Solution to HW 8

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Q5: ...Are ~~X~~ and Y independent? $\rightarrow$ ...Are Z and Y independent?

Heads up

This homework is more or less about the change of variable for single/multi variate p.d.f. Before you start, think about what tools you have at hand. You may use any that fit the problem.

- Change of Variable formula (CoV)
  (p.55 56 for single variable, p.60 61 for multi-variable) This method is mention also in RC7 as well. Just remember that you need to put a measure-correction term $\left|\frac{\partial x}{\partial y}\right|$ (for single variable case) or $|J_T|$ (for multivariable case)
  For single variable case, changing from x to y with transformation $T(y)$

$$g(y) = f(T(y)) \left|\frac{\partial x}{\partial y}\right|$$

  For 2 variable case (multi-variable is similar), changing a coordinate (x, y) to (u, v) with transformation $(x, y) = T(u, v)$, we can get the new p.d.f. $g(u, v)$ by the original p.d.f $f(x, y)$ by

$$g(u, v) = f(T(u, v)) |J_T|$$

- Cumulative distribution function (c.d.f.) (p.54 55, **only for single variable case**)
  As we know that for cont. r.v., the way to 'measure' the probability of an event is to integrate the p.d.f of that event. Hence, you can think of this method as a version that you **integrate first, then derivate** to get new p.d.f. We use (TBD) as examples.

Q1

Sol.
First, we have to determine the range of Y. You can check that $Y \in [1, \infty)$.
Using the c.d.f. method, we have (note that there should be two cases, $x \in [0, 0.5]$ and $x \in [0.5, 1]$)

$$P(Y \leq y) = P(\frac{1}{1+y} \leq X \leq 0.5) + P(0.5 \leq X \leq \frac{y}{1+y}) = \frac{y-1}{1+y}, \quad y \in [1, \infty)$$

Then, do derivative on both side we have

$$f_Y(y) = \frac{d}{dy}P(Y \leq y) = \begin{cases} \frac{2}{(1+y)^2}, & \text{if } y \in [1, \infty); \\ 0, & \text{ow} . \end{cases}$$

$\square$

Q2

Sol.

- $V = \frac{X}{1-X}$
  Same methods as in Q1.

- $W = X(1-X)$
  First, we have to determine the range of W. Since $X(1-X) = -(X-\frac{1}{2})^2 + \frac{1}{4}$, we have $W \in [0, \frac{1}{4}]$.

  Using c.d.f. method, we can write

  $$P(W \leq w) = P(X(1-X) \leq w) = P(X \leq \frac{1-\sqrt{1-4w}}{2}) + P(X \leq \frac{1+\sqrt{1-4w}}{2}), w \in [0, 1/4]$$

  which equals to

  $$P(W \leq w) = 1 - \sqrt{1-4w}, w \in [0, \frac{1}{4}]$$

  Do derivate on both sides, we have

  $$f_W(w) = \frac{d}{dw}P(W \leq w) = \begin{cases} \frac{2}{\sqrt{1-4w}}, & \text{if } w \in [0, \frac{1}{4}] \\ 0, & \text{ow} . \end{cases}$$

$\square$

Q3

Sol.
This is an obvious example that use CoV method.
Since $X \sim N(\mu, \sigma^2)$, the p.d.f. of X is $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/(2\sigma^2))$
If we let $U = \frac{X-\mu}{\sigma}$, the transformation $x = T(u)$ we need is $T(u) = \sigma(u) + \mu$. Before we proceed, it is important you remember **multiplying with scalar and linear shift will keep a Gaussian variable Gaussian, just with different parameter**.
By using the description for single variable case we have

$$f_U(u) = f_X(T(u)) \left| \frac{dT}{du} \right| = \frac{1}{\sqrt{2\pi}} \exp \frac{-u^2}{2}$$

By the definition of Guassian r.v., $U = \frac{X-\mu}{\sigma} \sim N(0, 1)$.
$\square$
$-X$ case is easy with $T(u) = -u$.

Q4

Sol.

This is an exercise of multi-variable integral. Do it yourself.

- The joint p.d.f. of (X, Y, Z)

  You may assume that they distribute uniformly.

- $P(X^2 > YZ)$

$$\int_0^1 (x^2 - x^2 \ln x^2)dx = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$$

- $P(X + Y < Z)$

$$\frac{1}{6}$$

- $P(\max(X, Y) > Z)$

$$\frac{2}{3}$$

Q5

Sol.

