自动控制原理 Automatic Control Theory

西南交通大学
电气工程学院

Chapter 5 Frequency domain analysis of control systems 控制系统频率域分析

5.1 Frequency Characteristic

Frequency Response: The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady state; it differs from the input waveform only in **amplitude** and **phase angle**.

频率响应—系统的频率响应定义为系统对正弦输入信号的稳态响应。在这种情况下，系统的输入信号是正弦信号，系统的内部信号以及系统的输出信号也都是稳态的正弦信号，这些信号频率相同，幅值和相角则各有不同。

5.1 Frequency Characteristic

- The **advantages** of the frequency response method

  频率响应法的优点:

1) The experimental determination of the frequency response of a system is easy and reliable.

   易于试验和测量,可用试验方法测量出系统的频率特性

2) Frequency response can be used for the stability analysis of the system (Nyquist Criterion).

   可用于系统的稳定性分析(应用Nyquist稳定性判据)

3) The magnitude and phase angle of $T(j\omega)$ can be represented by the graphical plots that provide significant insight into the analysis and design of control system.

   是一种图解法，形象直观揭示系统的内涵

5.1 Frequency Characteristic

- The **disadvantage** of the frequency response method

  频率响应法的缺点:

  The indirect link between the frequency domain and time domain, only for LTI system.

  频率域和时间域之间没有直接的联系，且仅适用于LTI系统

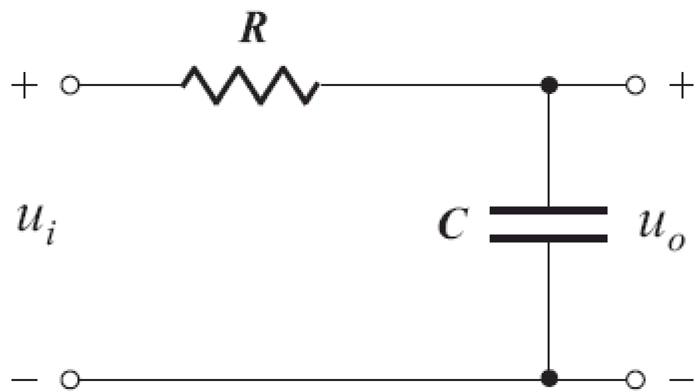
<Example> Analysis the frequency response of the *RC* filter

分析下图RC滤波电路频率响应

Input

输入

$$u_i(t) = U_m \sin \omega t$$



5.1 Frequency Characteristic

Output
输出：
$$\dot{U}_o = \dot{U}_i \frac{1/j\omega C}{R+1/j\omega C} = \frac{\dot{U}_i}{1+j\omega RC} = \frac{\dot{U}_i}{\sqrt{1+(\omega RC)^2}} \angle \varphi$$

where $\varphi = -tg^{-1}\omega RC$

$$\frac{\dot{U}_o}{\dot{U}_i} = \frac{1}{1+j\omega RC} = G(j\omega) = A(\omega)e^{j\varphi(\omega)}$$

Magnitude 辐值 $$\left|\frac{\dot{U}_o}{\dot{U}_i}\right| = A(\omega) = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

Phase angle 相角 $$\varphi(\omega) = -tg^{-1}\omega RC$$

The magnitude and the phase angle are the function of the input frequency ω

5.1 Frequency Characteristic

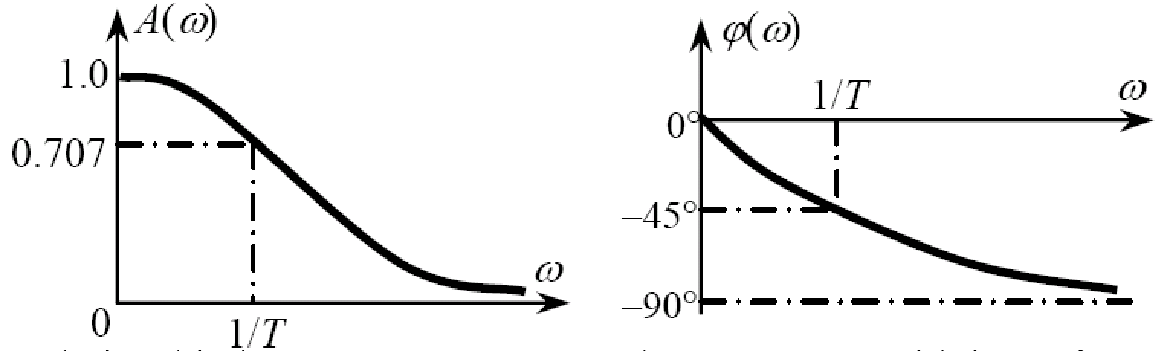


Fig. Relationship between **amplitude** and **phase angle** with input frequency

幅值、相位与频率关系图

Frequency Characteristic can be seen from the figure that:

由图可以得到频率特性如下:

As $\boldsymbol{\omega}$ ***increase*** ↑ $\begin{cases} \textbf{Amplitude}\ \textit{decrease} & \text{增益下降} \\ \textbf{Phase lag}\ \textit{increase}\ |\varphi| & \text{滞后增大} \end{cases}$

$\omega, R, C: \begin{cases} T=RC & \text{Time constant, system parameter 时间常数，系统参数} \\ \omega & \text{Frequency of input sinusoid signal 输入正弦信号的频率} \end{cases}$

5.1 Frequency Characteristic

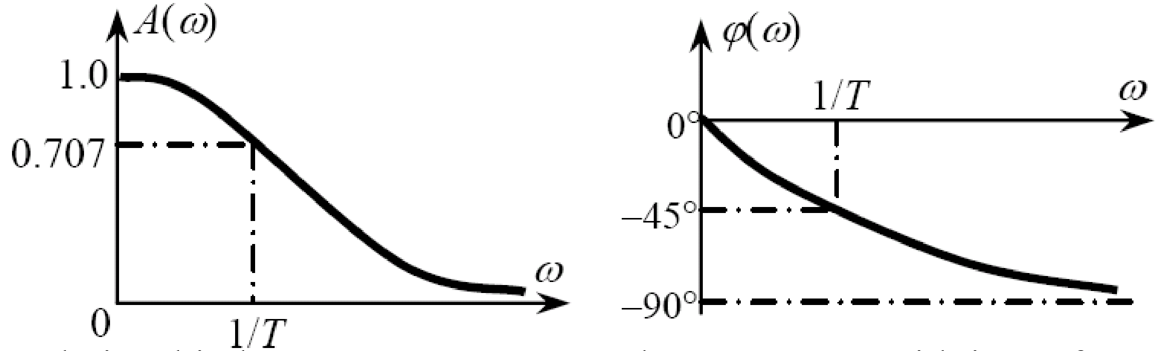


Fig. Relationship between **amplitude** and **phase angle** with input frequency

幅值、相位与频率关系图

定义：将频率特性的幅值下降到零频率幅值的0.707处的频率ω_b，称为系统的**带宽频率**；

$$\left|\frac{\dot{U}_o}{\dot{U}_i}\right| = \frac{1}{\sqrt{1+(\omega_b RC)^2}} = 0.707 = \frac{1}{\sqrt{2}} \Rightarrow 1+(\omega_b RC)^2 = 2 \Rightarrow \omega_b = 1/RC = 1/T$$

可以直接根据频率特性的形状及其特征量来分析系统的特性，而不必对系统的数学模型进行繁琐的求解，这正是频率响应法工程实用性的一个特点

5.1 Frequency Characteristic

Frequency characteristic of LTI system

线性定常系统的频率特性

Transfer function of the LTI system:

线性定常系统的传递函数：

$$G(s) = \frac{Y(s)}{R(s)} = \frac{M(s)}{N(s)} = \frac{M(s)}{\prod_{i=1}^{n}(s+p_i)}, p_i \neq p_j, i \neq j$$

Where $-p_i$ (i =1,2, …,n) are assumed to be distinct poles.

假设$-p_i$ (i =1,2, …,n)为不相等极点

输入 $r(t) = A\sin\omega t$

Input $R(s) = \dfrac{A\omega}{s^2+\omega^2} = \dfrac{A\omega}{(s+j\omega)(s-j\omega)}$

5.1 Frequency Characteristic

Output $$Y(s)=G(s)R(s)=\frac{M(s)}{N(s)}\frac{A\omega}{s^2+\omega^2}$$

$$=\frac{\beta}{s+j\omega}+\frac{\beta^*}{s-j\omega}+\sum_{i=1}^{n}\frac{\alpha_i}{s+p_i}$$

$$y(t)=\beta e^{-j\omega t}+\beta^* e^{j\omega t}+\sum_{i=1}^{n}\alpha_i e^{-p_i t} \qquad (5.1)$$

If $G(s)$ contains m_i poles at $s=-p_i$, thus $y(t)$ contains

如果$G(s)$含有m_i重极点$s=-p_i$, 则 $y(t)$ 中含有

$$t^{hi}e^{-s_i t}\,(h_i=0,1,2,\cdots,m-1)$$

5.1 Frequency Characteristic

Steady state response (Stable system)

稳态响应（稳定系统）

$$y_s(t) = L^{-1}\left[\frac{\beta}{s+j\omega} + \frac{\beta^*}{s-j\omega}\right] = \beta e^{-j\omega t} + \beta^* e^{j\omega t} \qquad (5.2)$$

where $$\beta = G(s)\frac{A\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} = -\frac{A}{2j}G(-j\omega) \qquad (5.3)$$

$$\beta^* = G(s)\frac{A\omega}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} = \frac{A}{2j}G(j\omega) \qquad (5.4)$$

β and β^* are conjugate

可以知道 β 与 β^* 互为一对共轭复数

5.1 Frequency Characteristic

Let $G(j\omega) = P(\omega) + jQ(\omega)$

that $G(-j\omega) = P(\omega) - jQ(\omega)$

$$\beta = -\frac{A}{2j}G(-j\omega) = \frac{A}{2}\left[Q(\omega) + jP(\omega)\right]$$

$$\beta^* = \frac{A}{2j}G(j\omega) = \frac{A}{2}\left[Q(\omega) - jP(\omega)\right]$$

$$\beta + \beta^* = AQ(\omega)$$

Where $AQ(\omega)$ is a real number 实数

$G(j\omega)$ can be written as

$G(j\omega)$可以表示为

$$G(j\omega) = P(\omega) + jQ(\omega) = A(\omega)e^{j\varphi(\omega)}$$

5.1 Frequency Characteristic

$$\begin{cases} A(\omega)=|G(j\omega)|=\sqrt{P^2(\omega)+Q^2(\omega)} & \text{—— } G(j\omega)\text{的幅值} \quad (5.5) \\ \varphi(\omega)=\angle G(j\omega)=tg^{-1}\dfrac{Q(\omega)}{P(\omega)} & \text{—— } G(j\omega)\text{的辐角} \quad (5.6) \end{cases}$$

$$G(-j\omega)=|G(-j\omega)|e^{-j\varphi(\omega)}=|G(j\omega)|e^{-j\varphi(\omega)}=A(\omega)e^{-j\varphi(\omega)}$$

thus
$$y_s(t)=\beta e^{-j\omega t}+\beta^* e^{j\omega t}$$

$$=A|G(j\omega)|\left[\frac{e^{j(\omega t+\varphi(\omega))}-e^{-j(\omega t+\varphi(\omega))}}{2j}\right]$$

$$=A|G(j\omega)|\sin[\omega t+\varphi(\omega)] \quad (5.7)$$

5.1 Frequency Characteristic

Frequency characteristic, *that is transfer function in the frequency domain, is the ratio of the output to the input signal where the input is a sinusoid. It is expressed as G(jω).*

频率特性，也称频率特性函数，是指在正弦输入信号作用下，输出与输入的傅立叶变换之比，用$G(j\omega)$表示。

$$\frac{Y(j\omega)}{R(j\omega)} = |G(j\omega)|e^{j\phi} = |G(j\omega)|e^{j\angle G(j\omega)} = G(j\omega)$$

$$\boxed{G(j\omega) = G(s)\big|_{s=j\omega}}$$

5.2 Frequency Response Plots

Frequency characteristic

$$G(j\omega) = P(\omega) + jQ(\omega) = \sqrt{P^2(\omega) + Q^2(\omega)} \angle tg^{-1}\frac{Q(\omega)}{P(\omega)} = A(\omega)e^{j\varphi(\omega)}$$

The most widely used graphical tools for analyzing and designing control system are **Bode Plot** and **Polar Plot**

工程上应用最广泛的频率特性图是Bode图(对数坐标图)和极坐标图(Nyquist图、幅相频率特性图)

5.2 Frequency Response Plots – Polar plot

- **Polar plot**： *is a plot of the real part of $G(j\omega)$ versus the imaginary part of $G(j\omega)$* .

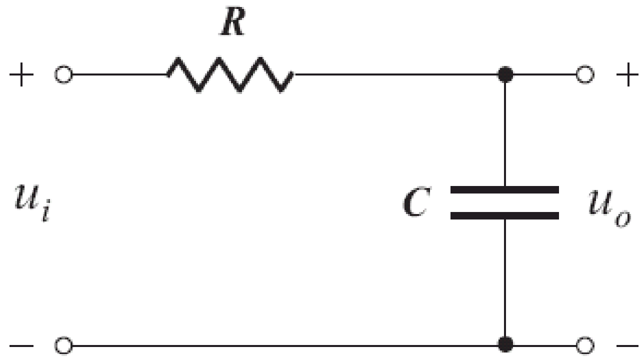
  极坐标图 是$G(j\omega)$ 的实部与虚部的关系图

<E5.1> Plot the polar plot of the *RC* filter

$$G(s)=\frac{1}{RCs+1}$$

$$G(j\omega)=\frac{1}{j\omega(RC)+1}=\frac{1}{j\omega\tau+1}$$

where $\tau = RC$



5.2 Frequency Response Plots – Polar plot

The polar plot can be obtained from:

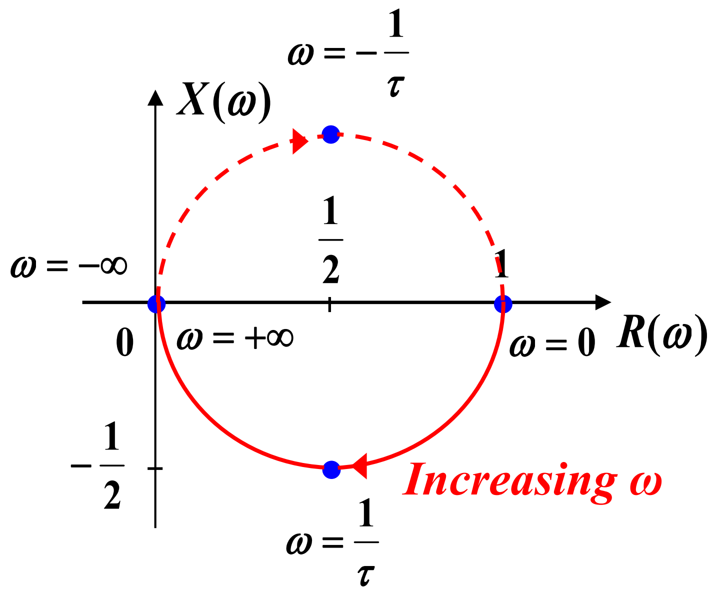
$$\boxed{G(j\omega) = R(\omega) + jX(\omega)}$$

$$G(j\omega) = \frac{1}{1+j\omega\tau} = \frac{1}{1+\omega^2\tau^2} - \frac{j\omega\tau}{1+\omega^2\tau^2}$$

$$\omega = 0 \quad R(\omega) = 1 \quad X(\omega) = 0$$

$$\omega = \frac{1}{\tau} \quad R(\omega) = \frac{1}{2} \quad X(\omega) = -\frac{1}{2}$$

$$\omega = +\infty \quad R(\omega) = 0 \quad X(\omega) = 0$$



5.2 Frequency Response Plots – Polar plot

The polar plot can also be obtained from:

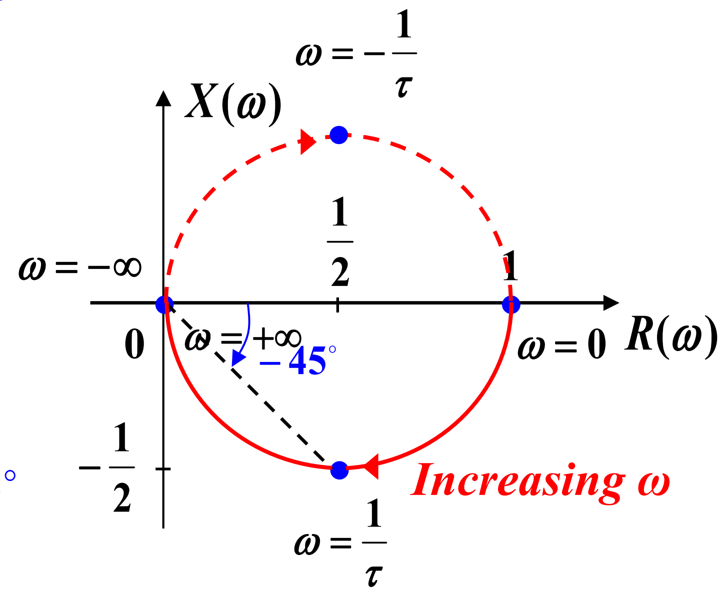
$$\boxed{G(j\omega) = A(\omega)e^{j\varphi(\omega)}}$$

$$G(j\omega) = \frac{1}{1+j\omega\tau} = A(\omega)e^{j\varphi(\omega)} = \frac{1}{\sqrt{1+\omega^2\tau^2}}e^{-jtg^{-1}\omega\tau}$$

$$\omega = 0 \quad |G(j\omega)| = 1 \quad \varphi(\omega) = 0$$

$$\omega = \frac{1}{\tau} \quad |G(j\omega)| = \frac{1}{\sqrt{2}} \quad \varphi(\omega) = -45^\circ$$

$$\omega \to +\infty \quad |G(j\omega)| \to 0 \quad \varphi(\omega) = -90^\circ$$



5.2 Frequency Response Plots – Bode plot

Bode Plot: *The logarithm of the magnitude of the frequency characteristic $G(j\omega)$ is plotted versus the logarithm of ω. The phase, φ, of the frequency characteristic $G(j\omega)$ is separately plotted versus the logarithm of the frequency.*

波特图： 频率特性$G(j\omega)$的对数幅值与对数频率之间的关系图以及$G(j\omega)$的相角φ与对数频率之间的关系图。

5.2 Frequency Response Plots – Bode plot

- **Bode Plot**:将频率特性分为Amplitude characteristic幅频特性和Phase characteristic相频特性，分别绘于(半)对数坐标上;

a) 频率ω(横)坐标：用lgω分度;

b) 幅值$A(\omega)$用$20\lg A(\omega)$[dB]分度： $20\lg A(\omega) \sim \lg\omega$

c) 相角$\varphi(\omega)$用线性分度： $\varphi(\omega) \sim \lg\omega$

Relationship between ω & lgω

ω	1	2	3	4	5	6	7	8	9
lgω	0	0.301	0.477	0.602	0.699	0.778	0.845	0.903	0.954
ω	10	20	30	40	50	60	70	80	90
lgω	1	1.301	1.477	1.602	1.699	1.778	1.845	1.903	1.954

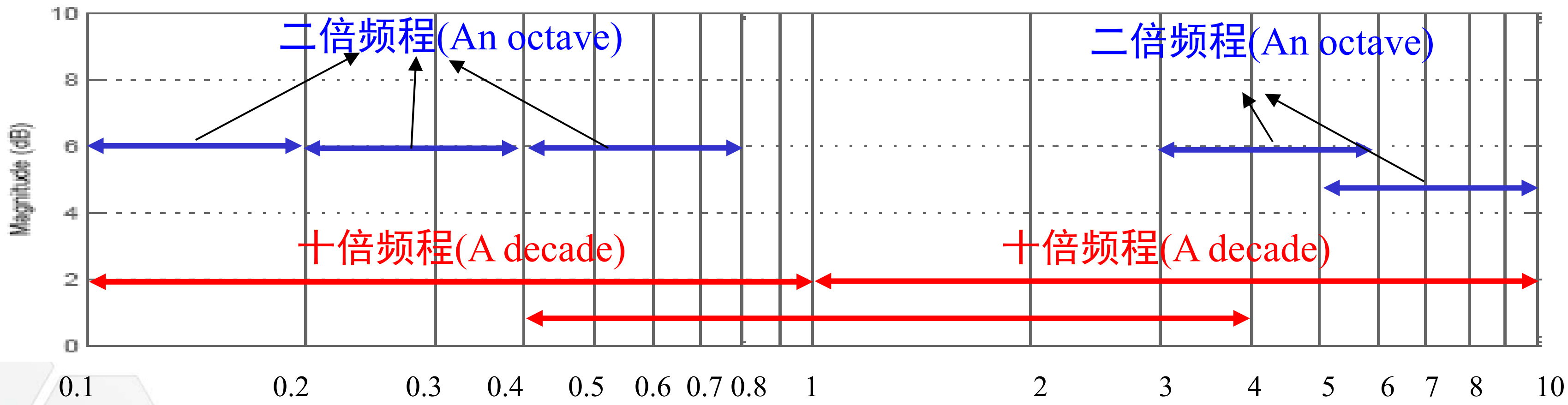
5.2 Frequency Response Plots – Bode plot

- A decade 十倍频程

A decade is an interval of two frequencies on x-coordinate of Bode plot with a ratio equal to 10.

