Problems for midterm

Chapter 1.

- **1. (Birthday problem)**
  Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). Write down the expression of probability that there exists at least a pair of student that share the same birthday

- **2. (Probability axiom)**
  (Modified from Panchenko Exercise 1.2.15) Let $\Omega = \{a, b, c, d\}$ and the probability function $P : \Omega \to [0, 1]$. Suppose $P(\{a, b\}) = 0.6, P(\{b, c\}) = 0.3, P(\{c, d\}) = 0.4$, Calculate $P(\{a\}), P(\{b\}), P(\{c\}), P(\{d\})$

- **3. (independence)**
  We roll a die three times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events A_{12}, A_{23}, A_{13} are pairwise independent but not independent events.

Chapter 2.

- **1. (Poisson + Conditioning)** In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time.
  (a) Compute $\mathbb{P}(H = h \mid N = n)$, where H is the total number of heads.
  (b) Show that the total number of heads has the Poisson distribution with parameter λp.

- **2. (Refer to Homework 4.4.)** You and your opponent both roll a fair die. If one get a greater number than the other one, and that number > 3, then the game ends and whoever rolls the larger number wins. Otherwise, we repeat the game.
  (a) Let N be the number of rounds in this game. Write down the p.m.f. of N.
  Solution (a). $\mathbb{P}(N = n) = (1 - p)^{n-1} p$ where $p = \frac{2}{3}$. □
  (b) What is $P(\text{you win})$?

- **3. (Normalize constant)**
  Consider a function f defined on $\mathbf{2}, 3, 4, \dots$ such that $f(x) = C\frac{1}{x(x+1)}$, where C is a constant. Please find C such that f is a pmf.

Chapter 3.

- **1. (Birthday problem II.)** Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). What is the expectation of number of distinct birthday?

- **2. (Expectation and variance of matchings)** Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $Var(S_n)$.

- **3. (Refer to Homework 6.11.)** Let $(X_i)_{1\le i\le n}$ be a sequence n *i.i.d.* random variables with
$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$
Define $S_k = X_1 + X_2 + \cdots + X_k$ for $1 \le k \le n$ as the k-th partial sum.
(a) Let N be a random variable taking values from $\{1, \cdots, n\}$ with equal probability, independent to $(X_i)_{1\le i\le n}$. What is the variance of the random sum, $Var(S_N)$?
(b) Let M be a random variable that has the same distribution as N in (a), but independent to N and $(X_i)_{1\le i\le n}$. What is $Cov(S_N, S_M)$? (You may encounter the calculation of $1^2 + 2^2 + \cdots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$)
Solution (a). Write $S_N = S_N \mathbb{1}_{\{N=1\}} + \cdots + S_N \mathbb{1}_{\{N=n\}}$, then by linearity of expectation,
$$\mathbb{E}\big(S_N^2\big) = \sum_{k=1}^{n} \mathbb{E}\big(S_N^2 \mathbb{1}_{\{N=k\}}\big) = \sum_{k=1}^{n} \mathbb{E}\big(S_k^2 \mathbb{1}_{\{N=k\}}\big)$$
Since $(X_i)_{1\le i\le n}$ and N are independent, by RHS we have
$$\mathbb{E}\big(S_N^2\big) = \sum_{k=1}^{n} \mathbb{E}\big(S_k^2\big)\mathbb{E}(\mathbb{1}_{\{N=k\}}) = \sum_{k=1}^{n} k\mathbb{P}(N = k) = \sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2}.$$
It's easy to see $\mathbb{E}(S_N) = 0$ by this method. Hence $Var(S_N) = \mathbb{E}\big(S_N^2\big) = \frac{n+1}{2}$. □
Solution (b). Similarly, write $S_N S_M = S_N S_M \mathbb{1}_{\{N=M\}} + S_N S_M \mathbb{1}_{\{N\ne M\}}$ and compute the expectation respectively. Since N and M are *i.i.d.*,
$$\mathbb{E}(S_N S_M \mathbb{1}_{\{N\ne M\}}) = \mathbb{E}(S_N S_M \mathbb{1}_{\{N<M\}}) + \mathbb{E}(S_N S_M \mathbb{1}_{\{N>M\}}) = 2\mathbb{E}(S_N S_M \mathbb{1}_{\{N<M\}}),$$
and it easy to see that $\mathbb{1}_{\{N<M\}} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{1}_{\{N=i,\, M=j\}}$, thus
$$\begin{aligned}
\mathbb{E}(S_N S_M \mathbb{1}_{\{N<M\}}) &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{E}(S_N S_M \mathbb{1}_{\{N=i,\, M=j\}}) \\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{E}(S_i S_j \mathbb{1}_{\{N=i,\, M=j\}}) \\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{E}(S_i S_j)\mathbb{E}(\mathbb{1}_{\{N=i,\, M=j\}}) \\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{E}(S_i S_j)\mathbb{P}(N = i, M = j).
\end{aligned}$$
Notice that since $(X_i)_{1\le i\le n}$ is an *i.i.d.* sequence,
$$\mathbb{E}(S_i S_j) = \mathbb{E}(S_i^2) + \mathbb{E}(S_i(X_{i+1} + \cdots + X_j)) = \mathbb{E}(S_i^2) + \mathbb{E}(S_i)\mathbb{E}(X_{i+1} + \cdots + X_j),$$

which is $\mathbb{E}(S_i^2) + 0 = i$. So the summation above becomes

$$\mathbb{E}(S_N S_M \mathbb{1}_{\{N<M\}}) = \sum_{i=1}^{n-1} \frac{(n-i)i}{n^2} = \frac{n-1}{2} - \frac{n(n-1)(2n-1)}{6n^2} = \frac{n^2-1}{6n}.$$

So $\mathbb{E}(S_N S_M \mathbb{1}_{\{N\neq M\}}) = \frac{n^2-1}{3n}$. Together with

$$\mathbb{E}(S_N S_M \mathbb{1}_{\{N=M\}}) = \sum_{i=1}^{n} \mathbb{E}(S_i^2)\mathbb{P}(N=i, M=i) = \frac{n(n+1)}{2n^2},$$

we have

$$\mathbb{E}(S_N S_M) = \frac{n}{3} + \frac{1}{2} + \frac{1}{6n} = Cov(S_N, S_M).$$

$\square$