

Oh, What a Bot!

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1 Introduction

This article describes the development of a control system model for an reaction wheel pendulum, shown in Figure 1. A reaction wheel pendulum is a variation of a simple pendulum, balanced upright, with rotating wheels at the top in place of the pendulum bob. Each wheel is driven by a separate motor. The pendulum is inherently unstable in the upright position - gravity will pull the pendulum over if it is not perfectly balanced. Thus, a control system is needed to accelerate the reaction wheels, applying torques as needed to the pendulum so as to keep it upright. The reaction wheels are mounted at right angles to each other, providing the ability to correct the alignment of the pendulum in any direction.

The pendulum physical structure includes a rod, an accelerometer, and two flywheels. Each flywheel is comprised of a motor mount, a DC brushed motor, and a reaction wheel. The flywheel assemblies are mounted at the top of the rod at right angles to each other and the rod. The DC motors are wired to a motor controller and an Arduino. The Arduino is also wired to the accelerometer, which is located at the top of the rod.



Figure 1: Sketch of the Pendulum

2 Pendulum Dynamics

Figure 2 shows a simplified sketch of the inverted pendulum. Reference [2] contains a summary of the model dynamics for the balancing pencil. The model dynamics can be summarized as

$$mgl \sin \theta - I_c \ddot{\theta} = I_f \ddot{\phi} \quad (1)$$

where:

m = total mass of inverted pendulum (kg)

g = gravitational constant (m s^{-2})

l = distance from the pivot point to the center of pendulum mass (m)

θ = pendulum angle (rad)

I_c = moment of inertia of the pendulum (kg m^2)

I_f = moment of inertia of the reaction wheel (kg m^2)

ϕ = flywheel angle (rad)

Other references may have alternate formulations using different definitions for θ and ϕ .¹

After linearizing the equation by noting that $\sin \theta \approx \theta$ for small angles, the transfer function for the pendulum is then

$$m g l \theta - I_c \ddot{\theta} = I_f \ddot{\phi} \quad (2)$$

The reaction wheel can be continuously revolving, making the reaction wheel angle, ϕ , less useful as a control parameter. In its place we will use $\omega = \dot{\phi}$.

$$m g l \theta - I_c \ddot{\theta} = I_f \dot{\omega} \quad (3)$$

For ease of use in later analysis, it is beneficial to split (3) into two parts:

$$m g l \theta + u(t) = I_c \ddot{\theta} \quad (4)$$

$$u(t) = -I_f \dot{\omega} \quad (5)$$

¹Reference [1] has a slightly different formulation for the model dynamics; the sign of the gravitational term is positive rather than negative. This difference is caused by Reference [1] defining the angle from the downward, resting position while Reference [2] defines the angle from the vertical. The cosines of angles differing by π have opposite signs.

where $u(t)$ is the control torque applied to the pendulum. The sign convention is important; a positive torque $u(t)$ will accelerate the pendulum in the direction of the positive pendulum angle. However the sign convention in (5) indicates that a positive acceleration of the rotor will result in a *negative* acceleration of the pendulum.

Note that if the pendulum is not vertical (i.e. $\theta = 0$) gravity will begin to pull the pendulum over. Correcting this requires accelerating the flywheel to apply torque to the pendulum. A flywheel turning at a constant velocity applies no torque to the pendulum and will not alter the movement of the pendulum.

