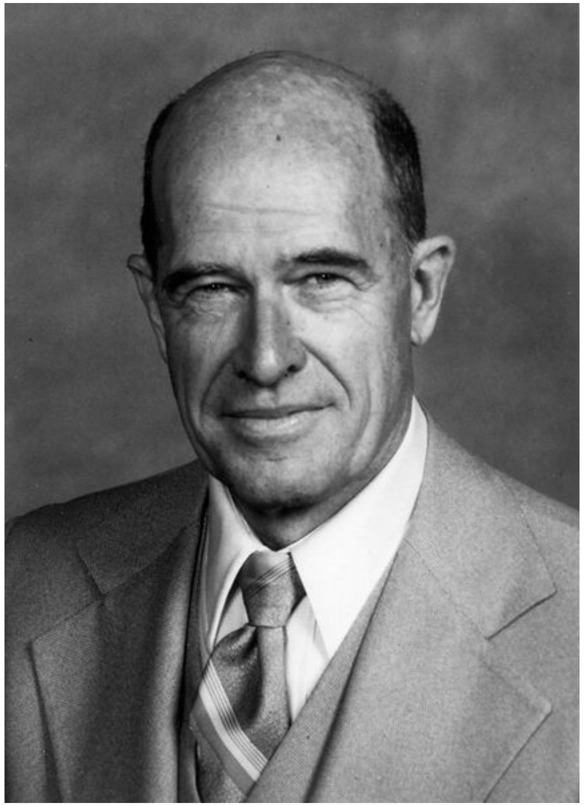
自动控制原理 Automatic Control Theory

西南交通大学

电气工程学院

Chapter 4 Root Locus Method

- **Walter Richard Evans,**
- (January 15, 1920 - July 10, 1999) was a noted American control theorist and the inventor of the root locus method in 1948.
- He was the recipient of the 1987 American Society of Mechanical Engineers Rufus Oldenburger Medal and the 1988 AACC Richard E. Bellman Control Heritage Award.



4.1 Introduction 引言

Stability ← **Roots** of the characteristic equation

Response ← **Roots** of the characteristic equation and **zeros**

The Roots of the characteristic equation determine:

- The stability of a system.
- The response of a system.

- 1948, W.R.Evans proposed Root Locus Methods：

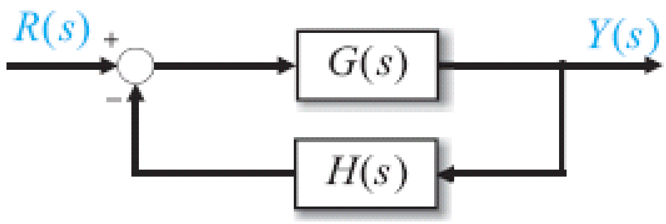
  The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter is changed.

  当开环增益或别的某个参数变化时特征根的轨迹图——找特征根的简单的图解法。

返回

4.2 The root locus concept

Closed-loop transfer function:



$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

Characteristic equation

$$1+G(s)H(s)=0 \qquad (4.1)$$

Loop transfer function

$$G(s)H(s)=K_g\frac{P(s)}{Q(s)}=K_g\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})} \qquad (4.2)$$

K_g ： 传递系数(开环根轨迹增益)

$-z_{oi}$ ： 开环(传函的)零点，$i=1,2,\ldots,m$.

$-p_{oj}$ ： 开环(传函的)极点，$j=1,2,\ldots,n$.

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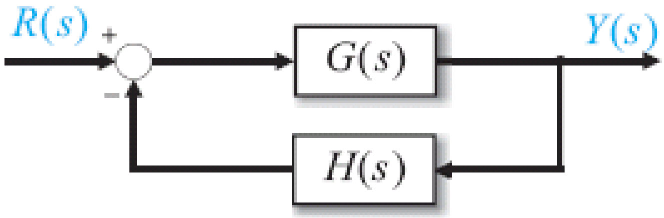
4.2 The root locus concept

So, characteristic equation



$$1+G(s)H(s)=1+K_g\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}=0 \qquad (4.3)$$

- **Root locus method：** According to the open loop poles and zeros, draw the trace of roots of the characteristic equation on the s-plane as the gain is changed. 根据开环传函(开环零点、极点)，找出开环增益(或别的某个参数)由0→∞变化时，闭环系统特征根的轨迹。
- **Basic principle of root locus method：** 开环传函等于-1的*s*值，必为特征根。

4.2 The root locus concept

- **Magnitude Criterion and Angle Criterion**
  幅角条件与幅值条件
  Characteristic equation (4.1) can be written as

$$G(s)H(s) = -1 \tag{4.4}$$

Loop transfer function is a complex number

开环传函$G(s)H(s)$为复数，故由(4.4),有

$$\begin{cases} \text{幅角条件} \quad : \angle G(s)H(s) = \pm(2k+1)180^\circ, k = 0,1,2,\cdots & (4.5) \\ \text{幅值条件} \quad : |G(s)H(s)| = 1 & (4.6) \end{cases}$$

- Characteristic roots are poles of closed-loop transfer function satisfy the magnitude criterion and angle criterion.
  满足幅角条件、幅值条件的 s 值就是特征方程的根，即闭环极点。

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4.2 The root locus concept

Characteristic equation can be written as:

$$\frac{P(s)}{Q(s)}=\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}=A(s)e^{j\theta(s)}=-\frac{1}{K_g} \qquad (4.7)$$

- **Angle Criterion**

$$\theta(s)=\angle G(s)H(s)$$

$$=\sum_{i=1}^{m}\angle(s+z_{oi})-\sum_{j=1}^{n}\angle(s+p_{oj})=\pm 180°(2k+1) \qquad (4.8)$$

$\angle(s+z_{oi})=\alpha_i$，开环有限零点到 s的矢量的幅角；

$\angle(s+p_{oj})=\beta_j$，开环有限极点到 s的矢量的幅角；

(矢量的幅角以逆时针方 向为正)

4.2 The root locus concept

Characteristic equation can be written as:

$$\frac{P(s)}{Q(s)}=\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}=A(s)e^{j\theta(s)}=-\frac{1}{K_g} \qquad (4.7)$$

- **Magnitude Criterion**

$$A(s)=\frac{\prod_{i=1}^{m}|s+z_{oi}|}{\prod_{j=1}^{n}|s+p_{oj}|}=\frac{1}{K_g} \qquad (4.9)$$

$|s+z_{oi}|=l_i$，开环有限零点到s的矢量长度之积

$|s+p_{oj}|=L_j$，开环有限极点到s的矢量长度之积

4.2 The root locus concept

- 由(4.8)和(4.9)给出了根轨迹的**基本原理**：

1) 以K_g为可变参数，s平面上满足幅角条件的点构成的曲线就是根轨迹；

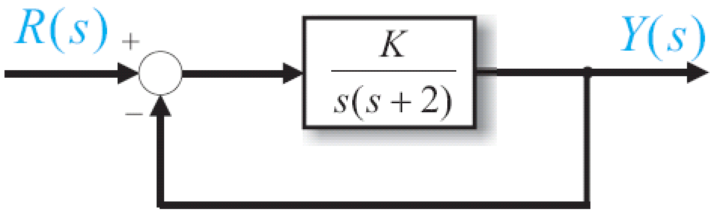
2) 根轨迹上各点的K_g值可由幅值条件确定。

- 由(4.8)和(4.9)也反映了根轨迹的**几何意义**。故在分析和绘制根轨迹时，幅角和幅值应可进行图解测量，故：

  横坐标和纵坐标采用同样的尺度等分

4.2 The root locus concept

<E 4.1>:Sketch the root locus of a second order system.

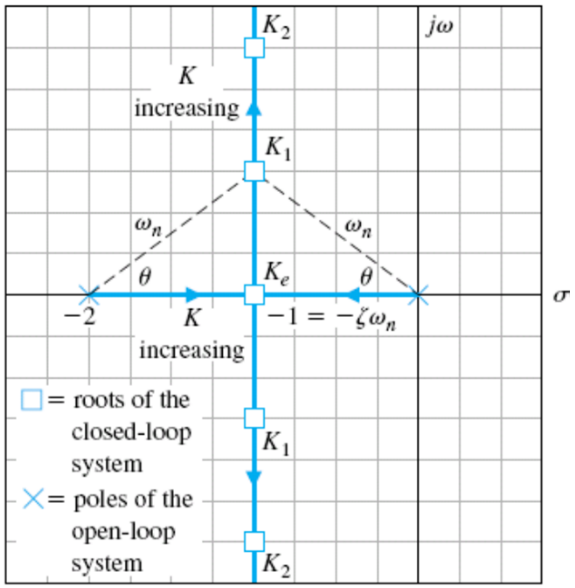


Characteristic equation:

$$s^2 + 2s + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Characteristic roots:

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -1 \pm \sqrt{1 - K}$$

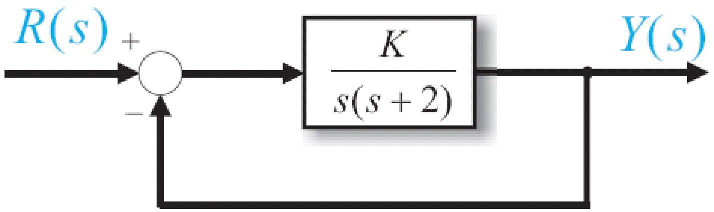


As K changes from $0 \rightarrow 1$, the roots s_1,s_2:

$K = 0$, $s_1 = 0$, $s_2 = -2$;

$K = 1$, $s_1 = s_2 = -1$ ($\zeta = 1$) ;

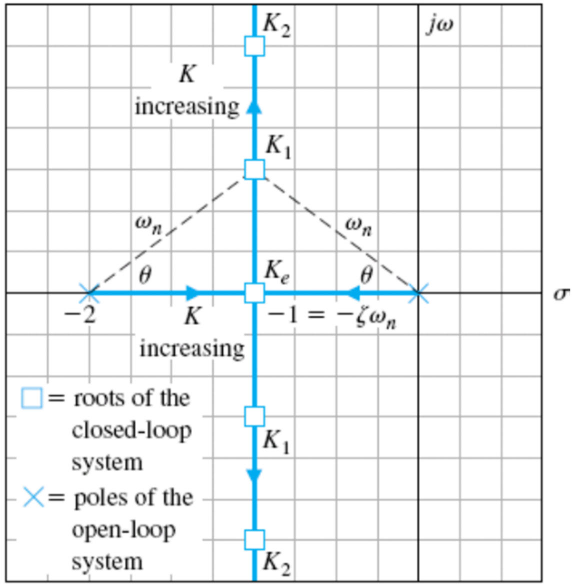
4.2 The root locus concept



When $0<K<1$, $(z>1)$,

$s1, s2$: are two real numbers

$$K:0 \to 1.\begin{cases} s_1 : 0 \to -1; \\ s_2 : -2 \to -1; \end{cases}$$



When $1<K<\infty$, $(0<\zeta<1)$,

s_1, s_2: are complex conjugation

$$s_1, s_2 = -1 \pm j\sqrt{K-1}$$

$$K:1 \to \infty.\begin{cases} s_1 : -1 \to -1 + j\infty; \\ s_2 : -1 \to -1 - j\infty; \end{cases}$$

4.2 The root locus concept

Any point on the root locus satisfies angle criterion :

Any point s_1 , obviously

$\angle s_1 + \angle (s_1+2) = 180^\circ$

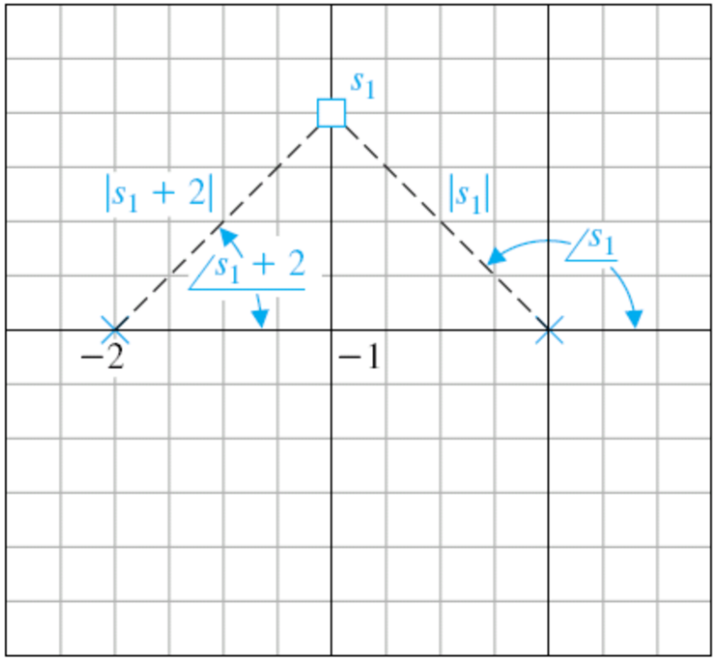
The corresponding gain K at

$$s_1 = -1 + j\omega :$$

$$|s(s+2)|_{s=s_1} = 1+\omega^2 = K$$

$$\omega = 1, s_1 = -1 + j1, K = 2$$

$$\omega = 2, s_1 = -1 + j2, K = 5$$



返回

4.3 The root locus procedure绘制根轨迹的基本规则

开环传函 (开环零点、极点) →闭环系统根轨迹

根轨迹性质→作图规则→特殊点→根轨迹(手工绘制根轨迹概略图)

Transform the **loop transfer** function as:

$$G(s)H(s) = K_g \frac{P(s)}{Q(s)} = K_g \frac{\prod_{i=1}^{m}(s + z_{oi})}{\prod_{j=1}^{n}(s + p_{oj})} \qquad (4.2)$$

Characteristic equation

$$1 + G(s)H(s) = 1 + K_g \frac{P(s)}{Q(s)} = 0$$

4.3 The root locus procedure

Characteristic equation can be written as:

$$\frac{P(s)}{Q(s)}=\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}=A(s)e^{j\theta(s)}=-\frac{1}{K_g} \qquad (4.7)$$

or

$$\prod_{j=1}^{n}(s+p_{oj})+K_g\prod_{i=1}^{m}(s+z_{oi})=0 \qquad (4.10)$$

As K_g: $0\rightarrow\infty(K_g\geq0)$, draw the trace of the characteristic roots of closed-loop transfer function.

