Quiz 1 Solution

1.

Solution. By the axiom of probability, $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1 \cup (A_2 \setminus A_1)) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1)$. And $\mathbb{P}(A_2) = \mathbb{P}((A_2 \setminus A_1) \cup (A_2 \cap A_1)) = \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_2 \cap A_1)$. Therefore, $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_2 \cap A_1)$. With this formula,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cup A_2) \cap A_3)\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_3) - \mathbb{P}((A_1 \cap A_2) \cup (A_1 \cap A_3))\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_3)\\ &\quad - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3).\end{aligned}$$

□

2.

Solution. Let A_i denotes the event that the i-th card is placed in the i-th box. Then the desired probability is

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = 1 - \mathbb{P}(A_1 \cup \cdots \cup A_n).$$

Also, for any distinct indices $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $1 \le k \le n$,

$$\mathbb{P}(A_{i_1} A_{i_2} \cdots A_{i_k}) = \frac{(n-k)!}{n!}.$$

Then by the inclusion-exclusion formula, the desired probability is

$$\mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = 1 - \mathbb{P}(A_1 \cup \cdots \cup A_n) = 1 - \sum_{k=1}^{n} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

And the limiting probability is

$$\lim_{n\to\infty} \mathbb{P}(A_1^c \cap \cdots \cap A_n^c) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = e^{-1}.$$

□

3.

Solution. By **2.**, we can know that when we put m cards into m boxes, the number of distributions such that there is no match is $m! \times \mathbb{P}(\text{no match}) = m! \sum_{k=0}^{m} (-1)^k \frac{1}{k!}$. So the total number of distributions such that there has exactly m matches is

$$\#\{\text{ways to choose } m \text{ matches}\} \times \#\{\text{ways to place remaining } n-m \text{ cards with no match}\}.$$

Divided by the total possible outcomes, we get

$$\mathbb{P}(\text{exactly } m \text{ matches}) = \frac{\binom{n}{m}(n-m)!\sum_{k=0}^{n-m}(-1)^k\frac{1}{k!}}{n!} = \frac{1}{m!}\sum_{k=0}^{n-m}(-1)^k\frac{1}{k!}.$$

□

4.

Solution. The total number of choose $2r$ socks from n pairs is $\binom{2n}{2r}$. The number of the desired choices can be count as follows: First choose the i pairs from n pairs, hence $\binom{n}{i}$ choices. For the remaining $2r-2i$ choices, we can't choose any pair from the remaining $n-i$ pairs, so we can at most pick one sock from each remaining pairs, therefore $\binom{n-i}{2r-2i} \times 2^{2r-2i}$. Hence the probability of choose exactly i pairs is $\frac{\binom{n}{i}\binom{n-i}{2r-2i}2^{2r-2i}}{\binom{2n}{2r}}$. □

5.

Solution. Define A_i as the event that the number i occurs exactly 6 times in ten rolls, $1 \le i \le 6$. Then the desired probability is $\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_6)$. Also, for each $1 \le i \le 6$, $\mathbb{P}(A_i) = \binom{10}{6}(6-1)^{10-6}/6^{10}$, hence by the inclusion-exclusion formula,

$$\begin{aligned}\mathbb{P}(A_1 \cup \cdots \cup A_6) &= \sum_{i=1}^{6}\mathbb{P}(A_i) - \sum_{i,j}\mathbb{P}(A_iA_j) + \sum_{i,j,k}\mathbb{P}(A_iA_jA_k) - \cdots \\ &= 6 \times \frac{\binom{10}{6}5^4}{6^{10}} = \frac{\binom{10}{6}5^4}{6^9}.\end{aligned}$$

□

6.

Solution.