在波特图的横坐标上，若两个频率之比为10，则其间隔为一个十倍频程，用*dec*来表示。

- An octave 二倍频程



5.2 Frequency Response Plots – Bode plot

- **Advantages of using Bode Plot**:

The use of a logarithmic scale of frequency is convenient. The primary advantage of the **Bode plot** is the conversion of *multiplicative factors*（因式相乘） into *addictive factors*（因式相加） by virtue of the definition of logarithmic gain.

$$G(j\omega)=\frac{K_b\prod_{i=1}^{Q}(1+j\omega\tau_i)}{(j\omega)^N\prod_{m=1}^{M}(1+j\omega T_m)\prod_{k=1}^{R}\left[1+(2\zeta_k/\omega_{nk})j\omega+(j\omega/\omega_{nk})^2\right]}$$

Transfer function include Q zeros, N poles at the origin, M poles on the real axis and R pairs of complex conjugate poles

5.2 Frequency Response Plots – Bode plot

The logarithmic magnitude of $G(j\omega)$:

$$20\lg\left|G(j\omega)\right| = 20\lg K_b + 20\sum_{i=1}^{Q}\lg\left|1 + j\omega\tau_i\right|$$

$$-20\lg\left|(j\omega)^N\right| - 20\sum_{m=1}^{M}\lg\left|1 + j\omega T_m\right|$$

$$-20\sum_{k=1}^{R}\lg\left|1 + (2\zeta_k/\omega_{nk})j\omega + (j\omega/\omega_{nk})^2\right|$$

The phase angle of $G(j\omega)$:

$$\varphi(\omega) = \sum_{i=1}^{Q}\tan^{-1}\omega\tau_i - N(90^\circ) - \sum_{m=1}^{M}\tan^{-1}\omega T_m$$

$$-\sum_{k=1}^{R}\tan^{-1}\left(\frac{2\zeta_k\omega_{nk}\omega}{\omega_{nk}^2 - \omega^2}\right)$$

5.2.1 The Bode plot of basic factors

1. Constant gain 比例环节/常数增益项

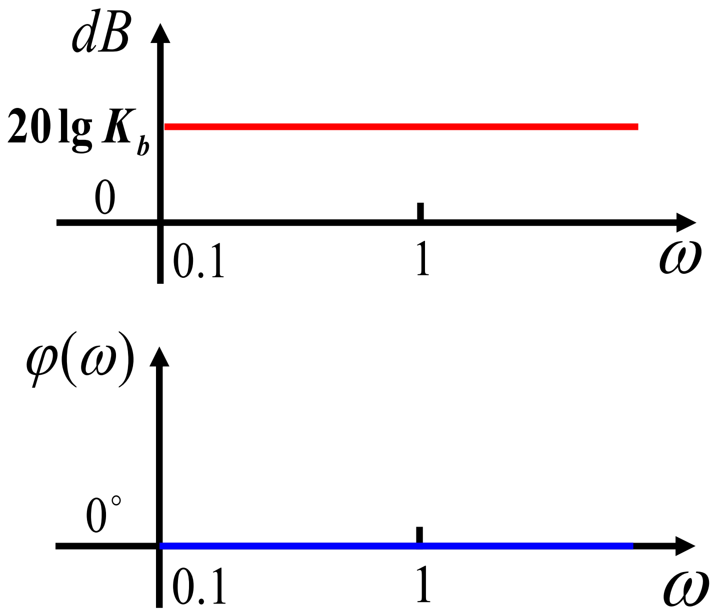
$$G(j\omega) = K_b \tag{5.8}$$

The logarithmic magnitude:

$$20\lg\left|G(j\omega)\right| = 20\lg K_b$$

The phase angle:

$$\varphi(\omega) = 0$$



Bode Plot

5.2.1 The Bode plot of basic factors

2. Poles and Zeros at the origin 积分环节和微分环节

$$G(j\omega)=(j\omega)^{\mp 1}$$

1) Poles at the origin 原点处极点项/积分环节

$$G(j\omega)=\frac{1}{j\omega} \qquad (5.9)$$

- The logarithmic magnitude

$$20\lg|G(j\omega)|=20\lg\left|\frac{1}{j\omega}\right|=-20\lg\omega \qquad \text{(dB)}$$

$$\omega=1,\quad 20\lg A(\omega)=0 \qquad \text{(dB)}$$

$$\omega=10,\quad 20\lg A(\omega)=-20 \quad \text{(dB)}$$

The slope of the straight line $-20\mathrm{dB/dec}$

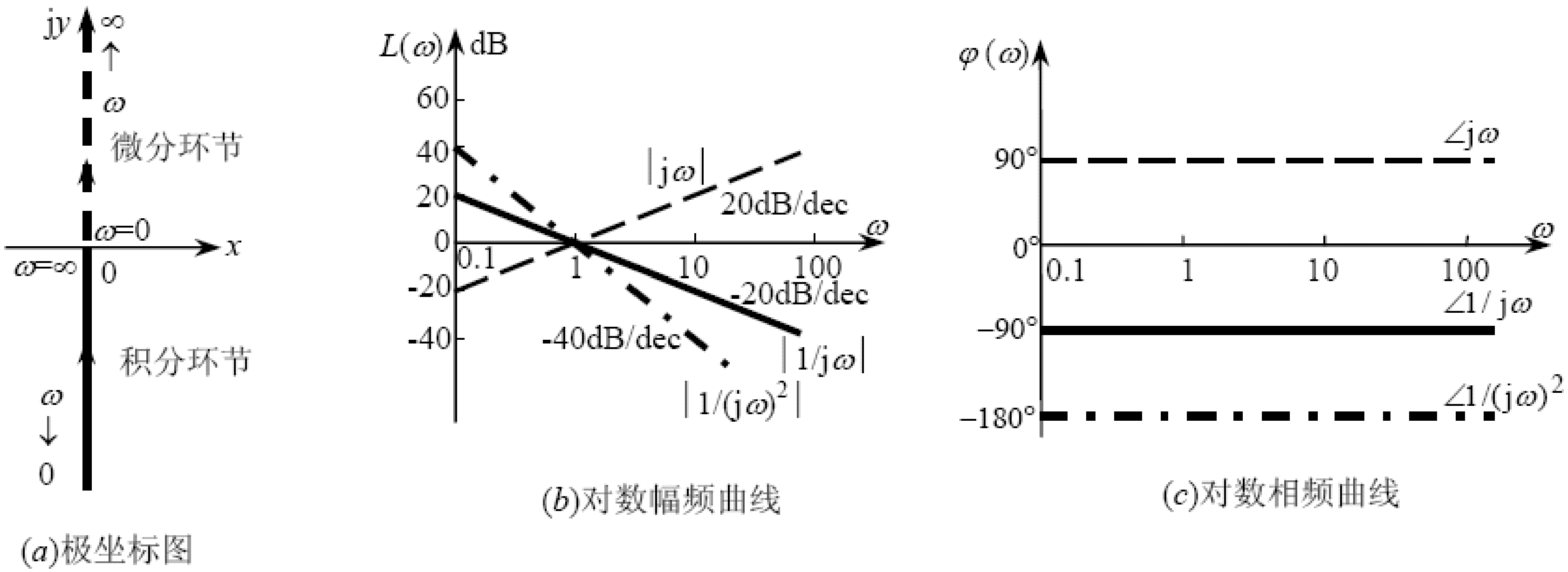
- The phase angle: $\varphi(\omega)=-90^\circ$

5.2.1 The Bode plot of basic factors

2) Zeros at the origin 原点处零点项/微分环节

$$G(j\omega)=j\omega \tag{5.10}$$

微分环节的Bode图与积分环节的Bode图关于横轴对称



5.2.1 The Bode plot of basic factors

3. Poles and Zeros on the real axis

实轴上的极点和零点 / 惯性环节和一阶微分环节

$$G(j\omega)=(1+j\omega T)^{\mp 1}$$

1) Poles on the real axis 实轴上的极点/惯性环节

$$G(j\omega)=\frac{1}{1+j\omega T} \tag{5.11}$$

- The logarithmic magnitude :

$$20\lg A(\omega)=20\lg\left|\frac{1}{1+j\omega T}\right|=-20\lg\sqrt{1+\omega^2T^2}$$

5.2.1 The Bode plot of basic factors

$$20\lg A(\omega) = 20\lg\left|\frac{1}{1+j\omega T}\right| = -20\lg\sqrt{1+\omega^2 T^2}$$

a) $\omega << 1/T$ 低频段 $20\lg A(\omega) \approx -20\lg 1 = 0(\mathrm{dB})$

b) $\omega >> 1/T$ 高频段 $20\lg A(\omega) \approx -20\lg \omega T(\mathrm{dB})$

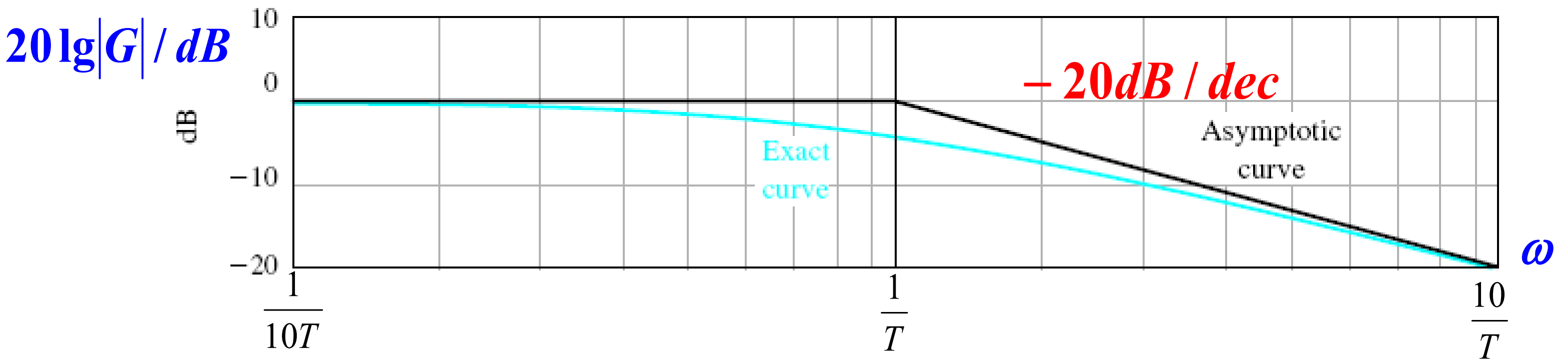
Break/Corner Frequency 转折频率 $\omega = 1/T = \omega_n$

Asymptotic Curve 渐近线

$$\omega < \frac{1}{T}, \quad 20\lg A(\omega) = 0 \quad \mathrm{dB};$$

$$\omega > \frac{1}{T}, \quad 20\lg A(\omega) = -20\lg(\omega T) \quad \mathrm{dB};$$

5.2.1 The Bode plot of basic factors



Error between exact values and approximation values:

- At $\omega=1/T$ *break frequency*

$$20\lg A(\frac{1}{T}) = -20\lg\sqrt{2} = -3.01 \quad (\text{dB})$$

- At $\omega=10/T$

$$20\lg A(\frac{10}{T}) = -20\lg\sqrt{1+100} = -20.043 \quad (\text{dB}) \approx -20 \quad (\text{dB})$$

- At $\omega=0.1/T$

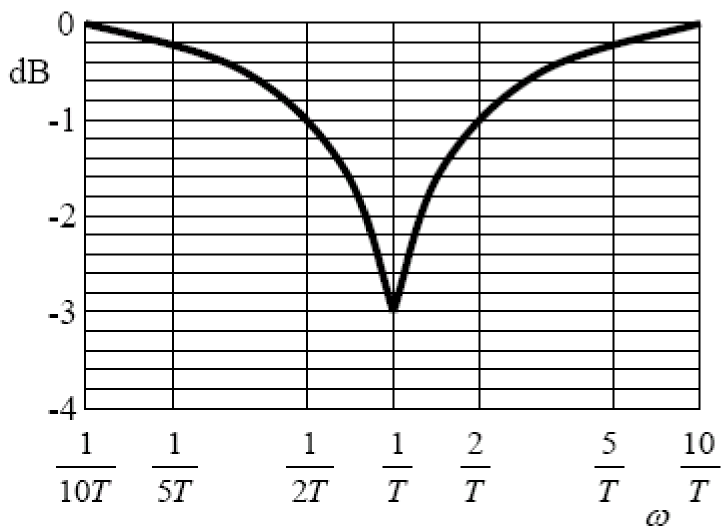
$$20\lg A(\frac{0.1}{T}) = -20\lg\sqrt{1+0.01} = -0.043 \quad (\text{dB}) \approx 0 \quad (\text{dB})$$

5.2.1 The Bode plot of basic factors

幅频特性误差修正表

频率相对值 $\frac{\omega}{1/T}=\omega T$	0.1	0.25	0.5	1	2	4	10
误差 $\Delta L(\omega)/\,\mathrm{dB}$	-0.04	-0.26	-0.97	-3.01	-0.97	-0.26	-0.04

幅频特性误差修正曲线



5.2.1 The Bode plot of basic factors

- The Phase Angle $\varphi(\omega) = -tg^{-1}\omega T$

$$\varphi(0) = 0°, \quad \varphi(\infty) = -90°, \quad \varphi(\frac{1}{T}) = -45°$$

相频特性的几个函数值

频率相对值 $\frac{\omega}{1/T} = \omega T$	0.01	0.05	0.1	0.2	0.5	1	2	5	10	20	100
相位移 $\varphi(\omega)$ （度）	-0.06	-2.9	-5.7	-11.3	-26.6	-45	-63.4	-78.7	-84.3	-87.1	-89.4

Asymptote $\omega < \frac{0.1}{T}, \quad \varphi(\omega) = 0°;$

$$\omega > \frac{10}{T}, \quad \varphi(\omega) = -90°;$$

$$\frac{0.1}{T} < \omega < \frac{10}{T}, \quad \text{Slope} \quad -45°/\text{dec}$$

5.2.1 The Bode plot of basic factors

最大误差出现在ω=0.1/T 和ω=10/T处，$\triangle\varphi = 5.7°$

次大误差出现在ω=0.4/T 和ω=2.5/T处，$\triangle\varphi = 5.3°$

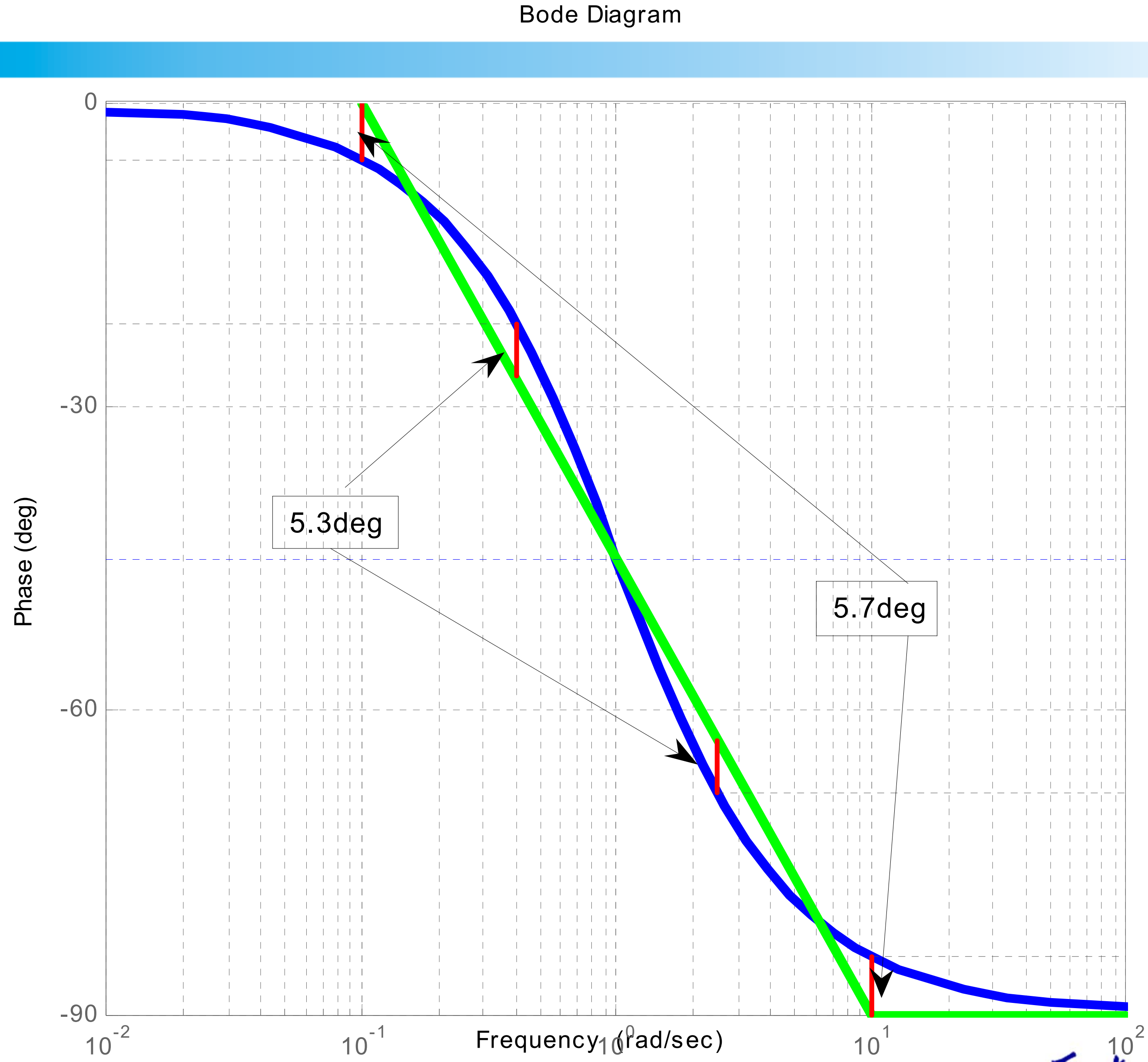
按等距分度：

0.4/T 距0.1/T 的相对距离：$\lg(0.4/0.1) = 0.6021 \approx 0.6$

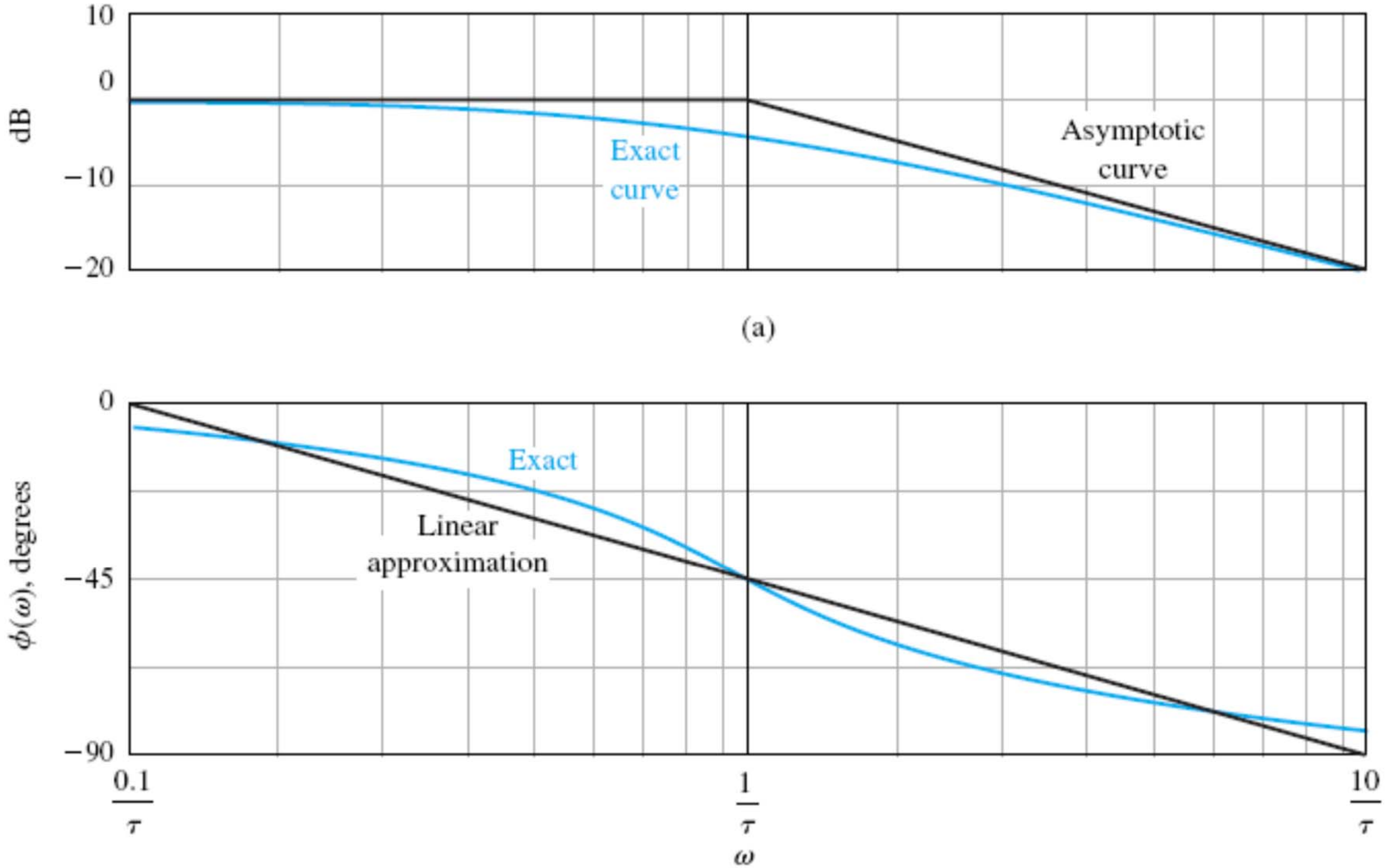
2.5/T 距 1/T 的相对距离：$\lg(2.5/1) = 0.3979 \approx 0.4$

与折线相交：

ω=0.16/T 和 ω=6.3/T


Bode Diagram
0
-30
-60
-90
Phase (deg)
5.3deg
5.7deg
Frequency (rad/sec)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

5.2.1 The Bode plot of basic factors

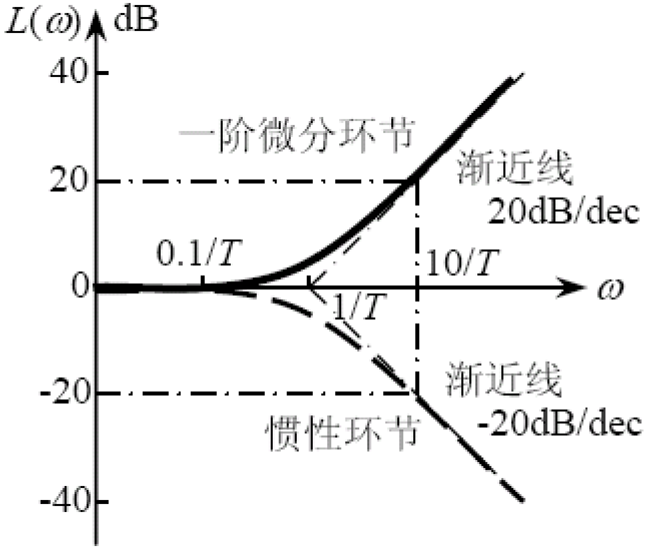


5.2.1 The Bode plot of basic factors

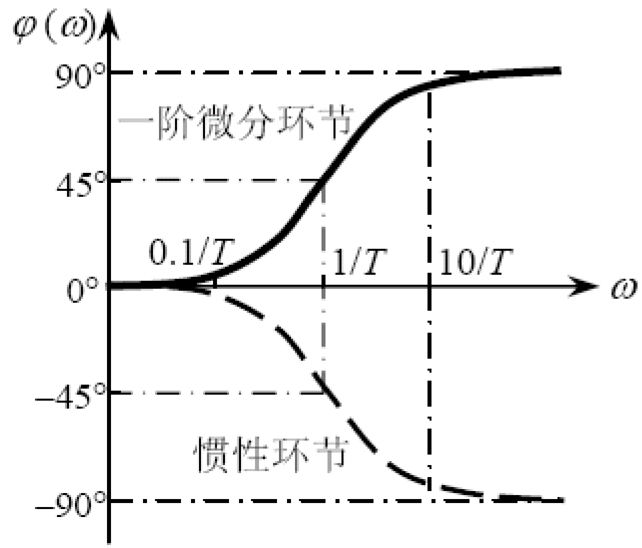
2) Zeros on the real axis 实轴上的零点/一阶微分环节

$$G(j\omega) = 1 + j\omega T \tag{5.12}$$

其Bode图与一阶惯性环节的Bode图关于横轴对称



(a)对数幅频曲线



(b)对数相频曲线

5.2.1 The Bode plot of basic factors

3. Complex conjugate Poles/ Complex conjugate Zeros 振荡环节和二阶微分环节

1) Complex conjugate Poles 二阶振荡环节

$$G(j\omega)=\frac{1}{1+\left(\frac{2\zeta}{\omega_n}\right)j\omega+\left(\frac{j\omega}{\omega_n}\right)^2} \tag{5.13}$$

$$=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\frac{\omega}{\omega_n}} \qquad (0\le\zeta<1)$$

