonal* to call/return matching; integrating existing techniques [1, 26, 31] in CFA2 should raise no difficulties. For example, CFA2 can be extended with call-strings polyvariance, like k-CFA, to produce a family of analyses CFA2.0, CFA2.1 and so on. Then, any instance of CFA2.k would be strictly more precise than the corresponding instance of k-CFA.

We believe that pushdown models are a better tool for higher-order flow analysis than control-flow graphs, and are working on providing more empirical support to this thesis. We plan to use CFA2 for environment analysis and stack-related optimizations. We also plan to add support for `call/cc` in CFA2.

References

[1] Ole Agesen. The Cartesian Product Algorithm: Simple and Precise Type Inference of Parametric Polymorphism. In *European Conference on Object-Oriented Programming*, pages 2–26, 1995.

[2] Rajeev Alur, Michael Benedikt, Kousha Etessami, Patrice Godefroid, Thomas W. Reps, and Mihalis Yannakakis. Analysis of Recursive State Machines. *Transactions on Programming Languages and Systems*, 27(4):786–818, 2005.

[3] Ahmed Bouajjani, Javier Esparza, and Oded Maler. Reachability Analysis of Pushdown Automata: Application to Model-Checking. In *International Conference on Concurrency Theory*, pages 135–150, 1997.

[4] Satish Chandra, Stephen J. Fink, and Manu Sridharan. Snugglebug: a powerful approach to weakest preconditions. In *Programming Language Design and Implementation*, pages 363–374, 2009.

[5] Swarat Chaudhuri. Subcubic Algorithms for Recursive State Machines. In *Principles of Programming Languages*, pages 159–169, 2008.

[6] William D. Clinger and Lars Thomas Hansen. Lambda, the Ultimate Label or a Simple Optimizing Compiler for Scheme. In *LISP and Functional Programming*, pages 128–139, 1994.

[7] Saumya K. Debray and Todd A. Proebsting. Interprocedural Control Flow Analysis of First-Order Programs with Tail-Call Optimization. *Transactions on Programming Languages and Systems*, 19(4):568–585, 1997.

[8] Christopher Earl, Matthew Might, and David Van Horn. Pushdown Control-Flow Analysis of Higher-Order Programs. In *Workshop on Scheme and Functional Programming*, 2010.

[9] Manuel Fähndrich and Jakob Rehof. Type-based flow analysis and context-free language reachability. *Mathematical Structures in Computer Science*, 18(5):823–894, 2008.

[10] Alain Finkel, Bernard Willems, and Pierre Wolper. A direct symbolic approach to model checking pushdown systems. *Electronic Notes in Theoretical Computer Science*, 9, 1997.

[11] Cormac Flanagan, Amr Sabry, Bruce F. Duba, and Matthias Felleisen. The Essence of Compiling with Continuations. In *Programming Language Design and Implementation*, pages 237–247, 1993.

[12] Jörgen Gustavsson and Josef Svenningsson. Constraint Abstractions. In *Programs as Data Objects*, pages 63–83, 2001.

[13] Nevin Heintze. *Set-based program analysis*. PhD thesis, Carnegie-Mellon Univ., 1992.

[14] Michael Hind. Pointer analysis: haven't we solved this problem yet? In *Program Analysis For Software Tools and Engineering*, pages 54–61, 2001.

[15] Naoki Kobayashi. Types and higher-order recursion schemes for verification of higher-order programs. In *Principles of Programming Languages*, pages 416–428, 2009.

[16] David Kranz. *ORBIT: An Optimizing Compiler for Scheme*. PhD thesis, Yale University, 1988.

[17] David Melski and Thomas Reps. Interconvertibility of a Class of Set Constraints and Context-Free-Language Reachability. *Theoretical Comp. Sci.*, 248(1-2):29–98, 2000.

[18] Jan Midtgaard and Thomas Jensen. Control-flow analysis of function calls and returns by abstract interpretation. In *International Conference on Functional Programming*, pages 287–298, 2009.

[19] Matthew Might. *Environment Analysis of Higher-Order Languages*. PhD thesis, Georgia Institute of Technology, 2007.

[20] Matthew Might and Olin Shivers. Improving Flow Analyses via ΓCFA: Abstract Garbage Collection and Counting. In *International Conference on Functional Programming*, pages 13–25, 2006.

[21] Christian Mossin. *Flow Analysis of Typed Higher-Order Programs*. PhD thesis, DIKU, Department of Computer Science, University of Copenhagen, 1996.

[22] Jakob Rehof and Manuel Fähndrich. Type-Based Flow Analysis: From Polymorphic Subtyping to CFL-Reachability. In *Principles of Programming Languages*, pages 54–66, 2001.

[23] Thomas W. Reps, Susan Horwitz, and Shmuel Sagiv. Precise Interprocedural Dataflow Analysis via Graph Reachability. In *Principles of Programming Languages*, pages 49–61, 1995.

[24] Amr Sabry and Matthias Felleisen. Reasoning About Programs in Continuation-Passing Style. In *LISP and Functional Programming*, pages 288–298, 1992.

[25] Micha Sharir and Amir Pnueli. Two Approaches to Interprocedural Data Flow Analysis. In *Program Flow Analysis, Theory and Application*. Prentice Hall, 1981.

[26] Olin Shivers. *Control-Flow Analysis of Higher-Order Languages*. PhD thesis, Carnegie-Mellon University, 1991.

[27] Olin Shivers. Higher-Order Control-Flow Analysis in Retrospect: Lessons Learned, Lessons Abandoned. In *Best of PLDI*, pages 257–269, 2004.

[28] Manu Sridharan and Rastislav Bodík. Refinement-based context-sensitive points-to analysis for Java. In *Programming Language Design and Implementation*, pages 387–400, 2006.

[29] Guy L. Steele. Rabbit: A Compiler for Scheme. Master's thesis, MIT, 1978.

[30] David Van Horn and Harry G. Mairson. Deciding k-CFA is complete for EXPTIME. In *International Conference on Functional Programming*, pages 275–282, 2008.

[31] Andrew Wright and Suresh Jagannathan. Polymorphic Splitting: An Effective Polyvariant Flow Analysis. *Transactions on Programming Languages and Systems*, 20(1):166–207, 1998.

Appendix A.

We use the notation $\pi_i(\langle e_1, \ldots, e_n \rangle)$ to retrieve the ith element of a tuple $\langle e_1, \ldots, e_n \rangle$. Also, we write $\mathcal{L}(g)$ to get the label of a term g.

In section 1, we mentioned that labels in a program can be split into disjoint sets according to the innermost user lambda that contains them. The "label to label" map $LL(\psi)$ returns the labels that are in the same set as ψ. For example, in the program $(\lambda_1(\text{x k1}) \ (\text{k1} \ (\lambda_2(\text{y k2}) \ (\text{x y} \ (\lambda_3(\text{u}) \ (\text{x u k2})^4))^5))^6)$, these sets are $\{1, 6\}$ and $\{2, 3, 4, 5\}$, so we know $LL(4) = \{2, 3, 4, 5\}$ and $LL(6) = \{1, 6\}$.

Definition A.1. For every term g, the map $BV(g)$ returns the variables bound by lambdas which are subterms of g. The map has a simple inductive definition:
$BV(\llbracket (\lambda_\psi (v_1 \ldots v_n) \, call) \rrbracket) = \{v_1, \ldots, v_n\} \cup BV(call)$
$BV(\llbracket (g_1 \ldots g_n)^\psi \rrbracket) = BV(g_1) \cup \cdots \cup BV(g_n)$
$BV(v) = \emptyset$ ☐

We assume that CFA2 works on an alphatized program, *i.e.*, a program where all variables have distinct names. Thus, if $(\lambda_\psi (v_1 \ldots v_n) \, call)$ is a term in such a program, we know that no other lambda in that program binds variables with names $v_1, \ldots, v_n$. (During execution of CFA2, we do not rename any variables.) The following lemma is a simple consequence of alphatization.

Lemma A.2. *A concrete state ς has the form $(\ldots, ve, t)$.*
(1) *For any closure $(lam, \beta) \in \text{range}(ve)$, it holds that $\text{dom}(\beta) \cap BV(lam) = \emptyset$.*
(2) *If ς is an Eval with call site call and environment β, then $\text{dom}(\beta) \cap BV(call) = \emptyset$.*
(3) *If ς is an Apply, for any closure (lam, β) in operator or argument position, then $\text{dom}(\beta) \cap BV(lam) = \emptyset$.*

Proof. We show that the lemma holds for the initial state $\mathcal{I}(pr)$. Then, for each transition $\varsigma \to \varsigma'$, we assume that ς satisfies the lemma and show that ς' also satisfies it.
- $\mathcal{I}(pr)$ is a *UApply* of the form $((pr, \emptyset), (lam, \emptyset), halt, \emptyset, \langle \rangle)$. Since ve is empty, (1) trivially holds. Also, both closures have an empty environment so (3) holds.
- The [UEA] transition is:
  $(\llbracket (f \, e \, q)^l \rrbracket, \beta, ve, t) \to (proc, d, c, ve, l :: t)$
  $proc = \mathcal{A}(f, \beta, ve)$
  $d = \mathcal{A}(e, \beta, ve)$
  $c = \mathcal{A}(q, \beta, ve)$

  The ve doesn't change in the transition, so (1) holds for ς'.
  The operator is a closure of the form (lam, β'). We must show that $\text{dom}(\beta') \cap BV(lam) = \emptyset$. If $Lam_?(f)$, then $lam = f$ and $\beta' = \beta$. Also, we know
  $\text{dom}(\beta) \cap BV(\llbracket (f \, e \, q)^l \rrbracket) = \emptyset$
  $\Rightarrow \text{dom}(\beta) \cap (BV(f) \cup BV(e) \cup BV(q)) = \emptyset$
  $\Rightarrow \text{dom}(\beta) \cap BV(f) = \emptyset$.
  If $Var_?(f)$, then $(lam, \beta') \in \text{range}(ve)$, so we get the desired result because ve satisfies (1).
  Similarly for d and c.
- The [UAE] transition is:
  $(proc, d, c, ve, t) \to (call, \beta', ve', t)$

$proc \equiv \langle [\![(\lambda_l (u\, k)\ call)]\!], \beta \rangle$
$\beta' = \beta[u \mapsto t][k \mapsto t]$
$ve' = ve[(u,t) \mapsto d][(k,t) \mapsto c]$

To show (1) for ve', it suffices to show that d and c don't violate the property. The user argument d is of the form (lam_1, β_1). Since ς satisfies (3), we know $\mathrm{dom}(\beta_1) \cap BV(lam_1) = \emptyset$, which is the desired result. Similarly for c.

