Quiz 1: Give yourself 50 minutes to solve 5 of the following 7 problems. Each problem weights 20 point scores

Problem 1

For any two events A and B in Ω, verify that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. For three events A, B and C, prove that

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \\ + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

Solution.

1. By the axiom of disjoint additivity for probability,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1 \cup (A_2 \setminus A_1)) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1), \qquad \text{and} \\ \mathbb{P}(A_2) = \mathbb{P}((A_2 \setminus A_1) \cup (A_2 \cap A_1)) = \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_2 \cap A_1).\end{aligned}$$

Therefore, $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_2 \cap A_1)$. □

2. With the above identity,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cup A_2) \cap A_3) \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cap A_2) \cup (A_1 \cap A_3)) \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_3) \\ &\quad - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3).\end{aligned}$$

□

Problem 2

(Matching problem*) There are a deck of n distinct cards and n distinct boxes. Shuffle the cards and placed them into the boxes (Only one card for each boxes), if the i-th card is placed at the i-th box, we say that there is a match. What is the probability of no match after a shuffling? Compute the limiting probability when $n \to \infty$.

Solution. Let A_i denotes the event that the i-th card is placed in the i-th box. Then the desired probability is

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = 1 - \mathbb{P}(A_1 \cup \cdots \cup A_n).$$

Also, for any distinct indices $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $1 \leq k \leq n$,

$$\mathbb{P}(A_{i_1} A_{i_2} \cdots A_{i_k}) = \frac{(n-k)!}{n!}.$$

Then by the inclusion-exclusion formula, the desired probability is

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = 1 - \mathbb{P}(A_1 \cup \cdots \cup A_n) = 1 - \sum_{k=1}^{n} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

And the limiting probability is

$$\lim_{n\to\infty} \mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = e^{-1}.$$

$\square$

Problem 3

(Matching problem, continuation**) What is the probability of exactly m matches, $1 \le m \le n$?

Solution. By the answer of problem 2, we can know that when we put m cards into m boxes, the number of distributions such that there is no match is $m! \times \mathbb{P}(\text{no match}) = m! \sum_{k=0}^{m} (-1)^k \frac{1}{k!}$. So the total number of distributions such that there has exactly m matches is

$$\#\{\text{ways to choose } m \text{ matches}\} \times \#\{\text{ways to place remaining } n-m \text{ cards with no match}\}.$$

Divided by the total possible outcomes $n!$, we get

$$\mathbb{P}(\text{exactly } m \text{ matches}) = \frac{\binom{n}{m}(n-m)! \sum_{k=0}^{n-m} (-1)^k \frac{1}{k!}}{n!} = \frac{1}{m!} \sum_{k=0}^{n-m} (-1)^k \frac{1}{k!}.$$

$\square$

Problem 4

You have n pairs of socks. If $2r$ socks was chosen randomly, what's the probability of getting exactly i pairs of socks?

Solution. The total number of choose $2r$ socks from n pairs is $\binom{2n}{2r}$. The number of the desired choices can be count as follows: First choose the i pairs from n pairs, hence $\binom{n}{i}$ choices. For the remaining $2r-2i$ choices, we can't choose any pair from the remaining $n-i$ pairs, so we can at most pick one sock from each remaining pairs, therefore $\binom{n-i}{2r-2i} \times 2^{2r-2i}$. Hence the probability of choose exactly i pairs is $\frac{\binom{n}{i}\binom{n-i}{2r-2i}2^{2r-2i}}{\binom{2n}{2r}}$. $\square$

Problem 5

Roll a fair die 10 times. Compute the probability that at least one number occurs exactly 6 times.

Solution. Define A_i as the event that the number i occurs exactly 6 times in ten rolls, $1 \le i \le 6$. Then the desired probability is $\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_6)$. Also, for each $1 \le i \le 6$, $\mathbb{P}(A_i) = \binom{10}{6}(6-1)^{10-6}/6^{10}$, hence by the inclusion-exclusion formula,

$$\begin{aligned}\mathbb{P}(A_1 \cup \cdots \cup A_6) &= \sum_{i=1}^{6} \mathbb{P}(A_i) - \sum_{i,j} \mathbb{P}(A_i A_j) + \sum_{i,j,k} \mathbb{P}(A_i A_j A_k) - \cdots \\ &= 6 \times \frac{\binom{10}{6} 5^4}{6^{10}} = \frac{\binom{10}{6} 5^4}{6^9}.\end{aligned}$$

□

Problem 6

Given two integers N and K. The task is to find the number of good permutations of the first N natural numbers. A permutation is called good if there exist at least $N-K$ indices i, $1 \le i \le N$, such that i is at the ith position. What is the probability of getting good permutations if $N = 6$, $K = 3$?

Solution. This is an application of the previous matching problem. Let A_i be the event of getting a permutation with exactly i correct positions. Then the desired probability is

$$\mathbb{P}(\text{at least } N-K \text{ correct positions}) = \mathbb{P}\Big(\bigcup_{l=N-K}^{N} A_l\Big) = \sum_{l=N-K}^{N} \mathbb{P}(A_l),$$

since the events are disjoint. Also, by Problem 4,

$$\mathbb{P}(A_l) = \frac{1}{l!} \sum_{k=0}^{N-l} (-1)^k \frac{1}{k!},$$

Thus $\mathbb{P}(A_3) = \frac{1}{18}$, $\mathbb{P}(A_4) = \frac{1}{48}$, $\mathbb{P}(A_5) = 0$, $\mathbb{P}(A_6) = \frac{1}{720}$. So the answer is 56/720. □

Another solution. A cheap way of calculating the answer is to enumerate the number of cases for A_3, A_4, A_5, A_6, and then divide the total number of cases by $|\Omega| = 6!$ to get the answer.

$$|A_l| = \underbrace{\binom{6}{l}}_{\#\text{l number at good position}} \times (\#\text{derangement for } (1, 2, \ldots, n-l))$$

The length of derangement in this problem can be counted directly.

Problem 7

In a school, three-quarters of students are involved in sports, half are involved in cultural activities, and one-eighth are involved in neither. Calculate the probability that a student is involved in

(a) both sports and cultural activities

(b) cultural activities but not sports.

Solution.

(a) By inclusion-exclusion principle, we have

$$\mathbb{P}(\text{both sports and cultural activities}) = \frac{3}{4} + \frac{1}{2} - \left(1 - \frac{1}{8}\right) = \frac{3}{8}.$$

(b)

$$\mathbb{P}(\text{cultural activities but no sport}) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}.$$

$\square$