4.3 The root locus procedure

Rule 1: The root loci must be continuous and symmetrical with respect to the horizontal real axis.

根轨迹是连续的，且对称于实轴；

Rule 2: The locus of the roots of the characteristic equation begins at the poles of $GH(s)$ and ends at the zeros of $GH(s)$ as K increases form 0 to infinity. The number of separate loci is equal to $\max\{n,m\}$, where n is the number of poles, m is the number of zeros.

根轨迹起始于开环极点，终止于开环零点；根轨迹的分支数(条数)为$\max\{n,m\}$， n为开环(有限)极点数， m为开环(有限)零点数；

4.3 The root locus procedure

$$\frac{P(s)}{Q(s)}=\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}=A(s)e^{j\theta(s)}=-\frac{1}{K_g} \qquad (4.7)$$

Beings: $K_g = 0$, Equ. (4.7), $s = -p_{oj}$, $j=1,\ldots,n$;

The poles of closed-loop transfer function are the poles of loop transfer function.

闭环极点 = 开环极点

Ends: $K_g \to \infty$, Equ. (4.7), $s = -z_{oi}$, $i=1,\ldots,m$;

The poles of closed-loop transfer function are the zeros of loop transfer function.

闭环极点 = 开环零点

4.3 The root locus procedure

When $n>m$,

- m zeros of loop transfer function determine the position of m poles of closed-loop transfer function.

  m个开环有限零点决定了m个闭环极点的位置;

- The other n-m poles of closed-loop transfer function lie at infinity in the s-plane.

  另$(n-m)$个闭环极点趋向于无穷远(开环无限零点).

Note: 如果包括无限零点,则$G(s)H(s)$的零点数和极点数相等.

4.3 The root locus procedure

Characteristic Equation:

$$1+K_g\frac{P(s)}{Q(s)}=0\Rightarrow P(s)+\frac{1}{K_g}Q(s)=0$$

where $P(s)=s^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0$

$$Q(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0$$

- Consider m **Poles of closed-loop transfer function**

With $K_g\rightarrow\infty, s\neq\infty\Rightarrow\frac{1}{K_g}Q(s)=0\Rightarrow P(s)=0$

Therefore

m个闭环极点 $=$ m个开环(有限)零点

4.3 The root locus procedure

Characteristic Equation:

$$1 + K_g \frac{P(s)}{Q(s)} = 0 \Rightarrow P(s) + \frac{1}{K_g} Q(s) = 0$$

- Consider the other *n-m* **Poles of closed-loop transfer function**

With $K_g \to \infty, s \neq -z_{oi} \Rightarrow P(s) \neq 0$

and $\left|\frac{P(s)}{Q(s)}\right| = \frac{1}{K_g} = 0 \Rightarrow Q(s) = \infty \Rightarrow s \to \infty$

Therefore

另$(n - m)$个无限零点决定了$(n - m)$个闭环极点的位置.

4.3 The root locus procedure

Usually $n \geq m$， the number of separate loci is equal to the number of poles of closed-loop transfer function

闭环极点数 = 开环极点数n = 系统阶次n

在绘制其它可变参数的根轨迹时，可能出现等效传函的$m>n$的情况，这时将有$(m-n)$条根轨迹起始于$(m-n)$个开环无限极点。

4.3 The root locus procedure

Rule 3： The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.

实轴上根轨迹段存在的区间的右侧，开环零点和开环极点之和为奇数。

1) 开环共轭复数零、极点到实轴上的点的幅角和为$2k\pi$，因此对实轴上的根轨迹的幅角条件无影响;

2) 实轴上根轨迹的左侧的开环零、极点到实轴上的点的幅角均为0°，因此对实轴上的根轨迹的幅角条件也无影响；

4.3 The root locus procedure

3) N_{zo} = 实轴上根轨迹右侧的开环零点数

N_{po} = 实轴上根轨迹右侧的开环极点数

$$\sum_{i=1}^{m}\alpha_i - \sum_{j=1}^{n}\beta_j = N_{oz}\pi - N_{op}\pi = (2k+1)\pi$$

$N_{oz} - N_{op} = 2k+1$(奇数) $\Rightarrow N_{oz} + N_{op}$为奇数

4.3 The root locus procedure

Rule 4： The $|n\text{-}m|$ branches of loci must end at zeros at infinity along asymptotes centered at σ_A and with angles φ_A

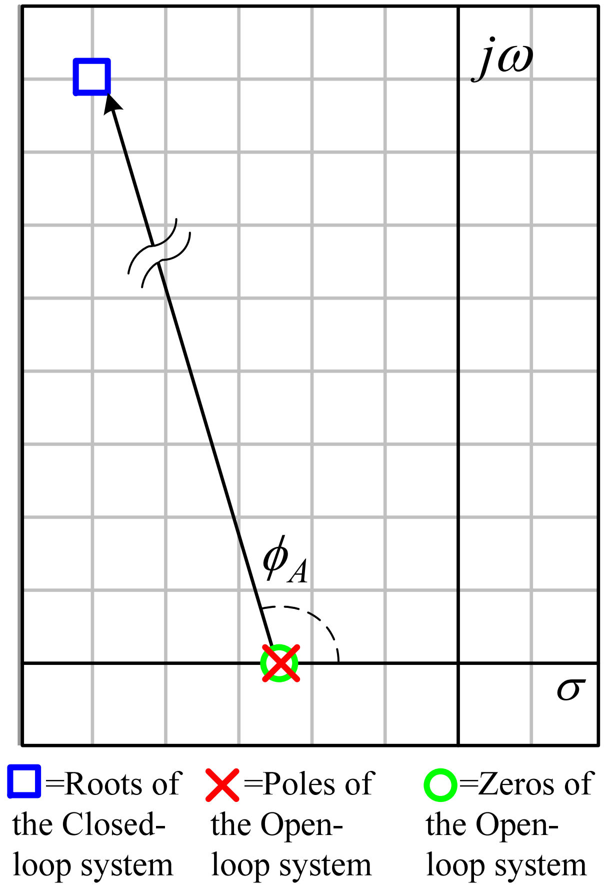
根轨迹有$|n\text{-}m|$条分支沿渐近线趋于(或始于)无穷远，这些渐近线的倾角ϕ_A及与实轴的交点σ_A分别为:

$$\varphi_A = \frac{2k+1}{n-m}\pi, k = 0,1,\cdots,|n-m|-1 \qquad (4.11)$$

$$\sigma_A = \frac{\sum_{j=1}^{n}(-p_{oj}) - \sum_{i=1}^{m}(-z_{oi})}{n-m} = -\frac{\sum_{j=1}^{n}p_{oj} - \sum_{i=1}^{m}z_{oi}}{n-m} \qquad (4.12)$$

4.3 The root locus procedure

1) The angles of asymptotes 渐近线倾角ϕ_A



when $s = s_k \to \infty$,

with angle criterion

$$\sum_{i=1}^{m} \alpha_i - \sum_{j=1}^{n} \beta_j = m\phi_A - n\phi_A$$

$$= (2k'+1)\pi = -(2k+1)\pi$$

$$\boxed{\phi_A = \frac{2k+1}{n-m}\pi}$$

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4.3 The root locus procedure

2) The center of asymptotes 渐近线与实轴交点σ_A

$$G(s)H(s) = K_g \frac{\prod_{i=1}^{m}(s + z_{oi})}{\prod_{j=1}^{n}(s + p_{oj})} = K_g \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \quad (4.14)$$

As $b_{m-1} = \sum_{i=1}^{m} z_{oi}$, $a_{n-1} = \sum_{j=1}^{n} p_{oj}$,

Use **polynomial long division** to rewrite Equ. (4.14)

$$G(s)H(s) = \frac{K_g}{s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots} \quad (4.15)$$

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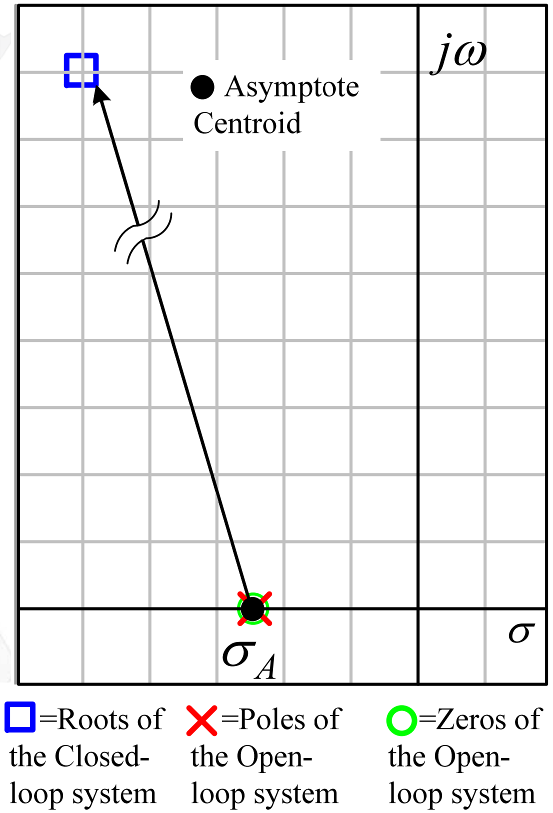
4.3 The root locus procedure

when $s = s_k \to \infty$, $-z_{oi} = -p_{oj} = \sigma_A$



$$G(s)H(s) = K_g \frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}$$

$$\xrightarrow{s\to\infty} \frac{K_g}{(s-\sigma_A)^{n-m}} \qquad (4.16)$$

Use **Binomial Theorem**

$$(s-\sigma_A)^{n-m} = s^{n-m} + (n-m)(-\sigma_A)s^{n-m-1} + \cdots$$

4.3 The root locus procedure

$$G(s)H(s)=\frac{K_g}{s^{n-m}+\boxed{(a_{n-1}-b_{m-1})}s^{n-m-1}+\cdots} \quad (4.15)$$

$$G(s)H(s)=K_g\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}\xrightarrow{s\to\infty}\frac{K_g}{(s-\sigma_A)^{n-m}} \quad (4.16)$$

$$=\frac{K_g}{s^{n-m}+\boxed{(n-m)(-\sigma_A)}s^{n-m-1}+\cdots}$$

$$\boxed{-(n-m)\sigma_A}=\boxed{a_{n-1}-b_{m-1}}=\sum_{j=1}^{n}p_{oj}-\sum_{i=1}^{m}z_{oi}$$

$$\sigma_A=-\frac{\sum_{j=1}^{n}p_{oj}-\sum_{i=1}^{m}z_{oi}}{n-m}$$

4.3 The root locus procedure

Rule 5： Breakaway point of root locus

$$\begin{cases} P'(s)Q(s) - Q'(s)P(s) = 0 \\ K_g \geq 0 \end{cases} \tag{4.17}$$

定义：两条(或两条以上，成对)根轨迹在某点相遇后又分开的点称为根轨迹的分离点(或汇合点，可统称为分离点)。

实轴上的根轨迹，相邻开环极点之间、相邻开环零点之间必存在分离点。相邻开环零点和极点之间，或不存在分离点，或存在成对的分离点。

4.3 The root locus procedure

分离点处，特征方程为重根。因此，对于特征方程

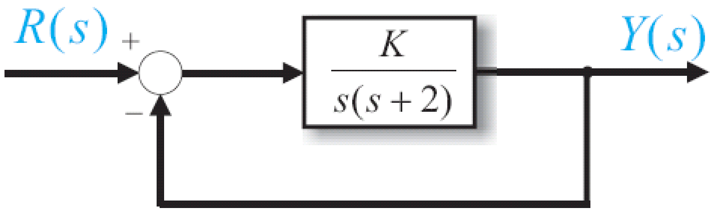
$$F(s)=1+G(s)H(s)=1+K_g\frac{P(s)}{Q(s)}=0$$

