Problems for midterm

Chapter 1.

1. (Birthday)

Chapter 2.

1. (Poisson + Conditioning) In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp.

Chapter 3.

1. (Birthday)

2. (Expectation and variance of matchings) Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $Var(S_n)$.

3.

Homework 6.

6.11. Use the method of indicators, for $i \neq j$, we can write

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}) &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}} + X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) \\ &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}).\end{aligned}$$

Adapt the indicator method again, you can calculate

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\, e(j)=l\}}) \\ &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_k X_l \mathbb{1}_{\{e(i)=k,\, e(j)=l\}}) \\ &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_k)\mathbb{E}(X_l)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=l\}}) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\, e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k,\, e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_k^2)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} 1\cdot\mathbb{P}(e(i)=k)\mathbb{P}(e(j)=k) \\ &= \sum_{k\neq i;\, k\neq j} \frac{1}{(n-1)^2} = \frac{n-2}{(n-1)^2}.\end{aligned}$$

Hence $\mathbb{E}(X_{e(i)}X_{e(j)}) = \frac{n-2}{(n-1)^2}$ for $i \neq j$.

When $i = j$, $\mathbb{E}\big(X_{e(i)}^2\big) = \sum_{k\neq i}\mathbb{E}(X_{e(i)}^2\mathbb{1}_{\{e(i)=k\}}) = \sum_{k\neq i}\mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}})$. Use the independence calculation again, you can see that $\mathbb{E}\big(X_{e(i)}^2\big) = (n-1)\cdot 1\cdot\frac{1}{(n-1)} = 1$.

Then the variance can be computed as

$$\begin{aligned}Var(X_{e(1)}+\cdots+X_{e(n)}) &= \sum_{i,j} Cov(X_{e(i)}, X_{e(j)}) = \sum_{i,j}\mathbb{E}(X_{e(i)}X_{e(j)}) - \mathbb{E}(X_{e(i)})\mathbb{E}(X_{e(j)}) \\ &= \sum_{i,j}\mathbb{E}(X_{e(i)}X_{e(j)}) = \sum_{i=j} 1 + \sum_{i\neq j}\frac{n-2}{(n-1)^2} = n + \frac{n(n-2)}{n-1}.\end{aligned}$$

You can check that $\mathbb{E}(X_{e(i)}) = 0$ with the similar method. □