

Problems for midterm

Chapter 1.

1. (Birthday)

Chapter 2.

1. (Poisson + Conditioning) In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ . You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp .

2. (Modified by Homework 4.4.) You and your opponent both roll a fair die. If one get a greater number than the other one, and that number > 3 , then the game ends and whoever rolls the larger number wins. Otherwise, we repeat the game. What is $P(\text{you win})$?

Chapter 3.

1. (Birthday)

2. (Expectation and variance of matchings) Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $\text{Var}(S_n)$.

3. (Random sum) Let $(X_i)_{1 \leq i \leq n}$ be a sequence n *i.i.d.* random variables with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Let N be a random variable taking value from $\{1, \dots, n\}$ with equal probability, independent to $(X_i)_{1 \leq i \leq n}$. Define $S_k = X_1 + X_2 + \dots + X_k$ for $1 \leq k \leq n$.

(a) What is the variance of the random sum, $\text{Var}(S_N)$?

(b) Let M be a random variable that has the same distribution as N , but independent to N and $(X_i)_{1 \leq i \leq n}$. What is $\text{Cov}(S_N, S_M)$? (You may encounter the calculation of $1^2 + 2^2 + \dots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$)

Homework

6.11. Use the method of indicators, for $i \neq j$, we can write

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}) &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) \\ &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}).\end{aligned}$$

Adapt the indicator method again, you can calculate

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k, e(j)=l\}}) \\ &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_kX_l\mathbb{1}_{\{e(i)=k, e(j)=l\}}) \\ &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_k)\mathbb{E}(X_l)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=l\}}) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k, e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k, e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_k^2)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} 1 \cdot \mathbb{P}(e(i) = k)\mathbb{P}(e(j) = k) \\ &= \sum_{k \neq i; k \neq j} \frac{1}{(n-1)^2} = \frac{n-2}{(n-1)^2}.\end{aligned}$$

Hence $\mathbb{E}(X_{e(i)}X_{e(j)}) = \frac{n-2}{(n-1)^2}$ for $i \neq j$.

When $i = j$, $\mathbb{E}(X_{e(i)}^2) = \sum_{k \neq i} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}}) = \sum_{k \neq i} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}})$. Use the independence calculation again, you can see that $\mathbb{E}(X_{e(i)}^2) = (n-1) \cdot 1 \cdot \frac{1}{(n-1)} = 1$.

Then the variance can be computed as

$$\begin{aligned}Var(X_{e(1)} + \dots + X_{e(n)}) &= \sum_{i,j} Cov(X_{e(i)}, X_{e(j)}) = \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) - \mathbb{E}(X_{e(i)})\mathbb{E}(X_{e(j)}) \\ &= \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) = \sum_{i=j} 1 + \sum_{i \neq j} \frac{n-2}{(n-1)^2} = n + \frac{n(n-2)}{n-1}.\end{aligned}$$

You can check that $\mathbb{E}(X_{e(i)}) = 0$ with the similar method. □