$$\frac{dF(s)}{ds}=K_g\frac{P'(s)Q(s)-P(s)Q'(s)}{Q^2(s)}=0$$

$F'(s)$是分离点的必要条件，不是充分条件。由(4.17)的两个条件确定（$K_g\geq 0$ 的）分离点。

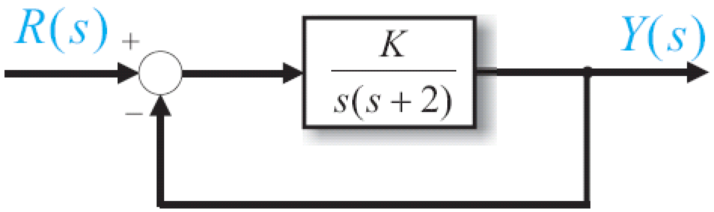
4.2 The root locus procedure

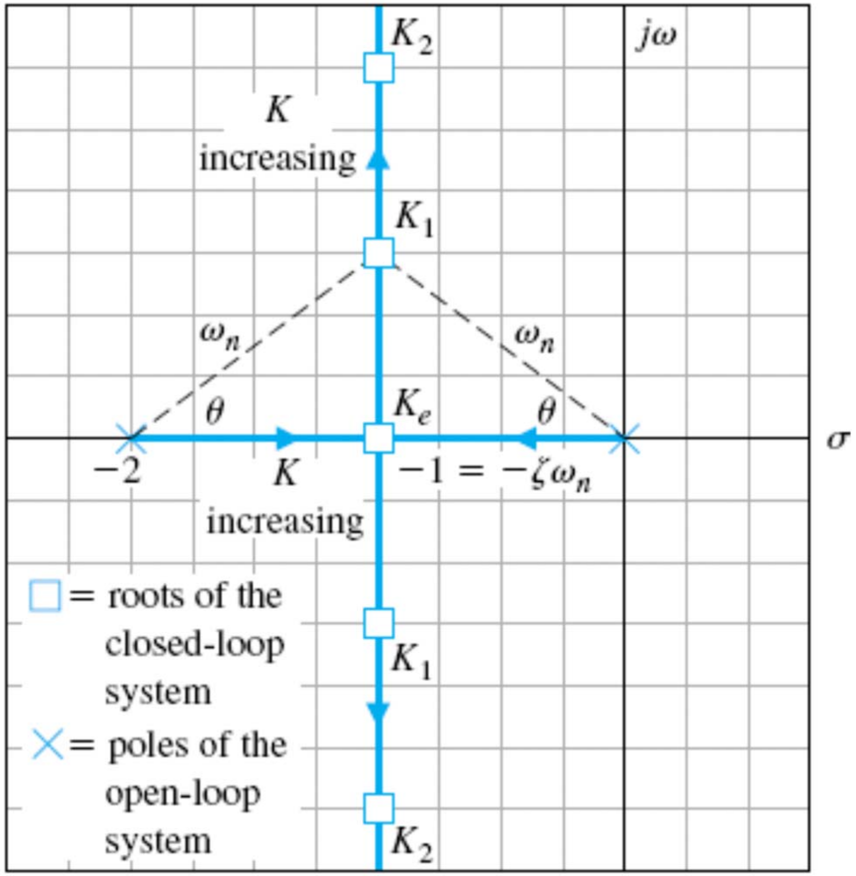
<E 4.1> Sketch the root locus ;



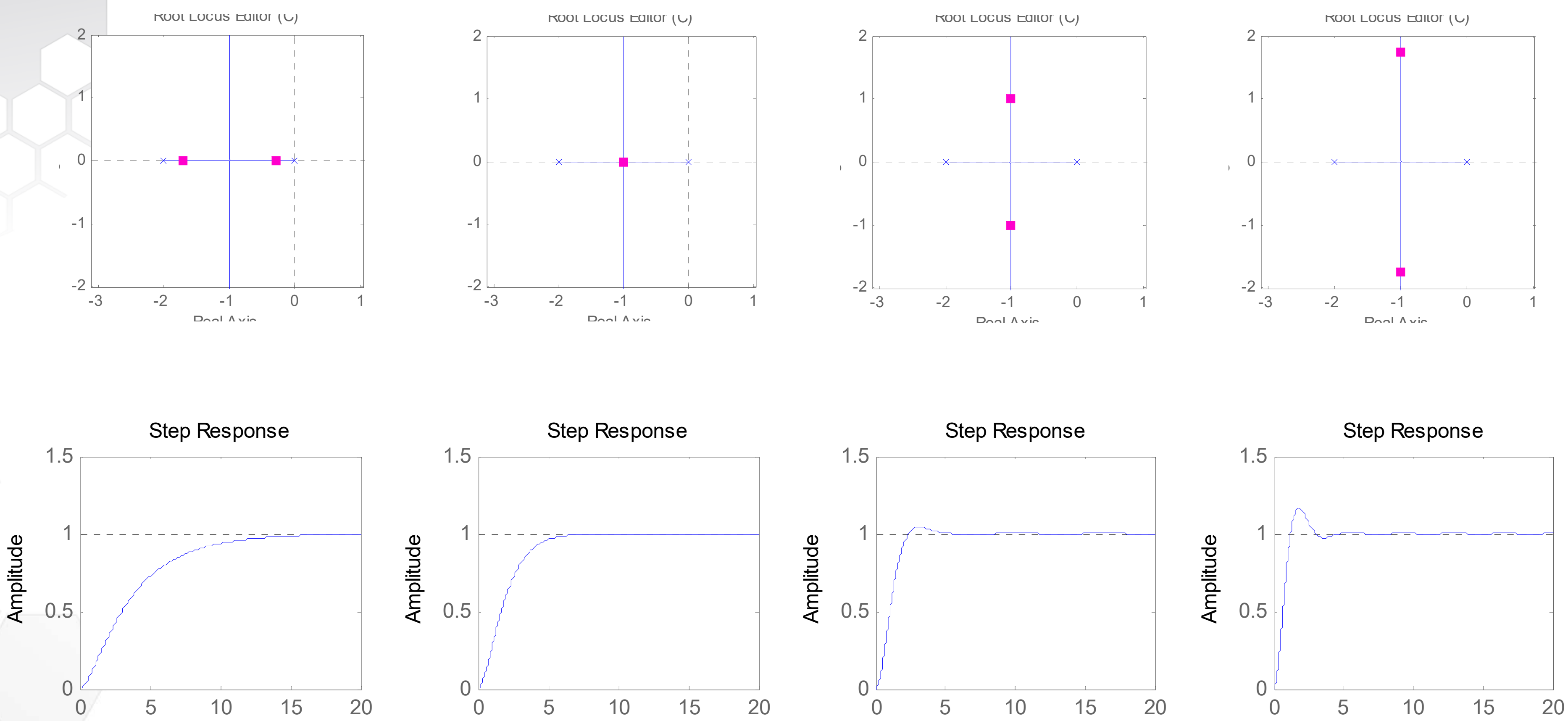
4.2 The root locus procedure

<E 4.1> Sketch the root locus ;





4.2 The root locus procedure



4.3 The root locus procedure

- **Rule 6**： The point at which the locus crosses the imaginary axis is critical stable point.

  根轨迹与虚轴的交点为闭环系统的临界稳定点。确定与虚轴交点和临界增益值的方法：

  a) Utilizing Routh Criterion Routh判据，确定临界稳定点；

  b) Utilizing characteristic equation 特征方程中，代入$s=j\omega$ 令实部和虚部分别等于0，解出与虚轴交点$\pm\omega$和临界增益值

$$s = j\omega$$

$$1 + G(j\omega)H(j\omega) = R(\omega) + jI(\omega) = 0$$

$$\begin{cases} R(\omega) = 0 \\ I(\omega) = 0 \end{cases} \Rightarrow \text{解出}\omega\text{和}K_g$$

4.3 The root locus procedure

- **Rule 7**： Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero . 根轨迹的出射角和入射角

根轨迹以开环复极点-p_{ol}出发的出射角为：

$$\theta_{p_{ol}} = \sum_{i=1}^{m} \angle(-p_{ol} + z_{oi}) - \sum_{j=1, j\neq l}^{n} \angle(-p_{ol} + p_{oj}) \pm 180^\circ(2k+1) \qquad (4.18)$$

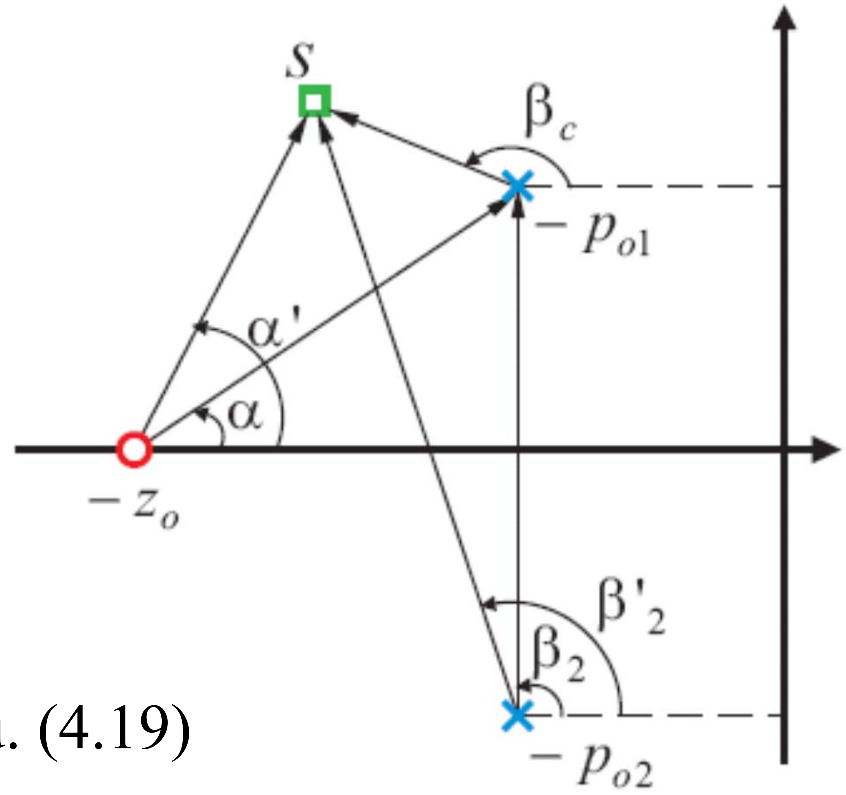
到达开环复零点-z_{ol}的入射角为：

$$\theta_{z_{ol}} = \sum_{j=1}^{n} \angle(-z_{ol} + p_{oj}) - \sum_{i=1, i\neq l}^{m} \angle(-z_{ol} + z_{oi}) \pm 180^\circ(2k+1) \qquad (4.19)$$

4.3 The root locus procedure

Determine the angle of departure of locus from $-p_{o1}$

要确定开环极点$-p_{o1}$处的出射角



Suppose s is on the root locus

$$\alpha' - (\beta_c + \beta_2') = (2k+1)\pi$$

$$|s + p_{o1}| \to 0(s \text{ close to} - p_{o1})$$

$$\Rightarrow \alpha' = \alpha, \quad \beta_2' = \beta_2$$

$$\therefore \beta_c = (2k+1)\pi + \alpha - \beta_2$$

Thus can get Equ.(4.18) and Equ. (4.19)

4.3 The root locus procedure

The angle of departure (or arrival) can be determined by:

Angle of departure:

$$\theta_{p_{ol}} = \beta_c = (2k+1)\pi - \sum_{j=1}^{n-1}\beta_j + \sum_{i=1}^{m}\alpha_i \qquad (4.20)$$

β_j：其余开环极点到被测极 点(起点)的矢量幅角；

α_i：开环有限零点到被测极 点(起点)的矢量幅角；

Angle of arrival:

$$\theta_{z_{ol}} = \alpha_r = (2k+1)\pi - \sum_{i=1}^{m-1}\alpha_i + \sum_{j=1}^{n}\beta_j \qquad (4.21)$$

α_i：其余开环有限零点到被 测零点(终点)的矢量幅角；

β_j：开环极点到被测零点 (终点)的矢量幅角；

4.3 The root locus procedure

- **Rule 8**： When n-m≥2 , the sum of closed-loop poles is a constant and it equals to the sum of open-loop poles.

当n-m≥2时，闭环极点之和为常数。因而，随着K_g的增大，一些根轨迹分支左行时，必有另一些根轨迹右行。

Since $$Q(s)+K_gP(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0$$

$$+K_g[s^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0]=0$$

When $n>m$,

$$Q(s)+K_gP(s)=\prod_{j=1}^{n}(s+p_j)=s^n+c_{n-1}s^{n-1}+\cdots+c_1s+c_0=0$$

where ，$-p_j$ is pole of transfer function， $c_{n-1}=\sum_{j=1}^{n}p_j$

4.3 The root locus procedure

When n-$m\geq 2$,

$$c_{n-1} = a_{n-1} = \sum_{j=1}^{n} p_{oj}$$

$$\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} p_{oj} = \text{constant, independent of } K_g$$

As K_g increase, some characteristic roots increase, the other characteristic decrease.

因此，随着K_g的增大，一些特征根增大，另一些特征根必减小。

4.3 The root locus procedure

- Using the information derived from these rules, the complete root locus sketch is obtained by filling in the sketch.

  根据这些规则，可确定根轨迹的一些特殊点。由这些特殊点可绘制出根轨迹的概略图。

- The other point of root locus can be derived from the angle criterion.

  根轨迹的其它点，根据幅角条件确定。

- In order to determine the root locus near the imaginary axis accurately, we can insert some points on the root near the imaginary axis by using the phase angle criterion.

  虚轴附近的根轨迹较为重要。可按需要补充一些点，以较精确地绘制出这部分根轨迹。

返回

4.4 Examples of root locus method

<E 4.2> Loop transfer function of a unit feedback system is:

$$G(s)=\frac{K}{s(s+1)(s+2)}, K>0$$

(1) Sketch the root locus;

(2) Find the value of K at the point where the domain roots with ζ=0.5.

