

Partial solution to HW 7

Q1

Explain in RC6 class. The key insight is that when you look at the proof of WLLN that use Chebyshev inequality. The only reason we need independent r.v. is that we want $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$. Hence, X_1, \dots, X_n being **uncorrelated** is enough.

Q2

Take $\epsilon = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{1}{\delta}}$ and apply the result of Q1.

Q3

Why is this problem important?

Because $\text{Var}(X_k)$ increases as k increases. In this case it is natural to think that maybe the average will be more 'spread out' and we will not have probability concentrated around $E[\bar{X}_n]$. Yet, the intuition is wrong and we actually can do so.

The question want you to show that the given independent r.v. sequence with same expectation has the average **converge in probability**.

Sol.

Hereon, we use \bar{X}_n to represent $\frac{1}{n} \sum_{i=1}^n X_i$.

Chebyshev inequality give you

$$P(|\bar{X}_n - E[\bar{X}_n]| \geq \epsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\epsilon^2}$$

(Notice that in this question $E[X_i] = 0, \forall i \leq n$) In order to let the bound on RHS work, $\text{Var}(\bar{X}_n)$ has to converge to 0 as $n \rightarrow \infty$.

From previous courses we know that for independent r.v. sequence X_1, \dots, X_n , the **variance of their sum is the sum of variances**, i.e

$$\text{Var}(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i)$$

(Do it your self) We know that $\text{Var}(X_k) = \frac{k}{\log(2k)}$ Also, it is true that $f'(x) > 0, \forall x \geq 1$ given $f(x) = \frac{x}{\log 2x}$

Then,

$$\frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} \sum_{k=1}^n \frac{k}{\log(2k)} \leq \frac{1}{n^2} n \left(\frac{n}{\log 2n} \right) \leq \frac{1}{\log 2n} \rightarrow 0$$

By Chebyshev inequality, \bar{X}_n converge to 0 in probability.

□

Q4

Direct application of Chebyshev inequality on Poisson r.v. Note that $E[X] = E[X^2] = \lambda$ for Poisson inequality.

Q5

Why may it be important?

This is a problem that generalize Markov inequality, Remember that Markov inequality requires r.v. X to be positive. This is true for all $\Phi(X)$.

Sol.

Since $\Phi(x)$ is strictly increasing, we have $\{X \geq x\} = \{\Phi(X) \geq \Phi(x)\}$ (strictly increasing function is 1-1). Therefore, using Markov inequality we have

$$P(X \geq x) = P(\Phi(X) \geq \Phi(x)) \leq \frac{E[\Phi(X)]}{\Phi(x)}$$

□

Note: In case you're wondering, The condition ' $\Phi(X)$ is integrable' is to ensure $E[\Phi(X)]$ exists. Sometime for continuous r.v. the first moment will not exists. Check out the textbook.

Q6

We went through this problem in RC5 class, so let me just give you some hint.

As we said, the ultimate WLLN is the version with \bar{X}_n and showing the covariance sum converge to zero, in our case you have to make sure that

$$\binom{n}{2}^{-2} \sum_{i < j, k < l} \text{Cov}(X_i X_j, X_k X_l) \rightarrow 0$$

as $n \rightarrow \infty$. You can break the sum to three case:

1. none out of four in i, j, k, l being the same. e.g. $\text{Cov}(X_1 X_2, X_3 X_4) = 0$
2. One out of four in i, j, k, l being the same. e.g. $\text{Cov}(X_1 X_2, X_1 X_3) = \mu^2(\sigma^2 - \mu^2)$
3. Two out of four in i, j, k, l being the same. e.g. $\text{Cov}(X_1 X_2, X_1 X_2) = \sigma^2(\sigma^2 + 2\mu^2)$

(Actually the exact number does not matter, we just need them to be some constant.)

The first has $2! \frac{4!}{2!2!} \binom{n}{4}$ many, the second has $6 \binom{n}{3}$ many, and the third has $\binom{n}{2}$ many.

Then show that the sum of all covariance will converge to zero as $n \rightarrow \infty$.

Why may it be important?

What make this different from the previous problems is that the cross-term may have non-zero contribution.