- The p.d.f. of $Z = \min(X, Y)$ Obviously, the range of Z is $[0, 1]$. We use c.d.f. method here.

$$P(Z \le z) = P(\min(X, Y) \le z) = 1 - P(X > z \cap Y > z) = 1 - (1 - z)^2, z \in [0, 1]$$

  Hence,

$$f_Z(z) = P'(Z \le z) = 2(1 - z), \quad z \in [0, 1]$$

$\square$

- Are Z and Y independent?

  It is unlikely right? A counterexample to consider is the events $Z > 0.5$ and $Y \le 0.1$.

Q6

Sol.

Since X and Y are indep. , the joint p.d.f. $f(x, y)$ is

$$f(x, y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

First way: Polar coordinate & CoV

Remember how you will solve a Gaussian integral wiki, We consider turning (X, Y) to

(r,θ) with $T(r,\theta) = (r\cos\theta, r\sin\theta), r \in \mathbb{R}^+, \theta \in (0, 2\pi)$ Then by CoV method we have the new joint p.d.f. as

$$h(r,\theta) = f(T(r,\theta))\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r\frac{1}{2\pi}e^{-\frac{r^2}{2}}$$

Then, the marginal p.d.f. of r is

$$h_r(r) = \int_0^{2\pi} h(r,\theta)d\theta = re^{-\frac{r^2}{2}}, r \in \mathbb{R}^+$$

We are not done yet! $r = \sqrt{X^2+Y^2}$ but we want X^2+Y^2. Let $u = r^2$, we can use CoV for single variable to acquire the new p.d.f. of u as

$$f_U(u) = h_r(\sqrt{u})\left|\frac{dr}{du}\right| = \frac{1}{2}e^{-\frac{u}{2}}, \quad u \in \mathbb{R}^+.$$

In other word, $U \sim \text{Exp}(\frac{1}{2})$.

□

Another way: Using method in p.61

You can also find the p.d.f. of X^2 and Y^2 first and multiplying to get the joint p.d.f. of (X^2, Y^2). It can work because if X, Y are indep. then $g(X), f(Y)$ is also indep. Finally, use the method for $X+Y$ introduced in p.61 .

Q7

Sol.

- $\min(X,Y) \sim \text{Exp}(\alpha+\beta)$

  Use the same method of c.d.f. in Q5 but in this case $P(X > z \cap Y > z)$ is different.

- $P(X < Y)$

  This is a direct computation.

$$P(X<Y) = \int_0^\infty \alpha e^{-\alpha x}\int_x^\infty \beta e^{-\beta y}dydx = \frac{\alpha}{\beta+\alpha}$$

Q8

Sol.

Base case: $n = 2$

$$P(X_1+X_2 \le s) = \int_0^s\int_0^{s-x_1} dx_2dx_1 = \frac{s^2}{2}$$

Induction hypothesis: The statement is true up to $K \in \mathbb{N}$.
Induction step: Consider $n = K + 1$

$$\begin{aligned}
&P((X_1 + \dots X_K) + X_{K+1} \leq s) \\
&= \int_0^s P((X_1 + \dots X_K) \leq s - x_{K+1}) dx_{K+1} \\
&= \int_0^s \frac{(s - x_{K+1})^K}{K!} dx_{K+1} \\
&= \frac{s^{K+1}}{(K+1)!}
\end{aligned}$$

Hence, the hypothesis is also true for $n = K + 1$. By mathematical induction, the statement is true for all $n \in \mathbb{N}$.
□

Q9

Sol.
It will only work for gaussian random variables that covariance equals zero implies independent (Prop. 14)
Hence, the goal of this problem is to show that $X + Y$ and $X - Y$ are uncorrelated.
For multiple normal r.v. problem, their joint p.d.f. will be a good starting point. Let $Z = (X, Y)$, the p.d.f of random vector Z is (p.73 or RC note Theorem 8.1.1)

$$h(z) = \frac{1}{(2\pi)(\det(\Sigma))^{\frac{1}{2}}} e^{-\frac{1}{2}(z)^T \Sigma^{-1}(z)}$$

where $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Next step, finding the transformation matrix T.

$$\begin{bmatrix} X+Y \\ X-Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

which means

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{T} \begin{bmatrix} X+Y \\ X-Y \end{bmatrix},$$

Hence, given $W = [X + Y, X - Y]^\top$, we have $z = Tw$, so

$$z^\top \Sigma^{-1} z = w^\top \underbrace{T^\top \Sigma^{-1} T}_{\Sigma_W^{-1}} w$$

where

$$\Sigma_W = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By Prop. 13, it implies $X+Y$ and $X-Y$ are independent with $\sigma^2 = 4$

