Magnetic Mirror Effect in Magnetron Plasma: Modeling of Plasma Parameters

1 Kinetic Theory of Plasma Physics

1.1 The Density Function

The kinetic theory describes the plasma as collection of particles whose state (position and velocity) is treated as a random variable with a density function $f(x, y, z, v_x, v_y, v_z, t)$ which describes the number of particles at position (x, y, z) at time t with velocities between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, v_z and $v_z + dv_z$ in directions x, y and z respectively. For a simpler notation $f(x, y, z, v_x, v_y, v_z, t)$ is denoted as $f(\mathbf{r}, \mathbf{v}, t)$. The expression

$$\int_{all\ v_x} dv_x \int_{all\ v_y} dv_y \int_{all\ v_z} dv_z f(\mathbf{r}, \mathbf{v}, t)$$

gives the number of particles at position $\mathbf{r}$, at time t. For a simple choice of notation, it if often written as

$$\int_{all\ \mathbf{v}} d^3 v f(\mathbf{r}, \mathbf{v}, t) \quad \text{or} \quad \int_{all\ \mathbf{v}} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t)$$

A density function is said to be normalized if

$$\int_{all\ \mathbf{v}} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) = 1$$

Such a density function is also denoted with a hat as $\hat{f}(\mathbf{r}, \mathbf{v}, t)$.
The average velocity $\bar{v}$ for a density function $\hat{f}(\mathbf{r}, \mathbf{v}, t)$ is calculated as

$$\int_{all\ \mathbf{v}} d\mathbf{v}\ v f(\mathbf{r}, \mathbf{v}, t)$$

Various other features of the distribution are: the Root Mean Square velocity v_{rms}, the average absolute velocity $|\bar{v}|$, the average velocity in z direction $\bar{v}_z$, etc.

One important density function often used in the kinetic theory of plasma is the Maxwell-Boltzmann distribution often called the Maxwellian which has the density function

$$\widehat{f_M} := \hat{f}(\mathbf{r}, \mathbf{v}, t) = \left(\frac{m}{2\pi KT} \right)^{\frac{3}{2}} \exp\left(-\frac{v^2}{v_{th}^2} \right) \tag{1}$$

where

$$v_{th}^2 = \frac{2KT}{m}$$

Some features of the Maxwellian are:

$$v_{rms} = \sqrt{\frac{3KT}{m}}, |\bar{v}| = 2\sqrt{\frac{2KT}{\pi m}}, |\bar{v}_z| = \sqrt{\frac{2KT}{\pi m}}, \bar{v}_z = 0$$

1.2 Equation of Motion

The dynamics of the system is described by change in the density function. In a one dimensional consideration, in a frame moving with each particle, this is described as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt}$$

as f is a function of the position x, the velocity v_x and time t in general. This is also written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x$$

replacing the expressions for velocity and acceleration. If there were no external disturbance to the system, the equation of motion would be

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x = 0$$

or simply

$$\frac{df}{dt} = 0 \tag{2}$$

In three dimensions it is easy to see that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt}$$

as f is a function of the position (x, y, z), the velocity (v_x, v_y, v_z) and time t in general. Substituting the expressions for velocity and acceleration, it is also written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial v_y}a_y + \frac{\partial f}{\partial z}v_z + \frac{\partial f}{\partial v_z}a_z$$

which is often written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_{\mathbf{v}} f$$

hence the equation of motion for the case with no external disturbance would be

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_{\mathbf{v}} f = 0$$

which is again the simple expression in equation (2).

A special case of this is when the force acting on the particles in the Lorentz force. In such a case this equation is written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \partial_{\mathbf{v}} f = 0 \tag{3}$$

and is called the Vlasov equation.

Including the external disturbances caused due to Electric and Magnetic Fields or any other force would have a non zero term on the right hand side of the equation (2). The equation

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_c \tag{4}$$

where the term on the right hand side describes the change of the density function due to collisions between particles is called the Boltzmann equation which is often written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \partial_\mathbf{v} f = \left(\frac{\partial f}{\partial t}\right)_c$$

The collision term is often used to describe electromagnetic Coulomb between two particles, collisions with uncharged particles, etc.

2 Magnetic Mirror

The magnetic mirror effect can first be illustrated with a single particle. For the simple case, it is assumed that there is no electric field acting on the particle, the magnetic field is cylindrically symmetric and the gradient of the magnetic field is only along the axial direction of the cylinder, the z direction. The magnetic moment

$$\mu = \frac{mv_\perp^2}{2B_z}$$

of the particle is an adiabatic invariant of the particle motion and hence is approximately conserved for a magnetic field whose z component B_z does not vary too much . The kinetic energy

$$KE = \frac{1}{2}mv^2$$

of the particle is also approximately conserved. Writing $v_\perp = vsin\theta$,

$$\frac{\mu}{KE} = \frac{sin^2\theta}{B_z}$$

is also a conserved quantity. This means that as B increases (within a small range), $sin^2\theta$ increases as well and so does $|sin\theta|$ which means that $|v_\perp|$ increases and since v is conserved, $v_\parallel$ decreases to zero. When $v_\parallel = 0$, the particle is no longer moving in the z direction, but is moving with velocity v in the plane perpendicular to z direction. The $v_\parallel$ then increases in the opposite direction as the particle moves towards decreasing B because of the Lorentz force. Such a particle is seen as being reflected due to the configuration of the magnetic field and hence such a configuration of the magnetic field is called a magnetic mirror. Particles whose $v_\parallel$ does not decrease to 0 while the value of B is decreasing, escape from the magnetic mirror configuration.