Deriving the transfer functions,

$$m g l \theta(s) + u(s) = I_c \theta(s) s^2 \quad (6)$$

$$u(s) = -I_f \omega(s) s \quad (7)$$

$$\frac{\theta(s)}{u(s)} = \frac{\frac{1}{I_c}}{s^2 - \frac{mgl}{I_c}} \quad (8)$$

$$\frac{u(s)}{\omega(s)} = -I_f s \quad (9)$$

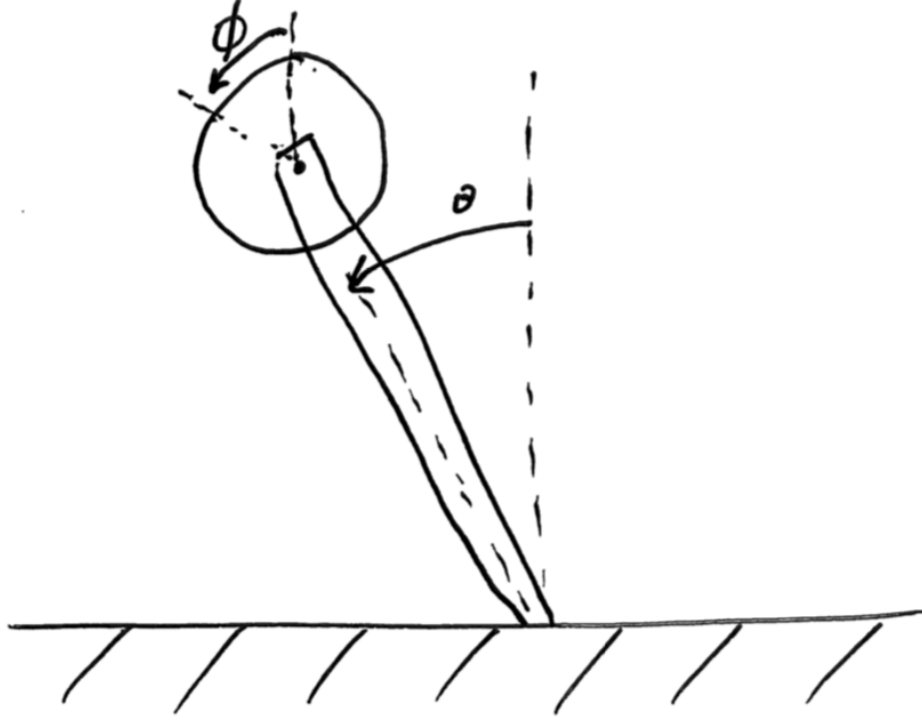


Figure 2: Simplified Pendulum Model

3 Motor and Flywheel Dynamics

Reference [2] models the motor as

$$I_f \ddot{\phi} \eta_m \eta_g = K_t i \quad (10)$$

where:

η_m = motor efficiency

η_g = gear efficiency

K_t = motor torque constant (N m A⁻¹)

i = motor current

This formulation appears to have problems, as a reduction in the motor and gear efficiencies results in a greater flywheel acceleration for a given motor current. Similarly, the gear ratio above unity would decrease the flywheel

acceleration for a specified motor current/torque, which is non-physical. The equation can be altered to correct these issues, resulting in the following:

$$I_f \ddot{\phi} = \eta_m \eta_g K_t i \quad (11)$$

However, this format may not be the easiest to implement, as the motor and gear efficiencies are not known. Reference [1] takes a different approach,

$$I_f \ddot{\phi} = T_{shaft} \quad (12)$$

Where the earlier format dealt with motor/gear efficiencies using η_m and η_g , Reference [1] develops a correlation of friction in terms of torque required to maintain a given $\dot{\phi}$. Here the torque will be reformulated as

$$T_{shaft} = T - T_{friction} \quad (13)$$

$$T_{friction} = A \operatorname{sgn}(\dot{\phi}) + B\dot{\phi} \quad (14)$$

$$T = K_t i \quad (15)$$

where

T = the total motor torque

T_{shaft} = torque available to the flywheel

$T_{friction}$ = torque lost to motor and gear friction

A = coulomb friction coefficient

B = rotational friction coefficient

Combining equations and adding the reduction gear ratio,

$$I_f \ddot{\phi} = K_t i - A \operatorname{sgn}(\dot{\phi}) - B\dot{\phi} \quad (16)$$

Again, using the notation $\omega = \dot{\phi}$,

$$I_f \dot{\omega} = K_t i - A \operatorname{sgn}(\omega) - B\omega \quad (17)$$

As will be seen, this equation is easier to implement because K_t , A , and B can be measured experimentally (see Appendixes C and D).