Characteristic equation:

$$\frac{K}{s(s+1)(s+2)}=-1$$

Angle criterion:

$$\angle G(s)=\angle\left(\frac{K}{s(s+1)(s+2)}\right)=-\angle s-\angle(s+1)-\angle(s+2)=(2k+1)\pi$$

Magnitude criterion:

$$|G(s)|=\left|\frac{K}{s(s+1)(s+2)}\right|=1\Rightarrow |s(s+1)(s+2)|=K$$

4.4 Examples of root locus method

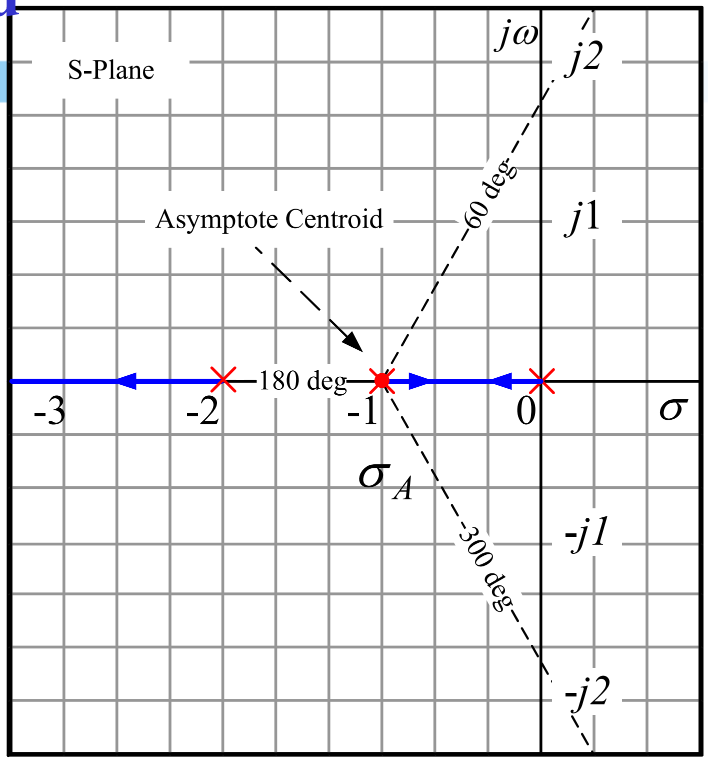
- Root loci on real axis
- Asymptotes

Angles of asymptotes:

$$\phi_A = \frac{2k+1}{n-m}\pi = \frac{2k+1}{3}\pi = \pm\frac{\pi}{3}, \pi$$

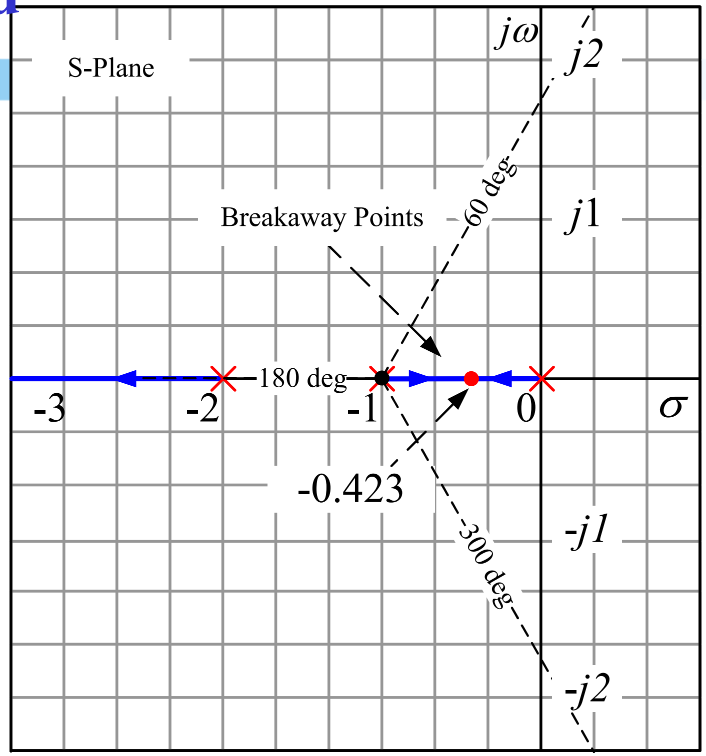
Center of asymptotes:

$$\sigma_A = -\frac{\sum p_{oj} - \sum z_{oi}}{n-m} = -1$$



4.4 Examples of root locus method

- Breakaway points



$$P(s)=1, P'(s)=0$$

$$Q(s)=s(s+1)(s+2)=s^3+3s^2+2s$$

$$Q'(s)=3s^2+6s+2$$

$$P(s)Q'(s)-P'(s)Q(s)=0$$

$$3s^2+6s+2=0$$

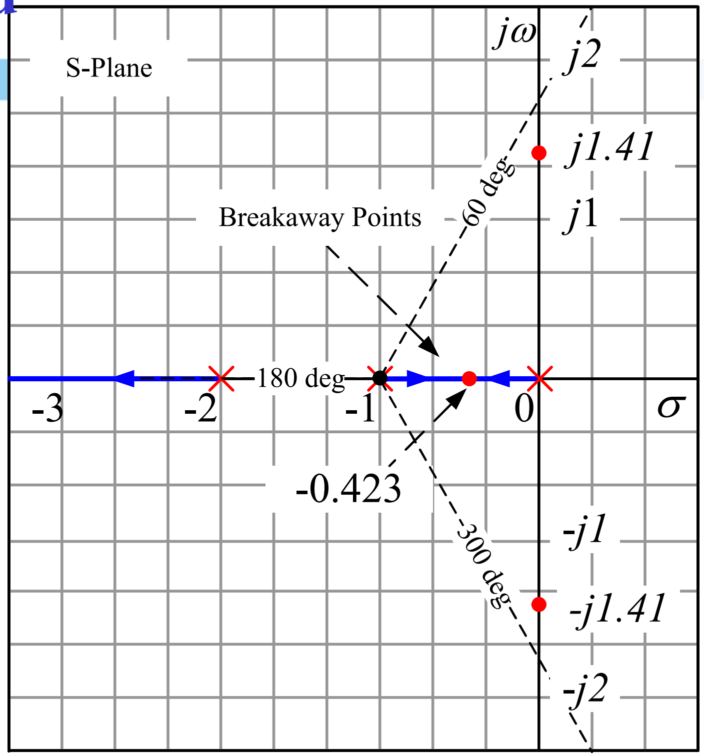
$$s_1=-0.423,\quad K=0.38,(\text{选取});$$

$$s_2=-1.577,\quad K=-0.38,(\text{舍去});$$

4.4 Examples of root locus method

- Points on the imaginary axis

$$s^3+3s^2+2s+K=0$$



Using Routh criterion,

$$\begin{array}{lcc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

6-K=0, thus K=6；Let auxiliary equation to 0

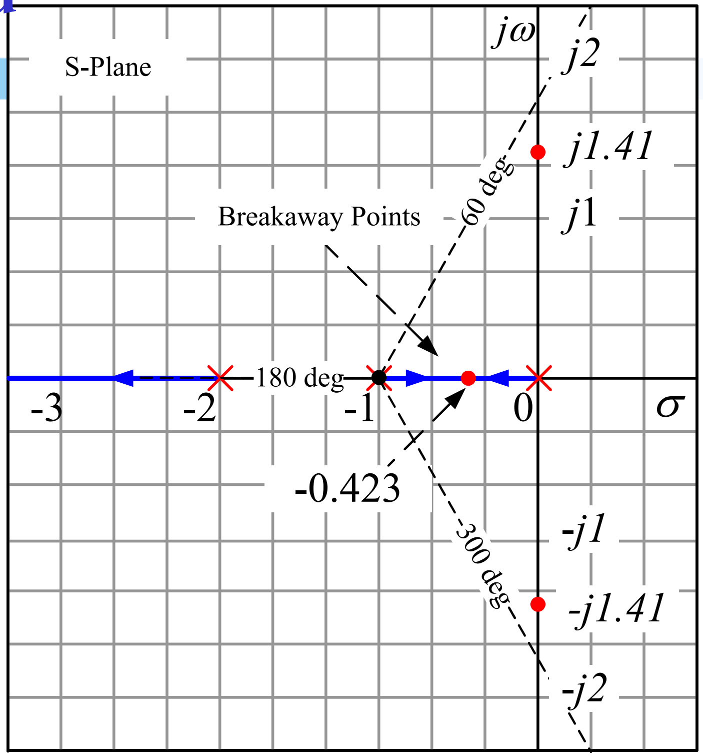
令辅助方程 $3s^2+K=0$

$$s^2=-\frac{K}{3}=-2, s=\pm j\sqrt{2}=\pm j1.414$$

(K=6时的另一个实根s=-3)

4.4 Examples of root locus method

- In order to determine the root locus near the imaginary axis accurately, we can insert some points on the root near the imaginary axis by using the phase angle criterion.



$$s = \sigma + j\omega$$

$$\theta(s) = tg^{-1}\frac{\omega}{1+\sigma} + tg^{-1}\frac{\omega}{2+\sigma} + 180° - tg^{-1}\left|\frac{\omega}{\sigma}\right|$$

$$= tg^{-1}\frac{\omega}{1+\sigma} + tg^{-1}\frac{\omega}{2+\sigma} + tg^{-1}\frac{\omega}{\sigma} + 180°, -1 < \sigma < 0$$

$s = -0.4 + j0.5 \quad \theta(s) = 196.2° \quad s = -0.355 + j0.5 \quad \theta(s) = 180.06°$

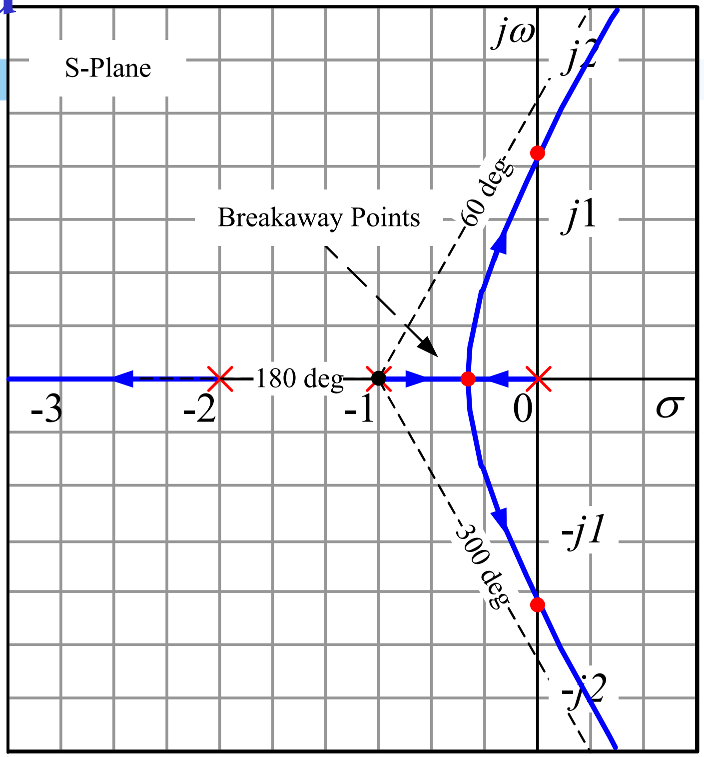
$s = -0.35 + j0.5 \quad \theta(s) = 179.4° \quad s = -0.354 + j0.5 \quad \theta(s) = 179.94°$

$s = -0.36 + j0.5 \quad \theta(s) = 180.7°$

4.4 Examples of root locus method

- Sketch the root locus;

- Determine the domain roots with ζ=0.5 确定ζ=0.5时的共轭复数主导极点



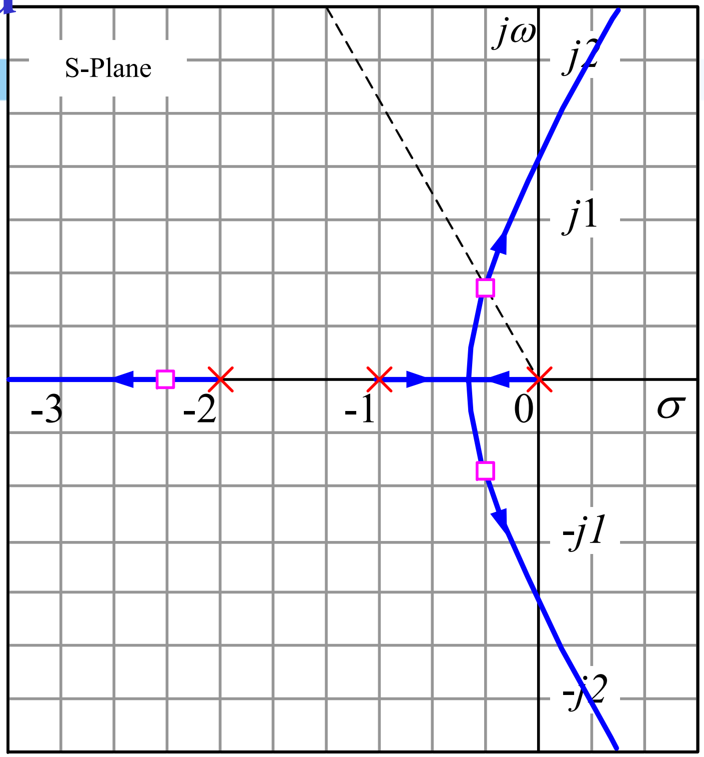
The damping angle is:

主导极点处向量与负实轴夹角：

$$\varphi = \cos^{-1}\frac{\omega_n\zeta}{\omega_n} = \cos^{-1}\zeta$$

$$= \cos^{-1}0.5 = 60°$$

4.4 Examples of root locus method



From root locus, determine the location of domain roots.