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This can be written as

$$\frac{sin^2\theta}{B_z} = \frac{sin^2\theta_0}{B_{z_0}}$$

TO DO 1
I am still a bit confused on how to use this for a dynamic velocity distribution to get an expression for reflected and lost flux. So things after here in this section are not entirely clear.

if the velocity of the particle (and hence $sin\theta$) and the magnetic field in the z direction B_z are known at some location to be $sin\theta_0$ and B_{z_0} respectively. The particle that either just escapes the magnetic mirror or is just reflected would have $v_\perp = v$ and $sin\theta = 1$

$$sin^2\theta = \frac{B_z}{B_{z_0}} \ sin^2\theta_0$$

Since

$$sin^2\theta \leq 1 \qquad\qquad \frac{B_z}{B_{z_0}} \ sin^2\theta_0 \leq 1$$

The reflected fraction is given by

$$f_{loss} = \frac{\displaystyle\int_{loss\ cone} f(\mathbf{r},\mathbf{v},t) \ d\mathbf{v}}{\displaystyle\int_{all\ \mathbf{v}} f(\mathbf{r},\mathbf{v},t) \ d\mathbf{v}} = \frac{\int_0^{2\pi} d\phi \left[\int_0^{\theta_0} sin\theta d\theta + \int_{\pi-\theta_0}^{\pi} sin\theta d\theta\right] \int_{allv} \frac{f(\mathbf{r},\mathbf{v},t) \ dv}{4\pi v^2}}{\int_0^{2\pi} d\phi \int_0^{\pi} sin\theta d\theta \int_{allv} \frac{f(\mathbf{r},\mathbf{v},t) \ dv}{4\pi v^2}} = 1-cos\theta_0$$

3 Parabolic Density Functions

The simplest density function to try would be to have all particles have the same velocity i.e. a Dirac delta function distribution. Another simple distribution would be a uniform distribution between two velocities, where a particle is equally likely to have any velocity in the range.

3.1 Definitions

The uniform distribution describes a collection of particles where a particle chosen at random is equally likely to have any velocity in the range.

$$f_0 = \begin{cases} \frac{1}{2v_a} & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

which is already normalized. The Dirac delta-like distribution would describe all the

particles having the same velocity.

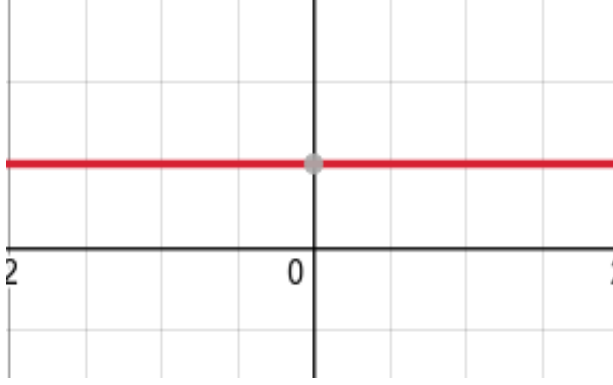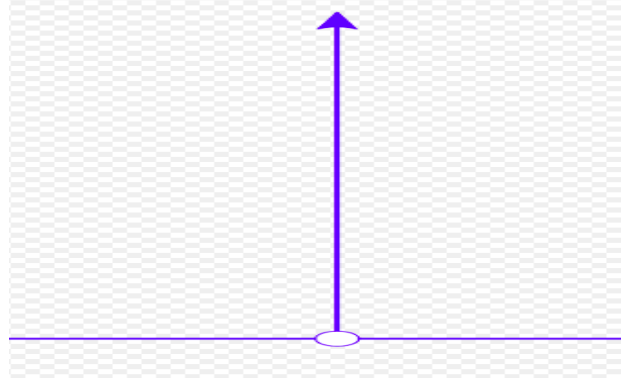$$f_\delta = \begin{cases} 1 & v = v_a \\ 0 & else \end{cases}$$

Figure 1: f_0

Figure 2: f_δ

The figure for f_0 was generated using desmos graphing calculator while the figure for f_δ was snipped from the image in wikipedia for the Dirac delta function.

