A Proof of Theorem 1

Proof. We illustrate the complexity of the problem with a simple example where $P = \{V\}$ in the function $f(\cdot)$. In this instance, the TSM problem simplifies to a co-exposure maximization problem, which is known to be NP-hard [3]. Consequently, the TSM problem itself is also NP-hard. □

B Proof of Theorem 2

Proof. For monotonicity, in general, $f(S_t) - f(S'_t) \geq 0$ holds constantly when $S'_t \subseteq S_t$. Thus, $f(\cdot)$ is monotonic. For non-submodularity, we consider it by figure **??** and let $S_r = \{v_0\}$ and $P = \{v_3\}$, while the dashed line indicates failed activation of the edge. Assume that edge (v_4, v_3) , (v_5, v_3) probabilities is 0 and other edges probabilities $p(e) = 1$. First, we consider $A = \emptyset$, $B = \{v_4\}$ and node v_5. It is straightforward to compute the score $s_X(A) = 0$ and $s_X(B) = 0$. Then, we consider the score rises to 1 when adding v_5 to B. Through calculation, we can obtain $s_X(B \cup \{v_5\}) = 1$ and $s_X(A \cup \{v_5\}) = 0$. Therefore, due to $s_X(B \cup \{v_5\}) - s_X(B) > s_X(A \cup \{v_5\}) - s_X(A)$ and $A \subseteq B$, the original objective function does not satisfy submodularity. □

C Proof of Lemma 1

Proof. $f(S'_t \cup \{v\}) - f(S'_t)$ represents the number of nodes in $f(v)$ that are not included in the union $\bigcup_{u \in S'_t} f(u)$. When $S'_t \subseteq S_t$, the number is at least as large as the number of nodes in $f(v)$ that are not in the larger union $\bigcup_{u \in S_t} f(u)$. Thus, it follows that $f(S'_t \cup \{v\}) - f(S'_t) \geq f(S_t \cup \{v\}) - f(S_t)$, which demonstrates the submodularity. □

D Proof of Lemma 2

Proof. Because we consider only the score of nodes activated by its neighbor, it becomes a traditional influence maximization problem. Thus, the lower bound function satisfies submodularity. □

E Proof of Lemma 5

Proof. Let $\Lambda_2(S^\star) = \frac{COV_{R_2}(S^\star)}{n}|P|$ be the weighted coverage of $S^\star$ in R_2 and $\theta_2 = |R_2|$. For any $\delta \in (0,1)$, we have

$$Pr[\sigma(S^\star) \geq ((\sqrt{\Lambda_2(S^\star) + \frac{25a}{36}} - \sqrt{a})^2 - \frac{a}{36})\frac{n}{\theta_2}] \geq 1 - \delta \tag{1}$$

where $a = \ln(\frac{1}{\delta})$ and $\theta_2 = \frac{(\epsilon - (1 - \frac{1}{e}\epsilon_1))^2 \mathrm{OPT}}{2(1-\frac{1}{e}) \log\left(\frac{\binom{n}{k}}{\delta_2}\right)}$.

First, we prove

$$
\begin{aligned}
&\Pr[\sigma(S^*) < ((\sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a})^2 - \frac{a}{36})\frac{n}{\theta_2}] \\
\leq &\Pr[\sigma(S^*) < ((\sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a})^2 - \frac{a}{36})\frac{n}{\theta_2}] \\
= &\Pr[\sigma(S^*) \cdot \frac{\theta_2}{n} < (\sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a})^2 - \frac{a}{36}] \\
= &\Pr[b + \frac{a}{36} < (\sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a})^2] \\
= &\Pr[\sqrt{b + \frac{a}{36}} < \sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a}] \\
&+ \Pr[\sqrt{b + \frac{a}{36}} < \sqrt{a} - \sqrt{\Lambda^2(S^*) + \frac{25a}{36}}] \\
= &\Pr[\sqrt{b + \frac{a}{36}} < \sqrt{\Lambda^2(S^*) + \frac{25a}{36}} - \sqrt{a}] \\
= &\Pr[\sqrt{b + \frac{a}{36}} + \sqrt{a} < \sqrt{\Lambda^2(S^*) + \frac{25a}{36}}] \\
= &\Pr[b + \sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3} < \Lambda^2(S^*)] \\
= &\Pr[\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3} < \Lambda^2(S^*) - b] \\
= &\Pr[\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3} < \Lambda^2(S^*) - \sigma(S^*) \cdot \frac{\theta_2}{n}] \\
\leq &\Pr[\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3} < \Lambda^2(S^*) - \sigma(S^*) \cdot \frac{\theta_2}{n}] \\
\leq &\exp(-\frac{(\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3})^2}{4\sigma(S^*) \cdot \frac{\theta_2}{n}\frac{|P|}{n} + \frac{2}{3}(\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3})}) \\
\leq &\exp(-\frac{(\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3})^2}{4\sigma(S^*) \cdot \frac{\theta_2}{n} + \frac{2}{3}(\sqrt{4ab + \frac{a^2}{9}} + \frac{a}{3})}) \\
= &\exp(-a) = \delta
\end{aligned}
$$