5.2.1 The Bode plot of basic factors

- The logarithmic magnitude:

$$20\lg A(\omega) = -20\lg\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}$$

a) When $\omega \ll \omega_n$ (低频段): $20\lg A(\omega) \approx -20\lg 1 = 0(\mathrm{dB})$

b) When $\omega \gg \omega_n$ (高频段): $20\lg A(\omega) \approx -20\lg\left(\frac{\omega}{\omega_n}\right)^2 = -40\lg\frac{\omega}{\omega_n}(\mathrm{dB})$

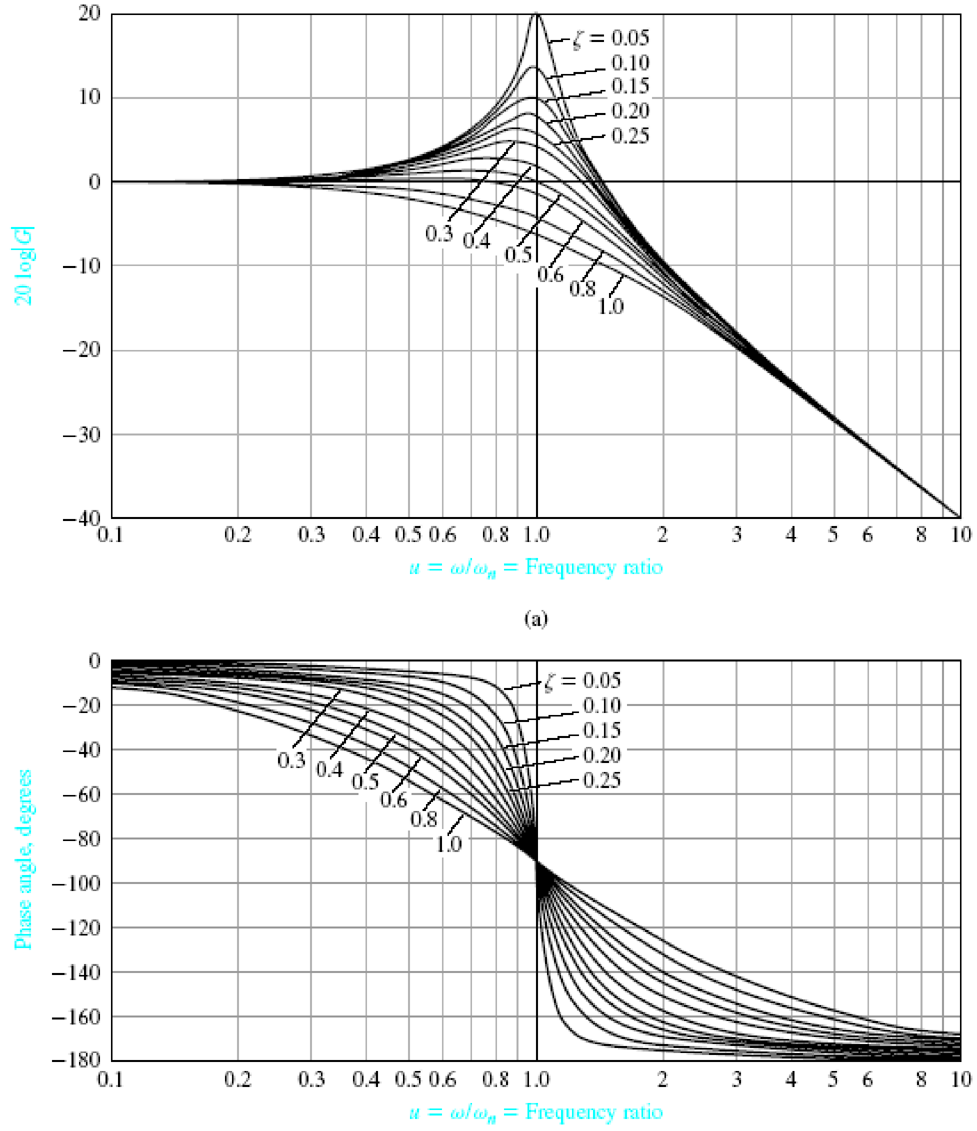
Break/Corner Frequency 转折频率 $\omega = 1/T = \omega_n$

The slope of the asymptote -40dB/dec.

5.2.1 The Bode plot of basic factors

- The Phase Angle :

$$\varphi(\omega) = -tg^{-1}\frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}$$



(a)

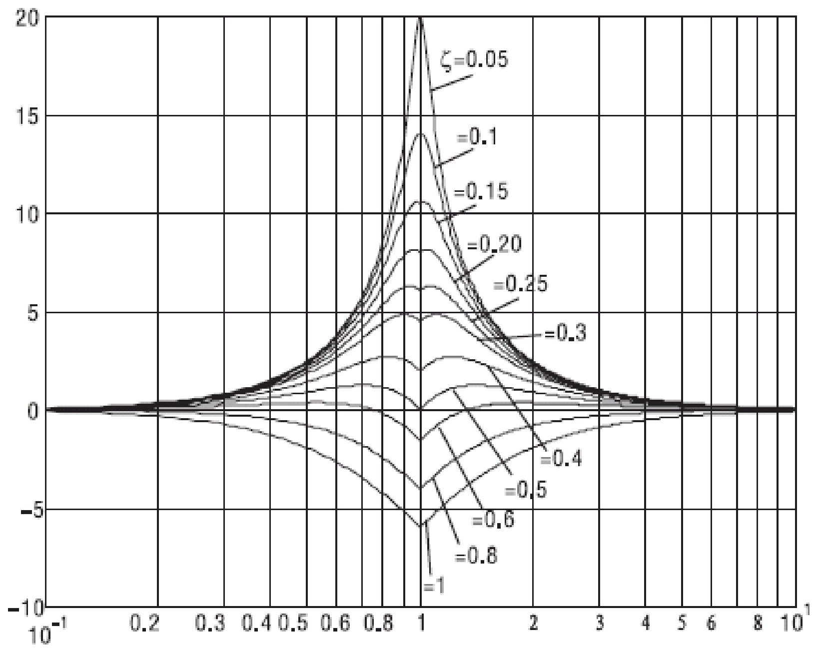
(b)

5.2.1 The Bode plot of basic factors

- 幅频特性误差修正曲线

幅频特性中, ζ=0.5~0.7, 渐近线(折线)近似效果较好；

相频特性， 渐近线(折线)近似效果不好；



5.2.1 The Bode plot of basic factors

2) ***Resonant Frequency*** 谐振频率ω_r

Resonant Peaks 谐振峰值$M_{p\omega}$

在ω_n附近，幅频特性出现谐振峰值$M_{p\omega}$，其大小与ζ有关。

Definition： ***Resonant Frequency*** ω_r is the frequency where the ***Resonant Peaks*** $M_{p\omega}$ occurs.

谐振频率ω_r，谐振峰值$M_{p\omega}$处的频率

$$A(\omega)=|G(j\omega)|=\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

Let $f(\omega)=\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2=\left(\frac{\omega}{\omega_n}\right)^4-2\left(\frac{\omega}{\omega_n}\right)^2(1-2\zeta^2)+1$

5.2.1 The Bode plot of basic factors

$$\frac{\mathrm{d}f(\omega)}{\mathrm{d}\omega}=\frac{1}{\omega_n{}^4}\left[4\omega^3-4\omega\omega_n{}^2(1-2\zeta^2)\right]=\frac{4\omega}{\omega_n{}^4}\left[\omega^2-\omega_n{}^2(1-2\zeta^2)\right]=0$$

$$\omega=\omega_n\sqrt{1-2\zeta^2},(\omega=0,\ \text{omited})$$

$$\frac{\mathrm{d}^2f(\omega)}{\mathrm{d}\omega^2}=\frac{4}{\omega_n{}^4}\left[3\omega^2-\omega_n{}^2(1-2\zeta^2)\right]$$

$$\left.\frac{\mathrm{d}^2f(\omega)}{\mathrm{d}\omega^2}\right|_{\omega=\omega_n\sqrt{1-2\zeta^2}}=\frac{8}{\omega_n{}^2}(1-2\zeta^2)$$

$$\text{when }\ 1-2\zeta^2>0,\left.\frac{\mathrm{d}^2f(\omega)}{\mathrm{d}\omega^2}\right|_{\omega=\omega_n\sqrt{1-2\zeta^2}}>0$$

$$\omega=\omega_n\sqrt{1-2\zeta^2},\qquad f(\omega)\min\Rightarrow A(\omega)=|G(j\omega)|\max$$

5.2.1 The Bode plot of basic factors

谐振频率 $\omega_r = \omega_n\sqrt{1-2\zeta^2}, 0 \le \zeta \le 0.707 \qquad (5.14)$

谐振峰值 $M_{p\omega} = \left|G(j\omega_{p\omega})\right| = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad (5.15)$

条件$1-2\zeta^2 > 0, \zeta < 0.707$

即$0 < \zeta < 0.707$

3) Complex Conjugate Zeros 二阶微分环节

$$G(j\omega) = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right), (0 \le \zeta < 1) \qquad (5.16)$$

其Bode图与二阶振荡环节的Bode图关于横轴对称

5.2.2 Plot the Bode diagram

Draw the Bode diagram according the basic factors.

由基本(典型)环节的幅频、相频曲线，绘制系统的Bode图曲线

<E5.1> $G(s)=\dfrac{2500(s+10)}{s(s+2)(s^2+30s+2500)}$ draw the Bode diagram

$$G(j\omega)=\frac{2500(j\omega+10)}{j\omega(j\omega+2)[(j\omega)^2+j30\omega+2500]}$$

$$=\frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)\left[1-\left(\dfrac{\omega}{50}\right)^2+j0.6\dfrac{\omega}{50}\right]}$$

5 different factors in the transfer function:

1、5; 2、$\dfrac{1}{j\omega}$; 3、$\dfrac{1}{1+j0.5\omega}$; 4、$1+j0.1\omega$; 5、$\dfrac{1}{1-\left(\dfrac{\omega}{50}\right)^2+j0.6\dfrac{\omega}{50}}$

5.2.2 Plot the Bode diagram

1、幅频特性（The logarithmic magnitude）

(1) 画每个环节(不包括比例环节)的渐近线(折线),代数相加;

(2) 20lg5=14(dB), 将0dB线下移14dB(即在原坐标上加14dB);

(3) 误差修正: (第(2)/(3)步可交换)

可计算几个点

(着重转折点)

记:

$$\omega_n = 2: \quad L_2 = -20\lg\sqrt{1+\left(\frac{\omega}{2}\right)^2}$$

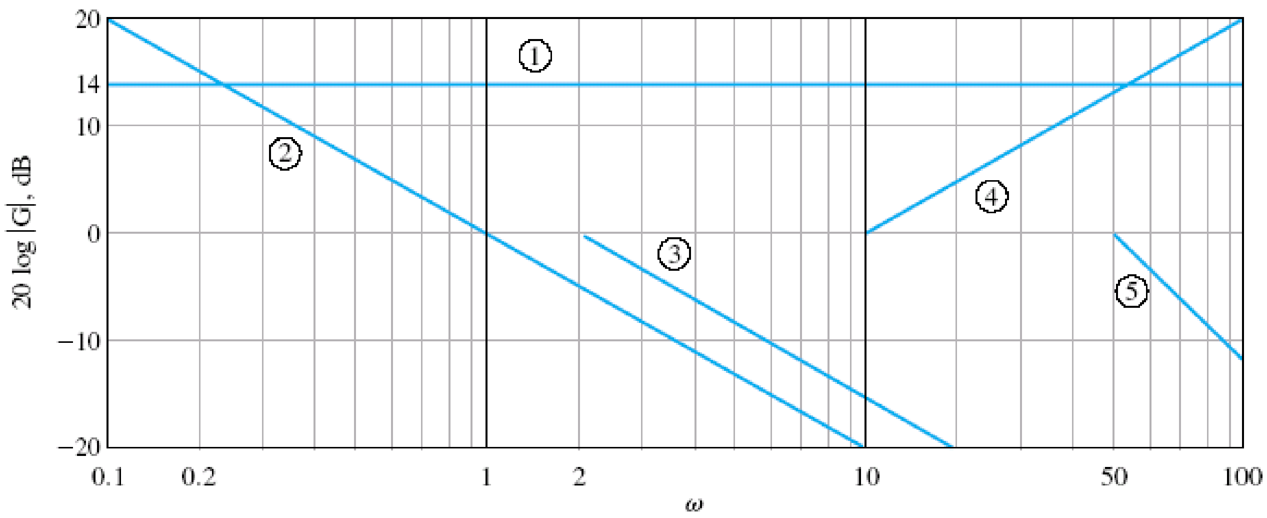
$$\omega_n = 10: \quad L_{10} = 20\lg\sqrt{1+\left(\frac{\omega}{10}\right)^2}$$

$$\omega_n = 50: \quad L_{50} = -20\lg\sqrt{\left[1-\left(\frac{\omega}{50}\right)^2\right]^2+\left(\frac{0.6\omega}{50}\right)^2}$$

$$积分\frac{1}{j\omega}: \quad L_I = -20\lg\omega$$

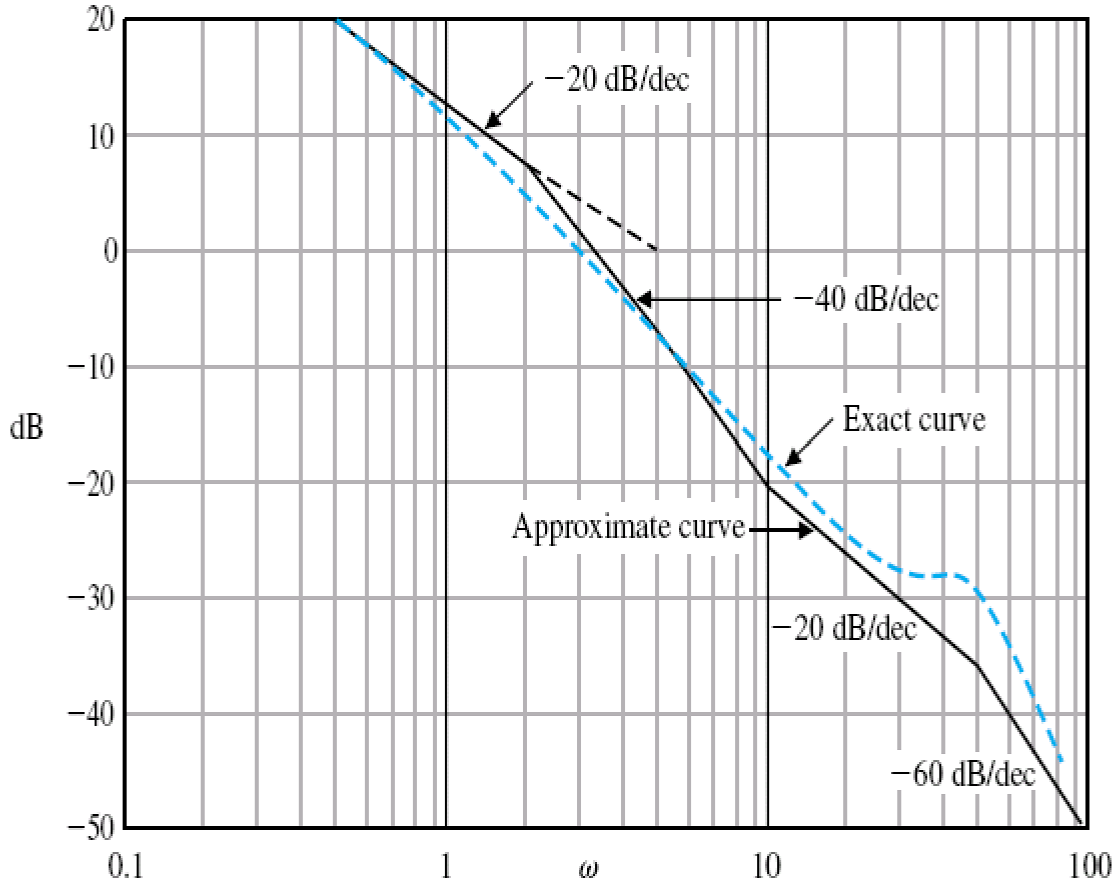
5.2.2 Plot the Bode diagram

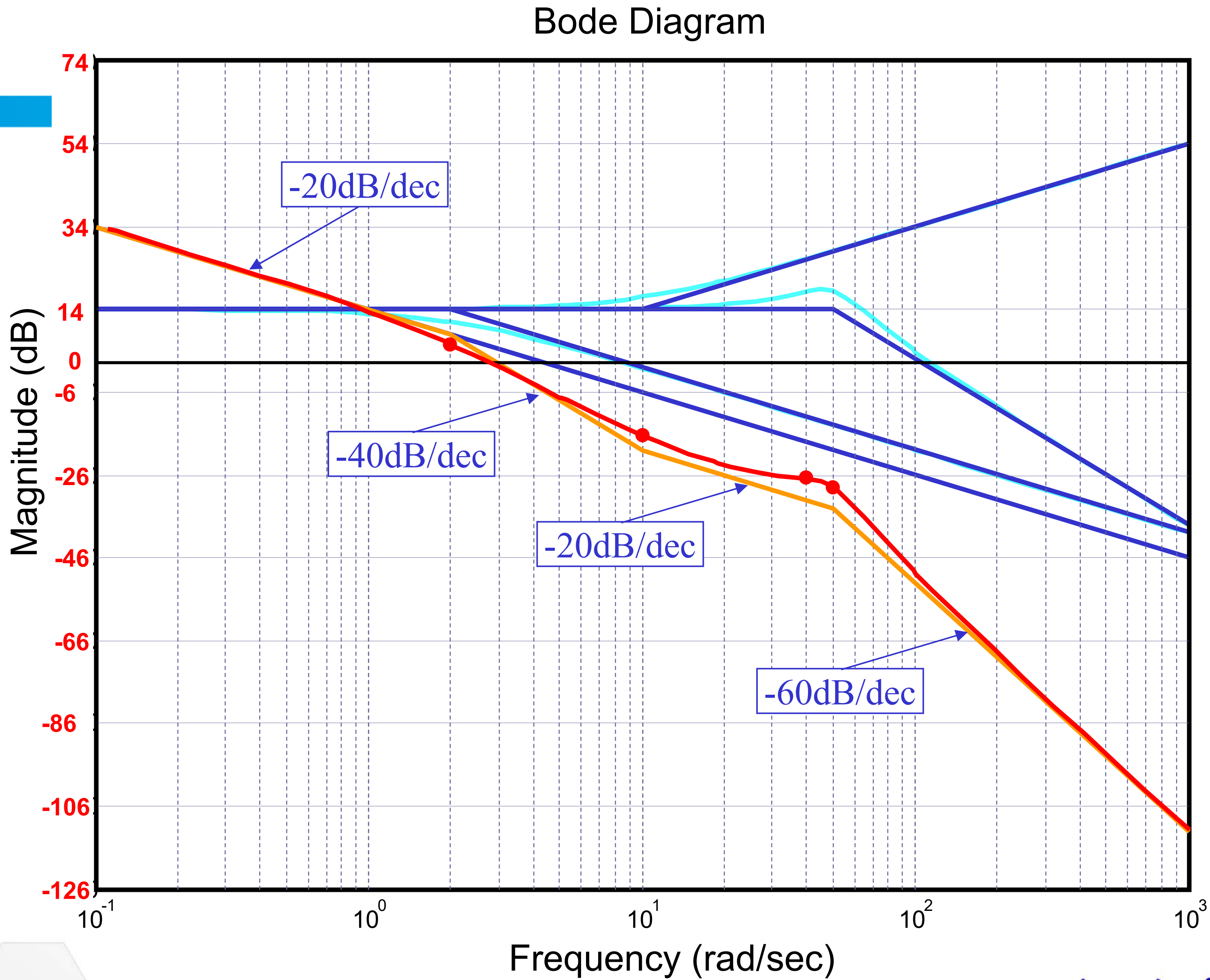
原坐标下	(a) $\omega = 2$ 处	(b) $\omega = 10$ 处	(c) $\omega = 50$ 处	(d) $\omega = 30$ 处
	$L_2 = -3.01$ $L_{10} = 0.17$ $L_{50} = 0.01$ $L_I = -6.02$	$L_2 = -14.15$ $L_{10} = 3.01$ $L_{50} = 0.29$ $L_I = -20$	$L_2 = -27.97$ $L_{10} = 14.15$ $L_{50} = 4.44$ $L_I = -33.98$	$L_2 = -23.54$ $L_{10} = 10$ $L_{50} = 2.68$ $L_I = -29.54$
+ 14	$L = -8.85$	$L = -30.85$	$L = -43.36$	$L = -40.40$
新坐标下	=5.15	= -16.85	= -29.36	= -26.40