4 Motor Control

The previous chapter determined the relationship between the motor current and the acceleration of the flywheel. However, the motor will be not be controlled via motor current. Instead, a motor controller will be used to control the motor by adjusting the motor voltage via a PWM signal. The equivalent, average voltage will then regulate the motor. Reference [1] give the relationship between motor current and voltage as

$$L \frac{di}{dt} + Ri = V - K_v \omega \quad (18)$$

where:

L = armature inductance (henry)

R = armature resistance (ohm)

V = voltage supplied to the motor (volt)

K_v = motor voltage constant (V rad⁻¹ sec)

ω = motor rotation speed (rad sec⁻¹)

Normally the inductance of the motor is much lower than the resistance, such that $L/R \sim 0.001$. In such cases it is acceptable to ignore the time dependence of the current and write

$$Ri = V - K_v \omega \quad (19)$$

Note that, provided mks units are used, K_v and K_t will have the same magnitude. This can be seen by equating the mechanical and electrical power

$$T\omega = Vi \quad (20)$$

where T = is the motor torque. Rearranging,

$$\frac{T}{i} = \frac{V}{\omega} \quad (21)$$

or

$$K_t = K_v \quad (22)$$

Back to the motor equation, we can solve (19) for i :

$$i = \frac{V}{R} - \frac{K_v \omega}{R} \quad (23)$$

Combining with (17)

$$I_f \dot{\omega} = K_t \left(\frac{V}{R} - \frac{K_v \omega}{R} \right) - A \operatorname{sgn}(\omega) - B\omega \quad (24)$$

Rearranging,

$$I_f \dot{\omega} + \left(B + \frac{K_t K_v}{R} \right) \omega + A \operatorname{sgn}(\omega) = \left(\frac{K_t}{R} \right) V \quad (25)$$

Recognizing that $K_t = K_v$ and replacing them with K ,

$$I_f \dot{\omega} + \left(B + \frac{K^2}{R} \right) \omega + A \operatorname{sgn}(\omega) = \left(\frac{K}{R} \right) V \quad (26)$$

Not all of the voltage results in acceleration of the reaction wheel. As the reaction wheel velocity increases, an increasing amount of the voltage (and the resulting torque) is consumed by friction and the back EMF.

We will assume that the control function can adjust the demand voltage by adding or subtracting a constant, depending on the current spin direction of the reaction wheel. This means that $-A \operatorname{sgn}(\dot{\phi})$ is folded into V for the purposes of this analysis. The transfer function for the motor and flywheel is then:

$$I_f \omega(s)s + \left(B + \frac{K^2}{R} \right) \omega(s) = \left(\frac{K}{R} \right) V(s) \quad (27)$$

$$\frac{\omega(s)}{V(s)} = \frac{\left(\frac{K}{I_f R} \right)}{s + \frac{BR+K^2}{I_f R}} \quad (28)$$

The characteristic equation for (28) is first order with a negative root, indicating that ω is controllable via V .

5 Modeling in the Frequency Domain

The primary objectives of a control system for the pendulum should be to stabilize the upright position of the pendulum and recovering from external forces on the pendulum. In addition, the control system should be design to deal with a number of physical limitations of the physical components of the pendulum. Specifically,

- the motor voltage is limited to maximum value,
- The rotor velocity has a practical upper limit.