By setting δ_2 and θ_2 which we will discuss next, we can obtain $\sigma^l(S^*) = \left(\left(\sqrt{\Lambda_2(S^\star) + \frac{25\ln\left(\frac{1}{\delta_2}\right)}{36}} - \sqrt{\ln\left(\frac{1}{\delta_2}\right)}\right)^2 - \frac{\ln\left(\frac{1}{\delta_2}\right)}{36}\right) \cdot \frac{(\epsilon-(1-\frac{1}{e}\epsilon_1))^2\mathrm{OPT}}{2(1-\frac{1}{e})\log\left(\frac{\binom{n}{k}}{\delta_2}\right)}$

Next, we conside the $\sigma^u(S^o)$. Let $\Lambda_1(S^\star) = \frac{COV_{R_1}(S^\star)}{n}|P|$ be the weighted coverage of $S^\star$ in R_1 and $\theta_1 = |R_1|$. We can obtain $\sigma(S^o)$ based on the weighted coverage of $S^\star$ in R_1 employing the property of the greedy algorithm that ensures $\Lambda_1(S^\star) \geq (1-\frac{1}{e})\Lambda_1(S_0)$. For any $\delta \in$ (0,1), we have

$$Pr[\sigma(S^o) \leq (\sqrt{\frac{\Lambda_1(S^\star)}{\frac{1}{e}}+a}+\sqrt{a})^2\frac{n}{\theta_2}] \geq 1-\delta \tag{2}$$

where $a=\ln(\frac{1}{\delta})$ and $\theta_1=\frac{4|P|log(\frac{1}{\delta_1})}{\epsilon_1^2 OPT}$,

$$\begin{aligned}
&\Pr[\sigma(S^o) > (\sqrt{\frac{\Lambda_1(S^*)}{1-1/e}+a}+\sqrt{a})^2\frac{n}{\theta_1}]\\
\leq &\Pr[\sigma(S^o) > (\sqrt{\Lambda_1(S^o)+a}+\sqrt{a})^2\frac{n}{\theta_1}]\\
= &\Pr[\sigma(S^o)\cdot\frac{\theta_1}{n} > (\sqrt{\Lambda_1(S^o)+a}+\sqrt{a})^2]\\
= &\Pr[b > (\sqrt{\Lambda_1(S^o)+a}+\sqrt{a})^2]\\
= &\Pr[\sqrt{b} > \sqrt{\Lambda_1(S^o)+a}+\sqrt{a}]\\
= &\Pr[\sqrt{b}-\sqrt{a} > \sqrt{\Lambda_1(S^o)+a}]\\
= &\Pr[b+a-2\sqrt{ab} > \Lambda_1(S^o)+a]\\
= &\Pr[-\sqrt{4ab} > \Lambda_1(S^o)-b]\\
= &\Pr[-\sqrt{4ab} > \Lambda_1(S^o)-\sigma(S^o)\cdot\frac{\theta_1}{n}]\\
\leq &\Pr[-\sqrt{4ab} > \Lambda_1(S^o)-\sigma(S^o)\cdot\frac{\theta_1}{n}]\\
\leq &\exp(\frac{-4a\sigma(S^o)\cdot\frac{\theta_1}{n}}{4\sigma(S^o)\cdot\frac{\theta_1}{n}\frac{|P|}{n}})\\
= &\exp(\frac{-a}{\frac{|P|}{n}})\\
\leq &\exp(-a)=\delta
\end{aligned}$$