5.2.2 Plot the Bode diagram

The logarithmic magnitude :





zhaoduo@home.swjtu.edu.cn

5.2.2 Plot the Bode diagram

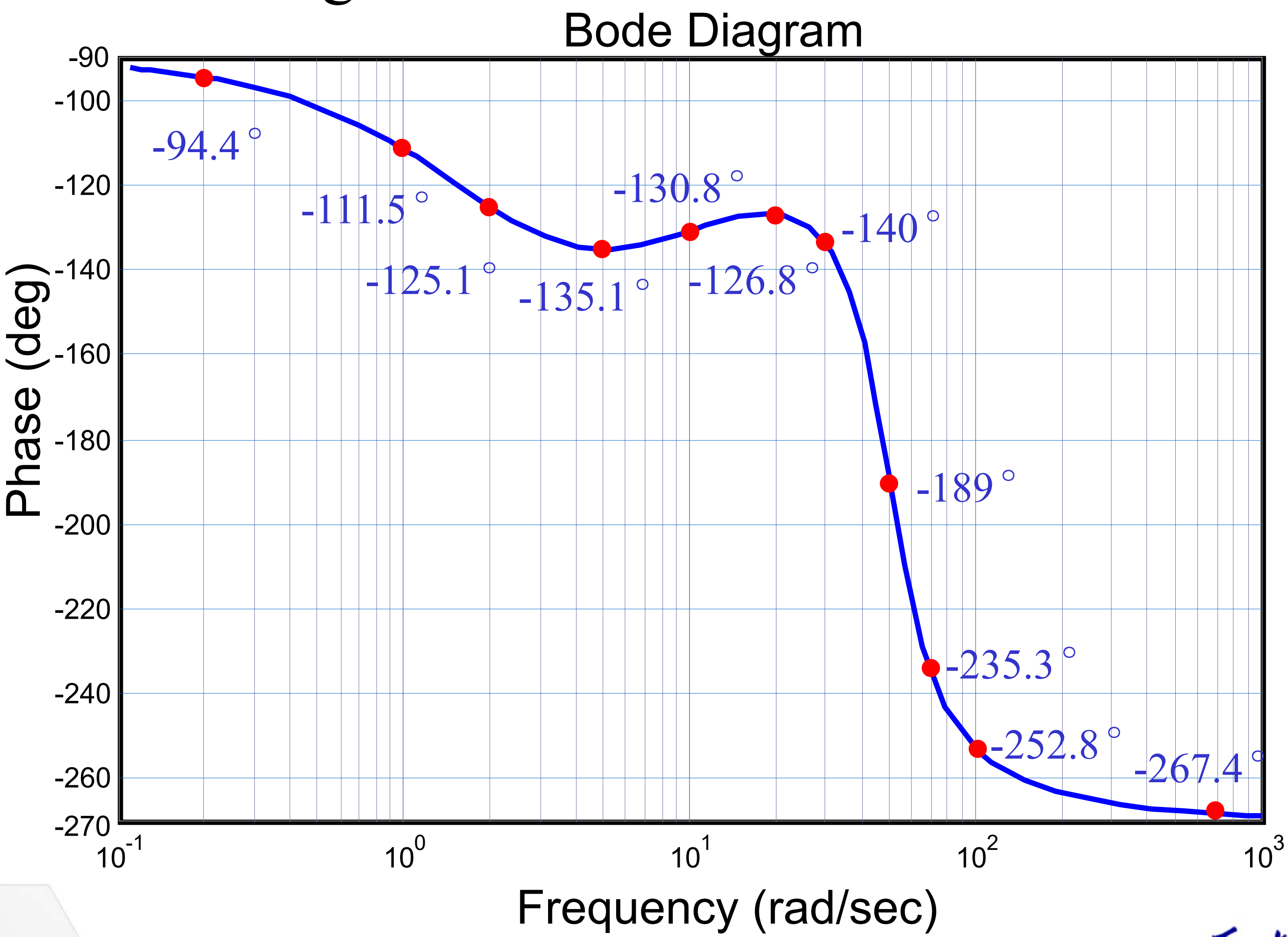
2、The phase angle

相频特性曲线可以直接计算几个点：

$\omega < 50$	$90° + \varphi(\omega) = tg^{-1}0.1\omega - tg^{-1}0.5\omega - tg^{-1}\dfrac{0.6\omega/50}{1-(\omega/50)^2}$							
ω	0.2	1	2	5	10	20	30	50
$\varphi(\omega)$	-94.4°	-111.5°	-125.1°	-135.1°	-130.8°	-126.8°	-140.0°	-189.0°
$\omega > 50$	$90° + \varphi(\omega) = tg^{-1}0.1\omega - tg^{-1}0.5\omega - 180° + tg^{-1}\dfrac{0.6\omega/50}{(\omega/50)^2 - 1}$							
ω	70	100	500					
$\varphi(\omega)$	-235.3°	-252.8°	-267.4°					

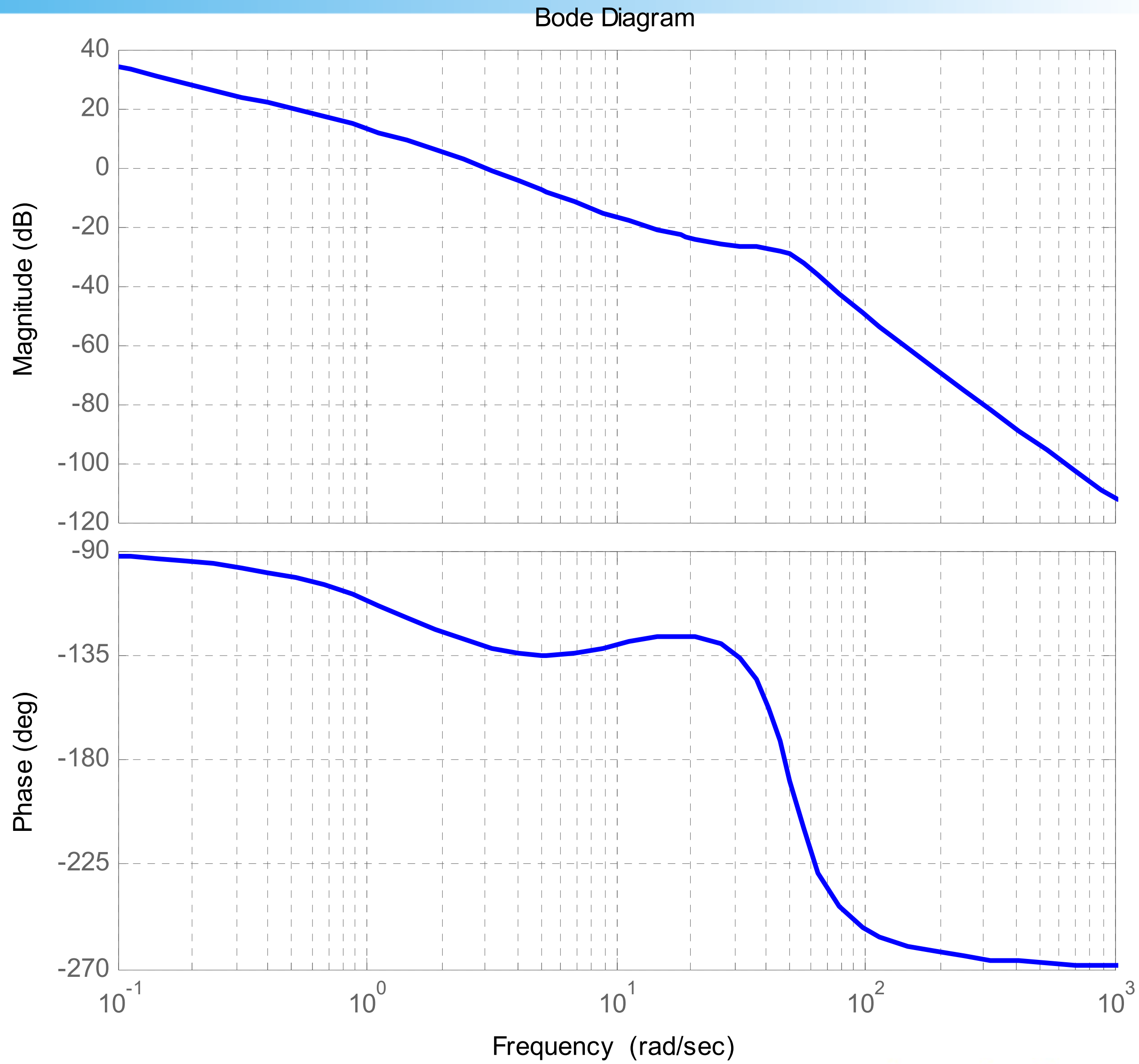
5.2.2 Plot the Bode diagram

The Phase angle



5.2.2 Plot the Bode diagram

- Bode Diagram of $G(j\omega)$



返回

5.3 Nyquist Stability Criterion 奈奎斯特稳定性判据

1932， H.Nyquist proposed the **Nyquist Criterion**.

The methods to determine the stability of a system *without* resolving the characteristic equation.

判断闭环系统的稳定性(不求特征根)的方法：

- Routh-Hurwitz： 适用于特征方程为代数方程的系统，不适用于时滞系统；
- Root Locus： 对于时滞系统有效，但很麻烦；
- Nyquist Criterion： 利用开环频率特性 $(G(j\omega)H(j\omega))$判断闭环系统稳定性的一种图解方法；

5.3 Nyquist Stability Criterion 奈奎斯特稳定性判据

Nyquist判据**特点**：

1) 应用方便：分析时滞系统的稳定性也较方便，也可推广到多变量系统，以及分析某类非线性系统的稳定性；

2) 开环频率特性可以通过试验测取，这对于不易建模的系统很有意义。

Nyquist判据判断特征方程$1+G(s)H(s)=0$在S右半平面内特征根的数目，其理论基础是复变函数的映射定理(Cauchy定理)。

5.3.1 Cauchy's Theorem

Let

$$F(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{b_m\prod_{i=1}^{m}(s+z_i)}{a_n\prod_{j=1}^{n}(s+p_j)}$$

$s=\sigma+j\omega$ is a complex number

Cauchy's Theorem: If a contour Γ_S in the s-plane encircles ***P*** poles and ***Z*** zeros of $F(s)$ and does **not** pass through any poles or zeros of $F(s)$ and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the $F(s)$-plane encircles the origin of the $F(s)$-plane ***N*=*Z*-*P*** times in the clockwise direction.

N的方向：顺时针方向为正，逆时针方向为负

5.3.1 Cauchy's Theorem

设

$$F(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{b_m\prod_{i=1}^{m}(s+z_i)}{a_n\prod_{j=1}^{n}(s+p_j)}$$

$s=\sigma+j\omega$ 是复变量

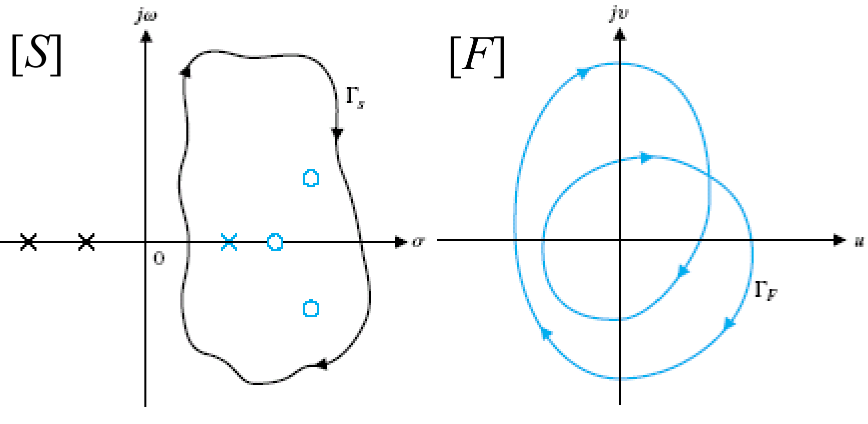
映射定理： 若$F(s)$在S平面上的闭曲线Γ_S的内部共有P个极点和Z个零点。设Γ_S**不经过**$F(s)$的任何零点和极点，则Γ_S唯一的映射到$F(s)$平面上的一条闭曲线Γ_F，当 s 按顺时针方向沿Γ_S变化一周时，在$F(s)$平面上，轨迹$F(s)$按顺时针方向沿Γ_F包围原点的周数N 等于Z-P

N 的方向：顺时针方向为正，逆时针方向为负

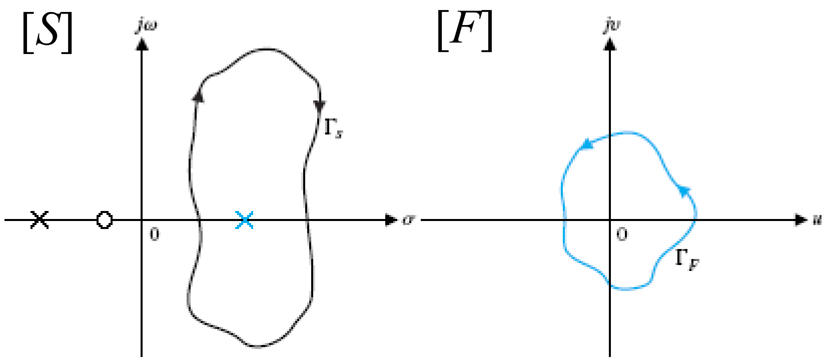
5.3.1 Cauchy's Theorem

According to the *Cauchy's theorem*, we can get the number of difference of the poles and zeros encircled in the contour Γ_s in the s-plane from the number that the corresponding contour in the $F(s)$-plane encircles the origin in the clockwise direction

根据*映射定理*，由F平面上Γ_F包围原点的周数，可知S平面上Γ_S中的零点数与极点数之差

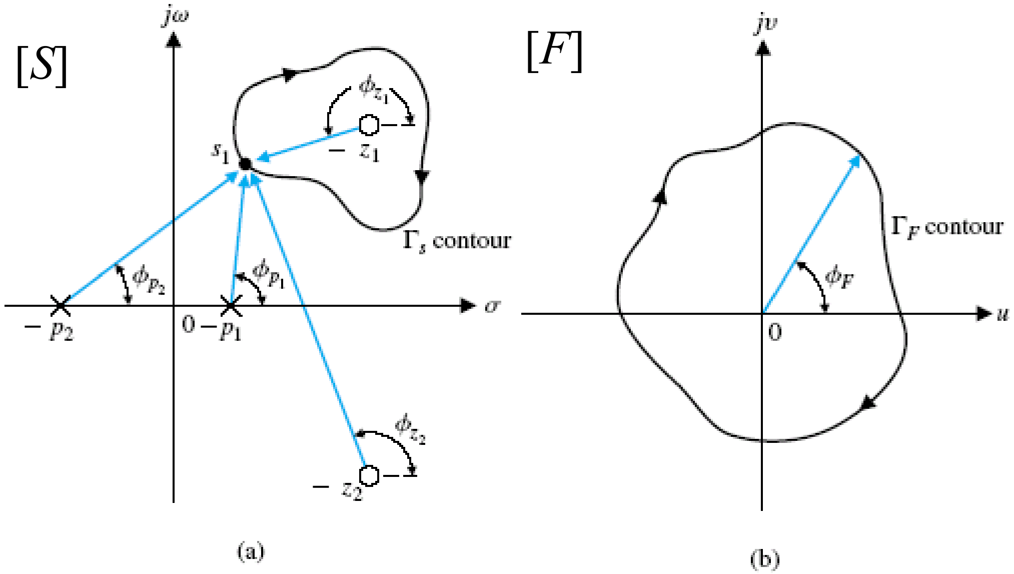


(a) (b)



(a) (b)

For example, let



$$F(s)=\frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$

$$F(s)=|F(s)|\angle F(s)$$

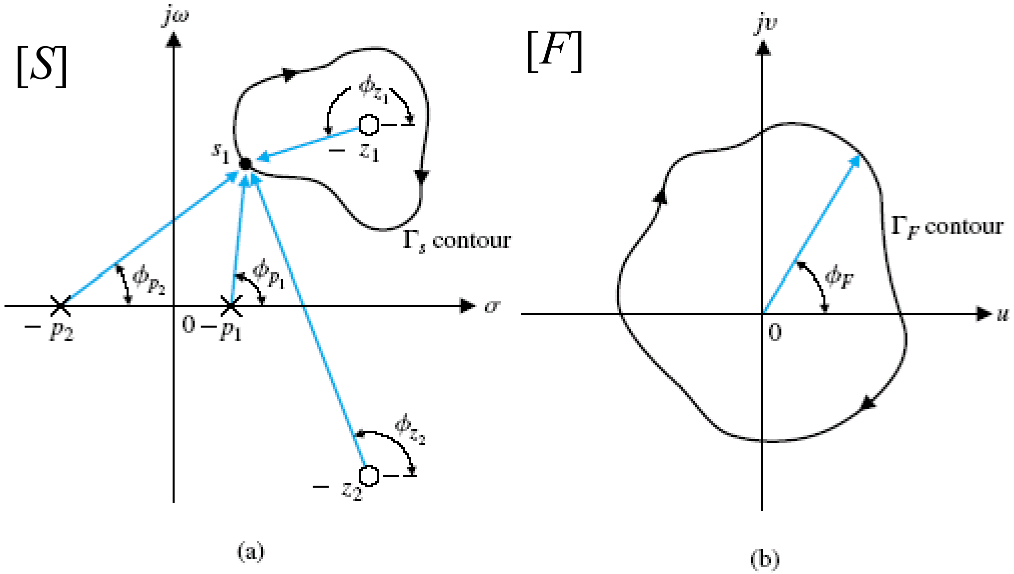
$$=\frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}[\angle(s+z_1)+\angle(s+z_2)-\angle(s+p_1)-\angle(s+p_2)]$$

$$=|F(s)|(\phi_{z1}+\phi_{z2}-\phi_{p1}-\phi_{p2})$$

In the figure: 1 zero is enclosed within Γ_S, as s traverses 360° around Γ_S clockwise, $\Delta\phi_{z1}=2\pi$, $\Delta\phi_{z2}=0$, $\Delta\phi_{p1}=0$, $\Delta\phi_{p2}=0$.

Thus the net angle $\Delta\angle F(s)=2\pi$, on [F]-plane Γ_F encircles the origin once in clockwise direction.

If $\varGamma_S$ encloses $\boldsymbol{Z}$ zeros and $\boldsymbol{P}$ poles.



The net angle of $\varGamma_F$ of the contour in the $F(s)$-plane is:

$$\Delta\phi_F = \Delta\phi_Z - \Delta\phi_P = 2\pi\boldsymbol{Z} - 2\pi\boldsymbol{P}$$

即当s沿$\varGamma_S$ 顺时针方向移动一周时，映射曲线$\varGamma_F$在$[F]$平面的相角变化为：

$$2\pi N = 2\pi\boldsymbol{Z} - 2\pi\boldsymbol{P}$$

The net number of encirclements of the $F(s)$-plane is:

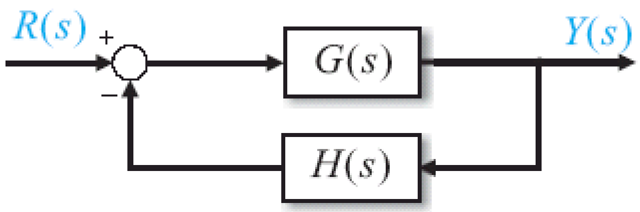
因此， $\varGamma_F$顺时针包围原点的周数为：

$$\boldsymbol{N} = \boldsymbol{Z} - \boldsymbol{P}$$

5.3.2 Nyquist Stability Criterion

根据开环幅相频率特性图判断闭环系统的稳定性

对于闭环控制系统：



设 $G(s)=\dfrac{K_1P_1(s)}{Q_1(s)}$ $H(s)=\dfrac{K_2P_2(s)}{Q_2(s)}$

开环传函： $$G(s)H(s)=\frac{K_1P_1(s)}{Q_1(s)}\frac{K_2P_2(s)}{Q_2(s)}=\frac{KP(s)}{Q(s)}=K\frac{\prod_{i=1}^{m}(s+z_{oi})}{\prod_{j=1}^{n}(s+p_{oj})}$$

闭环传函： $$\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{K_1P_1(s)Q_2(s)}{Q(s)+KP(s)}=\frac{K_1P_1(s)Q_2(s)}{D(s)}$$

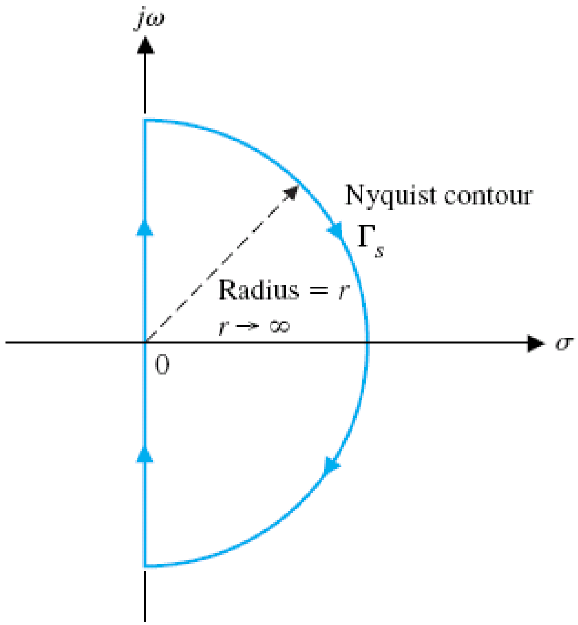
特征多项式： $D(s)=Q(s)+KP(s)$

——开环传函$G(s)H(s)$的分母与分子之和

5.3.2 Nyquist Stability Criterion

Let $F(s)=\dfrac{D(s)}{Q(s)}=1+\dfrac{KP(s)}{Q(s)}=1+G(s)H(s)$

The **Nyquist contour** that encloses the entire right-hand *s*-plane clockwise. Nyquist contour passes along the $j\omega$-axis and completed by a semicircular path of radius *r*, where *r* approaches infinity.