由图中读出，这时闭环主导极点为:

$$s_{1,2} = -0.33 + j0.58$$

The value of K

对应的K值:

$$K = \left| s(s+1)(s+2) \right|_{s=-0.33+j0.58}$$

$$= 1.0454$$

The position of the third pole with K=1.0454

对应的第三个极点位置:

$$s_3 = -2.34$$

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4.4 Examples of root locus method

<E 4.3> Characteristic equation of a system is

$$s(s+4)(s^2+4s+20)+K=0$$

Sketch the root locus of the system.

Equivalent characteristic equation 等效特征方程：

$$1+G(s)H(s)=1+\frac{K}{s(s+4)(s^2+4s+20)}=1+K\frac{P(s)}{Q(s)}$$

Equivalent loop transfer function 等效开环传递函数：

$$G(s)H(s)=\frac{K}{s(s+4)(s^2+4s+20)}=\frac{K}{s(s+4)[(s+2)^2+16]}$$

$$-p_{o1}=0,-p_{o2}=-4,-p_{o3,4}=-2\pm j4$$

4.4 Examples of root locus method

- Root loci on real axis
- Breakaway point

$$P(s)=1, P'(s)=0$$

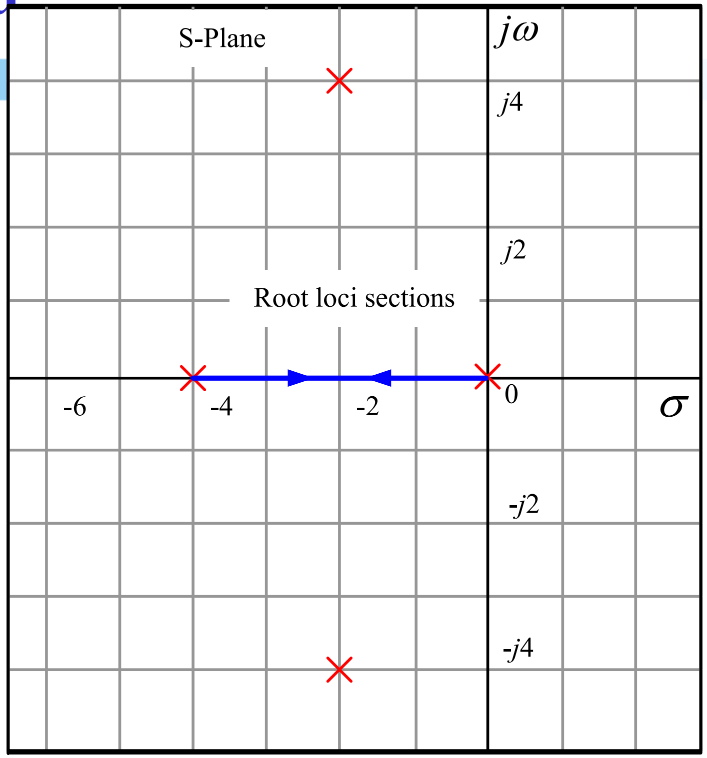
$$Q(s)=s(s+4)(s^2+4s+20)$$

$$=s^4+8s^3+36s^2+80s$$

$$Q'(s)=4(s^3+6s^2+18s+20)$$

$$P(s)Q'(s)-P'(s)Q(s)=0$$

$$F(s)=s^3+6s^2+18s+20=0$$



4.4 Examples of root locus method

Roots of the equation $F(s)=0$:

$$F(s)=20+s(18+s(6+s))$$

$$F(-2.5)=-3.125,\quad F(-1.5)=3.125,\quad F(-2)=0$$

$$s=-2;$$

$$F(s)=(s+2)(s^2+4s+10)=(s+2)[(s+2)^2+6]=0$$

$$s_1=-2,\quad s_{2,3}=-2\pm j\sqrt{6}=-2\pm j2.45$$

Examination with characteristic equation

$$s_1=-2,\quad K=64>0;$$

$$s_{2,3}=-2\pm j\sqrt{6}=-2\pm j2.45,\quad K=100>0.$$

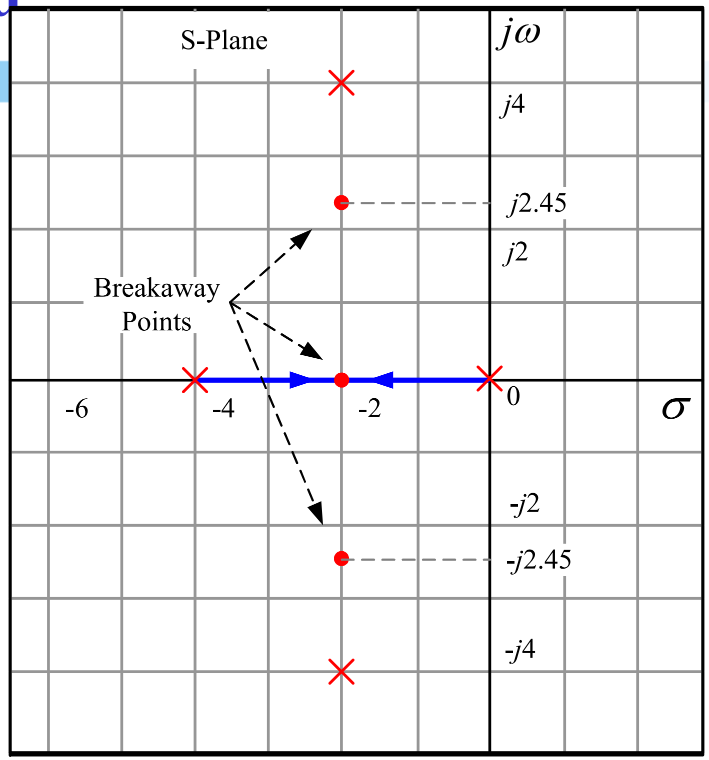
($s_1, s_{2,3}$ breakaway points)

4.4 Examples of root locus method

- Breakaway points

$$s_1 = -2, \quad K = 64;$$

$$s_{2,3} = -2 \pm j2.45, \quad K = 100.$$



4.4 Examples of root locus method

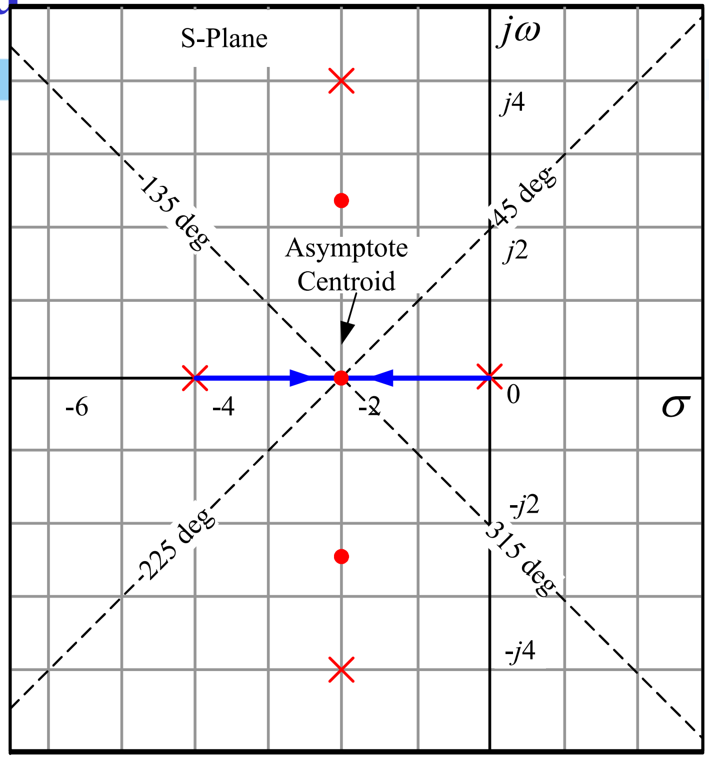
- Asymptotes

Asymptotes angles:

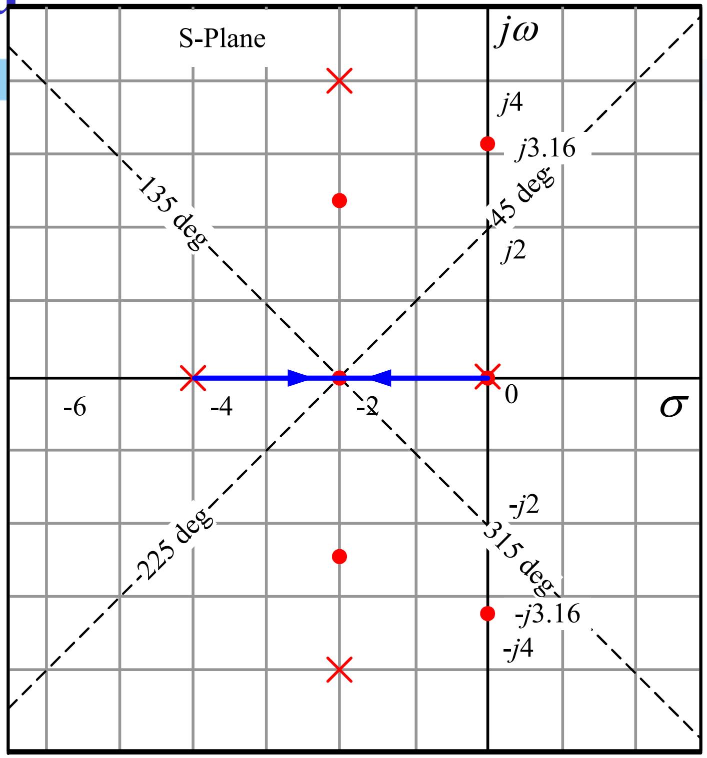
$$\phi_A = \frac{2k+1}{n-m}\pi = \frac{2k+1}{4}\pi = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

Asymptotes center:

$$\sigma_A = -\frac{\sum p_{oj} - \sum z_{oi}}{n-m} = -2$$



4.4 Examples of root locus method



- Points on the imaginary axis

Characteristic equation:

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

Utilizing characteristic equation,
let $s = j\omega$

$$\begin{cases} \omega^4 - 36\omega^2 + K = 0 \\ -8\omega^3 + 80\omega = 0 \end{cases}$$

$$\omega = \pm\sqrt{10} = \pm 3.16 \qquad K = 260;$$

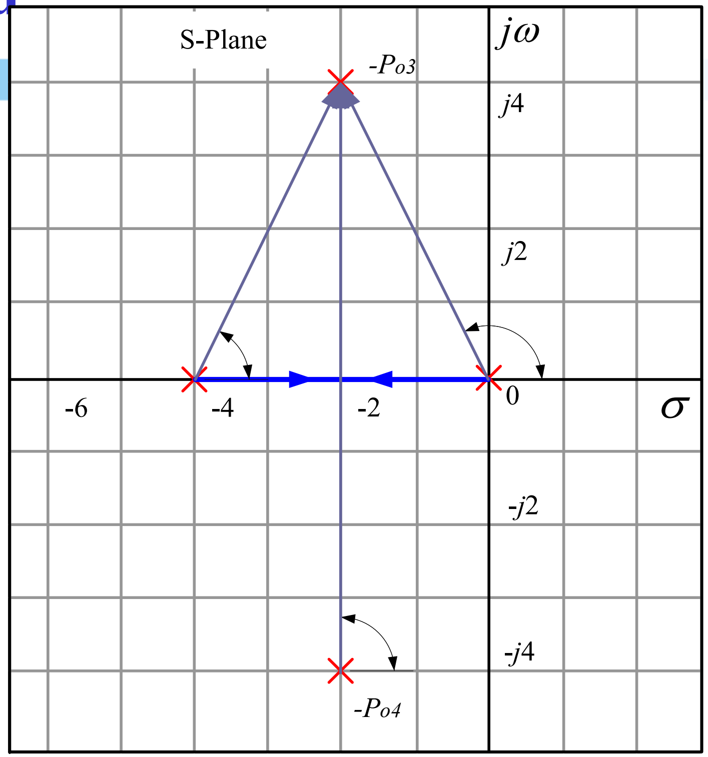
$$\omega = 0 \qquad K = 0;$$

(where $-p_{o1}$ the ploe of open-loop transfer function lies)

4.4 Examples of root locus method

- Angle of departure

$$\theta_{p_{o3}} = -[\angle(-p_{o3}) + \angle(-p_{o3}+4) + \angle(-p_{o3}+p_{o4})] \pm 180°(2k+1)$$
$$= \pm 180° - \angle(-2+j4) - \angle(2+j4) - \angle(j8)$$
$$= \pm 180° - 180° - 90° = -90°$$