Also, we must show that ς' satisfies (2). We know $\{u, k\} \cap BV(call) = \emptyset$ because the program is alphatized. Also, from property (3) for ς we know $\mathrm{dom}(\beta) \cap BV([\![(\lambda_l (u\, k)\ call)]\!]) = \emptyset$, which implies $\mathrm{dom}(\beta) \cap BV(call) = \emptyset$. We must show

$\mathrm{dom}(\beta') \cap BV(call) = \emptyset$
$\Leftrightarrow\ (\mathrm{dom}(\beta) \cup \{u, k\}) \cap BV(call) = \emptyset$
$\Leftrightarrow\ (\mathrm{dom}(\beta) \cap BV(call)) \cup (\{u, k\} \cap BV(call)) = \emptyset$
$\Leftrightarrow\ \emptyset \cup \emptyset = \emptyset.$

- Similarly for the other two transitions.   □

Theorem A.3 (Simulation). *If $\varsigma \to \varsigma'$ and $|\varsigma|_{ca} \sqsubseteq \hat{\varsigma}$, then there exists $\hat{\varsigma}'$ such that $\hat{\varsigma} \rightsquigarrow \hat{\varsigma}'$ and $|\varsigma'|_{ca} \sqsubseteq \hat{\varsigma}'$.*

Proof. By cases on the concrete transition.

a) Rule [UEA]
$([\![(f\, e\, q)^l]\!], \beta, ve, t) \to (proc, d, c, ve, l :: t)$
$proc = \mathcal{A}(f, \beta, ve)$
$d = \mathcal{A}(e, \beta, ve)$
$c = \mathcal{A}(q, \beta, ve)$

Let $ts = toStack(LV(l), \beta, ve)$. Since $|\varsigma|_{ca} \sqsubseteq \hat{\varsigma}$, $\hat{\varsigma}$ is of the form $([\![(f\, e\, q)^l]\!], st, h)$, where $|ve|_{ca} \sqsubseteq h$ and $ts \sqsubseteq st$.

The abstract transition is
$([\![(f\, e\, q)^l]\!], st, h) \rightsquigarrow (f', \hat{d}, \hat{c}, st', h)$
$f' \in \hat{\mathcal{A}}_u(f, l, st, h)$
$\hat{d} = \hat{\mathcal{A}}_u(e, l, st, h)$
$\hat{c} = \hat{\mathcal{A}}_k(q, st)$
$st' = \begin{cases} pop(st) & Var_?(q) \\ st & Lam_?(q) \wedge (H_?(l, f) \vee Lam_?(f)) \\ st[f \mapsto \{f'\}] & Lam_?(q) \wedge S_?(l, f) \end{cases}$

State $\hat{\varsigma}$ has many possible successors, one for each lambda in $\hat{\mathcal{A}}_u(f, l, st, h)$. We must show that one of them is a state $\hat{\varsigma}'$ such that $|\varsigma'|_{ca} \sqsubseteq \hat{\varsigma}'$.

The variable environment and the heap don't change in the transitions, so for ς' and $\hat{\varsigma}'$ we know that $|ve|_{ca} \sqsubseteq h$. We must show $\pi_1(proc) = f'$, $|d|_{ca} \sqsubseteq \hat{d}$, $|c|_{ca} \sqsubseteq \hat{c}$ and $ts' \sqsubseteq st'$, where ts' is the stack of $|\varsigma'|_{ca}$.

We first show $\pi_1(proc) = f'$, by cases on f:
- $Lam_?(f)$
  Then, $proc = (f, \beta)$ and $f' \in \{f\}$, so $f' = f$.

- $S_?(l, f)$

  Then, $proc = ve(f, \beta(f))$, a closure of the form (lam, β'). Since $ts(f) = |ve(f, \beta(f))|_{ca} = \{lam\}$ and $ts \sqsubseteq st$, we get $lam \in st(f)$. So, we pick f' to be lam.

- $H_?(l, f)$

  Then, $proc = ve(f, \beta(f))$, a closure of the form (lam, β'). Since $|ve|_{ca} \sqsubseteq h$ and $lam \in |ve|_{ca}(f)$, we get $lam \in h(f)$. So, we pick f' to be lam.

Showing $|d|_{ca} \sqsubseteq \hat{\mathrm{d}}$ is similar.

We now show $|c|_{ca} \sqsubseteq \hat{c}$, by cases on q:

- $Lam_?(q)$

  Then, $c = (q, \beta)$ and $\hat{c} = q$, so $|c|_{ca} \sqsubseteq \hat{c}$.

- $Var_?(q)$ and $c = ve(q, \beta(q)) = halt$

  Then, $ts(q) = halt$. Since $ts \sqsubseteq st$, we get $st(q) = halt$. Thus, $\hat{c} = halt$.

- $Var_?(q)$ and $c = ve(q, \beta(q)) = (lam, \beta')$

  Similar to the previous case.

It remains to show that $ts' \sqsubseteq st'$. We proceed by cases on q and f:

- $Var_?(q)$ and $c = ve(q, \beta(q)) = halt$

  Then, $ts' = \langle\rangle$. By $ts \sqsubseteq st$, we know that ts and st have the same size. Also, $st' = pop(st)$, thus $st' = \langle\rangle$. Therefore, $ts' \sqsubseteq st'$.

- $Var_?(q)$ and $c = ve(q, \beta(q)) = (lam, \beta')$

  By Fig. 5, we know that $ts' = toStack(LV(\mathcal{L}(lam)), \beta', ve) = pop(ts)$. Also, $st' = pop(st)$. Thus, to show $ts' \sqsubseteq st'$ it suffices to show $pop(ts) \sqsubseteq pop(st)$, which holds because $ts \sqsubseteq st$.

- $Lam_?(q) \wedge (Lam_?(f) \vee H_?(l, f))$

  Then, $ts' = ts$ and $st' = st$, so $ts' \sqsubseteq st'$.

- $Lam_?(q) \wedge S_?(l, f)$

  By $LV(\mathcal{L}(q)) = LV(l)$, we get that $ts' = ts$. Also, $proc = ve(f, \beta(f))$, a closure of the form (lam, β'). We pick f' to be lam. The stack of $\hat{\varsigma}'$ is $st' = st[f \mapsto \{lam\}]$. Since $pop(ts) \sqsubseteq pop(st)$, we only need to show that the top frames of ts' and st' are in $\sqsubseteq$. For this, it suffices to show that $ts'(f) \sqsubseteq st'(f)$ which holds because $ts'(f) = ts(f) = \{lam\}$.

b) Rule [UAE]

   $(proc, d, c, ve, t) \rightarrow (call, \beta', ve', t)$

   $proc \equiv \langle [\![(\lambda_l (u\, k)\, call)]\!], \beta \rangle$

   $\beta' = \beta[u \mapsto t][k \mapsto t]$

   $ve' = ve[(u, t) \mapsto d][(k, t) \mapsto c]$

   Let $ts = \begin{cases} \langle\rangle & c = halt \\ toStack(LV(\mathcal{L}(lam)), \beta_1, ve) & c = (lam, \beta_1) \end{cases}$

   Since $|\varsigma|_{ca} \sqsubseteq \hat{\varsigma}$, $\hat{\varsigma}$ is of the form $([\![(\lambda_l (u\, k)\, call)]\!], \hat{\mathrm{d}}, \hat{c}, st, h)$, where $|d|_{ca} \sqsubseteq \hat{\mathrm{d}}$, $|c|_{ca} = \hat{c}$, $ts \sqsubseteq st$ and $|ve|_{ca} \sqsubseteq h$.

   The abstract transition is

   $([\![(\lambda_l (u\, k)\, call)]\!], \hat{\mathrm{d}}, \hat{c}, st, h) \rightsquigarrow (call, st', h')$

   $st' = push([u \mapsto \hat{\mathrm{d}}][k \mapsto \hat{c}], st)$

   $h' = \begin{cases} h \sqcup [u \mapsto \hat{\mathrm{d}}] & H_?(u) \\ h & S_?(u) \end{cases}$

Let ts' be the stack of $|\varsigma'|_{ca}$. The innermost user lambda that contains $call$ is λ_l, therefore $ts' = toStack(LV(l), \beta', ve')$. We must show that $|\varsigma'|_{ca} \sqsubseteq \hat{\varsigma}'$, $i.e.$, $ts' \sqsubseteq st'$ and $|ve'|_{ca} \sqsubseteq h'$.

We assume that $c = (lam, \beta_1)$ and that $H_?(u)$ holds, the other cases are simpler. In this case, $|ve'|_{ca}$ is the same as $|ve|_{ca}$ except that $|ve'|_{ca}(u) = |ve|_{ca}(u) \sqcup |d|_{ca}$. Also, $h'(u) = h(u) \sqcup \hat{d}$, thus $|ve'|_{ca} \sqsubseteq h'$.