在*s*平面上做闭曲线Γ_S：整个虚轴和*s*右半平面上半径为无穷大的半圆—称为**Nyquist曲线**(按顺时针方向)，也称为“D形围线”(形状象字母D)

5.3.2 Nyquist Stability Criterion

According to Cauchy's theorem: the net number N of encirclements of the origin of the $F(s)$-plane as s traverses along Nyquist contour a circle (ω:-∞→ ∞), is:

由映射定理：s 顺时针沿着D形围线Γ_S变化一周时(ω:-∞→ ∞), $F(s)$在[F]平面上的轨迹Γ_F顺时针包围原点的周数N为：

$$N = Z - P \qquad (5.17)$$

$$F(s)=\frac{D(s)}{Q(s)}=1+\frac{KP(s)}{Q(s)}=1+G(s)H(s)$$

Z = **F(s)**在S右半平面的**零点数**

= 特征多项式$D(s)$在S右半平面的**零点**数(即在S右半平面的特征根数)

P = **F(s)**在S右半平面的**极点数**

= 开环传函在S右半平面的极点数($Q(s)$的**零点**)

5.3.2 Nyquist Stability Criterion

If we know P, $N \rightarrow$ we can get Z

Closed loop system stable — $Z = 0$,

- 闭环系统稳定的充要条件

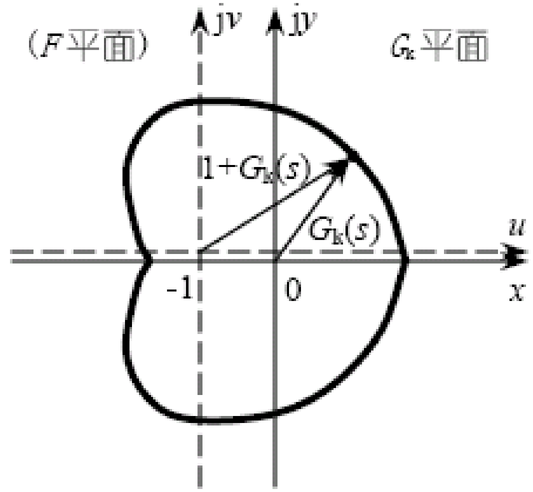
$$N = -P \tag{5.18}$$

即 s 顺时针沿 D 形围线 Γ_S 变化一周时，在$[1+GH]$平面上, $1+G(s)H(s)$ 的轨迹 Γ_F 须逆时针包围原点 P 周。这就是Nyquist稳定判据的基本内容。

5.3.2 Nyquist Stability Criterion

- 几点注记

1. 在[1+GH]平面上轨迹 1+$G(s)H(s)$ 对原点的包围周数，等于在[GH] 平面上轨迹$G(s)H(s)$对(-1,0)点的包围周数;



- **Nyquist稳定性判据** [$G(s)H(s)$ 在 $j\omega$ 轴上无零点、极点的情况]： $G(s)H(s)$ 在 s 右半平面有 ***P*** 个极点，且 $\lim_{s\to\infty} G(s)H(s) =$ 常量，闭环系统稳定的充要条件为，当 s 顺时针沿 D 形围线变化一周时，[GH]平面上$G(s)H(s)$ 的轨迹须逆时针包围 (-1,0) 点 ***P*** 周。

5.3.2 Nyquist Stability Criterion

2. $n>m$时， $\lim\limits_{s\to\infty} G(s)H(s)=0$ ，当s沿D形围线的无穷大半圆变化时，$G(s)H(s)$映射为$[GH]$平面上一点——原点。因此，当$n>m$时，只需要考虑s沿虚轴变化$(s=j\omega, -\infty<\omega<\infty)$时，$G(j\omega)H(j\omega)$的轨迹——用频率特性代替传函。并且，$G(j\omega)H(j\omega)$和$G(-j\omega)H(-j\omega)$关于实轴对称；

5.3.2 Nyquist Stability Criterion

3. A. 开环不稳定，$P\neq0$。要使闭环稳定,须$Z=0\rightarrow N=-P$，即$G(s)H(s)$轨迹须逆时针包围(-1,0)点P周；

B. 开环稳定，P=0。要使闭环稳定,须Z=0→N =0，即G(s)H(s)轨迹须不包围(-1,0)点；

对于闭环不稳定系统，由Nyquist判据可知s右半平面上的特征根数为：

$$Z = N + P \tag{5.19}$$

5.3.2 Nyquist Stability Criterion

4. $G(s)H(s)$在s平面的虚轴上有极点或者零点时

问题： D形围线不能通过$G(s)H(s)$的零点或极点。

处理方法：

对于$G(s)H(s)$在s平面上的原点或虚轴上有极(零)点，在s平面上作D形围线时应避开这些点——在这些点的右侧用半径为$\varepsilon(\varepsilon \to 0)$的半圆绕过这些点。(若在这些点的左侧画$\varepsilon$-半圆，则这些点要记入D形围线中的开环极(零)点数。)

<**E5.2**> The Open-loop transfer function of a unit feedback system is $G(s)H(s)=\dfrac{K}{(T_1s+1)(T_2s+1)}$, where $T_1,T_2>0$

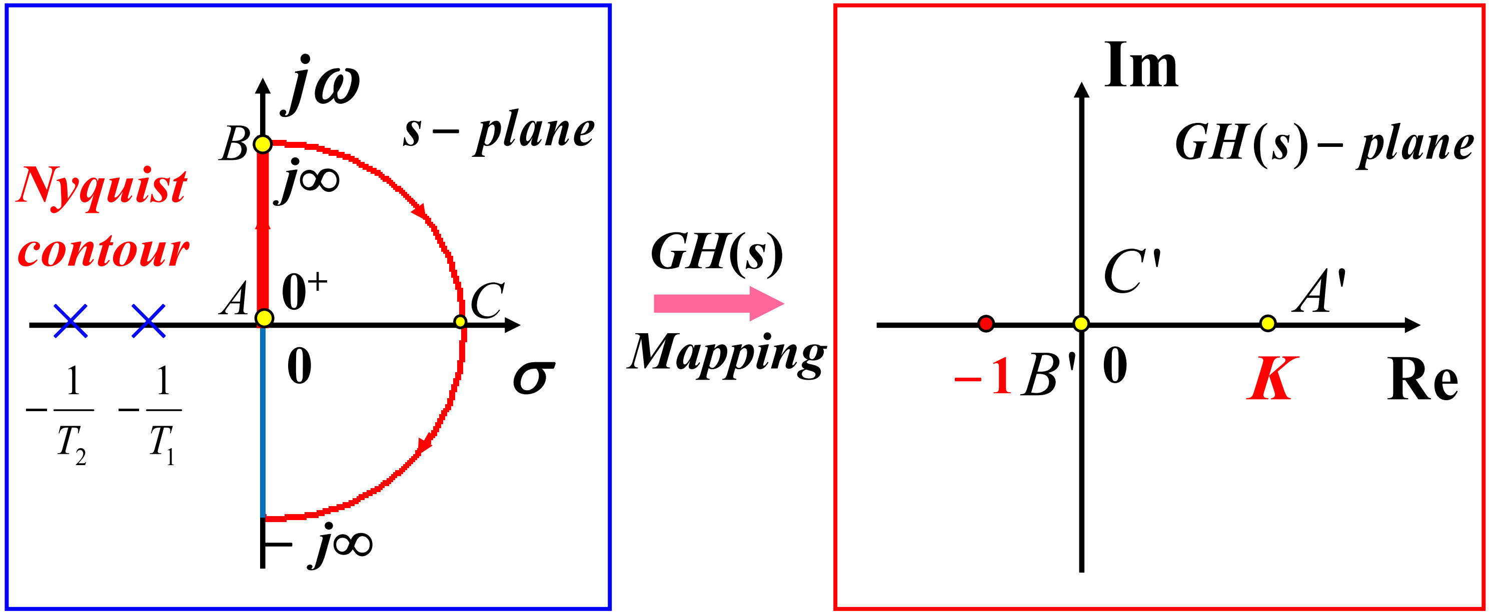
Determine whether the system is stable by using the Nyquist stability criterion.

Solution:

(1) **Determine the Γ_{GH}-contour:** ***Determine N, the number of encirclements of the* (-1,0) *point of the GH(s)-plane***

① *Select the Nyquist contour*

② *Map the Nyquist contour into the GH(s)-plane*

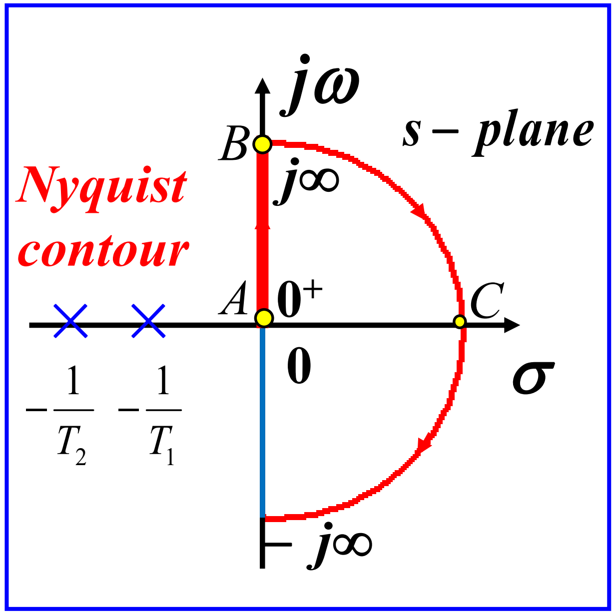


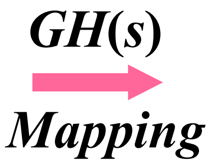
$$GH(s)=\frac{K}{(T_1s+1)(T_2s+1)}$$

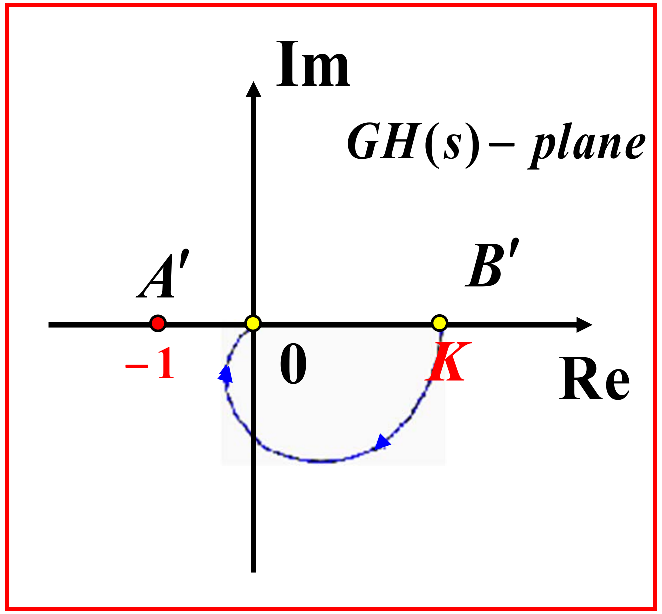
$$A: s=0e^{j90^\circ} \quad \rightarrow \quad A': GH(s)=Ke^{j0^\circ}$$

$$B: s=\infty e^{j90^\circ} \quad \rightarrow \quad B': GH(s)=\frac{K}{T_1T_2s^2}=\frac{K}{\infty e^{j90^\circ\times 2}}=0e^{-j180^\circ}$$

$$C: s=\infty e^{j0^\circ} \quad \rightarrow \quad C': GH(s)=0e^{j0^\circ}$$





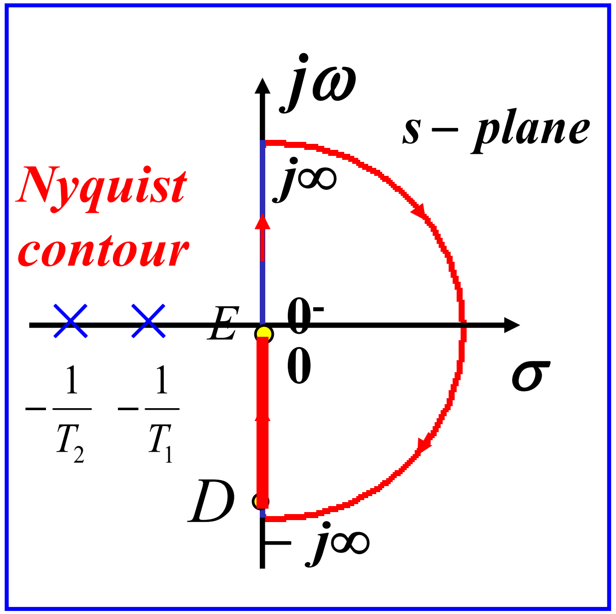


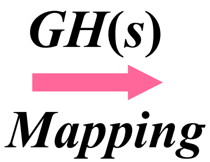
$$A \to B: \quad s = j\omega(\omega : 0^+ \to \infty)$$

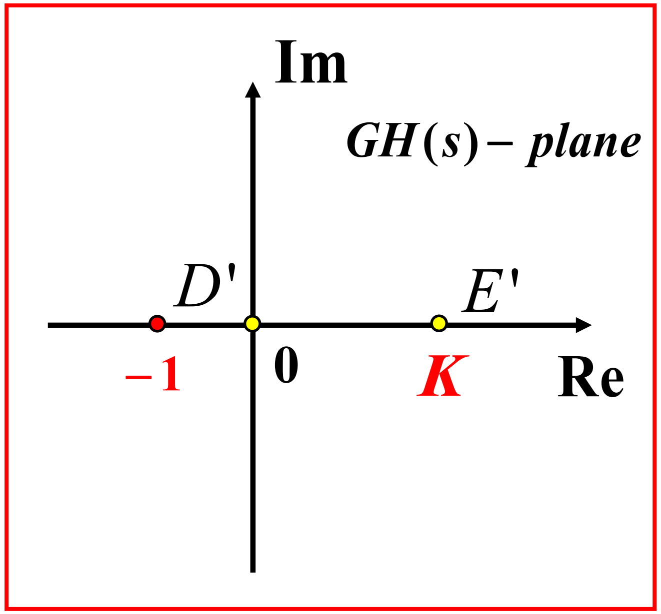
$$A' \to B': \quad GH(s) = \frac{K}{(1 + j\omega T_1)(1 + j\omega T_2)}$$

$$= \frac{K}{\sqrt{1 + (\omega T_1)^2}\sqrt{1 + (\omega T_2)^2}} \angle\left(-tg^{-1}\omega T_1 - tg^{-1}\omega T_2\right)$$

Locates on the third and fourth quadrants



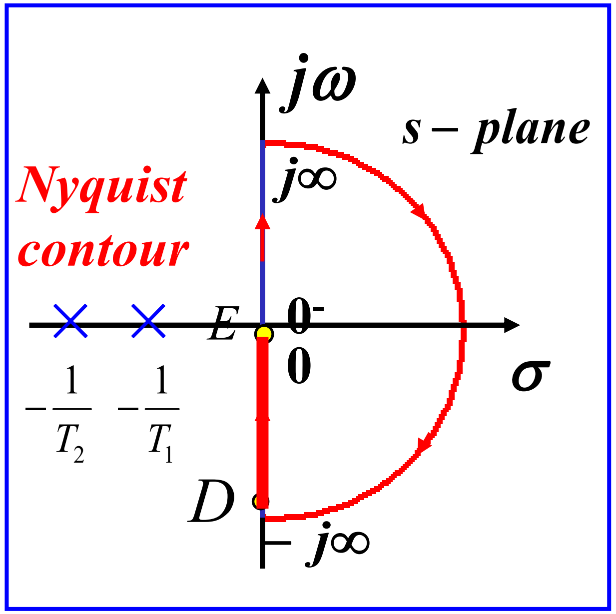




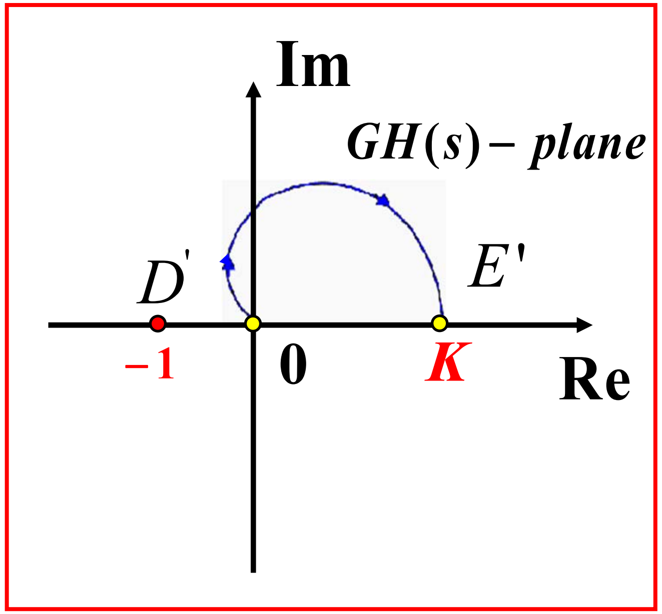
$$GH(s)=\frac{K}{(T_1 s+1)(T_2 s+1)}$$

$$D: s=\infty e^{-j90^\circ} \quad \Rightarrow \quad D': GH(s)=\frac{K}{T_1T_2s^2}=\frac{K}{\infty e^{-j90^\circ\times 2}}=0e^{j180^\circ}$$

$$E: s=0e^{-j90^\circ} \quad \Rightarrow \quad E': GH(s)=Ke^{j0^\circ}$$



GH(s)
Mapping



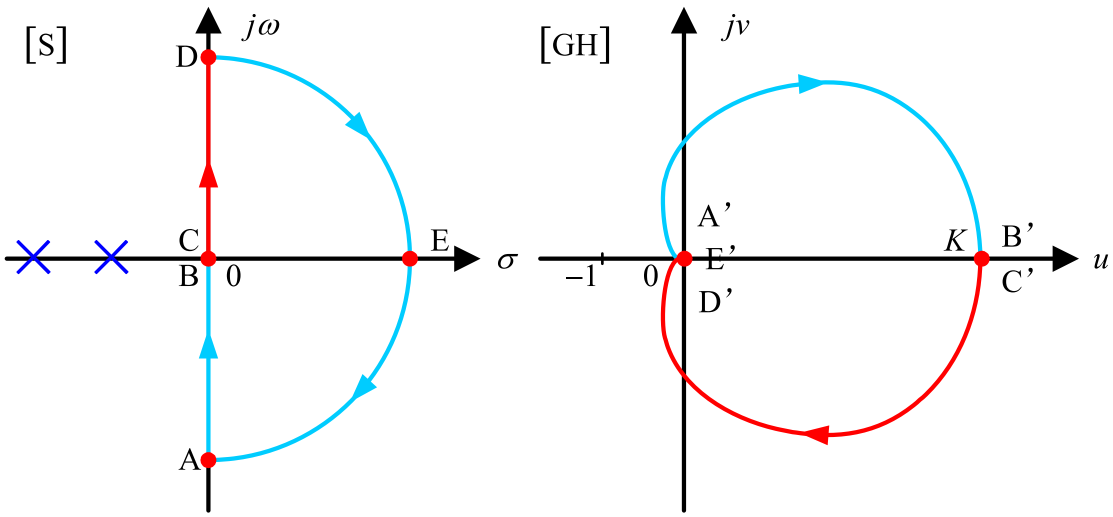
$$D \to E: \quad s = -j\omega(\omega: -\infty \to 0^-)$$

$$D' \to E': \quad GH(s) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle\left(-tg^{-1}\omega T_1 - tg^{-1}\omega T_2\right)$$

Locates on the 1st and 2nd quadrants

(2) Determine the stability of the system



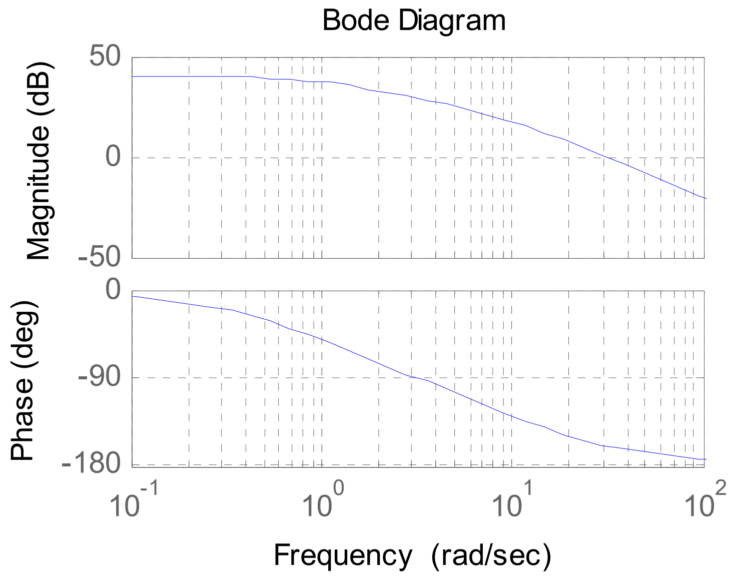
The open –loop transfer function has no poles in the right-hand s-plane ,therefore $\boldsymbol{P=0}$

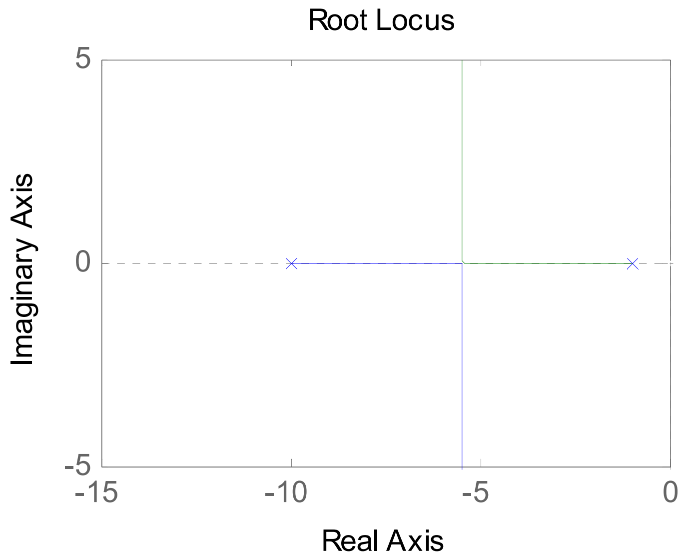
The GH(s) – contour does not encircle the -1 point, thus

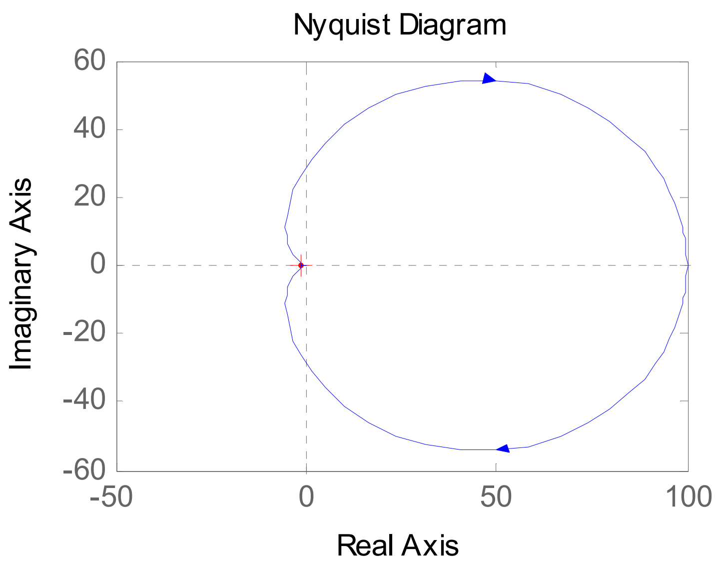
$$\boldsymbol{N=0}$$

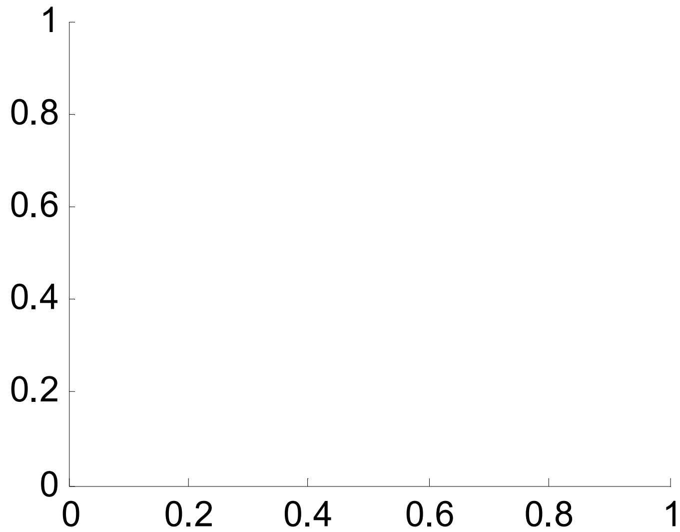
Therefore $\boldsymbol{Z=N+P=0}$

The system is stable.