4.4 Examples of root locus method

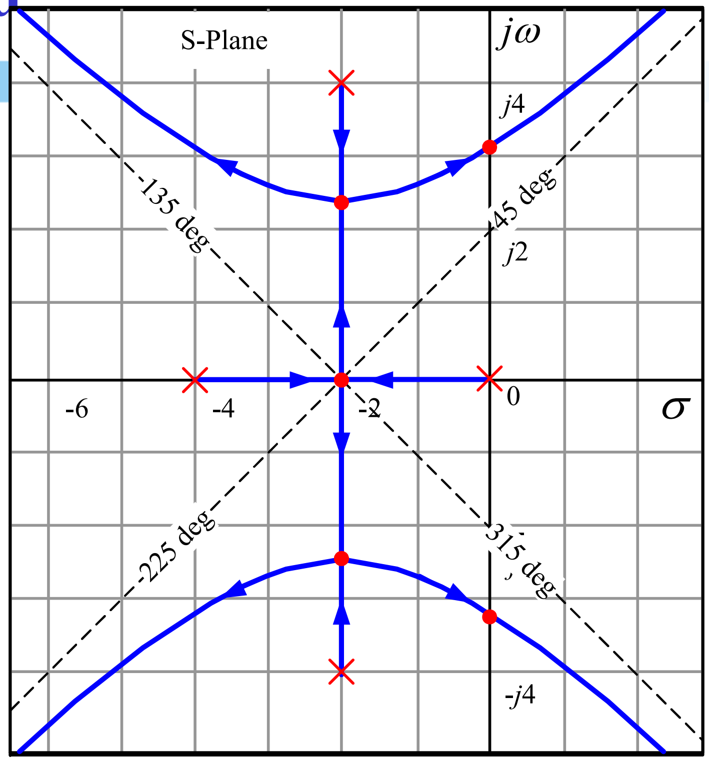
- 该根轨迹对称于过(-2,0)点的垂线：

由特征方程

$$d(s)=s^4+8s^3+36s^2+80s+K$$

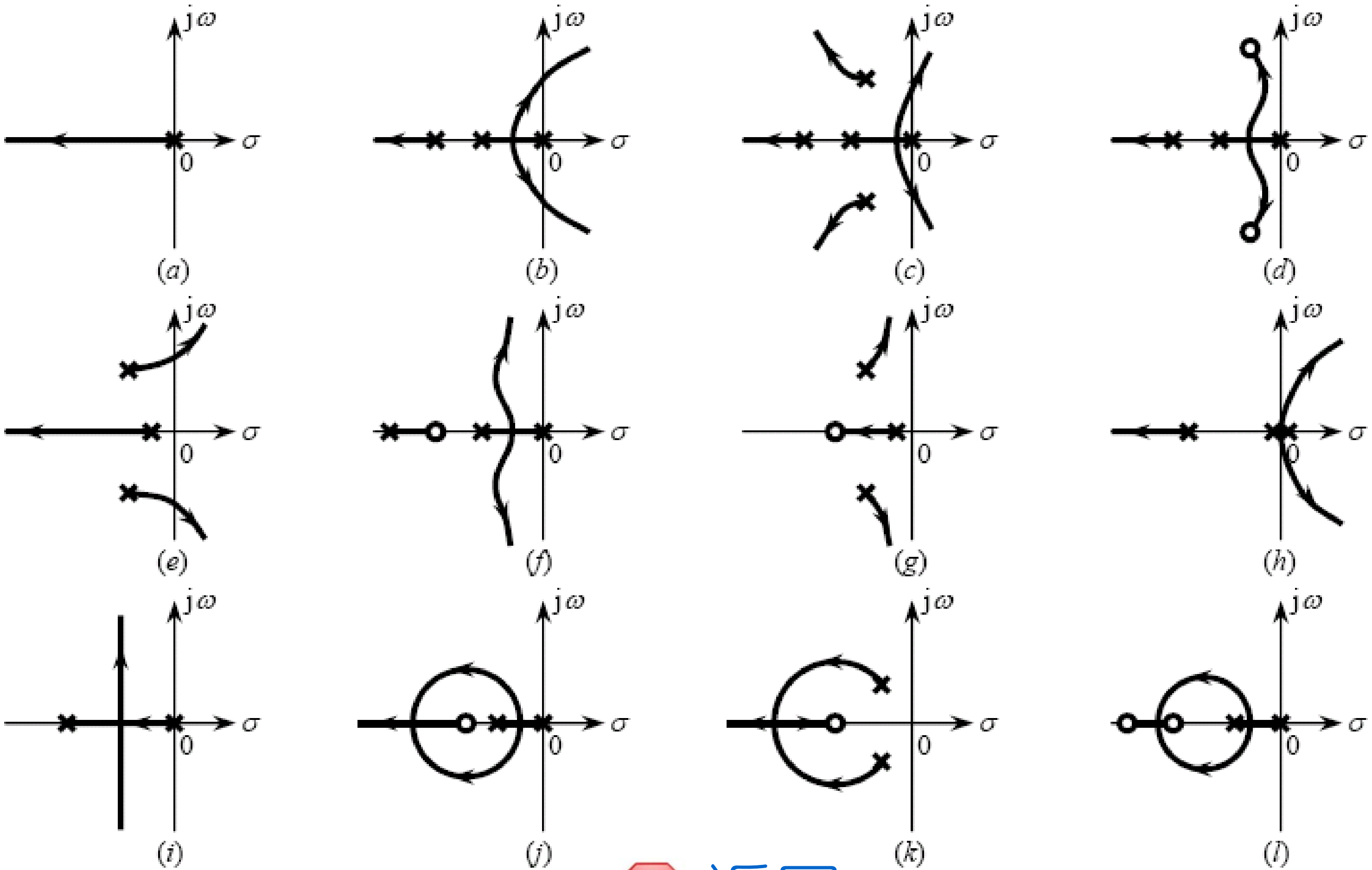
$$=[(s+2)^2-4][(s+2)^2+16]+K=0$$

可见



4.4 Examples of root locus method

- Root locus examples



返回

4.5 Parameter root locus

Parameter root locus : is the path of the roots of the characteristic equation traced out in the s-plane as other parameter is changed.

非增益参数变化的根轨迹(参数根轨迹)：是指系统中某一非增益参数从零变化到无穷时，特征方程的根在s平面上的变化轨迹

The first step is to construct an equivalent open-loop transfer function as following. 构造一等效的开环传递函数.

If the parameter is X, rewrite the characteristic equation as:

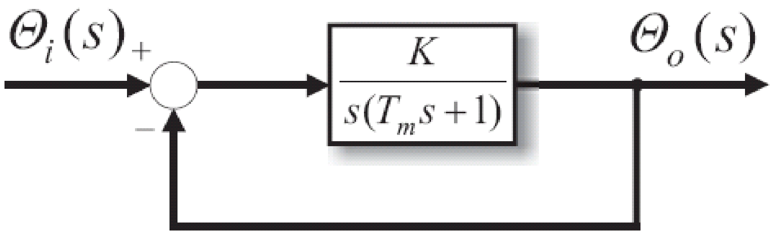
$$A(s) + XB(s) = 0 \qquad (4.22)$$

then transform the equation as:

$$1 + X\frac{B(s)}{A(s)} = 0 \quad \text{or} \quad X\frac{B(s)}{A(s)} = -1 \qquad (4.23)$$

4.5 Parameter root locus

<E4.4>: A unit feedback system is as following:

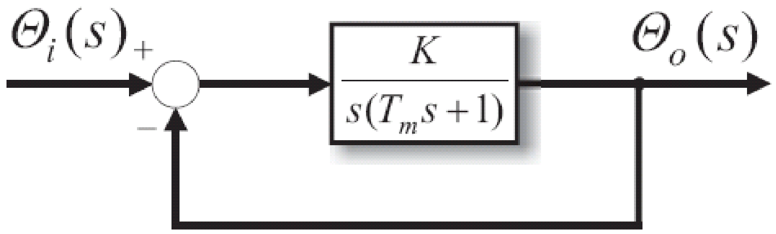


(1) Determine the roots distribution and the change of transient performance when the time constant T_m is varying.

(2) Determine the value of T_m, when K=29 , $\theta_i(t)=t$ and closed-loop poles is at $-p_{1,2}=-17.25\pm j26.521$.

(3) Calculate the performance specifications.

4.5 Parameter root locus



Solution:

(1) Characteristic equations:

$$T_m s^2 + s + K = 0$$

Rewrite as:

$$T_m \frac{s^2}{s+K} + 1 = 0$$

or

$$\frac{s^2}{s+K} = -\frac{1}{T_m}$$

Sketch the root locus by considering T_m ($0 \le T_m < \infty$) as the variable parameter.

作以T_m ($0\le T_m <\infty$)为变参数的根轨迹。

4.5 Parameter root locus

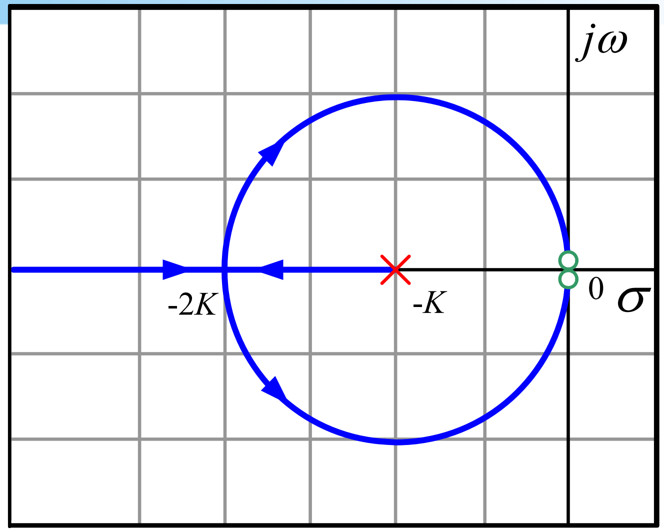
- Breakaway point

$$P(s)=s^2, P'(s)=2s$$

$$Q(s)=s+K, Q'(s)=1$$

$$P(s)Q'(s)-P'(s)Q(s)=0$$

$$s=-2K$$



The root locus of this system which is not on the real axis is a circle with diameter equals $2K$.

根轨迹是一个直径为 $2K$ 的圆 (实轴外)。

(考虑利用幅角条件证明根轨迹形状)

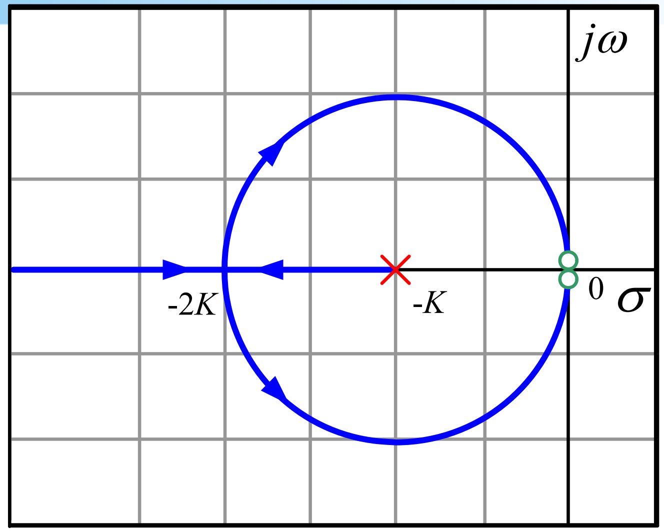
4.5 Parameter root locus

- Transient analysis:

  Breakaway point $s = -2K$,

  where T_m is:

$$T_m = \left|\frac{s+K}{s^2}\right|_{s=-2K} = \frac{1}{4K}$$



From the root locus :

When $0< T_m <\infty$, closed-loop system is stable ;

When $0< T_m <1/(4K)$, over-ramped system 过阻尼系统 ;

When $1/(4K) < T_m < \infty$, Under-damped system 欠阻尼系统

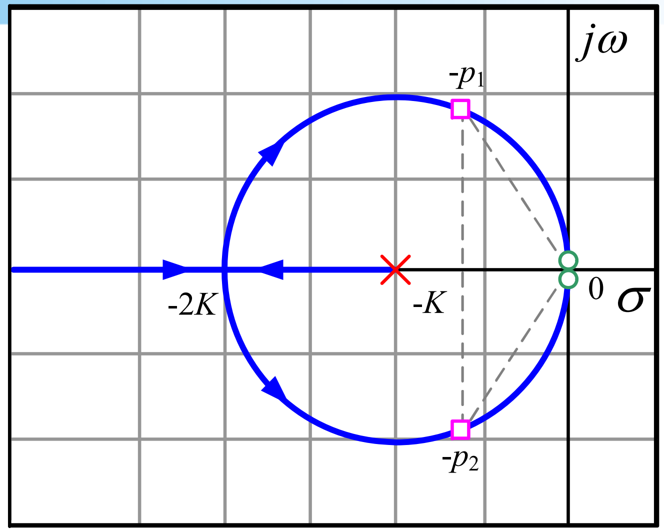
As T_m increase, ζ decrease ;

When $T_m=1/(4K)$, Critical damped system 临界阻尼系统 ;