The basic design of a control system for the pendulum is shown in Figure 3. The pendulum is controlled to a constant angle ($r=0$) intended to keep the pendulum vertical. The Controller block can be used to implement a form of feedback proportional to the error term. F represents a random environmental force that affects the pendulum. This sort of system design is often referred to as a Regulator Problem. Analysis of this design may be more clear if the schematic is rearranged as shown in Figure 4 to emphasize controlling θ in the presence of F .²

The stability of the system shown in Figure 4 can be evaluated by examining the open loop transfer function

$$T(s) = P(s)C(s)M(s)R(s) \quad (29)$$

where $P(s)$ is the pendulum transfer function (8), $C(s)$ is the control function, $M(s)$ is the motor transfer function (28), and $R(s)$ is the rotor transfer function (9). At this point in the analysis the control function will be assumed to be a unity gain.

Using (8), (9), and (28) results in

$$T(s) = \frac{-\left(\frac{K}{I_c R}\right)s}{s^3 + \left(\frac{BR+K^2}{I_f R}\right)s^2 - \left(\frac{mgl}{I_c}\right)s - \left(\frac{BR+K^2}{I_f R} - \frac{mgl}{I_c}\right)} \quad (30)$$

Substituting in values from the appendixes, the transfer function for the pendulum is

$$T(s) = \frac{-0.2464s}{s^3 + 4.7303s^2 - 34.8017s - 164.622} \quad (31)$$

The numerator of the transfer function is the characteristic equation that, when factored, is $(s - 5.8993)(s + 5.8993)(s + 4.7303)$.

Using the root locus technique and the open loop transfer function, Figure 5 shows the behavior of the closed loop poles as a function of feedback gain.

²Note that in Figure 3 the control signal and f are added, while in Figure 4 the control signal is subtracted. The "negative feedback" model in Figure 4 is achieved by noting that the rotor transfer function has a negative gain, resulting in a negative feedback.

Note that one pole is always in the right half of the plane, indicating that the system is unstable regardless of the magnitude of the gain. One possible method of stabilizing the system would be to use both proportional and derivative, or PD, feedback. The control function would be.

$$C(s) = 1 + rs \quad (32)$$

Rearranging,

$$C(s) = \frac{s + \frac{1}{r}}{r} \quad (33)$$

indicating a zero at $1/r$ and a gain of $1/r$. The results are sensitive to the placement of the zero. Placing the zero to the right of the pole at -5.8993 eliminates any oscillation, but does not remediate the pole in the right half of the plane. Placing the zero to the left of the pole at -5.8993 results in a much faster response, but can result in oscillations (Figure 6). Again, the pole in the right half of the plane is not remediated.

One possibility for resolving the issue of the zero at the origin would be to implement a PID controller.

$$C(s) = 1 + \frac{r}{s} + qs \quad (34)$$

Rearranging,

$$C(s) = \frac{qs^2 + s + r}{s} \quad (35)$$

This controller would have two zeros that can be placed as required and a pole that would cancel the zero at the origin. If the zeros are placed to the left of the pole at -4.7303, then the system may stabilize. However, in the real world the original zero may not be exactly at the origin and the poles in the left side of the axis may not be exactly where they are expected (due to nonlinearities, etc.) leading to a less-than robust control system.

Finally, it is possible to include feedback from the other monitored and controlled variable, the rotor velocity. Unfortunately, to continue to evaluate the system using a root locus analysis, the feedback from the rotor velocity will have to be directly factored into the transfer function.

Here we modify (19) to include the feedback from the rotor velocity:

$$Ri = V + K_\omega \omega - K_v \omega \quad (36)$$

where:

K_ω = rotor velocity feedback gain.