Bode Diagram
Magnitude (dB)
50
0
-50
Phase (deg)
0
-90
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)


Root Locus
Imaginary Axis
5
0
-5
-15
-10
-5
0
Real Axis


Nyquist Diagram
Imaginary Axis
60
40
20
0
-20
-40
-60
-50
0
50
100
Real Axis

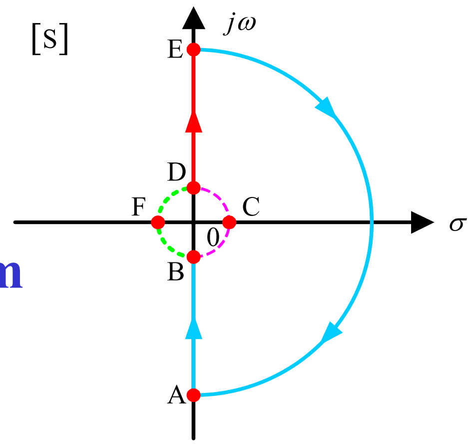

1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

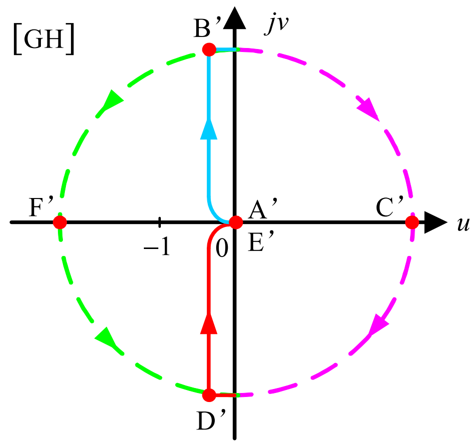
<E5.3> The Open-loop transfer function of a unit feedback system is

$$G(s)H(s)=\frac{K}{s(Ts+1)}$$

$$T>0, K>0$$

Determine whether the system is stable by using the Nyquist stability criterion.





$$A: s=\infty e^{-j90^\circ} \qquad A': G(s)H(s)=\frac{K}{Ts^2}=\frac{K}{\infty e^{-j90^\circ\times 2}}=0e^{j180^\circ}$$

$$B: s=\varepsilon e^{-j90^\circ} \qquad B': G(s)H(s)=\frac{K}{s}=\infty e^{j90^\circ}$$

$$C: s=\varepsilon e^{j0^\circ} \qquad C': G(s)H(s)=\infty e^{j0^\circ}$$

$$D: s=\varepsilon e^{j90^\circ} \qquad D': G(s)H(s)=\infty e^{-j90^\circ}$$

$$E: s=\infty e^{j90^\circ} \qquad E': G(s)H(s)=0e^{-j180^\circ}$$

$P=0, N=0,$

$Z=N+P=0$

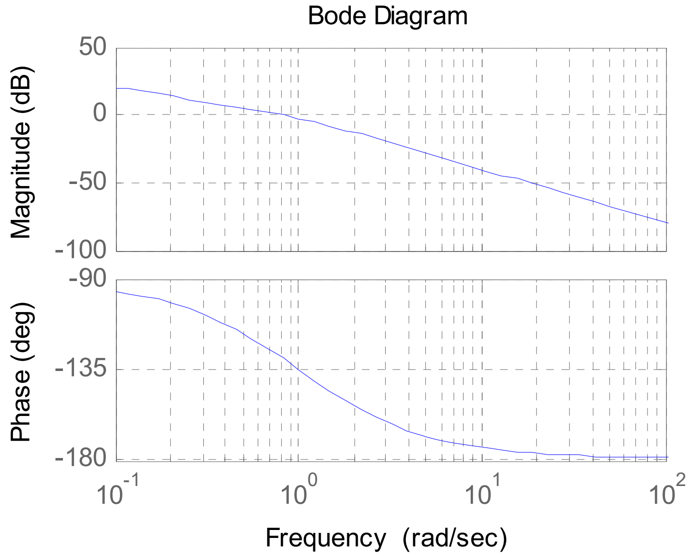
闭环系统稳定

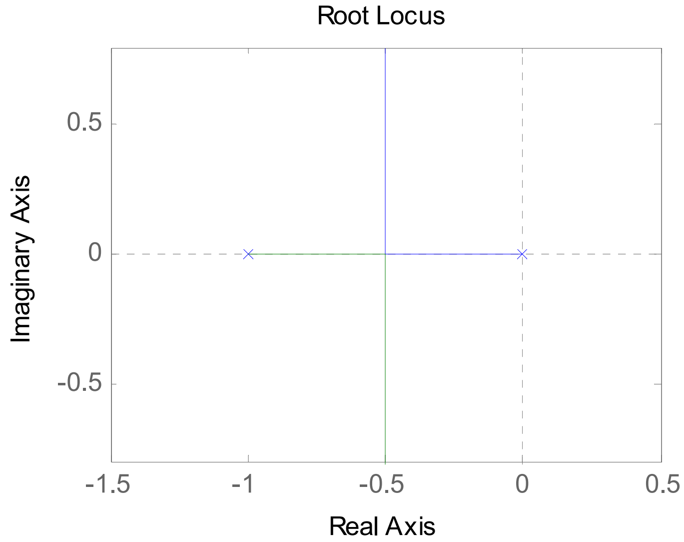
若： $F: s=\varepsilon e^{\pm j180^\circ}$

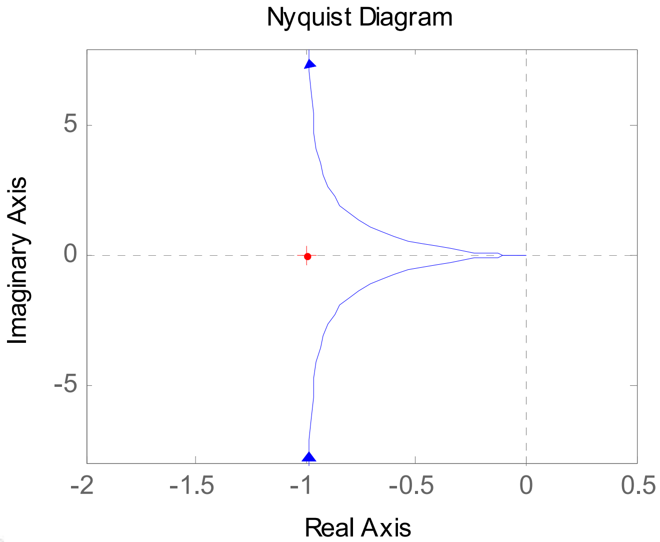
$F': G(s)H(s)=\infty e^{\mp j180^\circ}$

则 $P=1, N=-1,$

$Z=N+P=0$


Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

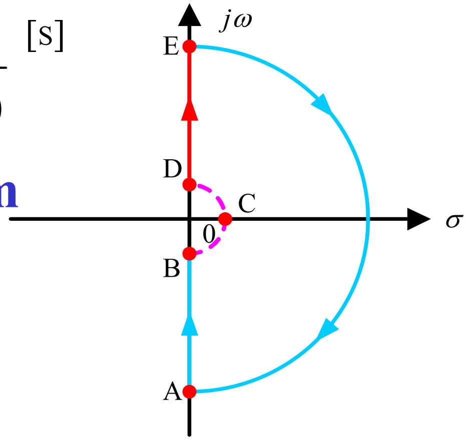

Root Locus
Imaginary Axis
0.5
0
-0.5
-1.5
-1
-0.5
0
0.5
Real Axis

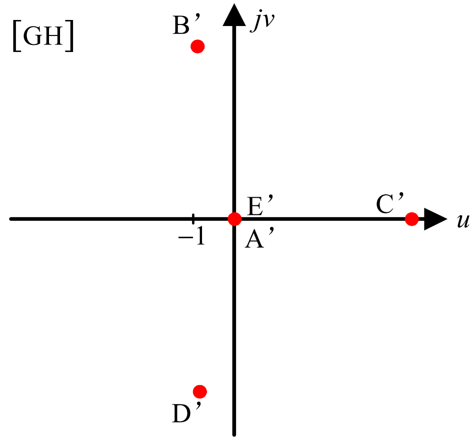

Nyquist Diagram
Imaginary Axis
5
0
-5
-2
-1.5
-1
-0.5
0
0.5
Real Axis

<E5.4> The Open-loop transfer function of a unit feedback system is

$$G(s)H(s)=\frac{K}{s(0.2s+1)(0.5s+1)}$$

Determine whether the system is stable by using the Nyquist stability criterion.





$$(1)\quad A: s=\infty e^{-j90^\circ} \qquad A': G(s)H(s)=\frac{K}{s^3}=0e^{j270^\circ}$$

$$(2)\quad B: s=\varepsilon e^{-j90^\circ} \qquad B': G(s)H(s)=\frac{K}{s}=\infty e^{j90^\circ}$$

(3) Determine the point where the *GH*(s)-locus intersects the real axis

Let $\quad g(s)=s(0.2s+1)(0.5s+1)=0.1s^3+0.7s^2+s$

$$v=\mathrm{Im}[g(j\omega)]=0$$

$$v=\mathrm{Im}[-0.1\,j\omega^3-0.7\omega^2+j\omega]=\omega(1-0.1\omega^2)=0$$

$$\omega=-\sqrt{10}\,,\ G(j\omega)H(j\omega)|_{\omega=-\sqrt{10}}=-K/7$$

(4) $C: s = \varepsilon e^{j0^\circ}$

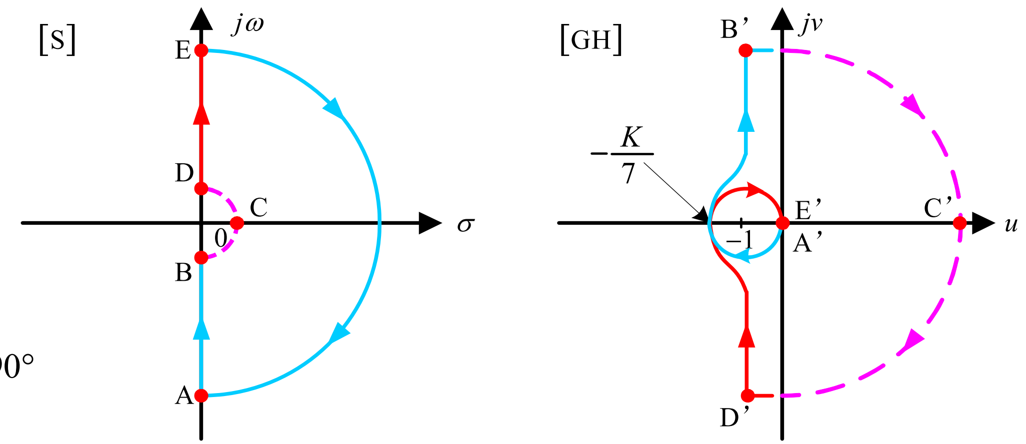
$C': G(s)H(s) = \infty e^{j0^\circ}$

(5) $D: s = \varepsilon e^{j90^\circ}$

$D': G(s)H(s) = \infty e^{-j90^\circ}$

(6) $E: s = \infty e^{j90^\circ}$

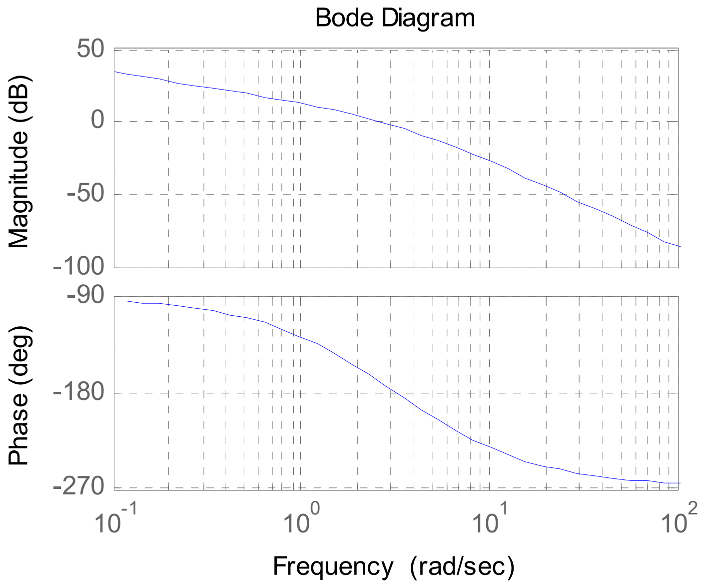
$E': G(s)H(s) = 0 e^{-j270^\circ}$

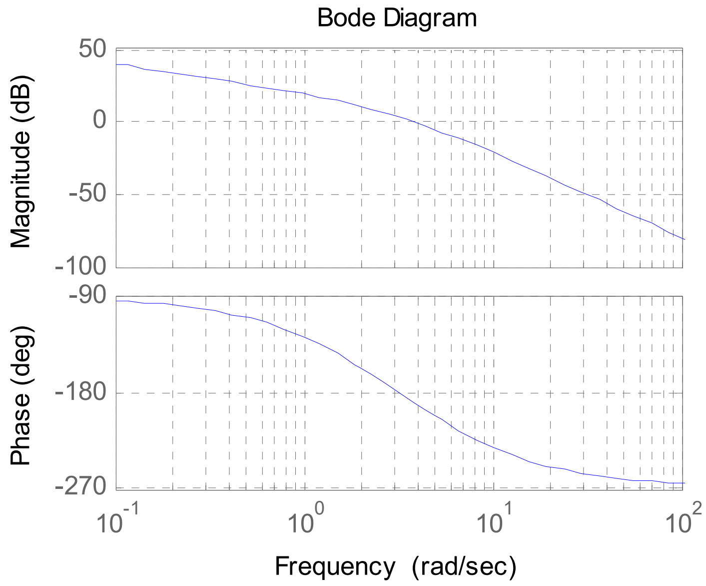


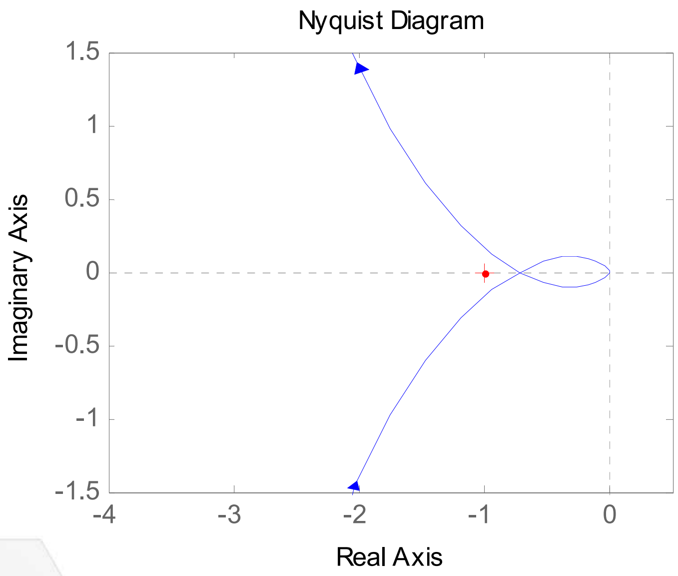
Determine the stability of the system:

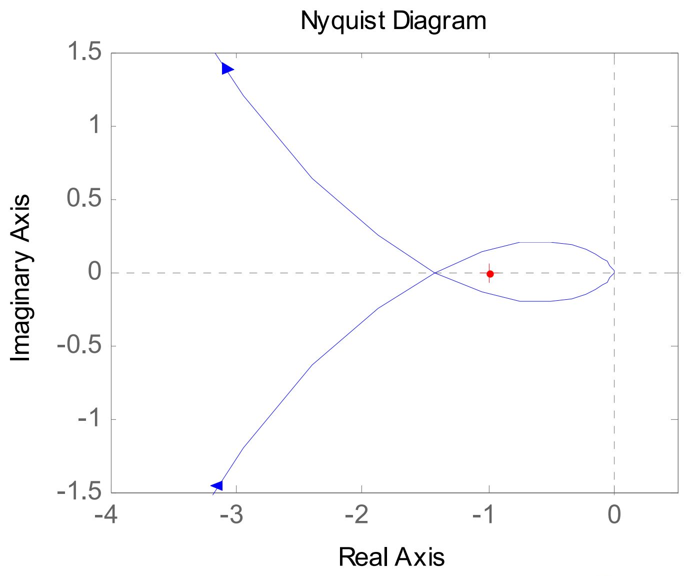
a) When K<7, ***N*=0** and ***P*=0**, thus ***Z*=*N*+*P*=0**, **System is stable**.

b) When K>7, *GH*(*s*) *contour encircles the* -1 *point twice*, thus ***N*=2** and ***P*=0**, thus ***Z*=*N*+*P*=2**, which means there are two poles on the right hand of s-plane, **system is unstable**. 有两个闭环极点(特征根)在s右半平面，闭环系统不稳定


Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-180
-270
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)


Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-180
-270
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

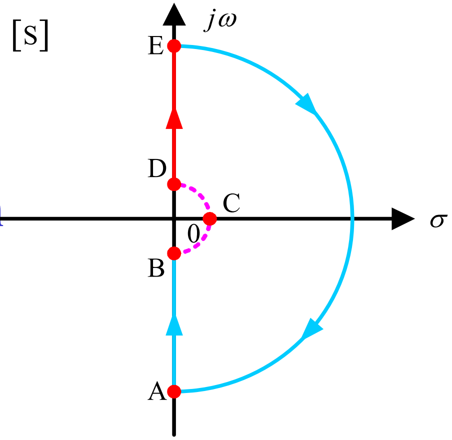

Nyquist Diagram
Imaginary Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-4
-3
-2
-1
0
Real Axis

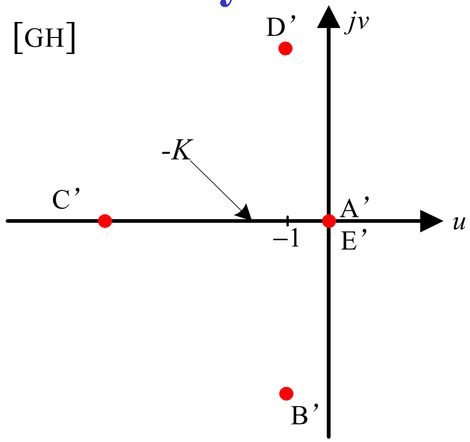

Nyquist Diagram
Imaginary Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-4
-3
-2
-1
0
Real Axis

<E5.5> The Open-loop transfer function of a unit feedback system is

$$G(s)H(s)=\frac{K(s+3)}{s(s-1)}$$

Determine whether the system is stable by using the Nyquist stability criterion.





(1) $A: s=\infty e^{-j90^\circ}$ $\quad A': G(s)H(s)=\dfrac{K}{s}=0e^{j90^\circ}$

(2) $B: s=\varepsilon e^{-j90^\circ}$ $\quad B': G(s)H(s)=-\dfrac{K}{s}=\infty e^{j270^\circ}$

(3) Determine the point where the *GH*(s)-locus intersects the real axis

$$G(j\omega)H(j\omega)=\frac{K(j\omega+3)}{j\omega(j\omega-1)}=-\frac{K(\omega-3j)(1+j\omega)}{\omega(\omega^2+1)}=-\frac{K}{\omega(\omega^2+1)}\left[4\omega+j(\omega^2-3)\right]$$

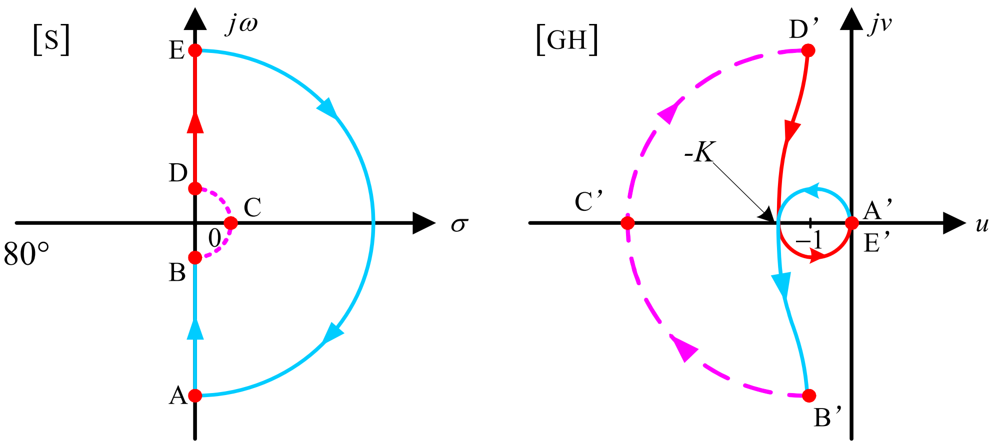
$$v=\mathrm{Im}[G(j\omega)H(j\omega)]=0,\omega^2-3=0,\omega=\pm\sqrt{3}$$

$$u=\mathrm{Re}[G(j\omega)H(j\omega)]\Big|_{\omega=\pm\sqrt{3}}=\frac{K4\omega}{\omega(\omega^2+1)}\Bigg|_{\omega=\pm\sqrt{3}}=-K$$