We know that β' contains bindings for u and k, and by lemma A.2 it doesn't bind any variables in $BV(call)$. Since $LV(l) \setminus \{u, k\} = BV(call)$, β' doesn't bind any variables in $LV(l) \setminus \{u, k\}$. Thus, the top frame of ts' is $[u \mapsto |d|_{ca}][k \mapsto |c|_{ca}]$. The top frame of st' is $[u \mapsto \hat{d}][k \mapsto \hat{c}]$, therefore the frames are in $\sqsubseteq$. To complete the proof of $ts' \sqsubseteq st'$, we must show that $pop(ts') \sqsubseteq pop(st')$

$\Leftrightarrow pop(ts') \sqsubseteq st$
$\Leftarrow pop(ts') = ts$.

We know $pop(ts') = toStack(LV(\mathcal{L}(lam)), \beta_1, ve')$, $ts = toStack(LV(\mathcal{L}(lam)), \beta_1, ve)$. By the temporal consistency of states ($cf.$ [19] definition 4.4.5), $pop(ts')$ won't contain the two bindings born at time t because they are younger than all bindings in β_1. This implies that $pop(ts') = ts$.

c) Rule [CEA]
$(\llbracket (q\,e)^\gamma \rrbracket, \beta, ve, t) \rightarrow (proc, d, ve, \gamma :: t)$
$proc = \mathcal{A}(q, \beta, ve)$
$d = \mathcal{A}(e, \beta, ve)$

Let $ts = toStack(LV(\gamma), \beta, ve)$. Since $|\varsigma|_{ca} \sqsubseteq \hat{\varsigma}$, $\hat{\varsigma}$ is of the form $(\llbracket (q\,e)^\gamma \rrbracket, st, h)$, where $|ve|_{ca} \sqsubseteq h$ and $ts \sqsubseteq st$. The abstract transition is

$(\llbracket (q\,e)^\gamma \rrbracket, st, h) \rightsquigarrow (q', \hat{d}, st', h)$
$q' = \hat{\mathcal{A}}_k(q, st)$
$\hat{d} = \hat{\mathcal{A}}_u(e, \gamma, st, h)$
$st' = \begin{cases} pop(st) & Var_?(q) \\ st & Lam_?(q) \end{cases}$

Let ts' be the stack of $|\varsigma'|_{ca}$. We must show that $|\varsigma'|_{ca} \sqsubseteq \hat{\varsigma}'$, $i.e.$, $|proc|_{ca} = q'$, $|d|_{ca} \sqsubseteq \hat{d}$, and $ts' \sqsubseteq st'$.

We first show $|proc|_{ca} = q'$, by cases on q:

- $Lam_?(q)$
  Then, $proc = (q, \beta)$ and $q' = q$. Thus, $|proc|_{ca} = q'$.
- $Var_?(q)$ and $proc = ve(q, \beta(q)) = (lam, \beta_1)$
  Since $q \in LV(\gamma)$ we get $ts(q) = lam$. From the latter and $ts \sqsubseteq st$, we get $st(q) = lam$, which implies $q' = lam$, which implies $|proc|_{ca} = q'$.
- $Var_?(q)$ and $proc = ve(q, \beta(q)) = halt$
  Similar to the previous case.

Showing $|d|_{ca} \sqsubseteq \hat{d}$ is similar, by cases on e.

Last, we show $ts' \sqsubseteq st'$, by cases on q:

- $Lam_?(q)$
  Then, $st' = st$. Also, $ts' = toStack(LV(\mathcal{L}(q)), \beta, ve)$ and $LV(\mathcal{L}(q)) = LV(\gamma)$. Thus, $ts' = ts$, which implies $ts' \sqsubseteq st'$.

- $Var_?(q)$ and $proc = ve(q, \beta(q)) = (lam, \beta_1)$
  Then, $ts' = toStack(LV(\mathcal{L}(lam)), \beta_1, ve) = pop(ts)$ and $st' = pop(st)$. To show $ts' \sqsubseteq st'$, it suffices to show $pop(ts) \sqsubseteq pop(st)$, which holds by $ts \sqsubseteq st$.
- $Var_?(q)$ and $proc = ve(q, \beta(q)) = halt$
  Similar to the previous case.

d) Rule [CAE]
   This case requires arguments similar to the previous cases.   □

Lemma A.4. *On an $\widehat{Eval\text{-}to\text{-}Apply}$ transition, the stack below the top frame is irrelevant. Formally,*

- *If $(\llbracket (f\ e\ lam)^l \rrbracket, tf :: st, h) \rightsquigarrow (ulam, \hat{d}, lam, tf' :: st, h)$ then for any st',*
  $(\llbracket (f\ e\ lam)^l \rrbracket, tf :: st', h) \rightsquigarrow (ulam, \hat{d}, lam, tf' :: st', h)$
- *If $(\llbracket (f\ e\ k)^l \rrbracket, tf :: st, h) \rightsquigarrow (ulam, \hat{d}, \hat{c}, st, h)$ then for any st',*
  $(\llbracket (f\ e\ k)^l \rrbracket, tf :: st', h) \rightsquigarrow (ulam, \hat{d}, \hat{c}, st', h)$
- *Similarly for rule $[\widehat{CEA}]$.*   □

Lemma A.5. *On an $\widehat{Apply\text{-}to\text{-}Eval}$ transition, the stack is irrelevant. Formally,*

- *If $(\llbracket (\lambda_l\ (u\ k)\ call) \rrbracket, \hat{d}, \hat{c}, st, h) \rightsquigarrow (call, [u \mapsto \hat{d}][k \mapsto \hat{c}] :: st, h')$ then for any st',*
  $(\llbracket (\lambda_l\ (u\ k)\ call) \rrbracket, \hat{d}, \hat{c}, st', h) \rightsquigarrow (call, [u \mapsto \hat{d}][k \mapsto \hat{c}] :: st', h')$
- *Similarly for rule $[\widehat{CAE}]$, where st' is any non-empty stack.*   □

Definition A.6 (Push Monotonicity).
Let $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}$ where $\hat{\varsigma}_e$ is an entry with stack st_e. The path p is **push monotonic** iff every transition $\hat{\varsigma}_1 \rightsquigarrow \hat{\varsigma}_2$ satisfies the following property:

> If the stack of $\hat{\varsigma}_1$ is st_e then the transition can only push the stack, it cannot pop or modify the top frame.

         □

Push monotonicity is a property of paths, not of individual transitions. A push monotonic path can contain transitions that pop, as long as the stack never shrinks below the stack of the initial state of the path. The following properties are simple consequences of push monotonicity.

Property A.7. The stack of the first state in a push-monotonic path is a suffix of the stack of every other state in the path.

Property A.8. In a push-monotonic path, the number of pushes is greater than or equal to the number of pops.

The following lemma associates entries with "same-level reachable" states. A state $\hat{\varsigma}$ is same-level reachable from an entry $\hat{\varsigma}_e$ if it is in the procedure whose entry is $\hat{\varsigma}_e$ or if it is in some procedure that can be reached from $\hat{\varsigma}_e$ through tail calls, *i.e.*, without growing the stack.

Lemma A.9 (Same-level reachability).
Let $\hat{\varsigma}_e = (\llbracket (\lambda_l\ (u\ k)\ call) \rrbracket, \hat{d}, \hat{c}, st_e, h_e)$, $\hat{\varsigma} = (\dots, st, h)$, and $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}$ where $\hat{\varsigma}_e \in CE_p^*(\hat{\varsigma})$. Then,

(1) *If $\hat{\varsigma}$ is an entry, $st = st_e$.*

(2) *If $\hat{\varsigma}$ is not an entry,*
  (a) *st is of the form $tf :: st_e$, for some frame tf.*
  (b) *there exists k' such that $tf(k') = \hat{c}$.*
  (c) *if $\hat{\varsigma}_e = CE_p(\hat{\varsigma})$ then $\mathrm{dom}(tf) \subseteq LV(l)$, $tf(u) \sqsubseteq \hat{d}$ and $tf(k) = \hat{c}$.*
     Moreover, if $\hat{\varsigma}$ is an $\widehat{Eval}$ over call site ψ then $\psi \in LL(l)$, and if $\hat{\varsigma}$ is a $\widehat{CApply}$ over $(\lambda_\gamma(u')call')$ then $\gamma \in LL(l)$.
(3) *p is push monotonic.*

Proof. By induction on the length $|p|$ of p. Note that (3) follows from the form of the stack in (1) and (2), so we won't prove it separately.
Basecase:
If $|p| = 0$, then $\hat{\varsigma} = \hat{\varsigma}_e$ so $st = st_e$.
Inductive step:
If $|p| > 0$, there are two cases; either $\hat{\varsigma}_e = CE_p(\hat{\varsigma})$ or $\hat{\varsigma}_e \neq CE_p(\hat{\varsigma})$.

a) $\hat{\varsigma}_e = CE_p(\hat{\varsigma})$
   Since $|p| > 0$, $\hat{\varsigma}$ is not an entry, so the second or the third branch of the definition of CE_p determine the shape of p.

a1) $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$

   Here, the predecessor $\hat{\varsigma}'$ of $\hat{\varsigma}$ is not a $\widehat{CEval}$ exit, and $\hat{\varsigma}_e = CE_p(\hat{\varsigma}')$. We proceed by cases on $\hat{\varsigma}'$. Note that $\hat{\varsigma}'$ cannot be a $\widehat{UEval}$ because then $\hat{\varsigma}$ is an entry, so $\hat{\varsigma} = CE_p(\hat{\varsigma})$, and our assumption that $\hat{\varsigma}_e = CE_p(\hat{\varsigma})$ breaks.

a1.1) $\hat{\varsigma}'$ is an inner $\widehat{CEval}$
   Then, $\hat{\varsigma}' = ([\![((\lambda_\gamma(u')call')\,e')^{\gamma'}]\!], st', h')$. By *IH*, $st' = tf' :: st_e$, $\mathrm{dom}(tf') \subseteq LV(l)$, $tf'(u) \sqsubseteq \hat{d}$, $tf'(k) = \hat{c}$ and $\gamma' \in LL(l)$. By the abstract semantics, $\hat{\varsigma} = ([\![(\lambda_\gamma(u')call')]\!], \hat{d}', st', h')$ where $\hat{d}' = \hat{\mathcal{A}}_u(e', \gamma', st', h')$. We know that $\gamma \in LL(l)$ because $\gamma' \in LL(l)$. Also, the stack is unchanged in the transition. Thus, (2a), (2b) and (2c) hold for $\hat{\varsigma}$.

