Partial solution to HW 9

Q4

There is a useful identity for the density function of standard normal distribution: Let $p_X(x)$ be the density for $X \sim N(0,1)$, then $p'_X(x) = xp_X(x)$.

The proof is based on induction. $k = 1$ cases are simple and we left them to you. Suppose $\mathbb{E}(X^{2k-1}) = 0$. Then by integration by part,

$$\begin{aligned} E(X^{2k+1}) = \int x^{2k+1} p_X(x)dx &= x^{2k} p_X(x)\Big|_{-\infty}^{\infty} + 2k \int x^{2k-1} p_X(x)dx \\ &= 0 + 2kE(X^{2k-1}) = 0. \end{aligned}$$

On the other hand, suppose $E(X^{2k}) = 1 \cdot 3 \cdot 5 \cdots (2k-1)$, then by integration by part again,

$$\begin{aligned} E(X^{2(k+1)}) = \int x^{2(k+1)} p_X(x)dx &= x^{2k+1} p_X(x)\Big|_{-\infty}^{\infty} + (2k+1) \int x^{2k} p_X(x)dx \\ &= 0 + (2k+1)E(X^{2k}) = (2k+1)!!. \end{aligned}$$

The by mathematical induction you can get the desired result. $\square$

Q5

The point is to see that $\sum_{i=1}^{n} a_i X_i \sim N(0,1)$. Then the result follows from **Q4**.

Let $\mathbf{a}_1 = (a_1, \cdots, a_n)$ be a vector in $\mathbb{R}^n$. In the orthogonal complement subspace $U = \{u \in \mathbb{R}^n : u^T \mathbf{a}_1 = 0\}$, we find orthonormal basis $u_1, \cdots, u_{n-1}$ for U. Then the matrix defined by

$$A = \begin{pmatrix} — & \mathbf{a}_1 & — \\ — & u_1 & — \\ & \vdots & \\ — & u_n & — \end{pmatrix}$$

is orthonormal. Let $Y = AX$, where $X = (X_1, \cdots, X_n)^T$ is the standard normal random vector, then

$$p_Y(y) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{\frac{1}{2}y^T(AA^T)^{-1}y} \frac{1}{|A|} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{\frac{1}{2}y^T y}.$$

Which is a standard normal density function. Note that the first component of Y is $\mathbf{a}_1^T X = \sum_{i=1}^{n} a_i X_i$. Now by Y is a standard normal r.v., we immediately have $\sum_{i=1}^{n} a_i X_i \sim N(0,1)$. Finally, by **Q4**, $E\big(\sum_{i=1}^{n} a_i X_i\big)^{2k} = (2k-1)!!$. $\square$

Q6

If you can find a 2×2 matrix A such that $AA^T = \Sigma$, then $AX \sim N(0, \Sigma)$. You can simply let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and solve for $AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. For example, $(a, b, c, d) = (1, 1, 1, 0)$ is one of the solution. $\square$

Q7

By the property of multivariate Gaussian, one can see that $AA^T = \Sigma$. Moreover, consider $\mathbf{a} = (a_1, \cdots, a_n)^T \in \mathbb{R}^n$, then $\Sigma = [a_i a_j]_{1 \leq i,j \leq n} = \mathbf{a}\mathbf{a}^T$. To construct a $n \times n$ matrix A such that $AA^T = \mathbf{a}\mathbf{a}^T$, one can simply put $\mathbf{a}$ as the first column vector of A, and then duplicate it to fill up the rest. That is, let $A = [\mathbf{a} \cdots \mathbf{a}] / \sqrt{n}$. The factor $\sqrt{n}$ is needed so that AA^T and Σ are equal element-wise. $\square$