2022·B1·A卷

10:30 ~ 12:30　　用时103分钟.

1. $x+3y+2z=6$　　$A(6,0,0)$
$B(0,2,0)$
$C(0,0,3)$

$V_{OABC} = \frac{1}{3}\cdot\frac{1}{2}\cdot 6\cdot 2\cdot 3 = 6$

$\vec{AB} = (-6,2,0)$

$\vec{AC} = (-6,0,3)$

$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ -6 & 2 & 0 \\ -6 & 0 & 3 \end{vmatrix} = (6,18,12)$

另解:
$S_{\triangle ABC} = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}\cdot 6\sqrt{14} = 3\sqrt{14}$

$\vec{n} = (1,3,2)$

$\vec{OA} = (6,0,0)$

$h = \left|\frac{\vec{OA}\cdot\vec{n}}{|\vec{n}|}\right| = \frac{6}{\sqrt{14}} \Rightarrow V = \frac{1}{3}S_{\triangle ABC}\cdot h$　　$S_{\triangle ABC} = \frac{3V}{h} = \frac{18}{\frac{6}{\sqrt{14}}} = 3\sqrt{14}$

$\Rightarrow$ 球心: $(3,1,\frac{3}{2})$　　$(x-3)^2+(y-1)^2+(z-\frac{3}{2})^2 = \frac{49}{4}$

2. (1) $\lim\limits_{(x,y)\to(0,0)} \frac{24\cos\sqrt{x^2+y^2}-24+12(x^2+y^2)}{(\tan\sqrt{x^2+y^2})^4}$

$1-\frac{1}{2}x^2+\frac{1}{4!}x^4$

换元 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$　　$(x,y)\to(0,0) \Leftrightarrow r\to 0^+$

$1-\cos x \sim \frac{1}{2}x^2+\frac{1}{24}x^4$

原式 $= \lim\limits_{r\to 0^+}\frac{24\cos r-24+12r^2}{(\tan r)^4} = \lim\limits_{r\to 0^+}\frac{24(-\frac{1}{2}r^2)+12r^2}{r^4} = \lim\limits_{r\to 0^+}\frac{24\cdot(\frac{1}{24}r^4)}{r^4} = 1.$

(2) $\lim\limits_{(x,y)\to(0,0)} (x+\ln(1+y))\cos\frac{1}{x^2+y^2}$

$x+\ln(1+y) \sim x+y \to 0$，$\cos\frac{1}{x^2+y^2}$ 为有界量.

知极限为0

(3) $\lim\limits_{(x,y)\to(0,0)} \frac{x\sin y}{(\sin x)^2+(\sin y)^2}$

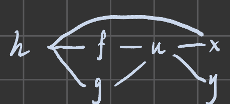
令 $y = kx$. 原式 $= \lim\limits_{x\to 0}\frac{x\sin kx}{(\sin x)^2+(\sin kx)^2} = \lim\limits_{x\to 0}\frac{kx^2}{(x+\frac{1}{3}x^3)^2+(kx+\frac{1}{3}k^2x^2)^2} = \frac{k}{1+k^2}$

当k取不同值时，极限不相等，故原极限发散，极限不存在.

3. $h(x,y) = xg(\frac{y}{x})+f(\frac{y}{x})$.

$x^2h_{xx}(x,y)+2xyh_{yx}(x,y)+y^2h_{yy}(x,y)$.

记 $u = \frac{y}{x}$



$h_x = g + x\cdot g_u\cdot(-\frac{y}{x^2}) + f_u\cdot(-\frac{y}{x^2})$

$h_y = x\cdot g_u\cdot\frac{1}{x} + f_u\cdot\frac{1}{x} = g_u+\frac{1}{x}f_u$

$$h_{xx} = g_u \cdot \left(-\frac{y}{x^2}\right) + g_u \cdot \left(-\frac{y}{x^2}\right) + x \cdot \left[g_{uu} \left(-\frac{y}{x^2}\right)^2 + g_u \cdot \frac{2y}{x^3} \right] + f_{uu} \left(-\frac{y}{x^2}\right) \cdot \left(-\frac{y}{x^2}\right) + f_u \cdot \frac{2y}{x^3}$$

$$= \frac{y^2}{x^3} \cdot g_{uu} + \frac{y^2}{x^4} f_{uu} + \frac{2y}{x^3} f_u$$

$$h_{yx} = g_{uu} \cdot \left(-\frac{y}{x^2}\right) + \left(-\frac{1}{x^2}\right) f_u + \frac{1}{x} \cdot f_{uu} \cdot \left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2} g_{uu} - \frac{1}{x^2} f_u - \frac{y}{x^3} f_{uu}$$

$$h_{yy} = g_{uu} \cdot \frac{1}{x} + \frac{1}{x} \cdot f_{uu} \cdot \frac{1}{x}$$

$$= \frac{1}{x} g_{uu} + \frac{1}{x^2} f_{uu}$$

$$\therefore x^2 h_{xx} + 2xy\, h_{yx} + y^2 h_{yy} = \frac{y^2}{x} g_{uu} + \frac{y^2}{x^2} f_{uu} + \frac{2y}{x} f_u$$

$$- \frac{2y^2}{x} g_{uu} - \frac{2y^2}{x^2} f_{uu} - \frac{2y}{x} f_u$$

$$+ \frac{y^2}{x} g_{uu} + \frac{y^2}{x^2} f_{uu}$$

$$= 0$$

4. $e^{xy} + xy + y^2 = 2$ 在 $(0,1)$ 处切线方程.