Solving for i ,

$$i = \frac{V}{R} - \frac{(K - K_\omega) \omega}{R} \quad (37)$$

and

$$\frac{\omega(s)}{V(s)} = \frac{\left(\frac{K}{I_f R}\right)}{s + \frac{BR - KK_\omega + K^2}{I_f R}} \quad (38)$$

Using (8), (9), and (38), the open loop transfer function is now

$$T(s) = \frac{-\left(\frac{K}{I_c R}\right)s}{s^3 + \left(\frac{BR - KK_\omega + K^2}{I_f R}\right)s^2 - \left(\frac{mgl}{I_c}\right)s - \left(\frac{BR - KK_\omega + K^2}{I_f R} - \frac{mgl}{I_c}\right)} \quad (39)$$

Figure 7 shows the root locus with $r = 1/4.5$ and $K_\omega = 0.15$. With a system gain of 510 the controlling poles are at $-2.24 \pm j2.5i$, a damping factor of 0.667, an overshoot of 6.01%, and a frequency of 3.35 rad/second.

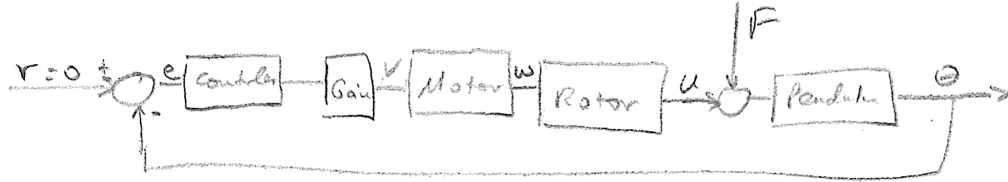


Figure 3: System Schematic with Controller

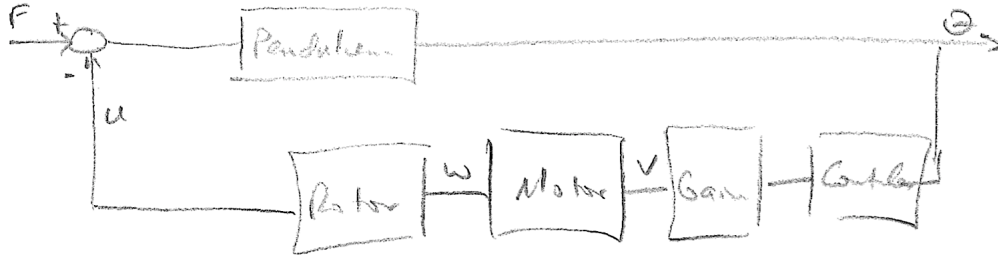


Figure 4: Modified System Schematic with Controller

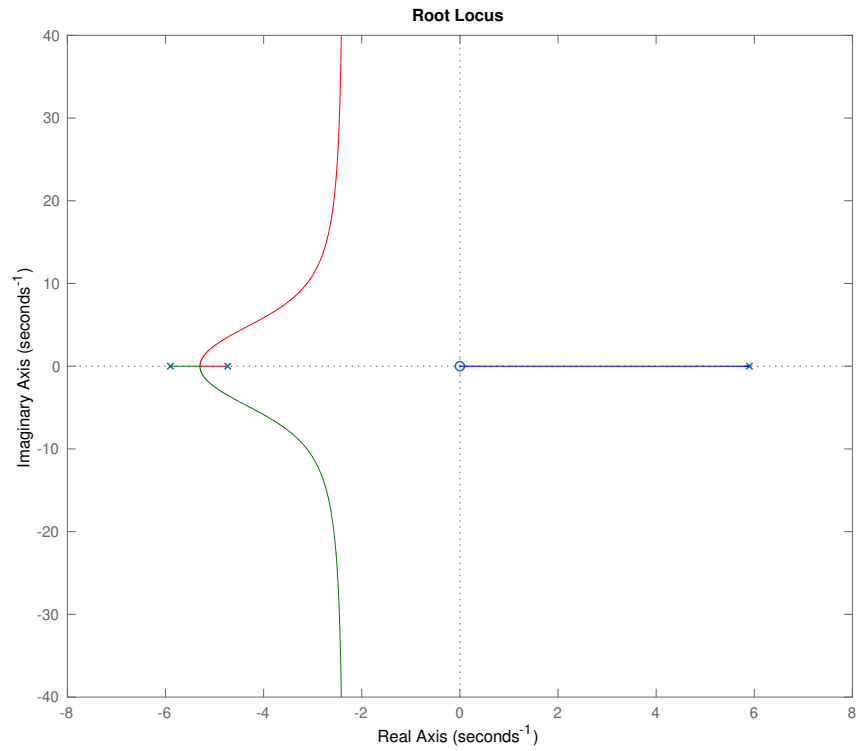


Figure 5: Root Locus Plot With Proportional Feedback

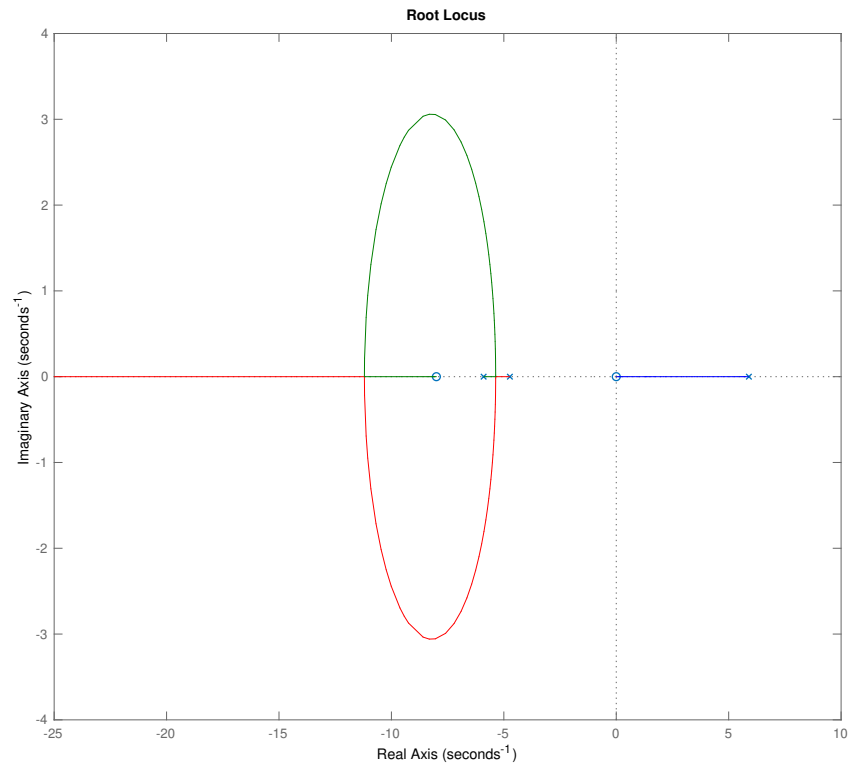


Figure 6: Root Locus Plot With PD Feedback

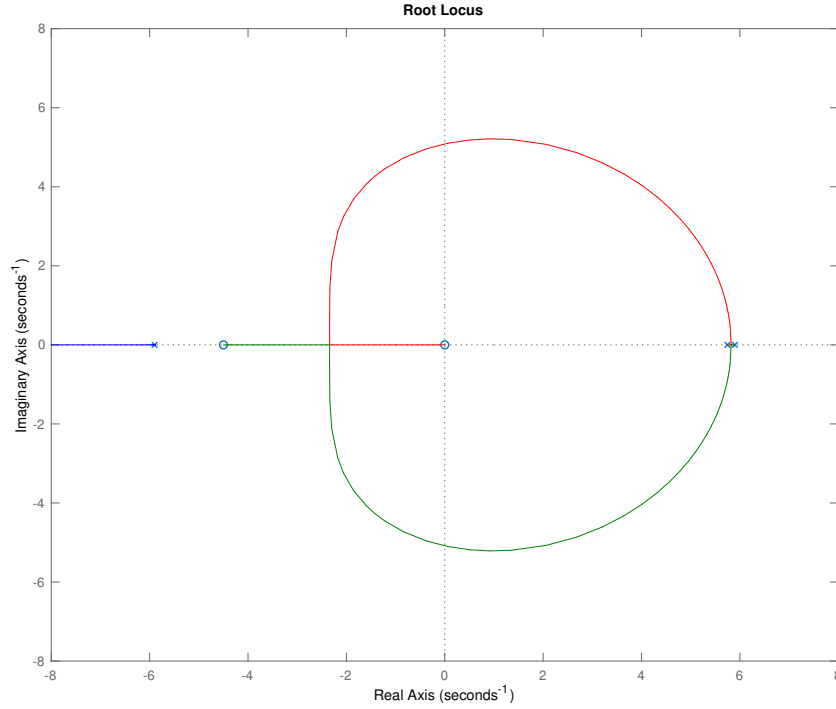


Figure 7: Root Locus Plot With PD and Rotor Velocity Feedback

References

- [1] Block, D. J., Astrom, K. J., and Spong, M. W. (2007). *The Reaction Wheel Pendulum*. Morgan & Claypool Publishers.