(4)

$$C: s = \varepsilon e^{j0°}$$

$$C': G(s)H(s) = -\frac{K}{s} = \infty e^{j180°}$$

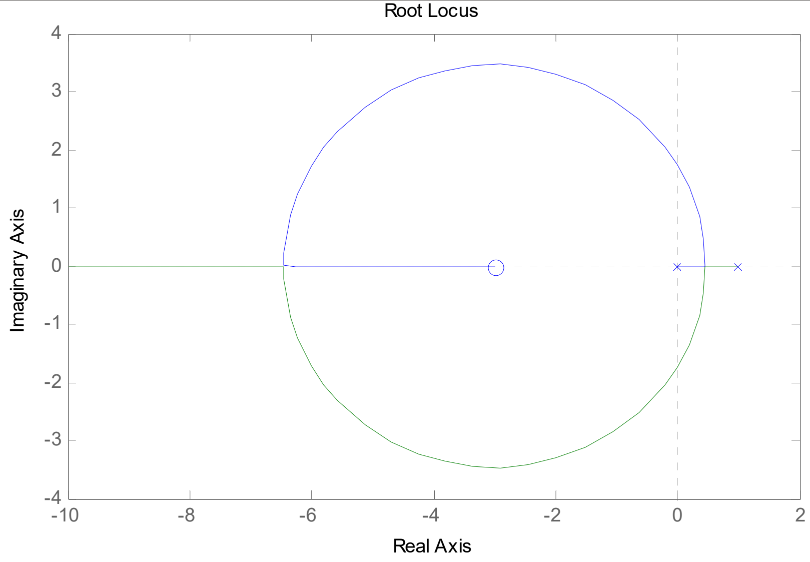


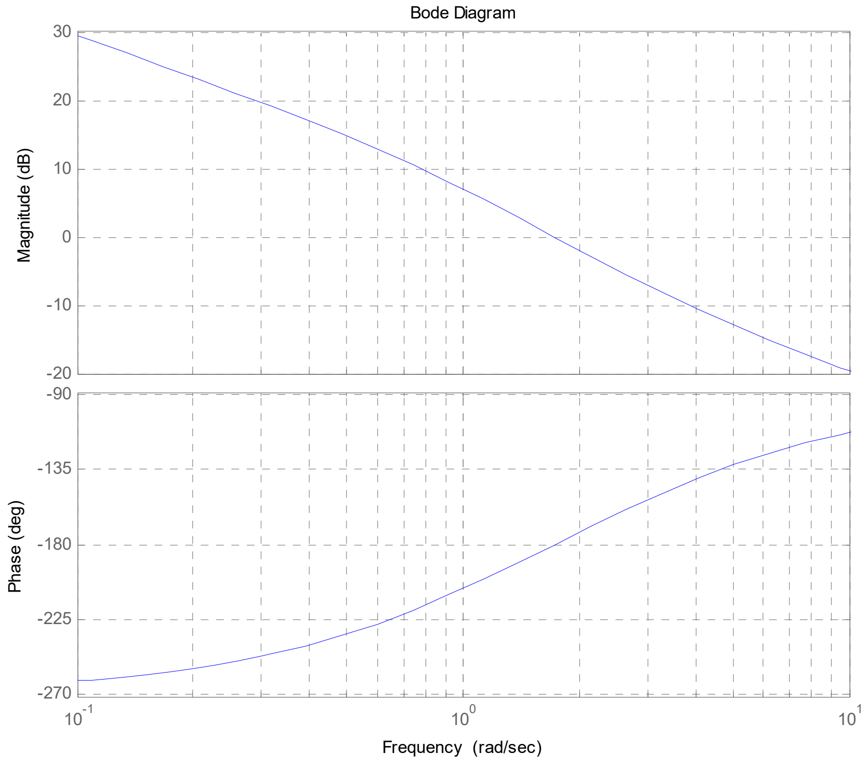
(5) Because the $GH(s)$ contour is symmetric to the real axis. 由 “对称于实轴”，可得到另一半Nyquist轨迹

Determine the stability of the system:

a) When K>1, ***N*=–1** and ***P*=1**,thus ***Z*=*N*+*P*=0**, open-loop system is unstable, the closed-loop system is stable

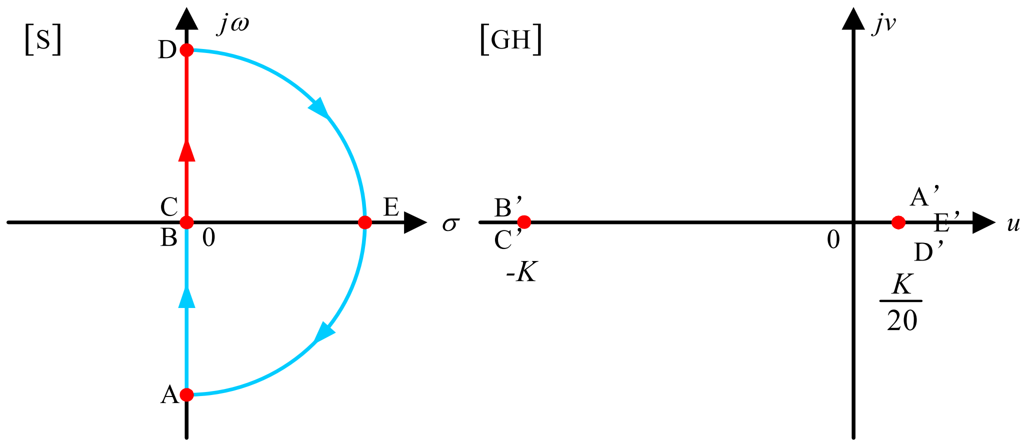
b) When K<1, ***N*=1** and ***P*=1**, thus ***Z*=*N*+*P*=2**, the closed-loop system is unstable

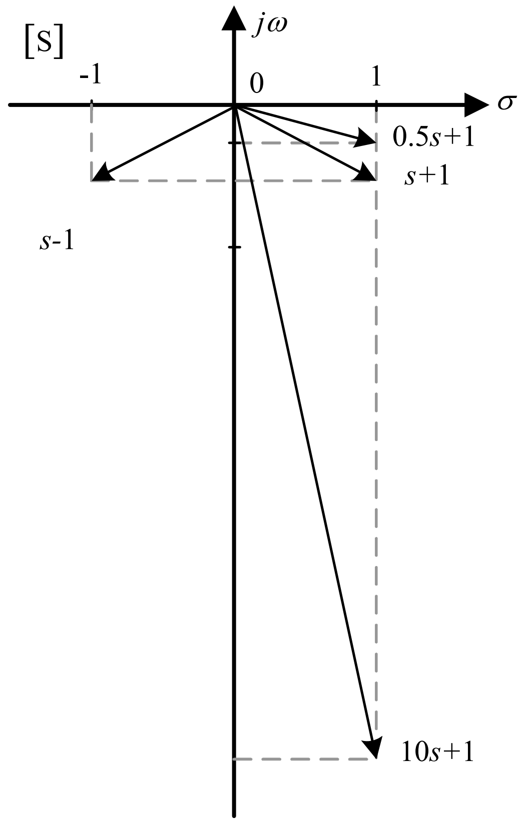

Root Locus
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-10
-8
-6
-4
-2
0
2


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
30
20
10
0
-10
-20
-90
-135
-180
-225
-270
10^{-1}
10^{0}
10^{1}

<E5.6> The Open-loop transfer function of a unit feedback system is

$$G(s)H(s)=\frac{K(0.5s+1)(s+1)}{(10s+1)(s-1)},$$ **Determine the range of K for which the system is stable by using the Nyquist criterion.**





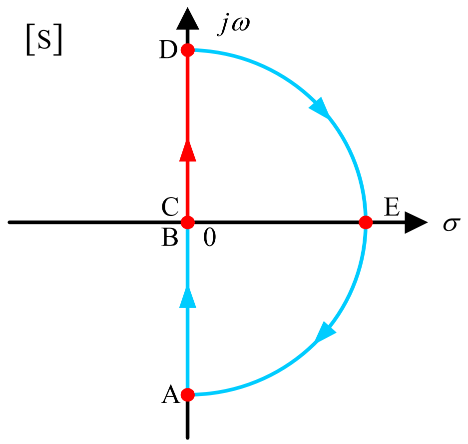
$$A: s=\infty e^{-j90^\circ} \qquad A': G(s)H(s)=\frac{K}{20}e^{j0^\circ}$$

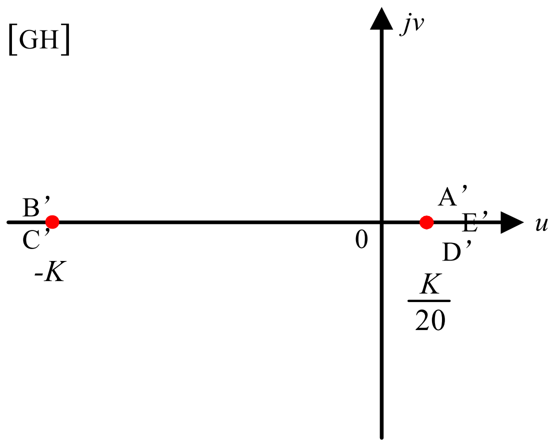
$$B: s=0e^{-j90^\circ} \qquad B': G(s)H(s)=-K=Ke^{j180^\circ}$$

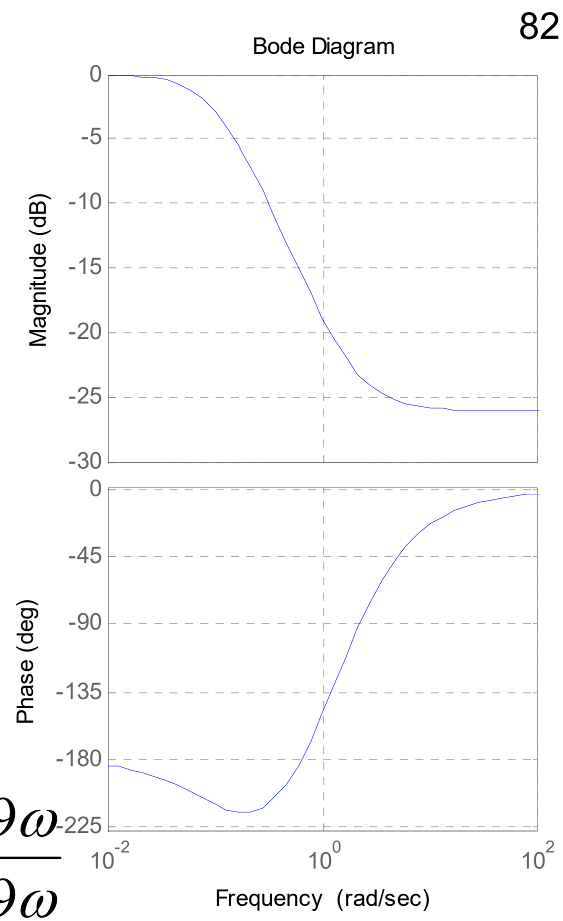
$$C: s=0e^{j90^\circ} \qquad C': G(s)H(s)=-K=Ke^{j180^\circ}$$

$$D: s=\infty e^{j90^\circ} \qquad D': G(s)H(s)=\frac{K}{20}e^{j0^\circ}$$

$$E: s=\infty e^{j0^\circ} \qquad E': G(s)H(s)=\frac{K}{20}e^{j0^\circ}$$







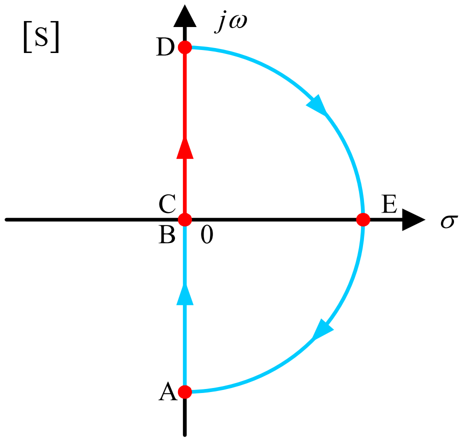
Determine the point that between A’-B’ where the GH(s)-locus intersects the real axis. A’ →B’之间，与实轴的交点：

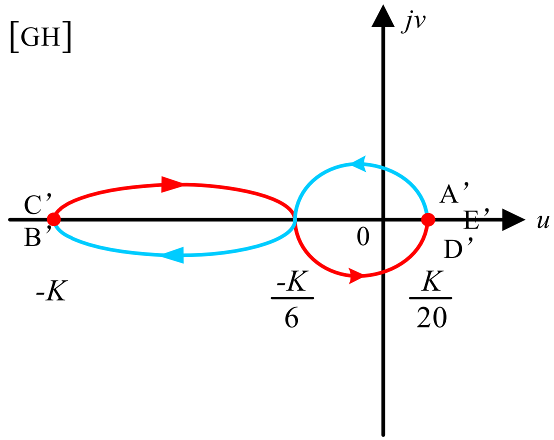
$$G(j\omega)H(j\omega)=\frac{0.5K(2-\omega^2+j3\omega)}{-(1+10\omega^2+j9\omega)}\cdot\frac{1+10\omega^2-j9\omega}{1+10\omega^2-j9\omega}$$

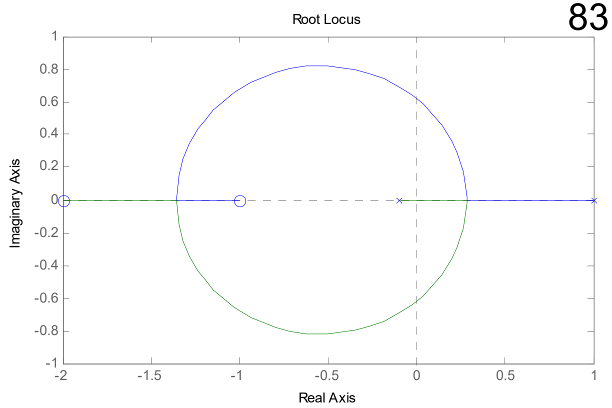
$$=\frac{0.5K[2+46\omega^2-10\omega^4+j\omega(39\omega^2-15)]}{(1+10\omega^2)^2+(9\omega)^2}$$

$$\varphi(\omega)=\tan^{-1}(0.5\omega)+\tan^{-1}(\omega)-\tan^{-1}(10\omega)-[\pi-\tan^{-1}(\omega)]$$

$$=\tan^{-1}(0.5\omega)+2\tan^{-1}(\omega)-\tan^{-1}(10\omega)-\pi$$







$$v = \mathrm{Im}\left[G(j\omega)H(j\omega)\right] = 0, \omega^2 = \frac{15}{39} = \frac{5}{13}$$

$$u = \mathrm{Re}\left[G(j\omega)H(j\omega)\right]_{\omega^2 = \frac{5}{13}} = \frac{0.5K[2 + 46\times\frac{5}{13} - 10\times\left(\frac{5}{13}\right)^2]}{(1+10\times\frac{5}{13})^2 + 81\times\frac{5}{13}} = -\frac{K}{6}$$

Determine the stability of the system:

Because ***P*=1**; when K>6, ***N*= –1**, ***Z*=*N*+*P*=0**.

So the range of K for which the system is stable is 6<K<∞

闭环系统稳定的K值的范围: 6<K<∞

返回

5.4 Stability margin of control system

- **Stability Margin:** is the measure of the relative stability of system which represents the "distance" between the critical stable work point.

- **稳定裕量：** 系统相对稳定性的一种度量，反映系统离临界稳定点的"距离"。

一个工程上可用的控制系统，不仅应稳定，而且应有相当的稳定裕量。

最小相位系统的稳定裕量与频率特性的关系是确定的。这里主要讨论最小相位系统的稳定裕量的计算。非最小相位系统可类似于最小相位系统进行定义和计算稳定裕量。

5.4.1 Minimum Phase System

Minimum Phase System: *A system with transfer function G(s) is called minimum phase if it has no pole or zero in the right half s-plane.*

最小相位系统：

在s右半平面没有极点和零点，且不含时滞环节的传递函数称为最小相位传递函数，反之称为非最小相位传递函数

具有最小相位传递函数的系统称为最小相位系统系统

5.4.1 Minimum Phase System

Key Words:

Minimum Phase System

Non-minimum Phase System

Stability Margin **（Phase Margin/Gain Margin）**

5.4.1 Minimum Phase System

The characteristics of a minimum phase system:

1) ***When ω varies from zero to infinity , the range of phase shift of a minimum phase system is the least possible corresponding to systems with same amplitude frequency characteristics.***

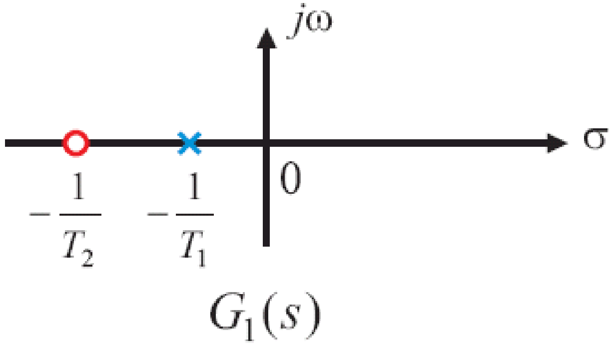
在具有相同幅频特性的系统中，ω:0→∞时，最小相位系统相角变化最小

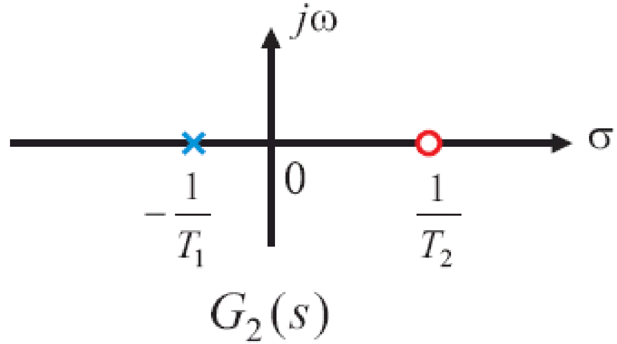
<E5.7> Consider the following two system.

$$G_1(s)=\frac{1+T_2 s}{1+T_1 s},\qquad G_2(s)=\frac{1-T_2 s}{1+T_1 s},\qquad T_1>T_2>0$$

A minimum phase system

A non-minimum phase system





The frequency characteristics of the two systems are

$$G_1(j\omega)=\frac{1+j\omega T_2}{1+j\omega T_1}\qquad G_2(j\omega)=\frac{1-j\omega T_2}{1+j\omega T_1}$$

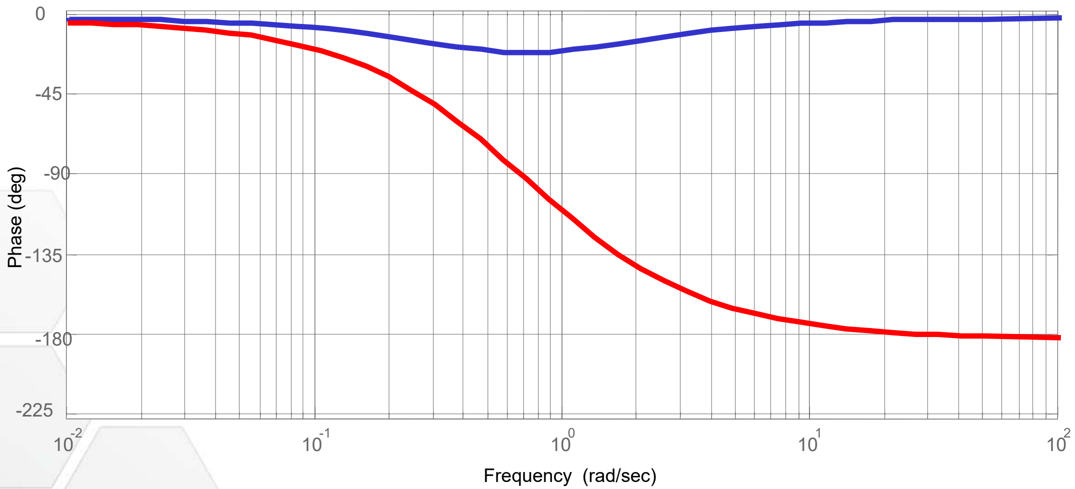
$$G_1(j\omega)=\frac{1+j\omega T_2}{1+j\omega T_1} \qquad G_2(j\omega)=\frac{1-j\omega T_2}{1+j\omega T_1}$$

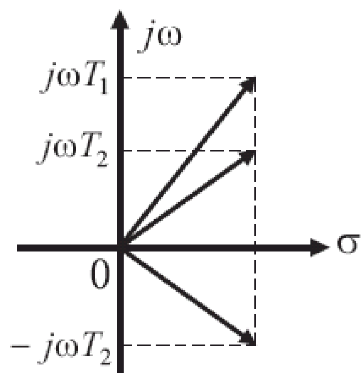
$$\begin{cases} |G_1(j\omega)|=\sqrt{\dfrac{1+\omega^2 T_2^{\,2}}{1+\omega^2 T_1^{2}}} \\ \phi_1(\omega)=tg^{-1}\omega T_2-tg^{-1}\omega T_1 \end{cases} \qquad \begin{cases} |G_2(j\omega)|=\sqrt{\dfrac{1+\omega^2 T_2^{\,2}}{1+\omega^2 T_1^{2}}} \\ \phi_2(\omega)=tg^{-1}(-\omega T_2)-tg^{-1}\omega T_1 \end{cases}$$

0° ~ -90° ***0° ~ -180°***

When $\omega: \ 0\rightarrow\infty$ *,we have*

$$|G_1(j\omega)|=|G_2(j\omega)| \qquad \Delta\varphi_1(\omega)<\Delta\varphi_2(\omega)$$



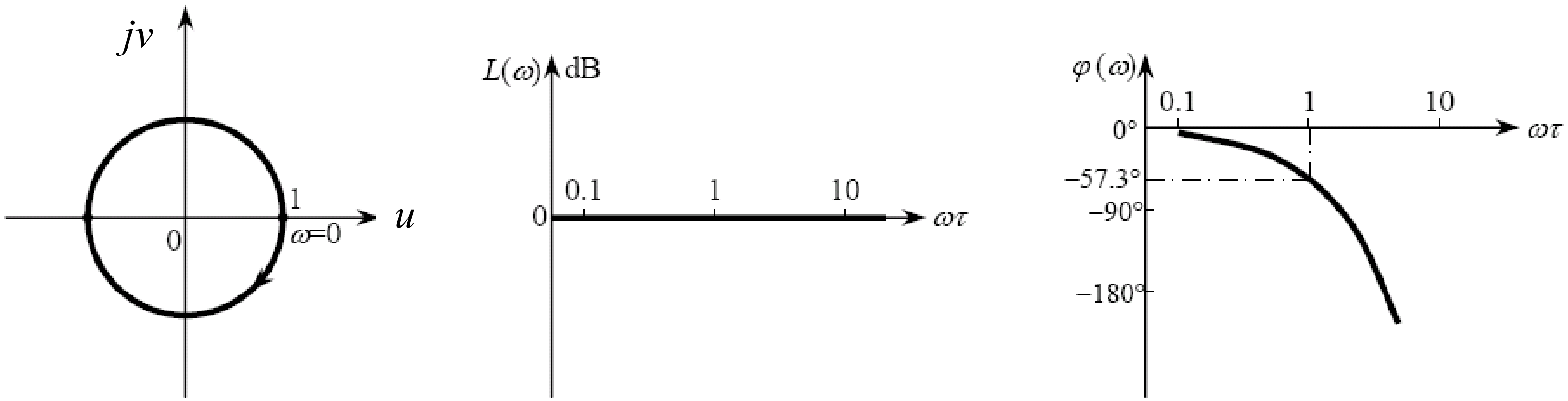


5.4.1 Minimum Phase System

时滞环节是非最小相位的，其频率特性

$$e^{-\tau s}\Big|_{s=j\omega} = e^{-j\omega\tau} = A(\omega)e^{j\varphi(\omega)}$$

$$A(\omega)=1, \varphi(\omega)=-\omega\tau(\text{弧度})=-57.3\omega\tau(\text{度})$$



(a)极坐标图　(b)对数幅频曲线　(c)对数相频曲线

系统的**相位滞后越大**，其稳定性问题就越复杂，所以控制系统尽可能避免具有非最小相位传函的元件

5.4.1 Minimum Phase System

2) ***When ω is going to infinity , the slope of logarithm amplitude characteristic and the phase angle of a minimum phase system are respectively:***

当$\omega\to\infty$时，最小相位系统对数幅频特性 的斜率和相角分别为：

Slope $\quad -20(n-m)dB/dec$

Phase angle $\quad -90^{\circ}(n-m)$

n: The order of the denominator polynomial of transfer function

m: The order of the numerator polynomial of transfer function

$\omega=\infty$	最小相位系统	非最小相位系统
幅频特性的斜率	$-20(n-m)$dB/dec	$-20(n-m)$dB/dec
相频	相角为$-90^{\circ}(n-m)$	相角滞后大于$90^{\circ}(n-m)$

5.4.1 Minimum Phase System

3) ***For minimum phase systems, the phase of the frequency response is uniquely determined by its magnitude.***

对于最小相位系统，其相频特性由它的幅频特性唯一确定

With the above characteristics, the phase of a minimum phase system can be sketched versus frequency using the information contained in the magnitude plot.

