Quiz 4 Solution

1. This is a problem about change of variable (CoV) for single variable

   (a) We may copy the c.d.f. method in HW8. First, observe that the range of U is $[0, \infty)$ not $\mathbb{R}$ now.

   To use the c.d.f. method, we consider

   $$P(U \leq u) = P(X^2 \leq u) = P(-\sqrt{u} \leq X \leq \sqrt{u}) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

   The integral does not have analytical form, but that's fine. We only want its derivative $f_U(u)$. By Leibniz integral rule

   $$\frac{d}{du} \int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2\sqrt{u}} (\frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}}) - \frac{-1}{2\sqrt{u}} (\frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}}) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{u}{2}}$$

   $\square$

   (b) By definition of expectation

   $$E[U] = \int_0^\infty u f_U(u) du = \frac{1}{\sqrt{2\pi}} \int_0^\infty u^{\frac{1}{2}} e^{-\frac{u}{2}} du \underbrace{=}_{u=2t^2} \frac{4}{\sqrt{\pi}} \int_0^\infty t^2 e^{-t^2} dt$$

   At this point you either recognize the integral as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, or do integrate by part as follows

   $$\int_0^\infty t^2 e^{-t^2} dt = \cancelto{0}{-\frac{t}{2} e^{-t^2} |_0^\infty} + \frac{1}{2} \int_0^\infty e^{-t^2} dt = \frac{\pi}{4}$$

   Hence, $E[U] = 1$.

   Note: If you notice $E[U] = E[X^2]$, which is given to be 1 since $1 = Var(X) = E[X^2] = E[X]^2 = E[X^2]$.

2. This is a problem that applies c.d.f. method, and that's why the problem is designed this way.

   (a) First step, we check the joint p.d.f. of X and Y. Since they are independent, the joint p.d.f. is the product of their respective p.d.f., i.e.

   $$f(x, Y) = \lambda^2 e^{-\lambda(x+y)}$$

Then, we compute the c.d.f. of Z.

$$\begin{aligned}
P(Z \le z) = P(X \le zY) &= \int_0^\infty \int_0^{zy} \lambda^2 e^{-\lambda(x+y)} dx dy \\
&= \int_0^\infty \lambda e^{-\lambda y}(1 - e^{-\lambda z y}) dy \\
&= 1 - \frac{1}{z+1}
\end{aligned}$$

□

(b) Getting p.d.f. from (a) is just a derivative away. Still, you have to be careful about the range of Z. In this case Z takes value in $[0, \infty)$ (or $\mathbb{R}^+$).

$$f_Z(z) = P'(Z \le z) = \frac{1}{(z+1)^2}$$

□

3. (a) This is a classic multi-variate change of variable problem. We will start with (if possible) finding T such that $(X, Y) = T(V, W)$.

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} V \\ W \end{bmatrix}$$

In this case we have T as a transformation matrix A. (b.t.w. it means $\left|\frac{\partial(x,y)}{\partial(v,w)}\right| = \frac{1}{2}$)
Then, since X, Y are i.i.d. standard normal, the joint p.d.f. is

$$f_(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

Finally, by multi-variate CoV (Please refer to HW8_sol), the joint p.d.f. of (V, W), g is

$$g(v, w) = f(T(v, w)) \left|\frac{\partial(x, y)}{\partial(v, w)}\right|$$

Which is

$$g(v, w) = \frac{1}{4\pi} e^{-\frac{(\frac{v+w}{2})^2 + (\frac{v-w}{2})^2}{2}} = \frac{1}{4\pi} e^{-\frac{v^2+w^2}{4}}$$

□

(b) If both variables have p.d.f. and they also have joint p.d.f. (which is the case we have here), then V W are independent if $g(v, w) = f_Z(z) f_W(w)$.

$$g(v, w) = \frac{1}{4\pi} e^{-\frac{v^2+w^2}{4}} = \underbrace{\frac{1}{2\pi\sqrt{2}} e^{-\frac{v^2}{2\cdot 2}}}_{f_Z(z)} \cdot \underbrace{\frac{1}{2\pi\sqrt{2}} e^{-\frac{w^2}{2\cdot 2}}}_{f_W(w)}$$

Hence, V, W are independent.

(c) The problem is equivalent to asking the marginal p.d.f. of Z. Due to the independence of V, W (as shown in (b)), this is just $f_Z(z)$.

Note: Since X, Y are independent normal variables, we may also use the **stability of normal r.v.** to get $X + Y \sim N(0, 1+1)$ directly.

4. This problem indents to find the transformation of X, A , such that the new **covariance matrix** is Σ. Since

$$\Sigma = AA^\top$$

is a semi-defninate matrix. It is good to start from eigenvalue decomposition.

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence, we may set $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

□

5. This is a rather weird question. Observe that we have

(a) $f_X(x) = 1, x \in [0, 1]$

(b) $P(X \leq x) = x, x \in [0, 1]$

(c) $f_Y(y) = e^{-y}, y \in \mathbb{R}^+$

(d) $P(Y \leq y) = 1 - e^{-y}, y \in \mathbb{R}^+$

Can we use c.d.f. method on this one? Let's try

$$1 - e^{-y} = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = g^{-1}(y)$$

(About the existance of inverse function, you may first assume it and check that g^{-1} increase with y continuously, or you can first observe that $P(X \leq g^{-1}(y))$)

Anyway, $x = g^{-1}(y) = 1 - e^{-y}$ gives us

$$y = g(x) = -\ln(1 - x)$$

□