- [2] Ramm, A., and Sjostedt, M. (2015). Reaction wheel balanced robot, Design and Sensor Analysis of Inverted Pendulum Robot. Technical Report, KTH Royal Institute of Technology.
- [3] Hellman, H. and Sunnerman, H. (2015). Two-Wheeled Self-Balancing Robot. Technical report, KTH Royal Institute of Technology.
- [4] Chauveau, G., Chazal, D., Nakayama, D., Olsen, E., and Palm, S., (2005). Controlling the Reaction Wheel Pendulum.

Appendices

Appendix A Motor and Gear Set Data

Key to much of the work so far are the constants associated with the motor and gear set, K_t , K_v , R , and L . Here we are using a Pololu 4.4:1 metal gear motor.

Pololu gives the following parameters for the 4.4:1 gear motor:

weight = 95g

gear ratio = 4.4:1

No-load speed @ 12V = 1700 rpm (178 rad/sec)

No-load current @ 12V = 0.2 A

Stall current @ 12V = 2.1 A

Stall torque @ 12V = 11 oz in (0.07768 Nm)

Encoder frequency = 211.2 counts/rev of gearbox shaft

A number of the motor constants can be calculated from the published Pololu parameters. However, it should be noted that, for inexpensive motors, the actual motor parameters may vary from the published data. In any event, the motor constants will be calculated here and then later compared with measured data.