5.4.2 Stability Margin

The stability margins of a stable minimum phase margin system can be represented as **Phase Margin** and **Gain Margin** respectively.

最小相位系统的稳定裕量，分为相角裕量和增益裕量。

1、 **Phase Margin** Φ_{pm}： *Phase margin is defined as the amount of phase shift of the GH(jω) at unity magnitude that will result in a marginally stable system with intersections of the (-1,0) point on the Nyquist diagram.*

相角裕量给出了保证系统稳定的最大冗余相角。

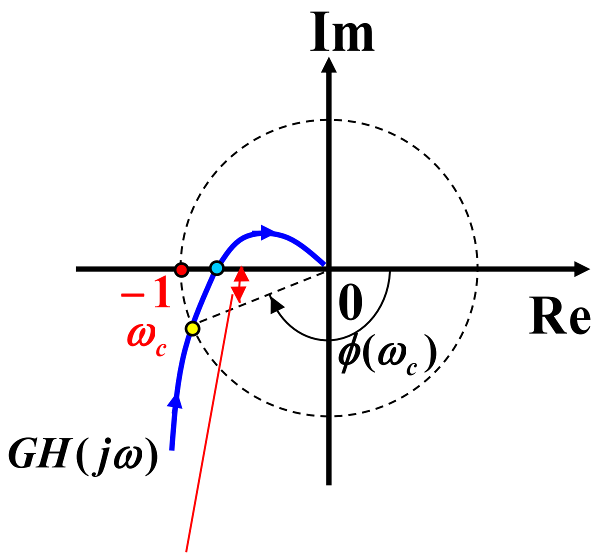
当开环频率特性的幅值等于$|G(j\omega)H(j\omega)|=1$时，其相角与$-180°$之差称为“系统的相角裕量Φ_{pm}”。

5.4.2 Stability Margin

$$\phi_{pm} = \varphi(\omega_c) - (-180°) = 180° + \varphi(\omega_c) \qquad (5.20)$$

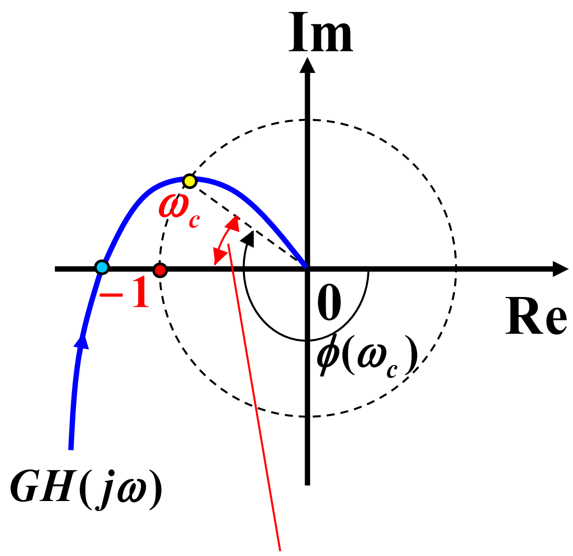
其中，$\varphi(\omega_c) = \angle G(j\omega_c)\,H(j\omega_c)$，由正实轴顺时针转到矢量$G(j\omega_c)\,H(j\omega_c)$的角度($\varphi(\omega_c)$ 是一个负角)

ω_c：幅穿频率，由 $|G(j\omega_c)H(j\omega_c)|=1$ 确定



Phase margin $\phi_{pm}>0$

闭环系统稳定



Phase margin $\phi_{pm}<0$

闭环系统不稳定

5.4.2 Stability Margin

2、 **Gain Margin** *GM*：*The gain margin is defined as the* ***reciprocal*** *of the gain* $|GH(j\omega)|$ *at the frequency at which the phase angle reaches* -180°.

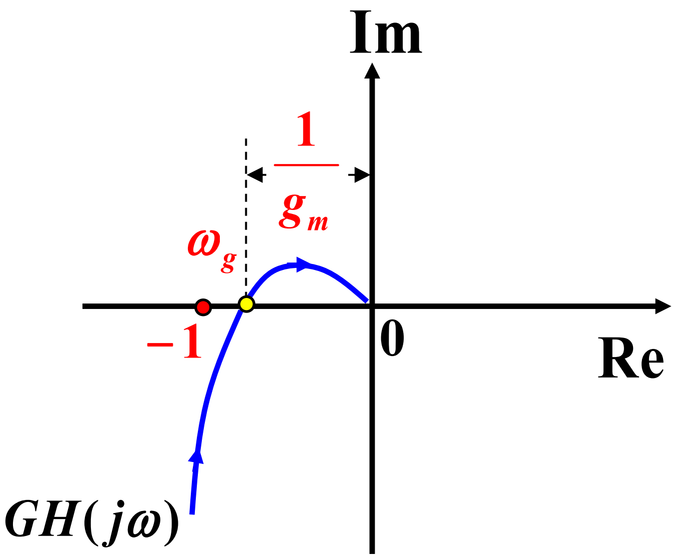
增益裕量定义为当$GH(j\omega)$的相角为-180°时$GH(j\omega)$幅值的倒数：

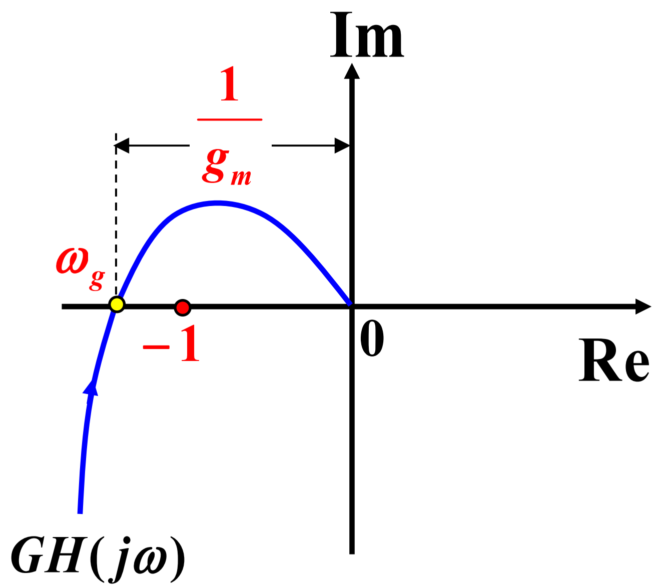
$$g_m = \frac{1}{\left|GH(j\omega_g)\right|} \qquad (5.21)$$

or

$$GM = 20\lg g_m = -20\lg\left|GH(j\omega_g)\right| \quad (dB) \qquad (5.22)$$

ω_g： 相穿频率,由 $\varphi(\omega_g) = \angle G(j\omega_g)H(j\omega_g) = -180^\circ$ 确定





Gain margin

$g_m>1$, $GM>0$

称为正增益裕量

$(20\lg|G(j\omega_g)H(j\omega_g)|<0)$

闭环系统稳定

Gain margin

$g_m<1$, $GM<0$

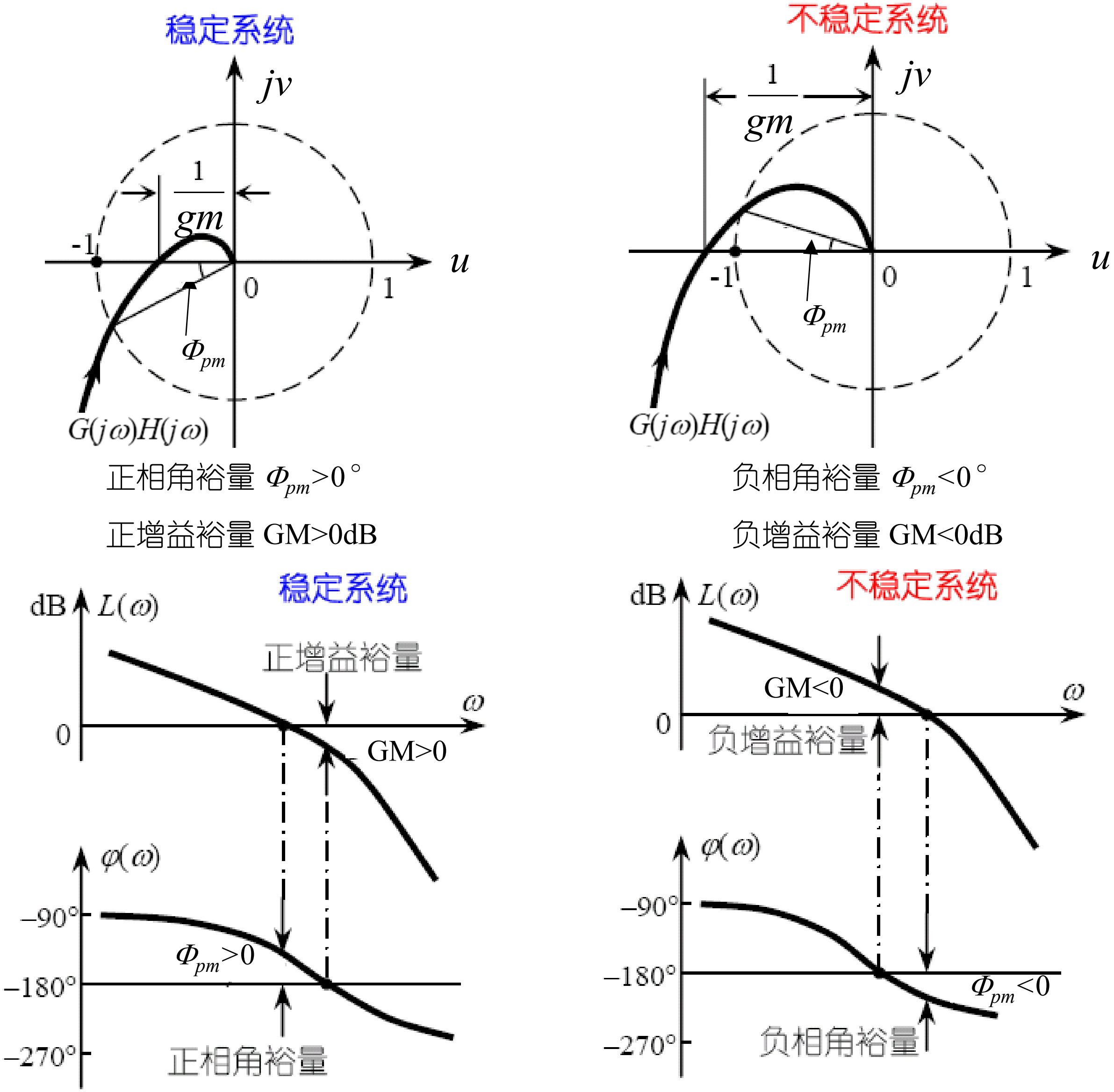
称为负增益裕量

$(20\lg|G(j\omega_g)H(j\omega_g)|>0)$

闭环系统不稳定

最小相位系统，相角裕量γ和 增益裕量g_m均为正值时，闭环系统稳定

5.4.2 Stability Margin



5.4.2 Stability Margin

<E5.7> *The open-loop transfer function of a unity feedback system is*

$$G(s)=\frac{K}{s(0.2s+1)(0.05s+1)}$$

Determine the phase margin and gain margin of the system when K=1.

Solution

When K=1, the open-loop frequency characteristic of the system is

$$GH(j\omega)=\frac{1}{j\omega(0.2j\omega+1)(0.05j\omega+1)}$$

5.4.2 Stability Margin

<E5.7> 某单位反馈控制系统其开环传递函数为

$$G(s)=\frac{K}{s(0.2s+1)(0.05s+1)}$$

当K=1时，确定该系统增益裕量和相位裕量.

解：

当K=1, 系统开环频率特性如下：

$$GH(j\omega)=\frac{1}{j\omega(0.2j\omega+1)(0.05j\omega+1)}$$

5.4.2 Stability Margin

(1) 确定系统的相位裕量 *Determine the phase margin*

$$GH(j\omega)=\frac{1}{j\omega(0.2j\omega+1)(0.05j\omega+1)}$$

$$|GH(j\omega_c)|=\frac{1}{\omega_c\sqrt{(0.2\omega_c)^2+1}\sqrt{(0.05\omega_c)^2+1}}=1$$

$$\Rightarrow\quad \omega_c\approx 1rad/\sec$$

$$\phi(\omega_c)=-90^\circ-tg^{-1}0.2\omega_c-tg^{-1}0.05\omega_c=-104.17^\circ$$

$$\phi_{pm}=180^\circ+\phi(\omega_c)\approx 76^\circ$$

5.4.2 Stability Margin

(2)确定系统的增益裕量 *Determine the gain margin*

$$GH(j\omega)=\frac{1}{j\omega(0.2j\omega+1)(0.05j\omega+1)}$$

$$\phi(\omega)=-90^\circ-tg^{-1}0.2\omega-tg^{-1}0.05\omega$$

$$\phi(\omega_g)=-90^\circ-tg^{-1}0.2\omega_g-tg^{-1}0.05\omega_g=-180^\circ$$

$$tg^{-1}0.2\omega_g+tg^{-1}0.05\omega_g=90^\circ$$

5.4.2 Stability Margin

对上式两边同时进行正切运算。Operating tangent at the both sides of the equation, we have

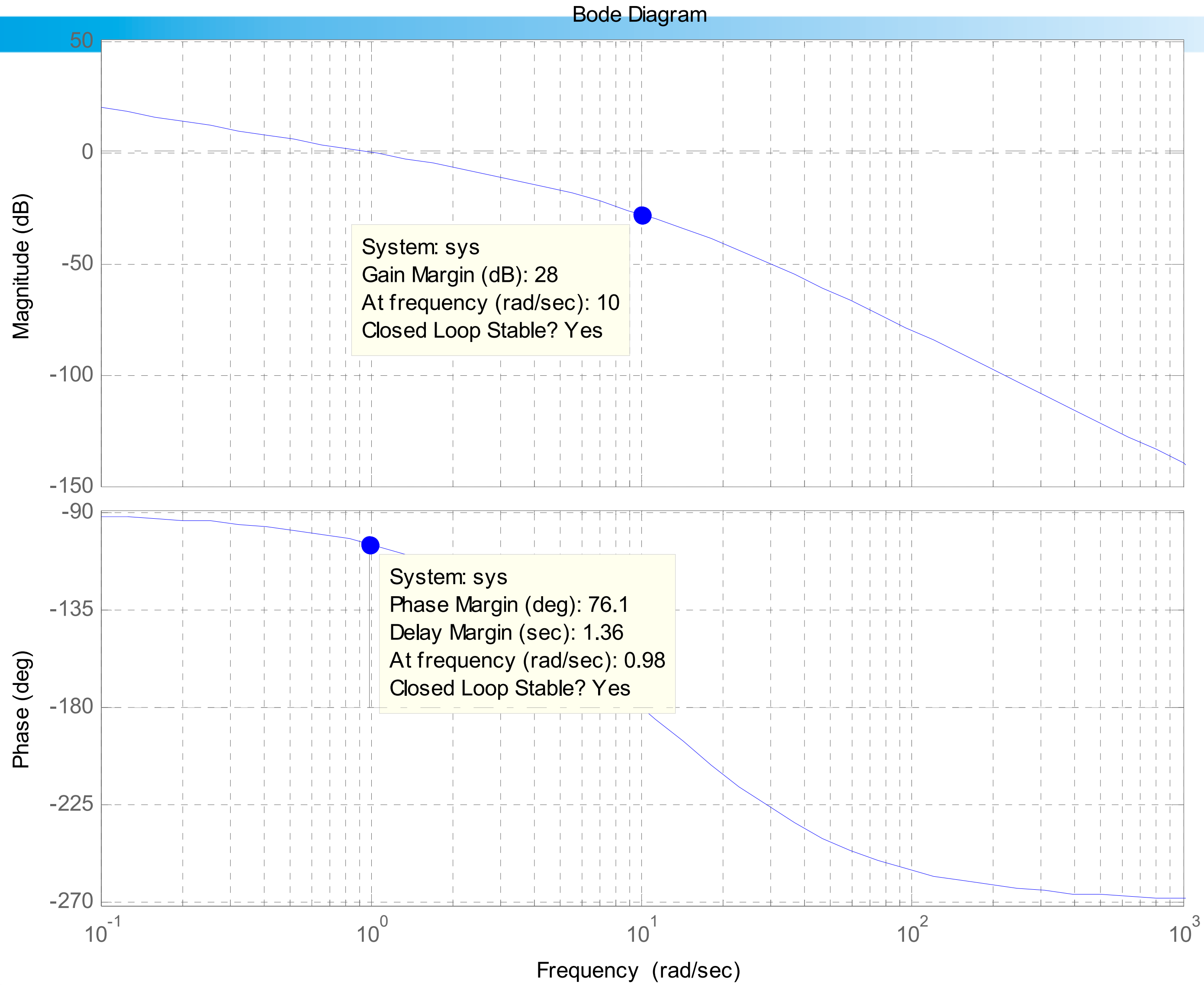
$$\frac{0.2\omega_g + 0.05\omega_g}{1 - 0.2\omega_g \times 0.05\omega_g} = \infty$$

$$1 - 0.2\omega_g \times 0.05\omega_g = 0$$

$$\Rightarrow \quad \omega_g = 10 rad/\sec$$

$$|GH(j\omega_g)| = \frac{1}{\omega_g\sqrt{(0.2\omega_g)^2+1}\sqrt{(0.05\omega_g)^2+1}} = 0.04$$

$$GM = -20\lg|GH(j\omega_g)| = 28dB$$


Bode Diagram
Magnitude (dB)
50
0
-50
-100
-150
System: sys
Gain Margin (dB): 28
At frequency (rad/sec): 10
Closed Loop Stable? Yes
Phase (deg)
-90
-135
-180
-225
-270
System: sys
Phase Margin (deg): 76.1
Delay Margin (sec): 1.36
At frequency (rad/sec): 0.98
Closed Loop Stable? Yes
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Summary

频率响应法是经典控制理论的重要组成部分，是控制系统分析和综合的一种实用工程方法。要求：

- 掌握频率响应法的概念；
- 熟悉系统Bode图；
- 熟练应用Nyquist稳定性判据；
- 熟练掌握$\gamma, g_m, \omega_c, \omega_g$的概念与定义，以及$M_r, \omega_r$的概念与定义

返回

Homework

1、绘制下列传递函数的Bode图

$$G(s)H(s)=\frac{1}{(1+0.5s)(1+2s)}$$

$$G(s)H(s)=\frac{(1+0.5s)}{s^2}$$

$$G(s)H(s)=\frac{s+10}{s^2+6s+10}$$

$$G(s)H(s)=\frac{30(s+8)}{s(s+2)(s+4)}$$

2、考虑题1给出的各个传递函数，用Nyquist判据判断每个系统的稳定性，并给出N,P,Z的取值。

3、考虑下列的两个开环传递函数，画出其极坐标图，并用Nyquist判据判断闭环系统的稳定性。针对稳定的系统，通过考察极坐标图与实轴的交点，判断K的取值范围：

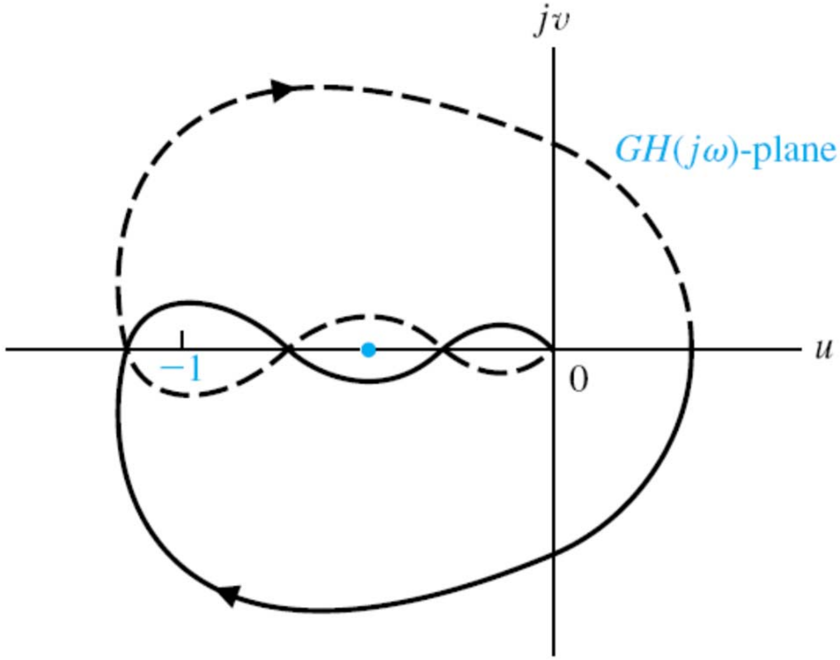
$$G(s)H(s)=\frac{K}{s(s^2+s+4)}$$

$$G(s)H(s)=\frac{K(s+2)}{s^2(s+4)}$$

4、某条件稳定的系统的极坐标图如下图所示

a、已知系统的在s右半平面上无极点，试判断系统是否稳定，并确定s右半平面上是否有闭环特征根，如果有，有多少个；

b、当图中圆点处表示-1时，请判断系统是否稳定；



5、某单位反馈系统的传递函数为：

$$G(s)=\frac{K}{s(s+1)(s+2)}$$

a、当K=4时，验证系统的增益裕度为3.5dB；

b、如果希望增益裕度为16dB，请求出对应的K值；

c、计算当$K=\sqrt{10}$时，系统的相角裕度；

zhaoduo@home.swjtu.edu.cn

6、某集成电路的Bode图如下图所示:

a、读图求出系统的增益裕量和相角裕量；

b、为了使相角裕量达到60度，系统的增益应该下降多少dB？