a1.2) $\hat{\varsigma}'$ is a $\widehat{CApply}$
   Then, $\hat{\varsigma}' = ([\![(\lambda_\gamma(u')call')]\!], \hat{d}', st', h')$. By *IH*, $st' = tf' :: st_e$, $\mathrm{dom}(tf') \subseteq LV(l)$, $tf'(u) \sqsubseteq \hat{d}$, $tf'(k) = \hat{c}$ and $\gamma \in LL(l)$.
   By the abstract semantics, $\hat{\varsigma} = (call', st, h)$ where $st = st'[u' \mapsto \hat{d}']$.
   So, $st = tf :: st_e$ which satisfies (2a). Also, $tf = tf'[u' \mapsto \hat{d}']$ where $u' \in LV(l)$ because $\gamma \in LL(l)$, and $u' \neq u$ because the program is α-tized. Thus, $\mathrm{dom}(tf) = \mathrm{dom}(tf') \cup \{u'\} \subseteq LV(l)$, and $tf(u) = tf'(u) \sqsubseteq \hat{d}$, and $tf(k) = tf'(k) = \hat{c}$. Last, the label of $call'$ is in $LL(l)$ because $\gamma \in LL(l)$.

a1.3) $\hat{\varsigma}'$ is a $\widehat{UApply}$
   Then, $\hat{\varsigma}' = \hat{\varsigma}_e$ because $\hat{\varsigma}_e = CE_p(\hat{\varsigma}')$. This case is simple.

a2) $p \equiv \hat{\varsigma}_e \rightsquigarrow^+ \hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$
   Here, the third branch of the definition of CE_p determines the shape of p, so $\hat{\varsigma}_2$ is a call, $\hat{\varsigma}_e = CE_p(\hat{\varsigma}_2)$, $\hat{\varsigma}'$ is a $\widehat{CEval}$ exit and $\hat{\varsigma}_3 \in CE_p^*(\hat{\varsigma}')$.
   By *IH* for $\hat{\varsigma}_e \rightsquigarrow^+ \hat{\varsigma}_2$ we get $\hat{\varsigma}_2 = ([\![(f_2\,e_2\,(\lambda_{\gamma_2}(u_2)call_2))^{l_2}]\!], st_2, h_2)$, where $st_2 \equiv tf_2 :: st_e$, $\mathrm{dom}(tf_2) \subseteq LV(l)$, $tf_2(u) \sqsubseteq \hat{d}$, $tf_2(k) = \hat{c}$ and $l_2 \in LL(l)$.
   By the abstract semantics for $\hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3$ we get
   $\hat{\varsigma}_3 = ([\![(\lambda_{l_3}(u_3\,k_3)call_3)]\!], \hat{d}_3, \hat{c}_3, st_3, h_2)$,
   where $[\![(\lambda_{l_3}(u_3\,k_3)call_3)]\!] \in \hat{\mathcal{A}}_u(f_2, l_2, st_2, h_2)$,

$\hat{d}_3 = \hat{\mathcal{A}}_u(e_2, l_2, st_2, h_2)$, $\hat{c}_3 = [\![(\lambda_{\gamma_2}(u_2)\, call_2)]\!]$ and
either $st_3 = st_2$, if $(Lam_?(f_2) \vee H_?(l_2, f_2))$ holds,
or $st_3 = st_2[f_2 \mapsto \{[\![(\lambda_{l_3}(u_3\, k_3)\, call_3)]\!]\}]$, if $S_?(l_2, f_2)$ holds.

a2.1) $S_?(l_2, f_2)$
Then, $st_3 = tf_2[f_2 \mapsto \{[\![(\lambda_{l_3}(u_3\, k_3)\, call_3)]\!]\}] :: st_e$.
By IH for $\hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}'$ we get $\hat{\varsigma}' = ([\![(k'\, e')^{\gamma'}]\!], st', h')$,
where $st' = tf' :: st_3$ and $tf'(k') = [\![(\lambda_{\gamma_2}(u_2)\, call_2)]\!]$.
Thus, by the abstract semantics for $\hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ we get
$\hat{\varsigma} = ([\![(\lambda_{\gamma_2}(u_2)\, call_2)]\!], \hat{d}', st_3, h')$.
Now, $\gamma_2 \in LL(l)$ follows from $l_2 \in LL(l)$.
Also, $st = tf :: st_e$ where $tf = tf_2[f_2 \mapsto \{[\![(\lambda_{l_3}(u_3\, k_3)\, call_3)]\!]\}]$.
Then, $\text{dom}(tf) = \text{dom}(tf_2) \cup \{f_2\} \subseteq LV(l)$ because $S_?(l_2, f_2)$ implies $f_2 \in LV(l)$.
Also, $tf(k) = tf_2(k) = \hat{c}$. Last, we take cases depending on whether u and f_2 are the same variable or not.

- $u = f_2$
  $tf(u) = \{[\![(\lambda_{l_3}(u_3\, k_3)\, call_3)]\!]\} \subseteq \hat{\mathcal{A}}_u(f_2, l_2, st_2, h_2) = st_2(f_2) = tf_2(f_2) = tf_2(u)$
  $\sqsubseteq \hat{d}$
- $u \neq f_2$
  $tf(u) = tf_2(u) \sqsubseteq \hat{d}$

a2.2) $Lam_?(f_2) \vee H_?(l_2, f_2)$
This case is simpler than the previous case because $st_3 = st_2$.

b) $\hat{\varsigma}_e \neq CE_p(\hat{\varsigma})$ (but $\hat{\varsigma}_e \in CE_p^*(\hat{\varsigma})$)
Then, the second branch of the definition of CE_p^* determines the shape of p;
$p \equiv \hat{\varsigma}_e \rightsquigarrow^+ \hat{\varsigma}_1 \rightsquigarrow \hat{\varsigma}_2 \rightsquigarrow^* \hat{\varsigma}$, where $\hat{\varsigma}_1$ is a tail call, $\hat{\varsigma}_2 = CE_p(\hat{\varsigma})$ and $\hat{\varsigma}_e \in CE_p^*(\hat{\varsigma}_1)$.
By IH for $\hat{\varsigma}_e \rightsquigarrow^+ \hat{\varsigma}_1$ we get $\hat{\varsigma}_1 = ([\![(f_1\, e_1\, k_1)^{l_1}]\!], st_1, h_1)$,
where $st_1 = tf_1 :: st_e$, $tf_1(k_1) = \hat{c}$.
By the abstract semantics, $\hat{\varsigma}_2 = ([\![(\lambda_{l_2}(u_2\, k_2)\, call_2)]\!], \hat{d}_2, \hat{c}, st_e, h_1)$.

b.1) $\hat{\varsigma}$ is an entry
Then, $\hat{\varsigma} = \hat{\varsigma}_2$ because $\hat{\varsigma}_2 = CE_p(\hat{\varsigma})$. So, $st = st_e$.

b.2) $\hat{\varsigma}$ is not an entry
By IH for $\hat{\varsigma}_2 \rightsquigarrow^* \hat{\varsigma}$ we get $st \equiv tf :: st_e$ and $tf(k_2) = \hat{c}$. This is the desired result for $\hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}$. □

Lemma A.10 (Local simulation)**.**
If $\hat{\varsigma} \rightsquigarrow \hat{\varsigma}'$ and $succ(|\hat{\varsigma}|_{al}) \neq \emptyset$, then $|\hat{\varsigma}'|_{al} \in succ(|\hat{\varsigma}|_{al})$.

Proof. By cases on the abstract transition.
We only show the lemma for $[\widehat{UEA}]$, the other cases are similar.
$([\![(f\, e\, q)^l]\!], st, h) \rightsquigarrow (f', \hat{d}, \hat{c}, st', h)$
$f' \in \hat{\mathcal{A}}_u(f, l, st, h)$
$\hat{d} = \hat{\mathcal{A}}_u(e, l, st, h)$
$\hat{c} = \hat{\mathcal{A}}_k(q, st)$
$$st' = \begin{cases} pop(st) & Var_?(q) \\ st & Lam_?(q) \wedge (H_?(l, f) \vee Lam_?(f)) \\ st[f \mapsto \{f'\}] & Lam_?(q) \wedge S_?(l, f) \end{cases}$$

A $\widehat{UEval}$ state has a successor only when its stack is not empty, so $st \equiv tf :: st''$.

Thus, $|st|_{al} = \{\, (v, tf(v)) \,:\, v \in \mathrm{dom}(tf) \wedge UVar_?(v)\,\}$.

Then, $|\hat{\varsigma}|_{al} = (\llbracket (f\, e\, q)^l \rrbracket, |st|_{al}, h)$. Also, $|\hat{\varsigma}'|_{al} = (f', \hat{\mathrm{d}}, h)$.