K_f can be calculated directly from the stall current, gear ratio, and torque: $K_f = 0.07768 \text{ Nm} / 2.1 \text{ A} = 0.0370 \text{ Nm/A}$. K_v can be set equal to K_f , or 0.0370 Vs.

The motor winding resistance can be determined from the rated voltage and the stall current, as no back-EMF is occurring at stall conditions. Thus, $R = 12 \text{ V} / 2.1 \text{ A} = 5.7 \text{ ohms}$.

Appendix B Pendulum Constants

A multitude of other values were calculated in MATLAB.

m 0.517327 kg

l 0.319038 m

I_c 0.046508 kg m²

I_f 0.000164 kg m²

g 9.81 m s⁻²

Appendix C Measuring Motor Constants

Measurements were made to determine a number of motor constants, including R , L , and K_v .

R was measured using an EXTEC LCR meter, on the R_{DC} setting, as 5.82Ω . L was measured with the same instrument as 5.994 mH at 1 kHz . This gives a motor electrical time constant L/R of 0.001 sec . This is the time taken by the current in the motor to go from rest to 63% of the final steady-state current.

K_v was measured by driving a motors with a constant voltage and varying loads while measuring V , i , and ω .

The motor was driven with a 12 V power supply. The current was measured by the power supply. Measuring current in this manner can have larger uncertainties, but comparing with measurements from a $\mu\text{Current}$ DVM adapter indicated that the power supply current measurement was reasonable.