4.5 Parameter root locus

(2) When K=29, poles

$-p_{1,2}=-17.25\pm j26.521$



$$T_m=\left|\frac{s+K}{s^2}\right|_{s=-p_1}=\frac{|11.75+j26.521|}{|-17.25+j26.521|^2}$$

$$=\frac{29.007}{1000.926}=\frac{1}{34.506}$$

Closed-loop transfer function

$$\frac{\Theta_o(s)}{\Theta_i(s)}=\frac{K/T_m}{s^2+s/T_m+K/T_m}=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$$

$$\omega_n=\sqrt{K/T_m}=31.633rad/s \qquad \zeta=\frac{1}{2\omega_n T_m}=\frac{1}{2\sqrt{T_m K}}=0.545$$

4.5 Parameter root locus

Performance specifications :

$$\sigma\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% = 12.95\%$$

$$T_s = \frac{4}{\zeta\omega_n} = 0.232(s)$$

$$T_r = \frac{\pi - \varphi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}} = 0.081(s)$$

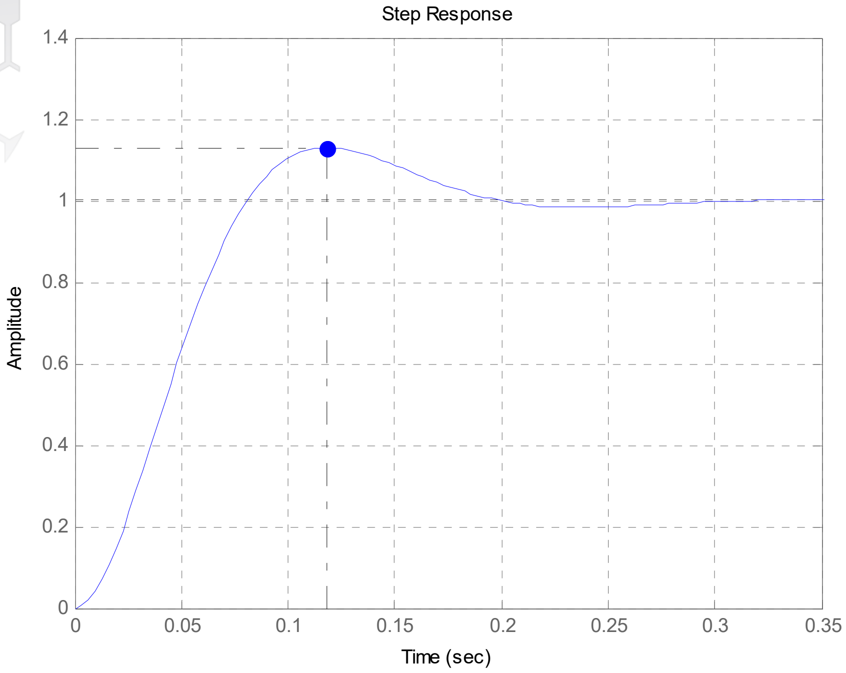
Steady state error : Type one system, with ramp input

$$K_v = K = 29,$$

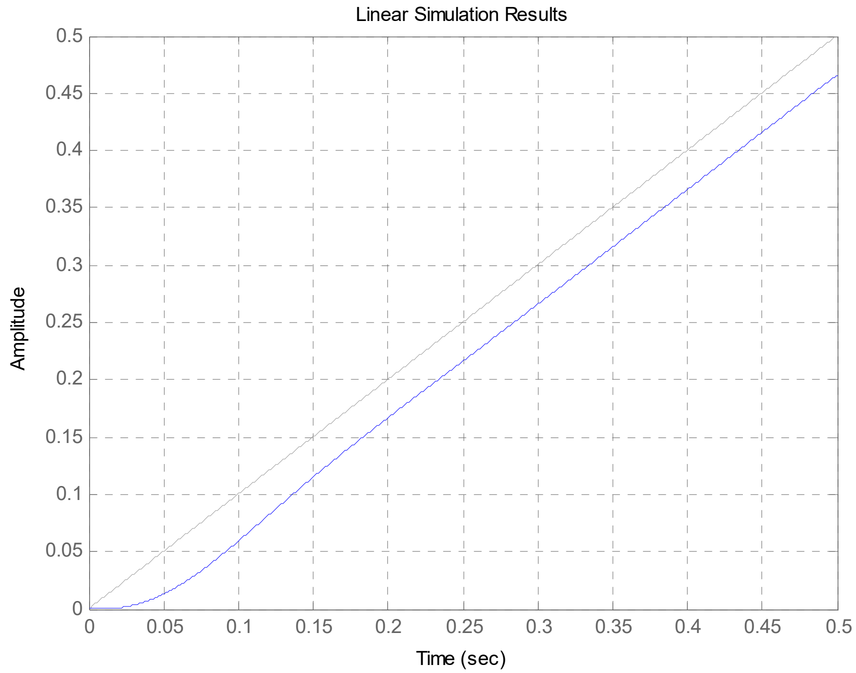
$$e_{ss} = 1/29 = 0.034$$

4.5 Parameter root locus

Unit step input response



Unit ramp input response



用根轨迹分析自动控制系统的方法和步骤：

(1)根据系统的开环传递函数和绘制根轨迹的基本规则绘制出系统的根轨迹图。

(2)由根轨迹在s平面上的分布情况分析系统的稳定性。如果全部根轨迹都位于s平面左半部，则说明无论开环根轨迹增益K_g为何值，系统都是稳定的；如根轨迹有一条（或一条以上）的分支全部位于s平面的右半部，则说明无论开环根轨迹增益K_g如何改变，系统都是不稳定的；如果有一条（或一条以上）的根轨迹从s平面的左半部穿过虚轴进入s面的右半部（或反之），而其余的根轨迹分支位于s平面的左半部，则说明系统是有条件的稳定系统，即当开环

用根轨迹分析自动控制系统的方法和步骤：

根轨迹增益K_g大于临界值时系统便由稳定变为不稳定（或反之）。此时，关键是求出开环根轨迹增益K_g的临界值。这为分析和设计系统的稳定性提供了选择合适系统参数的依据和途径。

(3)根据对系统的要求和系统的根轨迹图分析系统的瞬态响应指标。对于一阶、二阶系统，很容易在它的根轨迹上确定对应参数的闭环极点，对于三阶以上的高阶系统，通常用简单的作图法 （如作等阻尼比线等）求出系统的主导极点（如果存在的话），将高阶系统近似地简化成由主导极点（通常是一对共轭复数极点）构成的二阶系统，最后求出其各项性能指标。这种分析方法简单、方便、直观，在

用根轨迹分析自动控制系统的方法和步骤：

满足主导极点条件时，分析结果的误差很小。如果求出离虚轴较近的一对共轭复数极点不满足主导极点的条件，如它到虚轴的距离不小于其余极点到虚轴距离的五分之一或在它的附近有闭环零点存在等，这时还必须进一步考虑和分析这些闭环零、极点对系统瞬态响应性能指标的影响。

返回

Summary

- 本章介绍了根轨迹法
- 要求掌握根轨迹绘制规则，能熟练地绘制根轨迹概略图
- 注意应用根轨迹绘制规则时，将系统特征方程写成如下形式：

$$K_g \frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})} = -1$$

Summary

- 根轨迹是用较简单的等效开环传递函数，来分析研究闭环系统的特性；它是一种几何方法，形象直观；
- 根轨迹是以开环传递函数中的某个参数（一般是根轨迹增益）为参变量而画出的闭环特征方程式的根轨迹图。根据系统开环零、极点在s平面上的分布，按照绘制规则，就能方便的画出根轨迹的大致形状；

Summary

- 根轨迹图不仅使我们能直观的看到参数的变化对系统性能的影响，而且还可以用它求出指定参变量或指定阻尼比相对应的闭环极点。根据确定的闭环极点和已知的闭环零点，就能计算出系统的输出响应及其性能指标，从而避免了求解高阶微分方程的麻烦。

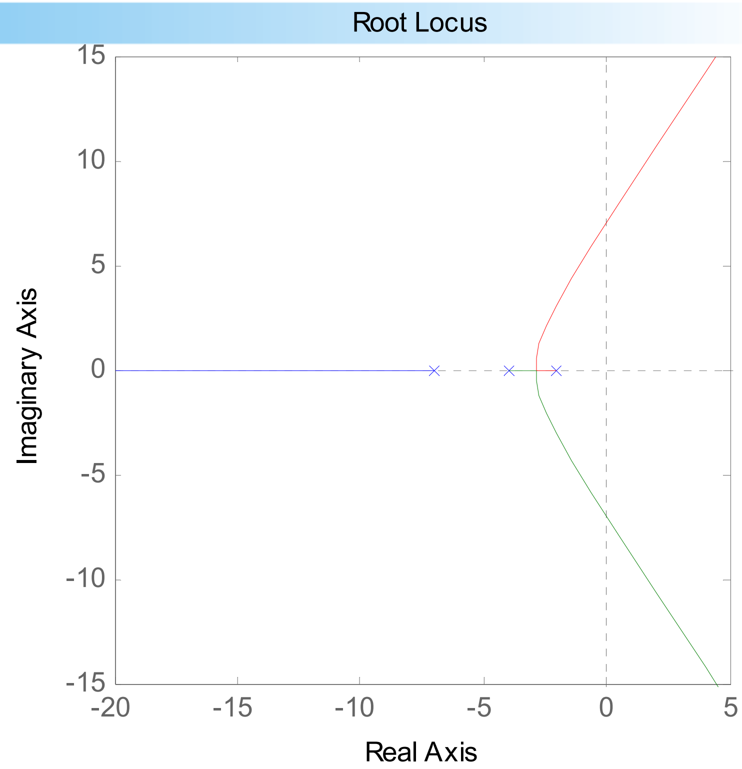
返回

Matlab绘制根轨迹和对系统分析

- Matlab绘制根轨迹函数
  - rlocus
  - pzmap
  - rlocfind
  - sgrid
  - rltool

rlocus

- Syntax
  - rlocus(sys)
  - rlocus(sys,k)
  - rlocus(sys1,sys2,...)
  - [r,k] = rlocus(sys)
  - r = rlocus(sys,k)

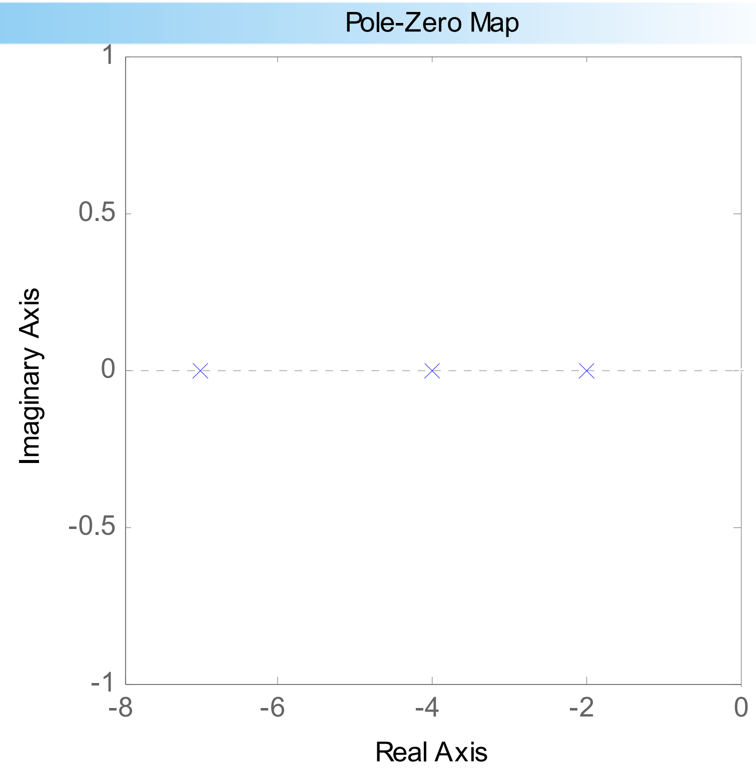


- Description

rlocus 求系统的根轨迹,绘制系统根轨迹,求解系统在特定开环增益k时所对应的闭环极点位置

pzmap

- Syntax
  - pzmap(sys)
  - [p,z] = pzmap(sys)

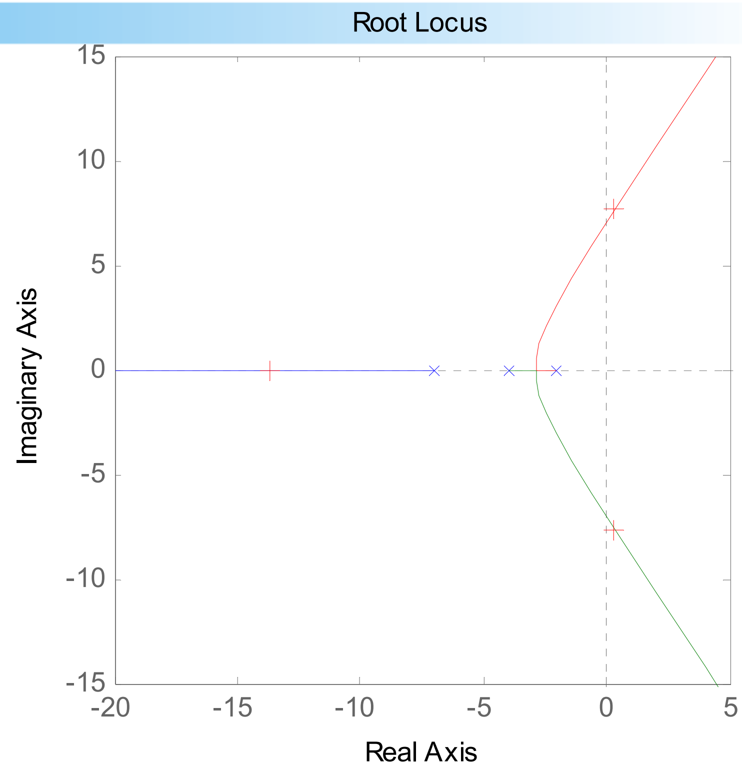


- Description

  绘制系统零极点分布图，并且返回系统零极点坐标位置；

rlocfind

- Syntax
  - [k,r] = rlocfind(sys)

