Homework 1

March 13, 2014

1. Assume the expectations of X and Y exist. Prove that, for g(x, y),

$$\mathbb{E}(\mathbb{E}(g(X,Y)\mid X)) = \mathbb{E}(g(X,Y)).$$

2. Assume X and Y are random variables. Prove that

$$Var(Y) = \mathbb{E}(Var(Y \mid X)) + Var(\mathbb{E}(Y \mid X)).$$

3. Assume $X = (X_1, \dots, X_m)^T \sim \mathcal{N}_m(\mathbf{0}, \mathbf{\Sigma})$. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix} \begin{matrix} p \\ q \end{matrix}, \quad \mathbf{\Theta} = \mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{\Theta}_{11} & \mathbf{\Theta}_{12} \\ \mathbf{\Theta}_{21} & \mathbf{\Theta}_{22} \end{pmatrix} \begin{matrix} p \\ q \end{matrix},$$

and $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Prove that $\Sigma_{11\cdot 2} = \Theta_{11}^{-1}$ and $\Sigma_{11}^{-1}\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$.

4. Suppose the multivariate t distribution has density

$$st_m\left(\mathbf{x}\mid\boldsymbol{\mu},\boldsymbol{\Sigma},\nu\right) = \frac{\Gamma\left[(\nu+m)/2\right]}{\Gamma\left(\nu/2\right)(\nu\pi)^{m/2}}|\boldsymbol{\Sigma}|^{-1/2}\left[1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(\nu+m)/2},$$

with $\nu > 0$. Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} egin{aligned} p \\ q \end{aligned}, \quad oldsymbol{\mu} = \begin{pmatrix} oldsymbol{\mu}_1 \\ oldsymbol{\mu}_2 \end{pmatrix} egin{aligned} p \\ q \end{aligned}, \quad oldsymbol{\Sigma} = \begin{pmatrix} oldsymbol{\Sigma}_{11} & oldsymbol{\Sigma}_{12} \\ oldsymbol{\Sigma}_{21} & oldsymbol{\Sigma}_{22} \end{pmatrix} egin{aligned} p \\ q \end{aligned}.$$

Compute $p(\mathbf{x}_1)$ and $p(\mathbf{x}_2 \mid \mathbf{x}_1)$.