$\square$

Note

We stopped at Σ_W, but if you want the p.d.f. of W, using the CoV method you have

$$\begin{aligned} h_W(w) &= h_Z(Tw)\left|\frac{\partial(X,Y)}{\partial(X+Y,X-Y)}\right| \\ &= h_Z(Tw)|\det(T)| \\ &= \frac{1}{(2\pi)(\det(\Sigma_W))^{\frac{1}{2}}} e^{-\frac{1}{2}(w)^T\Sigma_W^{-1}(z)} \end{aligned}$$

with $\det \Sigma_W = 16$

Q10

Sol.

- The joint p.d.f. of $V = X+Y$ and $W = \frac{X}{X+Y}$.
  Again starting with the joint p.d.f. of (X,Y), which is

  $$f(x,y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)}$$

  The first equality is due to the independence of X, Y.
  Since we have $(X,Y) = T(V,W) = (WV, V(1-W))$, the Jacobian is

  $$J_T = \begin{bmatrix} w & v \\ 1-w & -v \end{bmatrix}$$

  Hence, $|\det(J_T)| = v$. Using CoV method we may write the joint p.d.f. of V, W as

  $$h(v,w) = f(T(v,w))\,|\det(J_T)| = \lambda^2 v e^{-\lambda(v)}, \quad (v,w) \in [0,\infty)\times[0,1]$$

- Prove that V and W are independent.
  By the definition of independent for continuous r.v., we will show that $h(z,w) = f_Z(z)f_W(w)$.
  Therefore, we have to find $f_Z(z)$ and $f_W(w)$.

  - $f_Z(z)$:
    From $h(z,w)$ we can get $f_Z(z)$ by taking the marginal.

    $$f_Z(z) = \int_0^1 h(v,w)dw = \lambda^2 v e^{-\lambda v}, v \in [0,\infty)$$

- $f_W(w)$:
  Taking the marginal for w we have $f_W(w) = 1, w \in [0,1]$.

Finally, since $h(z,w)$ is the product of marignal p.d.f., V and W are independent.

Q11

Sol.

- The joint p.d.f. of (X^2, Y^2).
  Here since both V and W use the second moment, so it is convienient to get the distribution of X^2 and Y^2 and use the "Another way" mentioned in Q6. If A is an event in $\mathbb{R}^+$, the probability of $U = X^2 \in A$ is

$$\begin{aligned} P(U \le u) &= P(-\sqrt{u} \le X \le \sqrt{u}) \\ &= 2P(0 \le X \le \sqrt{u}) \end{aligned}$$

  The last equation does not have a exact form, and we don't need that either. Taking the derivative against u

$$f_U(u) = \frac{d}{du} 2P(0 \le X \le \sqrt{u}) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{u}{2}}$$

  On the other hand, since X, Y are independent X^2, Y^2 are indepent, too. Hence, the joint p.d.f. of (X^2, Y^2) is

$$f(x,y) = \frac{1}{2\pi\sqrt{xy}} e^{-\frac{1}{2}(x+y)}$$

  We can finally return to the original problem.

- The joint p.d.f. of (V, W)

$$\begin{bmatrix} X^2 \\ Y^2 \end{bmatrix} = \underbrace{\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{T} \begin{bmatrix} V \\ W \end{bmatrix}$$

  Here $|\det J_T| = \frac{1}{2}$. Consequently,

$$f_{V,W}(v,w) = \frac{1}{\pi\sqrt{v^2 - w^2}} e^{-\frac{1}{2}v}$$

  Becareful about the region, the region should be $\{(v,w) \in R^+ \times R^+, v \ge w\}$
  You can check that

$$\int_0^\infty \int_0^v f_{V,W}(v,w) dw dv = 1$$

- Are V and W independent?

  Given the weird joint p.d.f., it's unlikely right?

  Because the joint p.d.f. is quite complicated for the marginal of W, we can use conditional p.d.f. to check.

  i.e. Given v such that $f_V(v) > 0$, if the conditional p.d.f. $f_W(w|V = v) = \frac{f_{V,W}(v,w)}{f_V(v)}$ is not a function of solely w, then V, W are not independent.

  Since $f_V(v) = \frac{1}{2}e^{-\frac{v}{2}} > 0$, we have

  $$f_W(w|V = v) = \frac{2}{\pi\sqrt{v^2 - w^2}},$$

  which is a function of both v and w. That means W and V are not independent.