If suffices to show that $f' \in \tilde{\mathcal{A}}_u(f, l, |st|_{al}, h)$ and $\hat{\mathrm{d}} = \tilde{\mathcal{A}}_u(e, l, |st|_{al}, h)$; but these hold because $\tilde{\mathcal{A}}_u(v, \psi, |st|_{al}, h) = \hat{\mathcal{A}}_u(v, \psi, st, h)$ is true for any v (*uvar* or *ulam*). $\square$

Lemma A.11 (Converse of Local Simulation).
If $\tilde{\varsigma} \approx\!\!\gg \tilde{\varsigma}'$ then, for any $\hat{\varsigma}$ such that $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$, there exists a state $\hat{\varsigma}'$ such that $\hat{\varsigma} \rightsquigarrow \hat{\varsigma}'$ and $\tilde{\varsigma}' = |\hat{\varsigma}'|_{al}$ $\square$

Lemma A.12 (Path decomposition). *Let $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}$ be push monotonic and $\hat{\varsigma}_e = (\llbracket (\lambda_l\,(u\,k)\,call) \rrbracket, \hat{\mathrm{d}}, \hat{c}, st_e, h_e)$.*

- *if $\hat{\varsigma}$ is a $\widehat{CApply}$ of the form $(\hat{c}, \ldots, st_e, \ldots)$ then $CE_p(\hat{\varsigma})$ is not defined.*
- *Otherwise,*
  - *(1) $CE_p(\hat{\varsigma})$ is defined, i.e., $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$.*
  - *(2) Regarding the set $CE_p^*(\hat{\varsigma})$, p can be in one of four forms*
    - *(a) $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}$ where $\hat{\varsigma}_e = CE_p(\hat{\varsigma})$ and $CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}_e\}$*
    - *(b) $p \equiv e_1 \rightsquigarrow^+ c_1 \rightsquigarrow \ldots \rightsquigarrow e_k \rightsquigarrow^+ c_k \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}$, $k > 0$, where e_is are entries, c_is are tail calls, $e_1 = \hat{\varsigma}_e$, $e_i = CE_p(c_i)$, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$ and $CE_p^*(\hat{\varsigma}) = \{e_1, \ldots, e_k, \hat{\varsigma}_1\}$*
    - *(c) $p \equiv \hat{\varsigma}_e \rightsquigarrow^+ c \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}$ where c is a call, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$ and $CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}_1\}$*
    - *(d) $p \equiv \hat{\varsigma}_e \rightsquigarrow^+ c \rightsquigarrow e_1 \rightsquigarrow^+ c_1 \rightsquigarrow \ldots \rightsquigarrow e_k \rightsquigarrow^+ c_k \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}$, $k > 0$, where c is a call, e_is are entries, c_is are tail calls, $e_i = CE_p(c_i)$, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$ and $CE_p^*(\hat{\varsigma}) = \{e_1, \ldots, e_k, \hat{\varsigma}_1\}$*

Proof. By induction on the length of p.

Basecase: $\hat{\varsigma}_e \rightsquigarrow^0 \hat{\varsigma}_e$

Then, $\hat{\varsigma} = \hat{\varsigma}_e \Rightarrow \hat{\varsigma}_e = CE_p(\hat{\varsigma}) \Rightarrow CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}_e\} \Rightarrow$ (2a) holds

Inductive step: $\hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$

Cases on $\hat{\varsigma}'$:

a) $\hat{\varsigma}'$ is a Call

Then, $\hat{\varsigma}$ is an entry so $CE_p(\hat{\varsigma}) = \hat{\varsigma}$. Also, $CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}\}$ so (2c) holds.

b) $\hat{\varsigma}'$ is a Tail Call

Then, $\hat{\varsigma}$ is an entry so $CE_p(\hat{\varsigma}) = \hat{\varsigma}$.

To show (2), we take cases on whether (2a), (2b), (2c) or (2d) holds for $\hat{\varsigma}'$.

  b.1) (2a) holds for $\hat{\varsigma}'$, i.e.,

   $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ where $\hat{\varsigma}_e = CE_p(\hat{\varsigma}')$ and $CE_p^*(\hat{\varsigma}') = \{\hat{\varsigma}_e\}$. By the second branch of the definition of CE_p^*, $CE_p^*(\hat{\varsigma}') \subseteq CE_p^*(\hat{\varsigma})$. Hence, $CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}_e, \hat{\varsigma}\}$, which implies that (2b) holds for $\hat{\varsigma}$.

  b.2) (2b) holds for $\hat{\varsigma}'$

   By a similar argument, we find that (2b) holds for $\hat{\varsigma}$.

  b.3) (2c) holds for $\hat{\varsigma}'$

   By a similar argument, we find that (2d) holds for $\hat{\varsigma}$.

  b.4) (2d) holds for $\hat{\varsigma}'$

   By a similar argument, we find that (2d) holds for $\hat{\varsigma}$.

c) $\hat{\varsigma}'$ is a $\widehat{CApply} \equiv (\hat{c}, \ldots, st_e, \ldots)$

Then, in the transition $\hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ we modify the top frame of st_e, which means that p isn't push monotonic. Thus, this case can't arise.

d) $\hat{\varsigma}'$ is an inner $\widehat{CEval}$ or a $\widehat{CApply} \not\equiv (\hat{c}, \dots, st_e, \dots)$
   By *IH*, $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$.
   By the second branch of the definition of CE_p, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$.
   To show (2), we take cases on whether (2a), (2b), (2c) or (2d) holds for $\hat{\varsigma}'$. The reasoning
   is the same as in case (b).

e) $\hat{\varsigma}'$ is a $\widehat{CEval}$ exit
   By *IH*, $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$.
   Cases on (2a), (2b), (2c) or (2d) for $\hat{\varsigma}'$.

e.1) (2a) holds for $\hat{\varsigma}'$, *i.e.*
   $p \equiv \hat{\varsigma}_e \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ where $\hat{\varsigma}_e = CE_p(\hat{\varsigma}')$.
   By lemma A.9, the stack of $\hat{\varsigma}'$ is of the form $tf :: st_e$ and $tf(k) = \hat{c}$. Thus, $\hat{\varsigma}' \equiv$
   $(\hat{c}, \dots, st_e, \dots)$. The only way for $CE_p(\hat{\varsigma})$ to exist is by the third branch of the definition
   of CE_p, since $\hat{\varsigma}'$ is a $\widehat{CEval}$ exit. But there is no call leading to $\hat{\varsigma}_e$, thus $CE_p(\hat{\varsigma})$ can't
   exist.
   Similarly when (2b) holds for $\hat{\varsigma}'$.

e.2) (2c) holds for $\hat{\varsigma}'$, *i.e.*
   $p \equiv \hat{\varsigma}_e \rightsquigarrow^+ c \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ where c is a call and $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$.
   By *IH*, $CE_p(c)$ exists so p can be written $p \equiv \hat{\varsigma}_e \rightsquigarrow^* \hat{\varsigma}_2 \rightsquigarrow^+ c \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ where
   $\hat{\varsigma}_2 = CE_p(c)$. Then, by the third branch of the definition of CE_p, $CE_p(\hat{\varsigma}) = CE_p(c) =$
   $\hat{\varsigma}_2$.
   To show (2) for $\hat{\varsigma}$ we work as in the previous cases.

f) $\hat{\varsigma}'$ is an Entry
   This case is simple.   □

Lemma A.13 (Stack irrelevance).
Let $p \equiv \hat{\varsigma}_1 \rightsquigarrow \hat{\varsigma}_2 \rightsquigarrow \dots \rightsquigarrow \hat{\varsigma}_n$ be push monotonic, where $\hat{\varsigma}_1 = (ulam, \hat{\mathrm{d}}, \hat{c}, st_e, h_e)$. Also, $\hat{\varsigma}_n$ is
not a $\widehat{CApply}$ of the form $(\hat{c}, \dots, st_e, \dots)$. By property A.7, the stack of each $\hat{\varsigma}_i$ is of the
form $append(st_i, st_e)$.
For an arbitrary stack st' and continuation $\hat{c}'$, consider the sequence p' of states $\hat{\varsigma}'_1 \ \hat{\varsigma}'_2 \ \dots \ \hat{\varsigma}'_n$
where each $\hat{\varsigma}'_i$ is produced by $\hat{\varsigma}_i$ as follows:

- *if $\hat{\varsigma}_i$ is an entry with stack st_e then replace the continuation argument with $\hat{c}'$ and the*
  stack with st'.
- *if st_e is a proper suffix of the stack of $\hat{\varsigma}_i$ then the latter has the form $append(st'_i, \langle fr_i \rangle, st_e)$*
  for some stack st'_i. Change st_e to st' and bind the continuation variable in fr_i to $\hat{c}'$.

(Note: the map isn't total, but it should be defined for all states in p.)
Then,

- *for any two states $\hat{\varsigma}'_i$ and $\hat{\varsigma}'_{i+1}$ in p', it holds that $\hat{\varsigma}'_i \rightsquigarrow \hat{\varsigma}'_{i+1}$*
- *the path p' is push monotonic*

Proof. By induction on the length of p.
The basecase is simple.
Inductive step: $p = \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}_{n-1} \rightsquigarrow \hat{\varsigma}_n$
By *IH*, the transitions in the path $\hat{\varsigma}'_1 \rightsquigarrow^* \hat{\varsigma}'_{n-1}$ are valid with respect to the abstract semantics
and the path is push monotonic. We must show that $(\hat{\varsigma}'_{n-1}, \hat{\varsigma}'_n) \in \rightsquigarrow$ and that $\hat{\varsigma}'_1 \rightsquigarrow^* \hat{\varsigma}'_n$ is
push monotonic.
Cases on $\hat{\varsigma}_{n-1}$:

(1) $\hat{\varsigma}_{n-1}$ is a $\widehat{UEval}$, of the form $(\llbracket (f\,e\,q)^l \rrbracket, st, h)$

By lemma A.12, $CE_p(\hat{\varsigma}_{n-1})$ is defined and p can be in one of four forms. We consider only the first case, the rest are similar.

Let $p \equiv \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_{n-1} \rightsquigarrow \hat{\varsigma}_n$ where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}_{n-1})$.

By lemma A.9, st is of the form $tf :: st_e$ and the continuation variable in tf (call it k) is bound to $\hat{c}$.

  (a) q is a variable

   By the abstract semantics we have that $\hat{\varsigma}_n$ is $(ulam_n, \hat{\mathrm{d}}_n, \hat{c}, st_e, h)$. Also, the state $\hat{\varsigma}'_{n-1}$ is $(\llbracket (f\,e\,q)^l \rrbracket, tf[k \mapsto \hat{c}'] :: st', h)$, and it transitions to $(ulam_n, \hat{\mathrm{d}}_n, \hat{c}', st', h)$ which is $\hat{\varsigma}'_n$.