□

Similarly, by setting δ_1 and θ_1 which we will discuss next, we can obtain $\sigma^u(S^o)=\left(\sqrt{\frac{\Lambda_1(S^\star)}{1/e}+\ln\left(\frac{1}{\delta_1}\right)}+\sqrt{\ln\left(\frac{1}{\delta_1}\right)}\right)^2\frac{n\epsilon_1^2\mathrm{OPT}}{4|P|\log\left(\frac{1}{\delta_1}\right)}$

F Proof of Lemma 6

Proof. First, we consider obtaining the θ_1. Let $\delta_1 \in (0,1), \epsilon_1 > 0$ and

$$\theta_1=\frac{4|P|log(\frac{1}{\delta_1})}{\epsilon_1^2 OPT} \tag{3}$$

If $\theta \geq \theta_1$, then $n \cdot F_R(S^o) \geq (1-\epsilon_1) \cdot OPT$ holds with at least $1-\delta_1$ probability. Let $p = E[F_R(S^o)] = \sigma(S^o)/n = OPT/n$,

$$\begin{aligned}
&Pr[n \cdot F_R(S^o) \leq (1-\epsilon_1) \cdot OPT] \\
&\quad = Pr[n \cdot F_R(S^o) \leq (1-\epsilon_1) \cdot np] \\
&\quad = Pr[\theta \cdot F_R(S^o) \leq (1-\epsilon_1) \cdot \theta p] \\
&\quad \leq Pr[\theta \cdot F_R(S^o) \leq (1-\epsilon_1) \cdot \theta p] \\
&\quad = Pr[\sum_i Z_i(S^o) - \theta p \leq \epsilon_1 \theta p] \\
&\quad = Pr[\sum_i Z_i(S^o) - \sigma(S^o)\frac{\theta}{n} \leq \epsilon_1 \sigma(S^o)\frac{\theta}{n}] \\
&\quad \leq exp(\frac{(-\epsilon_1^2 \sigma(S^o)^2 (\frac{\theta}{n})^2}{4\sigma(S^o)\frac{\theta}{n}\frac{|P|}{n}}) \\
&\quad = exp(\frac{-\epsilon_1^2 \sigma(S^o)\theta}{4|P|}) \\
&\quad \leq \delta_1
\end{aligned} \tag{4}$$

Suppose that $nF_R(S^o) \geq (1-\epsilon_1) \cdot OPT$ holds, by the properties of the greedy approach,

$$\begin{aligned}
n \cdot F_R(S^\star) &\geq (1-1/e) n \cdot F_R(S^o) \\
&\geq (1-1/e)(1-\epsilon_1) \cdot OPT
\end{aligned} \tag{5}$$

Intuitively, this indicates that the expected mitigation of $S^\star$ is likely to be large, since $n \cdot F_R(S^\star)$ is an indicator of $\sigma(S^\star)$.

Next, we consider obtaining the θ_2. Let $\delta_2 \in (0,1), \epsilon_1 < \epsilon$ and

$$\theta_2 = \frac{2(1-1/e) \cdot nlog(\frac{\binom{n}{k}}{\delta_2})}{(\epsilon - (1-1/e\epsilon'))^2 \cdot OPT} \tag{6}$$

. If Equation 5 holds and $\theta \geq \theta_2$, then with at least $1-\delta_2$ probability, $\sigma(S^\star) \geq (1-1/e-\epsilon) \cdot OPT$.

Let S be an arbitrary size-k seed set. We say S is bad if $\sigma(S) < (1-1/e-\epsilon) \cdot OPT$. To prove this, we show that each bad size-k seed set has at most $\frac{\delta_2}{\binom{n}{k}}$ probability to be returned by applying the greedy algorithm to a collection of θ_2 RS sets. This suffices to establish the lemma because (1) there exist only $\binom{n}{k}$ bad size-k seed sets, and (2) if each of them has at most $\frac{\delta_2}{\binom{n}{k}}$ probability to be returned, then by the union bound, there is at least $1-\delta_2$ probability that none of them is output by the greedy algorithm. Consider any bad size-k seed set S.

