Homework 1

March 13, 2014

1. Assume the expectations of X and Y exist. Prove that, for $g(x,y)$,

$$\mathbb{E}(\mathbb{E}(g(X,Y) \mid X)) = \mathbb{E}(g(X,Y)).$$

2. Assume X and Y are random variables. Prove that

$$\mathrm{Var}(Y) = \mathbb{E}(\mathrm{Var}(Y \mid X)) + \mathrm{Var}(\mathbb{E}(Y \mid X)).$$

3. Assume $X = (X_1, \ldots, X_m)^T \sim \mathcal{N}_m(\mathbf{0}, \mathbf{\Sigma})$. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \mathbf{\Theta} = \mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{\Theta}_{11} & \mathbf{\Theta}_{12} \\ \mathbf{\Theta}_{21} & \mathbf{\Theta}_{22} \end{pmatrix} \begin{matrix} p \\ q \end{matrix},$$

and $\mathbf{\Sigma}_{11\cdot 2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$. Prove that $\mathbf{\Sigma}_{11\cdot 2} = \mathbf{\Theta}_{11}^{-1}$ and $\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12} = -\mathbf{\Theta}_{12}\mathbf{\Theta}_{22}^{-1}$.

4. Suppose the multivariate t distribution has density

$$st_m(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{\Sigma}, \nu) = \frac{\Gamma\left[(\nu+m)/2\right]}{\Gamma(\nu/2)\,(\nu\pi)^{m/2}} |\mathbf{\Sigma}|^{-1/2} \left[1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(\nu+m)/2},$$

with $\nu > 0$. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix} \begin{matrix} p \\ q \end{matrix}.$$

Compute $p(\mathbf{x}_1)$ and $p(\mathbf{x}_2 \mid \mathbf{x}_1)$.