  (b) q is a lambda and f is a stack reference

   Then, $\hat{\varsigma}_n$ is $(ulam_n, \hat{\mathrm{d}}_n, q, tf[f \mapsto \{ulam_n\}] :: st_e, h)$.

   Also, the state $\hat{\varsigma}'_{n-1}$ is $(\llbracket (f\,e\,q)^l \rrbracket, tf[k \mapsto \hat{c}'] :: st', h)$, and it transitions to $(ulam_n, \hat{\mathrm{d}}_n, q, tf[k \mapsto \hat{c}'][f \mapsto \{ulam_n\}] :: st', h)$ which is $\hat{\varsigma}'_n$.

  (c) q is a lambda and f is a heap reference

   Similarly.

(2) $\hat{\varsigma}_{n-1}$ is a $\widehat{CEval}$ exit

By lemma A.12, $CE_p(\hat{\varsigma}_{n-1})$ is defined and p can be in one of four forms.

  (a) $p \equiv \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_{n-1} \rightsquigarrow \hat{\varsigma}_n$ where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}_{n-1})$

   Then, by lemma A.9 and the abstract semantics, it is easy to see that $\hat{\varsigma}_n$ is of the form $(\hat{c}, \ldots, st_e, \ldots)$. Thus, this case isn't possible.

   Similarly when $\hat{\varsigma}_1 \neq CE_p(\hat{\varsigma}_{n-1})$ but is in $CE_p^*(\hat{\varsigma}_{n-1})$.

  (b) $p \equiv \hat{\varsigma}_1 \rightsquigarrow^+ c \rightsquigarrow \hat{\varsigma}'_e \rightsquigarrow^+ \hat{\varsigma}_{n-1} \rightsquigarrow \hat{\varsigma}$ where $\hat{\varsigma}'_e = CE_p(\hat{\varsigma}_{n-1})$ and c is a call:

   Then, $CE_p(c)$ is defined and its stack has st_e as a suffix. Hence, by lemma A.9, the stack of c is bigger than st_e by at least a frame. Since the stack of $\hat{\varsigma}'_e$ has the same size as the stack of c, the stack of $\hat{\varsigma}_{n-1}$ is bigger than st_e by at least two frames. By lemma A.4 we get the desired result.

   Similarly when $\hat{\varsigma}'_e \neq CE_p(\hat{\varsigma}_{n-1})$ but is in $CE_p^*(\hat{\varsigma}_{n-1})$.

(3) $\hat{\varsigma}_{n-1}$ is an inner $\widehat{CEval}$

Similarly to the previous cases.

(4) $\hat{\varsigma}_{n-1}$ is a $\widehat{UApply}$

Lemma A.12 gives the same four cases. We only consider one, the rest are similar.

Let $p \equiv \hat{\varsigma}_1 \rightsquigarrow^+ c \rightsquigarrow \hat{\varsigma}_{n-1} \rightsquigarrow \hat{\varsigma}_n$ where c is a call.

Then, $CE_p(c)$ is defined and its stack has st_e as a suffix. Hence, by lemma A.9, the stack of c is bigger than st_e by at least a frame. Since the stack of $\hat{\varsigma}_{n-1}$ has the same size as the stack of c, we don't change the continuation argument in $\hat{\varsigma}'_{n-1}$. By lemma A.5 we get the desired result.

(5) $\hat{\varsigma}_{n-1}$ is a $\widehat{CApply}$

Similarly to the previous cases. $\square$

Theorem A.14 (Soundness).

If $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}$ then, after summarization:

- *if $\hat{\varsigma}$ is not a final state then $(|CE_p(\hat{\varsigma})|_{al}, |\hat{\varsigma}|_{al}) \in Seen$*
- *if $\hat{\varsigma}$ is a final state then $|\hat{\varsigma}|_{al} \in Final$*

- *if $\hat{\varsigma}$ is a $\widehat{CEval}$ exit and $\hat{\varsigma}' \in CE_p^*(\hat{\varsigma})$ then $(|\hat{\varsigma}'|_{al}, |\hat{\varsigma}|_{al}) \in Seen$*

Proof. By induction on the length of p.
Basecase: $\hat{\mathcal{I}}(pr) \rightsquigarrow^0 \hat{\mathcal{I}}(pr)$
Then, $(\hat{\mathcal{I}}(pr), \hat{\mathcal{I}}(pr)) \in Seen$.

Inductive step: $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$
Cases on $\hat{\varsigma}$:

a) $\hat{\varsigma}$ is an Entry
   Then, $CE_p(\hat{\varsigma}) = \hat{\varsigma}$. Also, $\hat{\varsigma}'$ is a call or a tail call.
   By lemma A.12, $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$.
   By *IH*, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}'|_{al}) \in Seen$ which means that it has been entered in W and examined.
   By lemma A.10, $|\hat{\varsigma}|_{al} \in succ(|\hat{\varsigma}'|_{al})$ so in line 10 or 22 $(|\hat{\varsigma}|_{al}, |\hat{\varsigma}|_{al})$ will be propagated.

b) $\hat{\varsigma}$ is a $\widehat{CApply}$ but not a final state
   Then, $\hat{\varsigma} = (\llbracket (\lambda_\gamma (u)\ call) \rrbracket, \hat{d}, st, h)$ and $\hat{\varsigma}' = (\llbracket (q\ e)^{\gamma'} \rrbracket, st', h)$.

b.1) $Lam_?(q)$, *i.e.* $\hat{\varsigma}'$ is an inner $\widehat{CEval}$
   This case is simple.

b.2) $Var_?(q)$, *i.e.* $\hat{\varsigma}'$ is a $\widehat{CEval}$ exit
   The path $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$ satisfies part 2 of lemma A.12. It can't satisfy cases 2a or 2b because $\hat{\varsigma}$ would be a final state by lemma A.9. Thus, it satisfies 2c or 2d. Then, the path is of the form $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$
   where $\hat{\varsigma}_2$ is a call, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}_2)$ and $\hat{\varsigma}_3 \in CE_p^*(\hat{\varsigma}')$. Note that by the third branch of the definition of CE_p, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$. We must show that $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al}) \in Seen$.

   The state $\hat{\varsigma}_1$ is an entry of the form $\hat{\varsigma}_1 = (\llbracket (\lambda_{l_1} (u_1\ k_1)\ call_1) \rrbracket, \hat{d}_1, \hat{c}_1, st_1, h_1)$
   The state $\hat{\varsigma}_2$ is a call of the form $\hat{\varsigma}_2 = (\llbracket (f_2\ e_2\ q_2)^{l_2} \rrbracket, st_2, h_2)$, where q_2 is a *clam*.
   Lemma A.9 for $\hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_2$ gives $st_2 \equiv tf_2 :: st_1$.
   By the abstract semantics for $\hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3$, we get:
   $\hat{\varsigma}_3 = (ulam, \hat{d}_3, q_2, st_3, h_2)$, where
   either $st_3 = st_2$, if $(Lam_?(f_2) \vee H_?(l_2, f_2))$ holds,
   or $st_3 = st_2[f_2 \mapsto \{ulam\}]$, if $S_?(l_2, f_2)$ holds.
   i.e. $st_3 = tf_3 :: st_1$, and
   $$tf_3 = \begin{cases} tf_2 & Lam_?(f_2) \vee H_?(l_2, f_2) \\ tf_2[f_2 \mapsto \{ulam\}] & S_?(l_2, f_2) \end{cases}$$
   By lemma A.9 for $\hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}'$, we get $st' = tf' :: st_3$ and $tf'(q) = q_2$.
   Then, by the abstract semantics for $\hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$,
   $q_2 = \llbracket (\lambda_\gamma (u)\ call) \rrbracket$, $st = st_3$, and $\hat{d} = \hat{\mathcal{A}}_u(e, \gamma', st', h)$.
   The above information will become useful when dealing with the local counterparts of the aforementioned states.
   By *IH*, $(|\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$ was entered in W (at line 25) and later examined at line 13. Note that $\hat{\varsigma}_3 \neq \hat{\mathcal{I}}(pr)$ because $\hat{\varsigma}_2$ is between them, therefore `Final` will not be called at line 15.
   Also by *IH*, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al})$ was entered in W and later examined. Lemma A.10 implies that $|\hat{\varsigma}_3|_{al} \in succ(|\hat{\varsigma}_2|_{al})$ so $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al}, |\hat{\varsigma}_3|_{al})$ will go in *Callers*. We take cases on whether $(|\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$ or $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al})$ was examined first by the algorithm.

b.2.1) $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al})$ was examined first

Then, when $(|\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$ is examined, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al}, |\hat{\varsigma}_3|_{al})$ is in *Callers*.

Therefore, at line 18 we call $\texttt{Update}(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al}, |\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$.

By applying $|\cdot|_{al}$ to the abstract states we get

$|\hat{\varsigma}_1|_{al} = ([\![(\lambda_{l_1}\,(u_1\,k_1)\,call_1)]\!], \hat{d}_1, h_1)$

$|\hat{\varsigma}_2|_{al} = ([\![(f_2\,e_2\,q_2)^{l_2}]\!], tf_2, h_2)$,

where $q_2 = [\![(\lambda_\gamma\,(u)\,call)]\!]$.

$|\hat{\varsigma}_3|_{al} = (ulam, \hat{d}_3, h_2)$

$|\hat{\varsigma}'|_{al} = ([\![(q\,e)^{\gamma'}]\!], tf', h)$,

where $tf'(q) = [\![(\lambda_\gamma\,(u)\,call)]\!]$.

