Problems for midterm

Chapter 1.

1. (Birthday Paradox)
Suppose Write down the expression of probability that there exists at least a pair of student that share the same birthday **2. (Probability axiom)**
(Modified from Panchenko Exercise 1.2.15) Let $\Omega = \{a, b, c, d\}$ and the probability function $P : \Omega \to [0, 1]$. Suppose $P(\{a, b\}) = 0.6, P(\{b, c\}) = 0.3, P(\{c, d\}) = 0.4$, Calculate $P(\{a\}), P(\{b\}), P(\{c\}), P(\{d\})$
3. ()

Chapter 2.

1. (Poisson + Conditioning) In your pocket there is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp.

2. (Modified by Homework 4.4.) You and your opponent both roll a fair die. If one get a greater number than the other one, and that number > 3, then the game ends and whoever rolls the larger number wins. Otherwise, we repeat the game. What is $P(\text{you win})$? **3. (Normalize constant)** Consider a function f defined on $\mathbf{2}, 3, 4, \ldots$ such that $f(x) = C\frac{1}{x(x+1)}$, where C is a constant. Please find C such that f is a pmf.

Chapter 3.

1. (Birthday)

2. (Expectation and variance of matchings) Let S_n denotes the number of matchings of a random permutation of n cards. Compute $\mathbb{E}(S_n)$ and $Var(S_n)$.

3. (Random sum) Let $(X_i)_{1 \le i \le n}$ be a sequence n $i.i.d.$ random variables with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Let N be a random variable taking value from $\{1, \cdots, n\}$ with equal probability, independent to $(X_i)_{1 \le i \le n}$. Define $S_k = X_1 + X_2 + \cdots + X_k$ for $1 \le k \le n$.
(a) What is the variance of the random sum, $Var(S_N)$?
(b) Let M be a random variable that has the same distribution as N, but independent to N and $(X_i)_{1 \le i \le n}$. What is $Cov(S_N, S_M)$? (You may encounter the calculation of $1^2 + 2^2 + \cdots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$)

Homework 6.

6.11. Use the method of indicators, for $i \neq j$, we can write

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}) &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}} + X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) \\ &= \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) + \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}).\end{aligned}$$

Adapt the indicator method again, you can calculate

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)\neq e(j)\}}) &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\, e(j)=l\}}) \\ &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_k X_l \mathbb{1}_{\{e(i)=k,\, e(j)=l\}}) \\ &= \sum_{k\neq i;\, l\neq j;\, k\neq l} \mathbb{E}(X_k)\mathbb{E}(X_l)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=l\}}) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=e(j)\}}) &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_{e(i)}X_{e(j)}\mathbb{1}_{\{e(i)=k,\, e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k,\, e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} \mathbb{E}(X_k^2)\mathbb{E}(\mathbb{1}_{\{e(i)=k\}})\mathbb{E}(\mathbb{1}_{\{e(j)=k\}}) \\ &= \sum_{k\neq i;\, k\neq j} 1\cdot\mathbb{P}(e(i)=k)\mathbb{P}(e(j)=k) \\ &= \sum_{k\neq i;\, k\neq j} \frac{1}{(n-1)^2} = \frac{n-2}{(n-1)^2}.\end{aligned}$$

Hence $\mathbb{E}(X_{e(i)}X_{e(j)}) = \frac{n-2}{(n-1)^2}$ for $i \neq j$.

When $i = j$, $\mathbb{E}\big(X_{e(i)}^2\big) = \sum_{k\neq i}\mathbb{E}(X_{e(i)}^2\mathbb{1}_{\{e(i)=k\}}) = \sum_{k\neq i}\mathbb{E}(X_k^2\mathbb{1}_{\{e(i)=k\}})$. Use the independence calculation again, you can see that $\mathbb{E}\big(X_{e(i)}^2\big) = (n-1)\cdot 1\cdot\frac{1}{(n-1)} = 1$.

Then the variance can be computed as

$$\begin{aligned}Var(X_{e(1)}+\cdots+X_{e(n)}) &= \sum_{i,j} Cov(X_{e(i)}, X_{e(j)}) = \sum_{i,j}\mathbb{E}(X_{e(i)}X_{e(j)}) - \mathbb{E}(X_{e(i)})\mathbb{E}(X_{e(j)}) \\ &= \sum_{i,j}\mathbb{E}(X_{e(i)}X_{e(j)}) = \sum_{i=j} 1 + \sum_{i\neq j}\frac{n-2}{(n-1)^2} = n + \frac{n(n-2)}{n-1}.\end{aligned}$$

You can check that $\mathbb{E}(X_{e(i)}) = 0$ with the similar method. □