$F(x,y) = e^{xy} + xy + y^2 - 2 = 0$ ，隐函数 $y = y(x)$.

$F_x = e^{xy} \cdot y + y$

$F_x(0,1) = e^0 \cdot 1 + 1 = 2$

$F_y = e^{xy} \cdot x + x + 2y$ ， $F_y(0,1) = e^0 \cdot 0 + 0 + 2 \cdot 1 = 2 \neq 0$

切线斜率 $y'_x = \dfrac{-F_x}{F_y} = \dfrac{-2}{2} = -1$.

$\therefore$ 切线： $y - 1 = -(x - 0)$，即 $y = -x + 1$

5. $f(x,y,z) = \left(\frac{2x}{z}\right)^y \quad z \neq 0$. 求在$(\frac{1}{2}, 1, 1)$处最速降向单位向量.

即求单位化 $-\text{grad}\, f$

$\ln f = y \ln\left(\frac{2x}{z}\right)$.

$\frac{1}{f} \cdot f'_x = y \cdot \frac{z}{2x} \cdot \frac{2}{z}$

$f'_x = f \cdot \left(\frac{y}{x}\right) = \left(\frac{2x}{z}\right)^y \cdot \left(\frac{y}{x}\right) = 1^1 \cdot \frac{1}{\frac{1}{2}} = 2$

$f'_y = (1^y)' = 0$

$f'_z = \left[\left(\frac{1}{z}\right)^1\right]' = -\frac{1}{z^2} = -1$

$\therefore \text{grad}\, f(\frac{1}{2}, 1, 1) = (2, 0, -1)$

$-\text{grad}\, f(\frac{1}{2}, 1, 1) = (-2, 0, 1)$

单位向量： $\left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$.

6. $f(x,y)=\arctan\left(\frac{y}{x}\right)$ 在点 $(2,2)$ 处二阶泰勒.

$$f_x=\frac{1}{\frac{y^2}{x^2}+1}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}$$

$-(x^2+y^2)+y\cdot(2y)$.

$$f_y=\frac{1}{\frac{y^2}{x^2}+1}\cdot\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$f_{xx}=\frac{-(-y)\cdot 2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$$

$$f_{xy}=\frac{-(x^2+y^2)+y\cdot(2y)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_{yy}=\frac{-x\cdot 2y}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}.$$

代入 $(x,y)=(2,2)$

$f_x(2,2)=\frac{-2}{8}=-\frac{1}{4}$

$f_y(2,2)=\frac{1}{4}$

$f(2,2)=\arctan 1=\frac{\pi}{4}$

$f_{xx}(2,2)=\frac{8}{8^2}=\frac{1}{8}$

$f_{xy}(2,2)=0$

$f_{yy}(2,2)=-\frac{1}{8}$

$\therefore$ 泰勒多项式 $=\frac{\pi}{4}-\frac{1}{4}(x-2)+\frac{1}{4}(y-2)+\frac{1}{16}\cdot(x-2)^2-\frac{1}{16}(y-2)^2.\ =\frac{\pi}{4}-\frac{1}{2}x+\frac{1}{2}y+\frac{1}{16}x^2+\frac{1}{16}y^2$

7. $f(x)=(\sin x)^{\frac{2}{3}}+(\cos x)^{\frac{2}{3}}$ $\quad[-\frac{\pi}{2},\frac{\pi}{2}]$ 上最小值.

$f(-\frac{\pi}{2})=1$

$f(\frac{\pi}{2})=1$

$$f'_x=\frac{2}{3}(\sin x)^{-\frac{1}{3}}\cdot\cos x+\frac{2}{3}(\cos x)^{-\frac{1}{3}}\cdot(-\sin x).$$

$$=\frac{2}{3}\sin x^{-\frac{1}{3}}\cos x^{-\frac{1}{3}}(\cos x^{\frac{4}{3}}-\sin x^{\frac{4}{3}})$$

令 $f'_x=0$

$\therefore$ $x=0$ 或 $\pm\frac{\pi}{2}$ 或 $\frac{\pi}{4}$

$f(0)=1$

$f(\frac{\pi}{4})=2\sqrt[3]{\frac{1}{2}}=\sqrt[3]{\frac{1}{2}\cdot 8}=\sqrt[3]{4}>1$

$\therefore$ 最小值为1，最小值点：$\pm\frac{\pi}{2}$，0

8. $\forall k\in R.\ U(0)\ \exists W(0)\quad y=f(x)$.