Let $p = E[F_R(S)] = \sigma(S)/n$. We have

$$\begin{aligned} &Pr[n \cdot F_R(S) - \sigma(S) \geq \epsilon_2] \\ &=Pr[n \cdot F_R(S) \geq \sigma(S) + \epsilon_2 \cdot OPT] \\ &\geq Pr[n \cdot F_R(S) \geq (1-1/e)OPT + \epsilon_2 \cdot OPT] \\ &=Pr[n \cdot F_R(S) \geq (1-1/e+\epsilon) + \epsilon_2) \cdot OPT] \end{aligned} \tag{7}$$

We set ϵ_2 such that the multiplicative factor of OPT in Equation 3 is equal to the one in Equation 6.

$$\epsilon_2 = (\epsilon - (1 - 1/e\epsilon_1))^2 \tag{8}$$

Then, we can apply our Chernoff bounds by re-writing $Pr[n \cdot F_R(S) - \sigma(S) \geq \epsilon_2 \cdot OPT] = Pr[\theta \cdot F_R(S) - \sigma(S)\frac{\theta}{n} \geq 2\sigma(S) \cdot \frac{\theta}{n} \cdot OPT]$ and letting $\lambda = \sigma(S)2 \cdot \frac{\theta}{n} \cdot OPT$ to get

$$\begin{aligned} &Pr[n \cdot F_R(S) - \sigma(S) \geq \epsilon_2] \\ &=Pr[\theta \cdot F_R(S) - \sigma(S) \cdot \frac{\theta}{n} \geq \epsilon_2 \cdot \frac{\theta}{n} \cdot OPT] \\ &\leq exp(\frac{-\epsilon_2^2 \cdot \frac{\theta^2}{n^2} \cdot OPT}{\frac{2}{3}\epsilon_2 \cdot \frac{\theta}{n} \cdot OPT + 4\theta\frac{\sigma(S)}{n} \cdot \frac{|P|}{n}}) \\ &\leq exp(\frac{-\epsilon_2^2 \cdot \theta^2 \cdot OPT}{\frac{2}{3}\epsilon_2 \cdot \theta n \cdot OPT + 4\theta \cdot (1-1/e-\epsilon) \cdot OPT \cdot |P|}) \\ &\leq exp(-\frac{(\epsilon - (1-1/e\epsilon_1))^2 \cdot \theta \cdot OPT}{2(1-1/e) \cdot n}) \\ &\leq exp(-\frac{(\epsilon - (1-1/e\epsilon_1))^2 \cdot \theta_2 \cdot OPT}{2(1-1/e) \cdot n}) \\ &\leq \frac{\delta_2}{\binom{n}{k}} \end{aligned} \tag{9}$$

We can derive the third inequality through $\frac{2|P|}{n} \leq 1$

Then, due to $\sigma(S^\star) \geq (1-1/e-\epsilon) \cdot OPT$ holds with at least $1-\delta_2$ probability under the condition that Equation 5 holds. And Equation 5 holds with at least $1-\delta_1$ probability. By the union bound, $\sigma(S^\star) \geq (1-1/e-\epsilon) \cdot OPT$ holds with at least $1-\delta_1-\delta_2 \geq 1-\delta$ probability.

Then we consider how should we select ϵ_1, δ_1 and δ_2 to minimize θ? Assume that OPT is known. We are trying to minimize $\theta^\star = argmax\{\theta_1, \theta_2\}$ subject to $\delta_1 + \delta_2 \leq \delta$. Following an approach similar to that of [1], we set $\delta_1 = \delta_2 = \frac{1}{2}$ and set $\theta_1 = \theta_2$ to derive an approximately minimal value for $\theta^\star$.

$$\theta^\star \leq \frac{8n(1-1/e)[ln\frac{2}{\delta} + ln\binom{n}{k}]}{OPT\epsilon^2} \tag{10}$$

Let R be a collection of random RS sets and $S^\star$ be a size-k seed set generated by applying the greedy algorithm on R. For fixed ϵ, and δ, and

$$|R| \geq \frac{8n(1-1/e)[ln\frac{2}{\delta} + ln\binom{n}{k}]}{OPT\epsilon^2} \tag{11}$$

then $S^{\star}$ is a $(1-1/e-\epsilon)$-approximate solution with at least $1-\delta$ probability.

Thus, we can acquire the θ_{max} in algorithm 1:

$$\theta_{max} \geq \frac{8n(1-1/e)[ln\frac{2}{\delta} + ln\binom{n}{k}]}{LB\epsilon^2} \tag{12}$$

which is an upper bound on the number of RS sets needed to guarantee a $(1-1/e-\epsilon)$ approximation with at least $1-\delta$ probability when $LB \leq OPT$ is a lower bound of the optimal mitigation, we derive the LB from [2]. $\square$

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