By looking at $\texttt{Update}$'s code, we see that the return value is $\tilde{\mathcal{A}}_u(e, \gamma', tf', h) = \hat{\mathcal{A}}_u(e, \gamma', st', h) = \hat{d}$. The frame of the return state is

$$\begin{cases} tf_2 & Lam_?(f_2) \vee H_?(l_2, f_2) \\ tf_2[f_2 \mapsto \{ulam\}] & S_?(l_2, f_2) \end{cases}$$

which is equal to tf_3. The heap at the return state is h. Last, the continuation we are returning to is $[\![(\lambda_\gamma\,(u)\,call)]\!]$. Thus, the return state $\tilde{\varsigma}$ is equal to $|\hat{\varsigma}|_{al}$, and we call $\texttt{Propagate}(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$, so $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ will go in *Seen*.

b.2.2) $(|\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$ was examined first

Then, when $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al})$ is examined, $(|\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$ is in *Summary*, and at line 12 we call $\texttt{Update}(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}_2|_{al}, |\hat{\varsigma}_3|_{al}, |\hat{\varsigma}'|_{al})$.

Proceed as above.

c) $\hat{\varsigma}$ is a final state

Then, $\hat{\varsigma} = (halt, \hat{d}, \langle\rangle, h)$. We must show that $|\hat{\varsigma}|_{al}$ will be in *Final* after the execution of the summarization algorithm. By the abstract semantics for $\hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, $\hat{\varsigma}' = ([\![(k\,e)^\gamma]\!], st', h)$, where $st' = tf' :: \langle\rangle$, $tf'(k) = halt$, and $\hat{d} = \hat{\mathcal{A}}_u(e, \gamma, st', h)$.

By *IH* for $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$, we know that $(|\hat{\mathcal{I}}(pr)|_{al}, |\hat{\varsigma}'|_{al})$ was entered in W and *Summary* sometime during the algorithm. When it was examined, the test at line 14 was true so we called $\texttt{Final}(|\hat{\varsigma}'|_{al})$. Hence, we insert $\tilde{\varsigma} = (halt, \tilde{\mathcal{A}}_u(e, \gamma, tf', h), \emptyset, h)$ in *Final*. But, $\tilde{\mathcal{A}}_u(e, \gamma, tf', h) = \hat{\mathcal{A}}_u(e, \gamma, st', h) = \hat{d}$, hence $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$.

d) $\hat{\varsigma}$ is a $\widehat{CEval}$ exit

By lemma A.12 for $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$, $p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$. But $\hat{\varsigma}'$ is not a $\widehat{CEval}$ exit (it is an $\widehat{Apply}$ state), so by the second branch of the definition of CE_p we get $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$.

By *IH*, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}'|_{al})$ is entered in *Seen* and W; and examined at line 6. By lemma A.10, $|\hat{\varsigma}|_{al} \in succ(|\hat{\varsigma}'|_{al})$ so $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ will be propagated (line 7) and entered in *Seen* (line 25).

We need to show that for every $\hat{\varsigma}'' \in CE_p^*(\hat{\varsigma})$, $(|\hat{\varsigma}''|_{al}, |\hat{\varsigma}|_{al})$ will be inserted in *Seen*. The path $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$ satisfies part 2 of lemma A.12; proceed by cases:

d.1) $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$ satisfies 2a

Then, $\hat{\varsigma}_1 = \hat{\mathcal{I}}(pr)$ and $p \equiv \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$ and $CE_p^*(\hat{\varsigma}) = \{\hat{\varsigma}_1\}$. But we 've shown that $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ is entered in *Seen*.

d.2) $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}'$ satisfies 2b

Then, $p \equiv e_1 \rightsquigarrow^+ c_1 \rightsquigarrow \ldots \rightsquigarrow e_k \rightsquigarrow^+ c_k \rightsquigarrow \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}' \rightsquigarrow \hat{\varsigma}$, where $e_1 = \hat{\mathcal{I}}(pr)$, e_is are entries, c_is are tail calls, $e_i = CE_p(c_i)$, $CE_p^*(\hat{\varsigma}') = \{e_1, \ldots, e_k, \hat{\varsigma}_1\}$.

Hence, $CE_p^*(\hat{\varsigma}) = \{e_1, \ldots, e_k, \hat{\varsigma}_1\}$. To show that $(|e_k|_{al}, |\hat{\varsigma}|_{al})$ is entered in *Seen*, we proceed by cases on whether $(|e_k|_{al}, |c_k|_{al})$ or $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ was examined first by the algorithm.

d.2.1) $(|e_k|_{al}, |c_k|_{al})$ was examined first

By lemma A.10, $|\hat{\varsigma}_1|_{al}$ is in $succ(|c_k|_{al})$, hence $(|e_k|_{al}, |c_k|_{al}, |\hat{\varsigma}_1|_{al})$ will go in *TCallers*. Then, when $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ is examined, in line 19 we will call `Propagate`$(|e_k|_{al}, |\hat{\varsigma}|_{al})$, so $(|e_k|_{al}, |\hat{\varsigma}|_{al})$ will go in *Seen*.

d.2.2) $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ was examined first

When $(|e_k|_{al}, |c_k|_{al})$ is examined, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ will be in *Summary*, and by lemma A.10 we know $|\hat{\varsigma}_1|_{al} \in succ(|c_k|_{al})$. Thus, in line 24 we will call `Propagate` which will insert $(|e_k|_{al}, |\hat{\varsigma}|_{al})$ in *Seen*.

By repeating this process $k - 1$ times, we can show that all edges $(|e_i|_{al}, |\hat{\varsigma}|_{al})$ go in *Seen*.

d.3) $\hat{\mathcal{I}}(pr) \leadsto^* \hat{\varsigma}'$ satisfies 2c or 2d

These cases are similar to the previous cases. The only difference is that now $\hat{\mathcal{I}}(pr)$ is not in $CE_p^*(\hat{\varsigma}')$ (which doesn't change the proof).

e) $\hat{\varsigma}$ is a Tail Call (thus an exit)

By lemma A.12 for $\hat{\mathcal{I}}(pr) \leadsto^* \hat{\varsigma}'$, $p \equiv \hat{\mathcal{I}}(pr) \leadsto^* \hat{\varsigma}_1 \leadsto^* \hat{\varsigma}' \leadsto \hat{\varsigma}$, where $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}')$. But $\hat{\varsigma}'$ is not a $\widehat{CEval}$ exit (it is an $\widehat{Apply}$ state), so by the second branch of the definition of CE_p we get $\hat{\varsigma}_1 = CE_p(\hat{\varsigma})$.

By *IH*, $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}'|_{al})$ is entered in *Seen* and W; and examined at line 6. By lemma A.10, $|\hat{\varsigma}|_{al} \in succ(|\hat{\varsigma}'|_{al})$ so $(|\hat{\varsigma}_1|_{al}, |\hat{\varsigma}|_{al})$ will be propagated (line 7) and entered in *Seen* (line 25).

f) $\hat{\varsigma}$ is an inner $\widehat{CEval}$

This case is simple.

g) $\hat{\varsigma}$ is a Call

This case is simple. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem A.15 (Completeness).
After summarization:

- *For each $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ in Seen, there exist $\hat{\varsigma}_1$, $\hat{\varsigma}_2$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \leadsto^* \hat{\varsigma}_1 \leadsto^* \hat{\varsigma}_2$ and $\tilde{\varsigma}_1 = |\hat{\varsigma}_1|_{al}$ and $\tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al}$ and $\hat{\varsigma}_1 \in CE_p^*(\hat{\varsigma}_2)$*
- *For each $\tilde{\varsigma}$ in Final, there exist $\hat{\varsigma}$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \leadsto^+ \hat{\varsigma}$ and $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$ and $\hat{\varsigma}$ is a final state.*

Proof. By induction on the number of iterations. We prove that the algorithm maintains the following properties for *Seen* and *Final*.

(1) For each $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ in *Seen*, there exist $\hat{\varsigma}_1$, $\hat{\varsigma}_2$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \leadsto^* \hat{\varsigma}_1 \leadsto^* \hat{\varsigma}_2$ and $\tilde{\varsigma}_1 = |\hat{\varsigma}_1|_{al}$ and $\tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al}$ and, if $\tilde{\varsigma}_2$ is a $\widehat{CEval}$ exit then $\hat{\varsigma}_1 \in CE_p^*(\hat{\varsigma}_2)$ otherwise $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}_2)$

(2) For each $\tilde{\varsigma}$ in *Final*, there exist $\hat{\varsigma}$ and p such that $p \equiv \hat{\mathcal{I}}(pr) \leadsto^+ \hat{\varsigma}$ and $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$ and $\hat{\varsigma}$ is a final state.

Initially, we must show that the properties hold before the first iteration (at the beginning of the algorithm): *Final* is empty and W contains just $(\tilde{\mathcal{I}}(pr), \tilde{\mathcal{I}}(pr))$, for which property 1 holds.

Now the inductive step: at the beginning of each iteration, we remove an edge $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ from W. We assume that the properties hold at that point. We must show that, after we process the edge, the new elements of *Seen* and *Final* satisfy the properties.

- $\tilde{\varsigma}_2$ is an entry, a $\widetilde{CApply}$ or an inner $\widetilde{CEval}$

  $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ is in *Seen*, so by *IH*

  $\exists \hat{\varsigma}_1, \hat{\varsigma}_2, p.\ p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^* \hat{\varsigma}_2 \ \wedge\ \tilde{\varsigma}_1 = |\hat{\varsigma}_1|_{al} \ \wedge\ \tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al} \ \wedge\ \hat{\varsigma}_1 = CE_p(\hat{\varsigma}_2)$

  For each $\tilde{\varsigma}_3$ in $succ(\tilde{\varsigma}_2)$, $(\tilde{\varsigma}_1, \tilde{\varsigma}_3)$ will be propagated.

  If $(\tilde{\varsigma}_1, \tilde{\varsigma}_3)$ is already in *Seen* then property 1 holds by *IH* (in the following cases, we won't repeat this argument and will assume that the insertion in *Seen* happens now).

