Problems for midterm

Chapter 1.

- **1. (Birthday problem)**
  Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). Write down the expression of probability that there exists at least a pair of student that share the same birthday

- **2. (Probability axiom)**
  (Modified from Panchenko Exercise 1.2.15) Let $\Omega = \{a, b, c, d\}$ and the probability function $P : \Omega \to [0, 1]$. Suppose $P(\{a, b\}) = 0.6, P(\{b, c\}) = 0.3, P(\{c, d\}) = 0.4$, Calculate $P(\{a\}), P(\{b\}), P(\{c\}), P(\{d\})$

- **3. (independence)**
  We roll a die three times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events A_{12}, A_{23}, A_{13} are pairwise independent but not independent events.

Chapter 2.

- **1. (Poisson + Conditioning)** In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time.
  (a) Compute $\mathbb{P}(H = h \mid N = n)$, where H is the total number of heads.
  (b) Show that the total number of heads has the Poisson distribution with parameter λp.

- **2. (Modified by Homework 4.4.)** You and your opponent both roll a fair die. If one get a greater number than the other one, and that number > 3, then the game ends and whoever rolls the larger number wins. Otherwise, we repeat the game.
  (a) Let N be the number of rounds in this game. Write down the p.m.f. of N.
  (b) What is $P(\text{you win})$?

- **3. (Normalize constant)**
  Consider a function f defined on $\mathbf{2}, 3, 4, \ldots$ such that $f(x) = C\frac{1}{x(x+1)}$, where C is a constant. Please find C such that f is a pmf.

Chapter 3.

- **1. (Birthday problem II.)** Suppose there are n students in a class, and each has birthday equally likely to be 1 of 365 days (no leap year). What is the expectation of number of distinct birthday?

- **2. (Expectation and variance of matchings)** Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $Var(S_n)$.

- **3. (Random sum)** Let $(X_i)_{1 \leq i \leq n}$ be a sequence n i.i.d. random variables with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Let N be a random variable taking value from $\{1, \cdots, n\}$ with equal probability, independent to $(X_i)_{1 \leq i \leq n}$. Define $S_k = X_1 + X_2 + \cdots + X_k$ for $1 \leq k \leq n$.
(a) What is the variance of the random sum, $Var(S_N)$?
(b) Let M be a random variable that has the same distribution as N, but independent to N and $(X_i)_{1 \leq i \leq n}$. What is $Cov(S_N, S_M)$? (You may encounter the calculation of $1^2 + 2^2 + \cdots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$)

Homework 6.

6.11. Use the method of indicators, for $i \neq j$, we can write

$$\mathbb{E}(X_{e(i)}X_{e(j)}) = \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}} + X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}})$$
$$= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}).$$

Adapt the indicator method again, you can calculate

$$\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) = \sum_{k\neq i;\ l\neq j;\ k\neq l} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\,e(j)=l\}})$$
$$= \sum_{k\neq i;\ l\neq j;\ k\neq l} \mathbb{E}(X_k X_l \mathbb{1}_{\{e(i)=k,\,e(j)=l\}})$$
$$= \sum_{k\neq i;\ l\neq j;\ k\neq l} \mathbb{E}(X_k)\mathbb{E}(X_l)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=l\}})$$
$$= 0,$$

and

$$\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) = \sum_{k\neq i;\ k\neq j} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\,e(j)=k\}})$$
$$= \sum_{k\neq i;\ k\neq j} \mathbb{E}(X_k^2 \mathbb{1}_{\{e(i)=k,\,e(j)=k\}})$$
$$= \sum_{k\neq i;\ k\neq j} \mathbb{E}(X_k^2)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=k\}})$$
$$= \sum_{k\neq i;\ k\neq j} 1 \cdot \mathbb{P}(e(i)=k)\mathbb{P}(e(j)=k)$$
$$= \sum_{k\neq i;\ k\neq j} \frac{1}{(n-1)^2} = \frac{n-2}{(n-1)^2}.$$

Hence $\mathbb{E}(X_{e(i)}X_{e(j)}) = \frac{n-2}{(n-1)^2}$ for $i \neq j$.

When $i = j$, $\mathbb{E}(X_{e(i)}^2) = \sum_{k\neq i} \mathbb{E}(X_{e(i)}^2 \mathbb{1}_{\{e(i)=k\}}) = \sum_{k\neq i} \mathbb{E}(X_k^2 \mathbb{1}_{\{e(i)=k\}})$. Use the independence calculation again, you can see that $\mathbb{E}(X_{e(i)}^2) = (n-1) \cdot 1 \cdot \frac{1}{(n-1)} = 1$.

Then the variance can be computed as

$$Var(X_{e(1)} + \cdots + X_{e(n)}) = \sum_{i,j} Cov(X_{e(i)}, X_{e(j)}) = \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) - \mathbb{E}(X_{e(i)})\mathbb{E}(X_{e(j)})$$
$$= \sum_{i,j} \mathbb{E}(X_{e(i)}X_{e(j)}) = \sum_{i=j} 1 + \sum_{i\neq j} \frac{n-2}{(n-1)^2} = n + \frac{n(n-2)}{n-1}.$$

You can check that $\mathbb{E}(X_{e(i)}) = 0$ with the similar method. $\qquad\square$