$F(x,y)=e^{kx}+e^{ky}-2e^{x+y}=0$.

易知 $F(0,0)=0$

又 $F(x,y)=0$

$\therefore$ $F_x=ke^{kx}-2e^{x+y}$, $F_y=ke^{ky}-2e^{x+y}$ $\Rightarrow$ 在 $(0,0)$ 附近连续.

且 $F_y=ke^{ky}-2e^{x+y}$

若 $k\le 0$，$F_y\ne 0$

若 $k>0$，$F_y=0\Leftrightarrow ke^{ky}=2e^{x+y}$

$\ln k+ky=\ln 2+x+y$

$\Leftrightarrow \ln k+ky=\ln 2+y$

$\Leftrightarrow y=\frac{1}{k-1}\cdot(\ln 2-\ln k)$

令 $W=\{\,|y|<\left|\frac{1}{k-1}(\ln 2-\ln k)\right|\,\}$

补充讨论：

若 $k=2$

则 $F(x,y)=e^{2x}+e^{2y}-2e^{x+y}$

$=(e^x-e^y)^2=0$

$\Rightarrow e^x=e^y\Rightarrow x=y$

此时，$y=x$ 为唯一符合的函数，且 $W=R$

若 $k\neq 2$.
则由隐函数存在定理.
$y=f(x)$ 存在，且 $f'_x=-\frac{F_x}{F_y}=-\frac{ke^{kx}-2e^{x+y}}{ke^{ky}-2e^{x+y}}$.

9.(1) $D:\ U_r(0)$ $f: D\to\mathbb{R}$ $f\in C^3(D)$ $f(0,0)=0$ $df(0,0)=0$

$d^2f(0,0)=E(\Delta x)^2+2F\Delta x\Delta y+G(\Delta y)^2$

证: $\exists\ a,b: D\to\mathbb{R},\ \forall (x,y)\in D.\ f(x,y)=x\,a(x,y)+y\,b(x,y)$ $a(0,0)=b(0,0)=0$.

考虑点 $P_t(tx,ty)$. 当 $t\in[0,1]$ 时, P_t 在 O 与 (x,y) 连线上.

$f(x,y)$ 在 D 上可微 $\Leftrightarrow \varphi(t)=f(tx,ty)$ 可微.

$\frac{d\varphi}{dt}=\frac{\partial f}{\partial(tx)}\cdot x+\frac{\partial f}{\partial(ty)}\cdot y$

又 $\varphi(1)-\varphi(0)=\varphi'(t)$.

$\Leftrightarrow f(x,y)-f(0,0)=\frac{\partial f}{\partial(tx)}\cdot x+\frac{\partial f}{\partial(ty)}\cdot y$.

令 $a(x,y)=\frac{\partial f}{\partial(tx)}$, $b(x,y)=\frac{\partial f}{\partial(ty)}$. 即得.

由 $df(0,0)=0$, 知 $f_x(0,0)=f_y(0,0)=0$

即 $a(0,0)=b(0,0)=0$.

(2) 考虑 $f(x,y)$ 在 $(0,0)$ 处的二阶泰勒公式

$\therefore f(x,y)=f(0,0)+f_x(0,0)\cdot x+f_y(0,0)\cdot y+\frac{1}{2}f_{xx}(0,0)\cdot x^2+f_{xy}(0,0)\cdot xy+\frac{1}{2}f_{yy}(0,0)y^2+o(\rho^2)$

其中 $\rho=\sqrt{x^2+y^2}$

由题意, $f_x(0,0)x+f_y(0,0)y=0$, $E=\frac{1}{2}f_{xx}$, $F=\frac{1}{2}f_{xy}$, $G=\frac{1}{2}F_{yy}$.

$E>0$, $EG-F^2<0$ 知 $z=f(x,y)$ 在 $(0,0)$ 附近无极值.

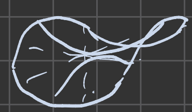
$z=\frac{1}{2}(Ex^2+2Fxy+Gy^2)$

$\begin{bmatrix}E & F\\ F & G\end{bmatrix}\to\begin{bmatrix}E & 0\\ 0 & G-\frac{F^2}{E}\end{bmatrix}$ $E>0 \Rightarrow G-\frac{F^2}{E}<0$

二次型. $\longrightarrow E(x')^2+(G-\frac{F^2}{E})(y')^2$

$\begin{bmatrix}1 & \\ -\frac{F^2}{E} & 1\end{bmatrix}$

知·充分近似于双曲抛物面.



$x^2-y^2-z=0$ 马鞍形

补充: 由二次型或配方可知存在

$x^2-y^2=z$ 存在.