ω was measured using the motor's encoder and an Arduino. The encoder measured the shaft speed, not the gear-set speed. However, a conversion coefficient of $211.2 \text{ counts/revolution}$ was used to convert the encoder pulse rate to gear set rotation speed.

The following table shows the results for three different loads. The first load was just rotor friction. The second load added the Pololu hub adapter. The third load included a plastic wheel attached to a propeller to increase the drag. The resulting measured values are as follows:

Case	v (volts)	i (A)	speed (counts/sec)	ω (RPM)	ω (sec^{-1})
1	11.95	0.0675	5,942.2	1,688.1	176.74
2	11.95	0.075	5,923.58	1,682.83	176.19
3	11.95	0.153	5,574.88	1583.77	165.82

Calculating K_v can be approached two ways. The first is to solve (19) using the measured value of R . The resulting values of K_v , one for each case, are:

Case	K_v (V sec)
1	0.0654
2	0.0653
3	0.0667

A second approach is to use two cases in (19) and solve simultaneously for R and K_v . Cases 1 and 3 were chosen because of the largest difference in i ,

attempting to ensure that the two cases are independent. This results in a values of $R = 8.24 \, \Omega$ and $K_v = 0.0645 \, \text{V sec}$.

The value of K_v is fairly consistent with the values determined in the first approach, but the value of R appears to be high. This may be caused by the limited change in i in the data. Use of a larger load torque might result in a larger value of i and a better estimate of R . However, for the purposes of this work the following values will be used:

R (Ω)	K_v (V sec)	K_t (N m A)
5.82	0.0667	0.0667

Appendix D Measuring Motor Friction Constants

Friction in the motor was evaluated by measuring the supply current as a function of motor speed. The friction force is related to the motor current as $F = K_t i$. The measurement was performed by using a power supply with a variable current limit. For various currents the speed of the motor was measured by monitoring the motor encoder output with an Arduino. The current was measured both at the power supply as well as with a μ Current DVM adapter.

The following data was collected:

Supply Voltage (volts)	Supply Current (amps)	μ Current (A)	Speed (pulse/sec)	Speed (rpm)	Speed radians/sec
10.90	0.068	0.0676	5417	1538.9	161.16
10.04	0.063	0.0627	5001	1420.7	148.78
9.09	0.058	0.0596	4520	1284.1	134.47
7.93	0.056	0.0562	3937	1118.5	117.13
6.99	0.055	0.0550	3465	984.4	103.08
6.00	0.051	0.0521	2953	838.9	87.85
5.00	0.047	0.0500	2437	692.3	72.50
4.04	0.046	0.0483	1960	556.8	58.31
3.57	0.046	0.0486	1701	483.2	50.60
3.02	0.047	0.0481	1421	403.7	42.27
2.52	0.045	0.0459	1169	332.1	34.78
2.02	0.042	0.0437	910	258.5	27.07
1.51	0.039	0.0409	662	188.1	19.69
1.02	0.034	0.0366	414	117.6	12.32
0.47	0.031	0.0328	148	42.0	4.40

The encoder speed, in pulses/sec, was converted to RPM using the encoder frequency of 211.2 counts/revolution.

The Excel regression function was used to fit the μ Current current measurement as a function of the speed in radians/sec. The regression intercept represents the fixed Coulomb friction and the slope represents the viscous friction. The regression results were 2.47E-3 N m and 1.20E-5 N m/(radian/sec), after converting from current using K_t . The coulomb friction represents approximately 3% of the motor stall torque. At the motor's no-load speed the viscous friction would double the total friction to approximately 6% of the motor stall torque. The viscous friction coefficient is significantly larger than the value of 9.06E-7 N m/(radian/sec) measured in Reference [4]. This may be due to the gearbox associated with the motor used in the current work. In any event, the friction will be included in the system modeling.

Appendix E Arduino Code for Measuring RPM