Final Exam

Part 1 (55 points)

1. Let $U \sim \text{Unif}([0,1])$. Find the p.d.f. of the following random variables:
   (a) $X = U^2$
   (b) $Y = e^U$
   (c) $Z = \sqrt{U}$
   (5 points for each problems)

2. Let $\{X_i\}_{i=1}^{\infty}$ be independent random variables having the exponential distribution with parameters λ.
   (a) (5 points) Find the density function of $X_1 + X_2$.
   (b) (10 points) Let $S_n = X_1 + \cdots + X_n$. Prove that the density of S_n is

$$f_{S_n}(s) = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s},\ s > 0.$$

   (c) (5 points) Now let $\overline{X}_n = S_n/n$. Calculate $E(\overline{X}_n)$ and Var$(\overline{X}_n)$.
   (d) (5 points) Prove the weak law of large numbers for $\overline{X}_n$. That is, show that for any $\epsilon > 0$,

$$P(|\overline{X}_n - E(\overline{X}_n)| \geq \epsilon) \to 0 \text{ as } n \to \infty.$$

3. Roll a die n times and let S_n be the number of times you roll 6 by time n. Assume that each rolls are independent. Let $\overline{X}_n = S_n/n$.
   (a) (5 points) Compute $E(S_n)$ and Var(S_n).
   (b) (10 points) Consider $\overline{X}_n = S_n/n$. We want to estimate $P(|\overline{X}_n - E(\overline{X}_n)| < \epsilon)$ for some small $\epsilon > 0$. Let $\Phi(x) = P(Z \leq x)$ be the c.d.f. of $Z \sim N(0,1)$. Use central limit theorem to write down an approximation of $P(|\overline{X}_n - E(\overline{X}_n)| < \epsilon)$. (Use Φ and n to express your answer)

Part 2 (45 points. Choose 3 of 6 problems to answer.)

4. (Gaussian distribution and integration by parts) Assume $X \sim N(0,1)$.
   (a) (5 points) Show that $E(X^{2k}) = (2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$.

(b) (10 points) Suppose f is continuously differentiable, prove that

$$E(Xf(X)) = E(f'(X)),$$

provided that both sides are well-defined.

5. Let X, Y have the normal distribution with unit variance and zero mean, and their covariance $\rho \neq 0$.
(a) (5 points) Write down the joint density function of (X, Y).
(b) (10 points) Let $U_\theta = X\cos\theta + Y\sin\theta$ and $V_\theta = -X\sin\theta + Y\cos\theta$, $\theta \in [0, \pi)$. Find all the possible θ such that U_θ and V_θ are independent.

6. Let X and Y be independent random variables with the same c.d.f F and p.d.f. f.
(a) (5 points) What is the c.d.f. of $V = \max\{X, Y\}$?
(c) (10 points) Derive the p.d.f. of $Z = \min\{X, Y\}$.

7. Suppose that (X, Y, Z) is a random point inside a unit cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$.
(a) What is the joint p.d.f. of (X, Y, Z)?
(b) Compute $P(X^2 > YZ)$.
(c) Compute $P(\max(X, Y) > Z)$ (Hint: Symmetry).
(5 points for each problems)

8. (Buffon's needle problem) Suppose the $\mathbb{R}^2$ plane was separated by infinitely many parallel lines $\{y = na\}_{n\in\mathbb{Z}}$ for some constant $a > 0$. You drop a needle of length $0 < r < a$ uniformly on the plane. We're going to estimate the probability that the needle crosses a line.
(a) (5 points) Given that the needle's orientation has an angle θ to the x-axis, where $\theta \in [0, \pi)$. Then what is the probability of crossing a line?
(b) (10 points) Using the result of (a), derive the probability of line crossing.

9. Find the value C for the following probability density functions.
(a) (5 points) $f(x) = C(\gamma^2 + x^2)^{-1}$, $x \in \mathbb{R}$.
(b) (5 points) $f(x) = \frac{Ce^{-x}}{(1+e^{-x})^2}$, $x \in \mathbb{R}$.
(c) (5 points) $\frac{C}{x}\exp(-\frac{1}{2}(\log(x) - \mu)^2)$, $x \in \mathbb{R}$. μ is a fixed real number.