Prob. RC Note

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Abstract

This is a note template, with all but minimal compilable files provided. Feel free to adjust for your usage.

Now let's start a simple demo for you to take fancy notes in LaTeX!

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Chapter 0

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22 Sep. 18:30

0.1 Basic rules

- Recitation: Thursday 12:30 to 13:30, 18:30 to 19:30
- My office hour is right after recitation.
- Grade distribution: (quiz correction + attend rc): 1×4; mid correction + attend rc: 2; Attend one rc: 4

0.2 Review of some high school comminatory and probability tricks

Early chapters are about high school counting things over again. We will swiftly go through the concept.

0.2.1 Permutation and Combination

- Permutation
  #ways to form an ordering of m out of n different things.
  $$P(n, m)$$
- Combination
  #ways to form a group of m with n different things.
  $$C(n, m), \quad \binom{n}{m}$$
- Multinomial coefficents [Gra21]
  #ways to divide a set of n elements into r (distinguishable) subsets of $n_1, n_2, \ldots, n_r$ elements.
  $$\frac{n!}{n_1! n_2! \ldots n_r!}$$

0.2.2 Set Operation

- De Morgan's Law

0.3 Axioms of Probability

0.3.1 Probability Space

The **probability space** is a triple $\Omega, \mathcal{F}, P$ that contains

- The **sample space** Ω contains all possible outcome.
- The σ-algebra $\mathcal{F}$ is the **event space**. It is a subset of the power set of Ω we are interested in.
- The **probability measure** P is a function $P : \mathcal{F} \to [0, 1]$ that satisfies the three axioms.
    1. $P(\Omega) = 1$
    2. Non-negative
    3. Countable additivity for disjoint sets in $\mathcal{F}$.

0.3.2 Standard process

A standard process of solving these problems (HW1,2) is to find the size of possible outcome first. Then, finding the size of desired event, and the ratio of the two is the prob.

Example. [Gra21] Example 3.11
You have 10 pairs of socks in the closet. Pick 8 socks at random. For every i, compute the probability that you get i complete pairs of socks.

- # outcome:
- # desirable outcome:
- the probability is :

Example. [Gra21] Problem 3.2 (HW2 problem 2)
Three married couples take seats around a table at random. Compute $P(\text{no wife sits next to her husband})$. Use Inclusion-Exclusion principle to compute the probability of its complement event.

0.3.3 Why do we need to set σ-algebra: Vitali set

Have you ever wonder: Why would I need $\mathcal{F}$ if I have Ω already? As the textbook said, you won't have any problem with this notion. However, things get messy when we encounter set of infinity size. The idea of "length" will not be clear then. We use **Vitali set** V as an example on $\mathbb{R}$ to show that we can't have a measure on V. This problem is one of the reason that we only put probability measure on $\mathcal{F}$.

For more information: How the Axiom of Choice Gives Sizeless Sets | Infinite Series

0.4 Homework Help

TBD

Chapter 1

RC 1

Recitation 1

1.1 Review

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1.1.1 Axioms of Probability

A **probability space** is a triple $\Omega, \mathcal{F}, P$ that contains

- The **sample space** Ω contains all possible outcome.
- The σ-algebra $\mathcal{F}$ is the **event space**. It is a subset of the power set of Ω we are interested in.
- The **probability measure** P

1.1.2 Probability (measure)

- The **probability measure** P is a function $P : \mathcal{F} \to [0, 1]$ that satisfies the three axioms.
  1. $P(\Omega) = 1$
  2. Non-negative
  3. Countable additivity for disjoint sets in $\mathcal{F}$.

1.1.3 σ-algebra

$\mathcal{F}$ is call a σ-algebra on a set Ω If

1. $\emptyset \in \mathcal{F}$
2. If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. If $A_1, A_2, \dots \in \mathcal{F}$, then $\cup_{i=1} A_i \in \mathcal{F}$

> **Example.** Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, find a minimal* σ-algebra that contains the sets $\{1, 2, 3\}, \{1\}$

Answer: [1]

1.1.4 Conditional Probability

For the general definition, take events A, B, and assume that $P(B) > 0$. The *conditional probability* of the event A given B equals

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

TBD (an example)

[1] $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{2, 3\}, \{1, 4, 5, 6\}\}$

1.1.5 Independence

Events $A_1, \ldots, A_n$ are independent if

$$P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$$

Please note that it means you can not just check $P(A_i \cap A_j) = P(A_i)P(A_j), i \neq j$. One example is teh following

Example (Pairwise Independence but not independent variables). ([Pan19] Exercise 1.4.2) Consider a regular tetrahedron die painted blue, red and green on three sides and painted in all three colours on the fourth side. If the die is equally likely to land on any side, show that the appearances of these colours on the side it lands on are pairwise-independent but not independent.

1.2 Problem

Problem 1. Put r distinguishable balls into n different boxes. What is the probability of all n boxes are occupied?

Probabilistic approach. $A_i = \{\text{Box } i \text{ is occupied}\}$ for all $i = 1, \cdots, n$.

$$\mathbb{P}(\text{No empty boxes}) = \mathbb{P}\Big(\bigcup_{i=1}^n A_i\Big). \tag{1.1}$$

(Inclusion-exclusion principle!)

Combinatorial approach. Define $A(r,n)$ as the number of distributions such that all n boxes are non-empty when you put r balls into them. Then the probability is $A(r,n)/n^r$. Knowing $A(r,n+1)$ can lead us to $A(r,n)$:

$$A(r,n) = \sum_{k=1}^{r-1} A(r-k, n-1). \tag{1.2}$$

Knowing this, we can prove that:

$$A(r,n) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (n-\nu)^r. \tag{1.3}$$

(Use Induction!)

(Think about it!) What is the probability of exactly m boxes are occupied?

$$\text{number of distributions } = \binom{n}{m} \times A(r,m). \tag{1.4}$$

$A(r,m) = \mathbb{P}(r \text{ balls into } m \text{ boxes and all of the boxes are occupied}) \times m^r$.

The probability that at least one cell is empty is given by (1.5), and hence we find for the *probability that all cells are occupied*

$$p_0(r, n) = 1 - S_1 + S_2 - + \cdots = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \left(1 - \frac{\nu}{n}\right)^r. \tag{2.3}$$

Consider now a distribution in which exactly m cells are empty. These m cells can be chosen in $\binom{n}{m}$ ways. The r balls are distributed among the remaining $n - m$ cells so that each of these cells is occupied; the number of such distributions is $(n-m)^r p_0(r, n - m)$. Dividing by n^r we find for the *probability that exactly m cells remain empty*

$$\begin{aligned} p_m(r, n) &= \binom{n}{m} \left(1 - \frac{m}{n}\right)^r p_0(r, n-m) = \\ &= \binom{n}{m} \sum_{\nu=0}^{n-m} (-1)^\nu \binom{n-m}{\nu} \left(1 - \frac{m+\nu}{n}\right)^r. \end{aligned} \tag{2.4}$$

Figure 1.1: A problem in Feller

Appendix

Appendix A

Additional Proofs

A.1 Proof of ??

We can now prove **??**.

Proof of ??. See here. ■

Bibliography

[Gra21] Janko Gravner. *Lecture Notes for Introductory Probability Introduction to Probability*. sbd, 2021. URL: https://www.math.ucdavis.edu/~gravner/MAT135A/resources/lecturenotes.pdf.

[Pan19] Dmitry Panchenko. *Introduction to Probability Theory*. 2019. URL: https://drive.google.com/file/d/1Rpkr-NCEyqygvypR65RznaZkb36KHB29/view.