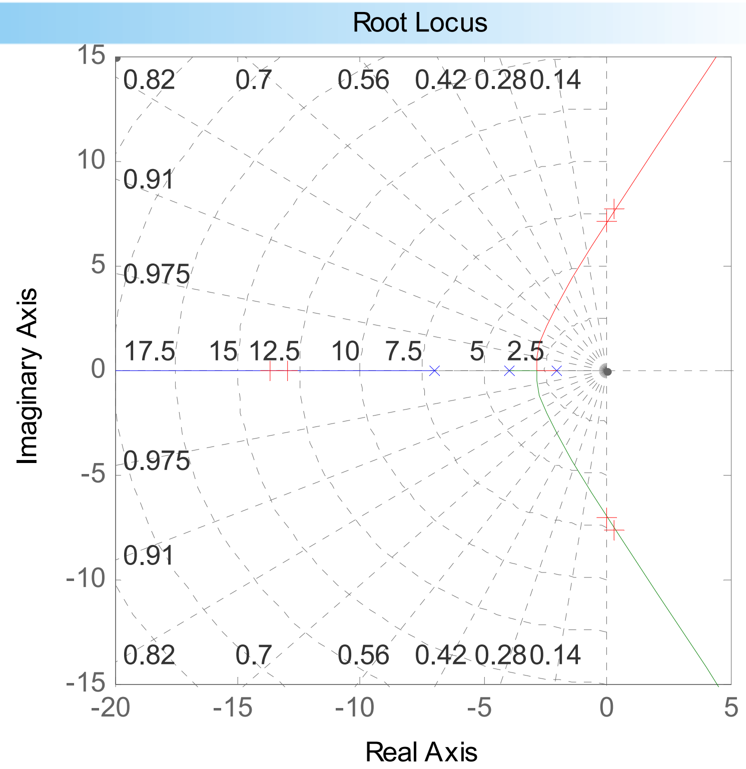


- Description

求取根轨迹上指定点的开环增益k,以及在该增益下系统的所有闭环极点；

Sgrid

- Syntax
  - sgrid
  - sgrid(z,wn)



- Description

在连续系统根轨迹图或零极点分布图中，绘制由等阻尼比线和等自然频率线所构成的网络

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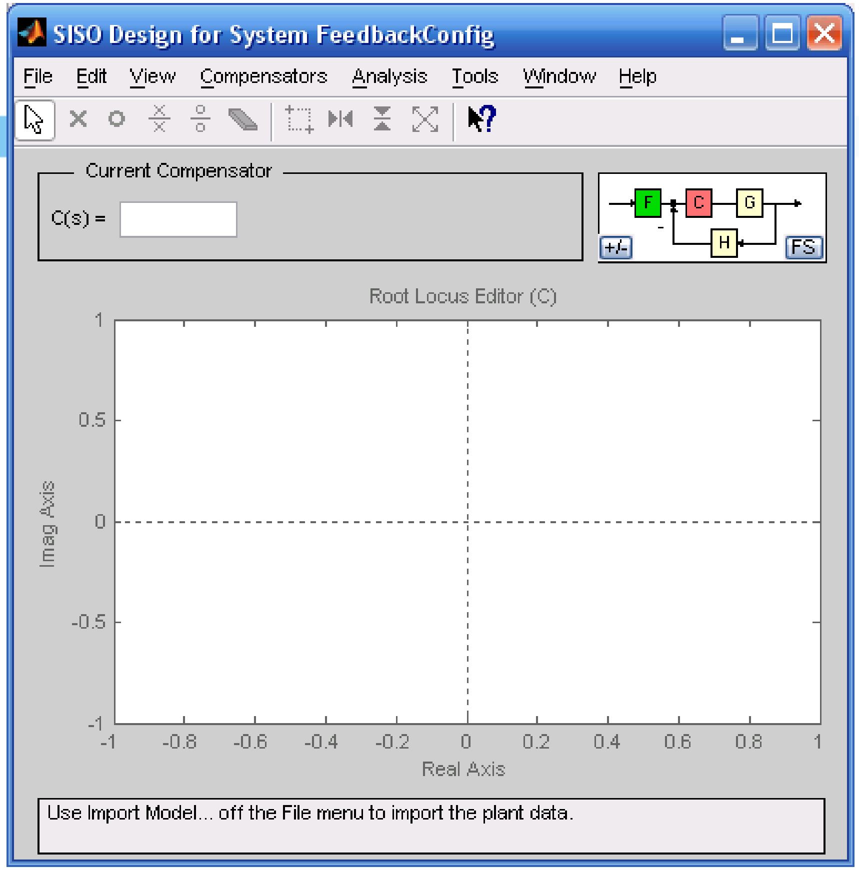
rltool

- Syntax
  - rltool
  - rltool(sys)
  - rltool(sys,comp)

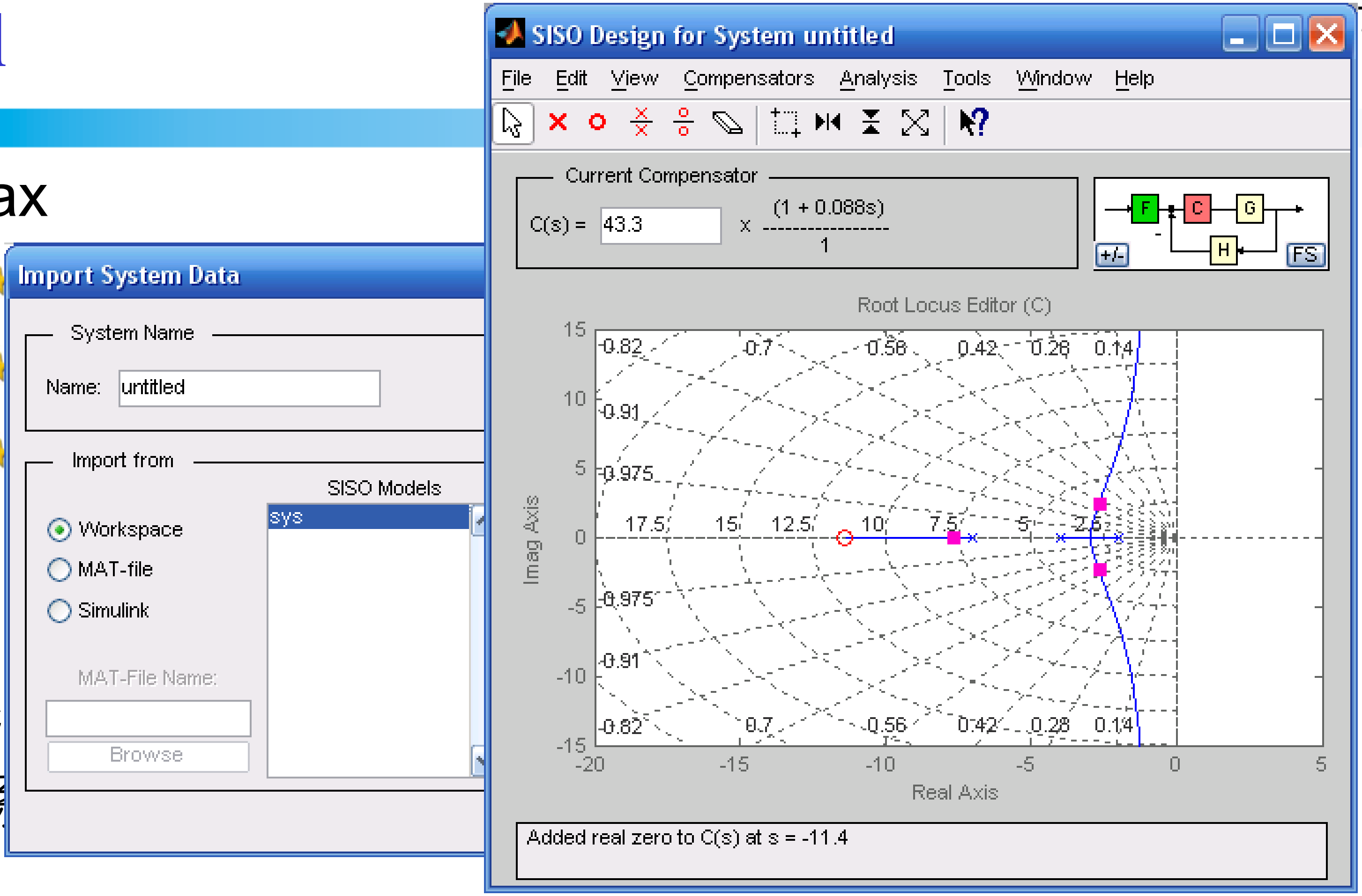


- Description

  激活系统的根轨迹设计工具

Rltool

- Syntax

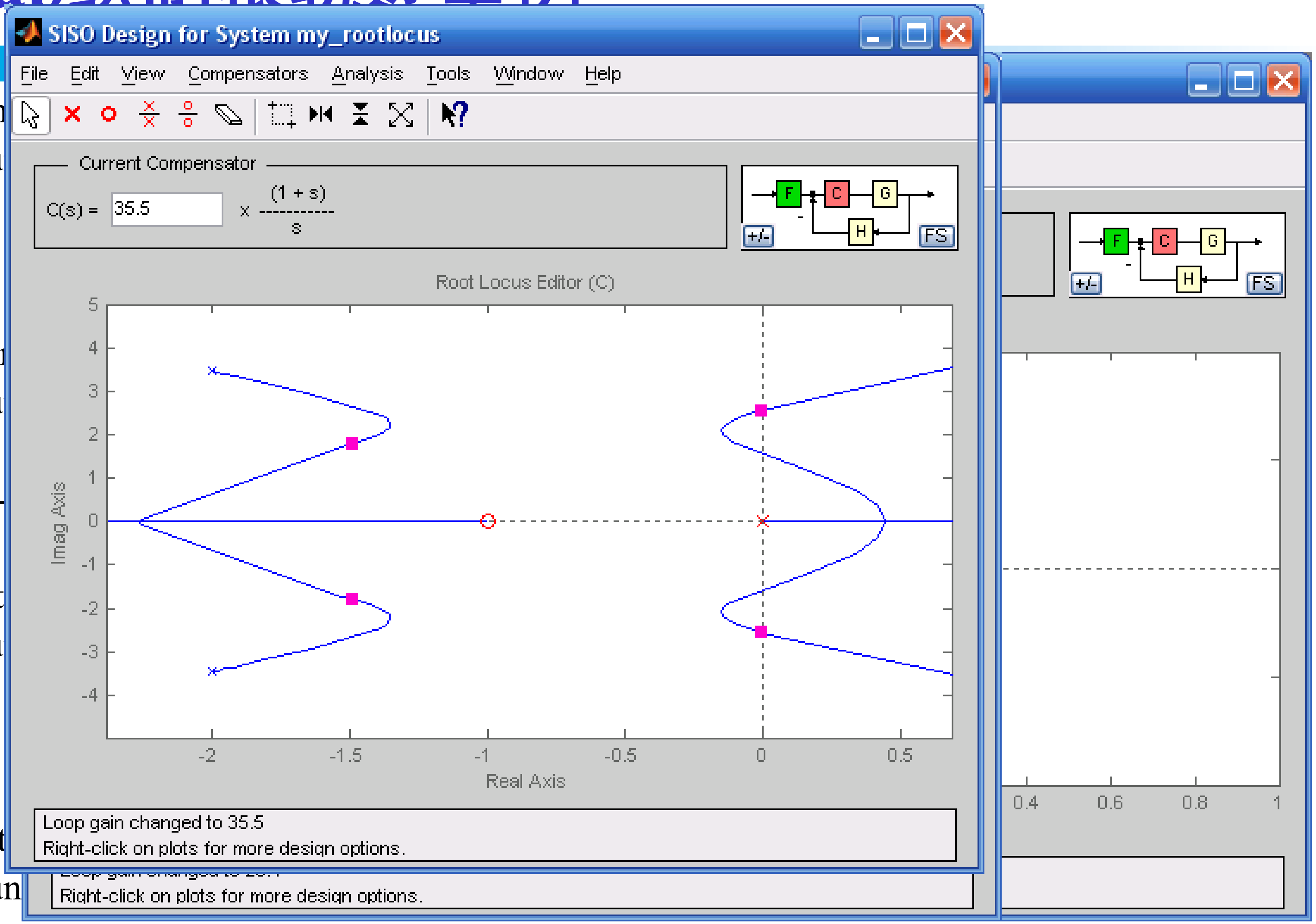


- Desc

激活

Matlab绘制根轨迹举例

>> sys_con
Transfer fu
s + 1
------
s
>> sys_plai
Transfer fu
1
----
s^2 + 4 s +
>> sys_h=t
Transfer fu
1
-----
s - 1
>> sys_f=t
Transfer fun
1



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