  Otherwise, we insert the edge at this iteration, at line 25. By lemma A.11,

  $\exists \hat{\varsigma}_3.\ \tilde{\varsigma}_3 = |\hat{\varsigma}_3|_{al} \ \wedge\ \hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3$

  By the second branch of the definition of CE_p, $\hat{\varsigma}_1 = CE_p(\hat{\varsigma}_3)$

- $\tilde{\varsigma}_2$ is a call

  Let $\tilde{\varsigma}_1 = (\llbracket (\lambda_1 (u_1\, k_1)\, call_1) \rrbracket, \hat{\mathrm{d}}_1, h_1)$ and $\tilde{\varsigma}_2 = (\llbracket (f_2\, e_2\, (\lambda_2 (u_2)\, call_2))^{l_2} \rrbracket, tf_2, h_2)$

  Also, assume $S_?(l_2, f_2)$ (the other cases are simpler).

  $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ is in *Seen*, so by *IH*

  $\exists \hat{\varsigma}_1, \hat{\varsigma}_2, p.\ p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_2 \ \wedge\ \tilde{\varsigma}_1 = |\hat{\varsigma}_1|_{al} \ \wedge\ \tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al} \ \wedge\ \hat{\varsigma}_1 = CE_p(\hat{\varsigma}_2)$

  Each entry $\tilde{\varsigma}_3$ in $succ(\tilde{\varsigma}_2)$ will be propagated. By lemma A.11,

  $\exists \hat{\varsigma}_3.\ \tilde{\varsigma}_3 = |\hat{\varsigma}_3|_{al} \ \wedge\ \hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3$

  Since $\hat{\varsigma}_3 = CE_p(\hat{\varsigma}_3)$, property 1 holds for $\tilde{\varsigma}_3$.

  If there is no edge $(\tilde{\varsigma}_3, \tilde{\varsigma}_4)$ in *Summary*, we are done.

  Otherwise, we call $\texttt{Update}(\tilde{\varsigma}_1, \tilde{\varsigma}_2, \tilde{\varsigma}_3, \tilde{\varsigma}_4)$ and we must show that property 1 holds for the edge inserted in *Seen* by $\texttt{Update}$.

  Let st_1 be the stack of $\hat{\varsigma}_1$. By lemma A.9, the stack of $\hat{\varsigma}_2$ is $tf_2 :: st_1$.

  Let $\tilde{\varsigma}_3 = (\llbracket (\lambda_3 (u_3\, k_3)\, call_3) \rrbracket, \hat{\mathrm{d}}_3, h_2)$ and $\tilde{\varsigma}_4 = (\llbracket (k_4\, e_4)^{l_4} \rrbracket, tf_4, h_4)$.

  (Note that tf_4 contains only user bindings.)

  We know *Summary* $\subseteq$ *Seen* so by *IH* for $(\tilde{\varsigma}_3, \tilde{\varsigma}_4)$ we get (note that $\tilde{\varsigma}_4$ is a $\widetilde{CEval}$ exit)

  $\exists \hat{\varsigma}_3', \hat{\varsigma}_4', p'.\ p' \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_3' \rightsquigarrow^+ \hat{\varsigma}_4' \ \wedge\ \tilde{\varsigma}_3 = |\hat{\varsigma}_3'|_{al} \ \wedge\ \tilde{\varsigma}_4 = |\hat{\varsigma}_4'|_{al} \ \wedge\ \hat{\varsigma}_3' \in CE_{p'}^*(\hat{\varsigma}_4')$

  Then, $\hat{\varsigma}_3' = (\llbracket (\lambda_3 (u_3\, k_3)\, call_3) \rrbracket, \hat{\mathrm{d}}_3, \hat{c}_3, st_3', h_2)$ and by lemma A.9,

  $\hat{\varsigma}_4' = (\llbracket (k_4\, e_4) \rrbracket, tf_4[k_4 \mapsto \hat{c}_3] :: st_3', h_4)$.

  But the path from $\hat{\varsigma}_3'$ to $\hat{\varsigma}_4'$ is push monotonic, so by lemma A.13 there exist states

  $\hat{\varsigma}_3 = (\llbracket (\lambda_3 (u_3\, k_3)\, call_3) \rrbracket, \hat{\mathrm{d}}_3, \llbracket (\lambda_2 (u_2)\, call_2) \rrbracket, st_3, h_2)$

  where $st_3 = tf_2[f_2 \mapsto \{\llbracket (\lambda_3 (u_3\, k_3)\, call_3) \rrbracket\}] :: st_1$, and $\hat{\varsigma}_4 = (\llbracket (k_4\, e_4) \rrbracket, st_4, h_4)$

  where $st_4 = tf_4[k_4 \mapsto \llbracket (\lambda_2 (u_2)\, call_2) \rrbracket] :: st_3$, such that $\hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}_4$.

  Thus, the path p can be extended to $\hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_1 \rightsquigarrow^+ \hat{\varsigma}_2 \rightsquigarrow \hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}_4$. By the abstract semantics, the successor $\hat{\varsigma}$ of $\hat{\varsigma}_4$ is $(\llbracket (\lambda_2 (u_2)\, call_2) \rrbracket, \hat{\mathcal{A}}_u(e_4, l_4, st_4, h_4), st_3, h_4)$.

  The state $\tilde{\varsigma}$ produced by $\texttt{Update}$ is $(\llbracket (\lambda_2 (u_2)\, call_2) \rrbracket, \tilde{\mathcal{A}}_u(e_4, l_4, tf_4, h_4), tf, h_4)$ where $tf = tf_2[f_2 \mapsto \{\llbracket (\lambda_3 (u_3\, k_3)\, call_3) \rrbracket\}]$. It is simple to see that $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$.

- $\tilde{\varsigma}_2$ is a $\widetilde{CEval}$ exit, $(\llbracket (k\, e)^{l_2} \rrbracket, tf_2, h_2)$

  If $\tilde{\varsigma}_1$ is $\tilde{\mathcal{I}}(pr)$ then $\texttt{Final}(\tilde{\varsigma}_2)$ is called and a local state $\tilde{\varsigma}$ of the form $(halt, \tilde{\mathcal{A}}_u(e, l_2, tf_2, h_2), \emptyset, h_2)$ goes in *Final*. We must show that property 2 holds.

  By *IH* for $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$, $\quad \exists \hat{\varsigma}_2, p.\ p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^+ \hat{\varsigma}_2 \ \wedge\ \tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al} \ \wedge\ \hat{\mathcal{I}}(pr) \in CE_p^*(\hat{\varsigma}_2)$.

(Note that $\hat{\varsigma}_1 = \hat{\mathcal{I}}(pr)$.) By lemma A.9, the stack st_2 of $\hat{\varsigma}_2$ is $tf_2[k \mapsto halt] :: \langle\rangle$. Hence, the successor $\hat{\varsigma}$ of $\hat{\varsigma}_2$ is $(halt, \hat{\mathcal{A}}_u(e, st_2, h_2), \langle\rangle, h_2)$, and $\tilde{\varsigma} = |\hat{\varsigma}|_{al}$ holds.

If $\tilde{\varsigma}_1 \neq \tilde{\mathcal{I}}(pr)$, for each triple $(\tilde{\varsigma}_3, \tilde{\varsigma}_4, \tilde{\varsigma}_1)$ in *Callers*, we call $\texttt{Update}(\tilde{\varsigma}_3, \tilde{\varsigma}_4, \tilde{\varsigma}_1, \tilde{\varsigma}_2)$. Insertion in *Callers* happens only at line 11, which means that $(\tilde{\varsigma}_3, \tilde{\varsigma}_4)$ is in *Seen*. Thus, by *IH*
$$\exists \hat{\varsigma}_3, \hat{\varsigma}_4, p.\ p \equiv \hat{\mathcal{I}}(pr) \rightsquigarrow^* \hat{\varsigma}_3 \rightsquigarrow^+ \hat{\varsigma}_4 \ \wedge\ \tilde{\varsigma}_3 = |\hat{\varsigma}_3|_{al} \ \wedge\ \tilde{\varsigma}_4 = |\hat{\varsigma}_4|_{al} \ \wedge\ \hat{\varsigma}_3 = CE_p(\hat{\varsigma}_4)$$
Also, $\tilde{\varsigma}_4 \approx\!\!\!> \tilde{\varsigma}_1$ thus by lemma A.11 $\exists \hat{\varsigma}_1.\ \hat{\varsigma}_4 \rightsquigarrow \hat{\varsigma}_1 \ \wedge\ \tilde{\varsigma}_1 = |\hat{\varsigma}_1|_{al}$

Using the *IH* for $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ and lemma A.13 we can show that the edge inserted by $\texttt{Update}$ satisfies property 1 (similar to the previous case).

For each triple $(\tilde{\varsigma}_3, \tilde{\varsigma}_4, \tilde{\varsigma}_1)$ in *TCallers*, we call $\texttt{Propagate}(\tilde{\varsigma}_3, \tilde{\varsigma}_2)$. We must show that property 1 holds for $(\tilde{\varsigma}_3, \tilde{\varsigma}_2)$. Insertion in *TCallers* happens only at line 23, which means that $(\tilde{\varsigma}_3, \tilde{\varsigma}_4)$ is in *Seen*. By *IH* for $(\tilde{\varsigma}_1, \tilde{\varsigma}_2)$ and $(\tilde{\varsigma}_3, \tilde{\varsigma}_4)$ and by lemma A.13, we can show that there are states $\hat{\varsigma}_3$ and $\hat{\varsigma}_2$ and path p' such that $\tilde{\varsigma}_3 = |\hat{\varsigma}_3|_{al}$, $\tilde{\varsigma}_2 = |\hat{\varsigma}_2|_{al}$ and $\hat{\varsigma}_3 \in CE^*_{p'}(\hat{\varsigma}_2)$. Hence, property 1 holds for $(\tilde{\varsigma}_3, \tilde{\varsigma}_2)$.

- $\tilde{\varsigma}_2$ is a tail call
  Similarly. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$