

Problems for midterm

Chapter 1.

- **1. (Birthday problem)**

Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). Write down the expression of probability that there exists at least a pair of student that share the same birthday

- **2. (Probability axiom)**

(Modified from Panchenko Exercise 1.2.15) Let $\Omega = \{a, b, c, d\}$ and the probability function $P : \Omega \rightarrow [0, 1]$. Suppose $P(\{a, b\}) = 0.6$, $P(\{b, c\}) = 0.3$, $P(\{c, d\}) = 0.4$, Calculate $P(\{a\})$, $P(\{b\})$, $P(\{c\})$, $P(\{d\})$

- **3. (independence)**

We roll a die three times. Let A_{ij} be the event that the i th and j th rolls produce the same number. Show that the events A_{12} , A_{23} , A_{13} are pairwise independent but not independent events.

Chapter 2.

- **1. (Poisson + Conditioning)** In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ . You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp .

- **2. (Modified by Homework 4.4.)** You and your opponent both roll a fair die. If one get a greater number than the other one, and that number > 3 , then the game ends and whoever rolls the larger number wins. Otherwise, we repeat the game. What is $P(\text{you win})$?

- **3. (Normalize constant)**

Consider a function f defined on $2, 3, 4, \dots$ such that $f(x) = C \frac{1}{x(x+1)}$, where C is a constant. Please find C such that f is a pmf.

Chapter 3.

- **1. (Birthday problem II.)** Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). What is the expectation of number of distinct birthday?

- **2. (Expectation and variance of matchings)** Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $Var(S_n)$.

- **3. (Random sum)** Let $(X_i)_{1 \leq i \leq n}$ be a sequence n *i.i.d.* random variables with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Let N be a random variable taking value from $\{1, \dots, n\}$ with equal probability, independent to $(X_i)_{1 \leq i \leq n}$. Define $S_k = X_1 + X_2 + \dots + X_k$ for $1 \leq k \leq n$.

(a) What is the variance of the random sum, $Var(S_N)$?

(b) Let M be a random variable that has the same distribution as N , but independent to N and $(X_i)_{1 \leq i \leq n}$. What is $Cov(S_N, S_M)$? (You may encounter the calculation of $1^2 + 2^2 + \dots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$)

Homework 6.

6.11. Use the method of indicators, for $i \neq j$, we can write

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}) &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) \\ &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}).\end{aligned}$$

Adapt the indicator method again, you can calculate

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i) \neq e(j)\}}) &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k, e(j)=l\}}) \\ &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_kX_l\mathbb{1}_{\{e(i)=k, e(j)=l\}}) \\ &= \sum_{k \neq i; l \neq j; k \neq l} \mathbb{E}(X_k)\mathbb{E}(X_l)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=l\}}) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k, e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k, e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} \mathbb{E}(X_k^2)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=k\}}) \\ &= \sum_{k \neq i; k \neq j} 1 \cdot \mathbb{P}(e(i) = k)\mathbb{P}(e(j) = k) \\ &= \sum_{k \neq i; k \neq j} \frac{1}{(n-1)^2} = \frac{n-2}{(n-1)^2}.\end{aligned}$$

Hence $\mathbb{E}(X_{e(i)}X_{e(j)}) = \frac{n-2}{(n-1)^2}$ for $i \neq j$.

When $i = j$, $\mathbb{E}(X_{e(i)}^2) = \sum_{k \neq i} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}}) = \sum_{k \neq i} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}})$. Use the independence calculation again, you can see that $\mathbb{E}(X_{e(i)}^2) = (n-1) \cdot 1 \cdot \frac{1}{(n-1)} = 1$.

Then the variance can be computed as

$$\begin{aligned}\text{Var}(X_{e(1)} + \cdots + X_{e(n)}) &= \sum_{i,j} \text{Cov}(X_{e(i)}, X_{e(j)}) = \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) - \mathbb{E}(X_{e(i)})\mathbb{E}(X_{e(j)}) \\ &= \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) = \sum_{i=j} 1 + \sum_{i \neq j} \frac{n-2}{(n-1)^2} = n + \frac{n(n-2)}{n-1}.\end{aligned}$$

You can check that $\mathbb{E}(X_{e(i)}) = 0$ with the similar method. □