However, uniform and delta distributions are not very realistic in a system of large number of particles. Parabolic functions in a given range also seem like interesting distributions and yet simple to work with. After some trial and error, two functions that seem interesting are:

$$f_1 = \begin{cases} 1 - \frac{v^2}{2v_a^2} & -v_a \le v \le v_a \\ 0 & else \end{cases}$$

$$f_2 = \begin{cases} 1 + \frac{v^2}{2v_a^2} & v_a \le v \le v_a \\ 0 & else \end{cases}$$

These functions can be normalized as

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \; c_0 \left(1 - \frac{v_x^2 + v_y^2 + v_z^2}{2v_a^2} \right) = 1 \qquad \text{and}$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \; c_0 \left(1 + \frac{v_x^2 + v_y^2 + v_z^2}{2v_a^2} \right) = 1 \qquad \text{giving}$$

$$\hat{f}_1 = \begin{cases} \frac{1}{4v_a^3} \left(1 - \frac{v^2}{2v_a^2} \right) & -v_a \le v \le v_a \\ 0 & else \end{cases}$$

$$\hat{f}_2 = \begin{cases} \frac{1}{12v_a^3} \left(1 + \frac{v^2}{2v_a^2} \right) & v_a \le v \le v_a \\ 0 & else \end{cases}$$

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The graph of these functions (considering v as a single variable) were generated using desmos graphing calculator. The figures represent $\hat{f}_1$ and $\hat{f}_2$ extrapolated to beyond the defined range.
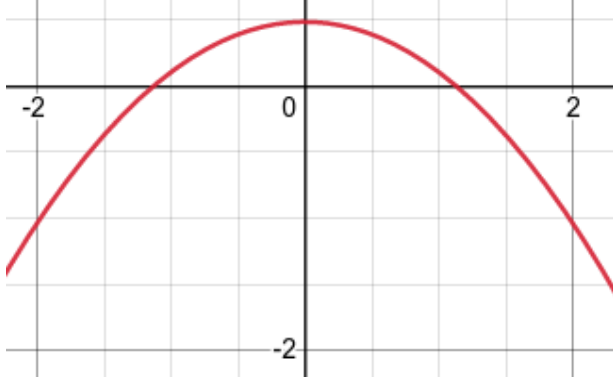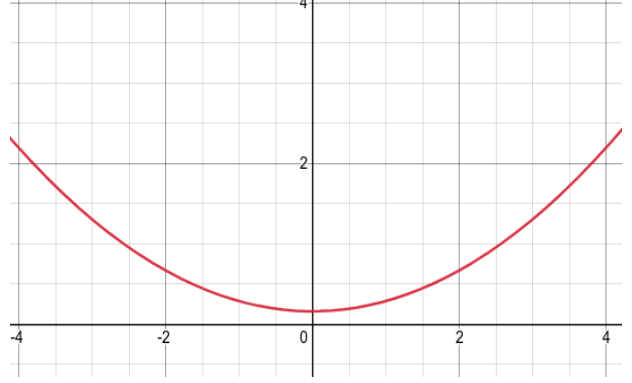

Figure 3: $\hat{f}_1$

Figure 4: $\hat{f}_2$

In the upper half of the plane, $\hat{f}_1$ looks approximately like a Gaussian distribution which the Maxwellian is for a distribution of velocity (not speed) for a fixed temperature. $\hat{f}_2$ is sort of the opposite of $\hat{f}_1$. But given that the functions are defined in a limited range, the integrals do not diverge.

3.2 Plugging into the Vlasov equation

One good exercise is to check what happens when plugging in $\hat{f}_1$, $\hat{f}_2$ and f_M in the collisionless Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \partial_\mathbf{v} f = 0$$

For all three of the density functions

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial z} = 0$$

as all three have no explicit dependence on position or time. The Vlasov equation then becomes

$$\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \partial_\mathbf{v} f = 0$$

which is

$$\left(E_x + v_y B_z - v_z B_y \right) \frac{\partial f}{\partial v_x} + \left(E_y + v_z B_x - v_x B_z \right) \frac{\partial f}{\partial v_y} + \left(E_z + v_x B_y - v_y B_x \right) \frac{\partial f}{\partial v_z} = 0$$

For the Maxwellian,

$$\widehat{f_M} := \hat{f}(\mathbf{r}, \mathbf{v}, t) = \left(\frac{m}{2\pi KT} \right)^{\frac{3}{2}} \exp\left(-\frac{v^2}{v_{th}^2} \right)$$

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$$\frac{\partial f_M}{\partial v_x} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_x}{v_{th}^2}\right), \qquad \frac{\partial f_M}{\partial v_y} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_y}{v_{th}^2}\right),$$

$$\frac{\partial f_M}{\partial v_z} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_z}{v_{th}^2}\right)$$

plugging in, the equation becomes

$$\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2}{v_{th}^2}\right)\left[v_x\left(E_x + v_y B_z - v_z B_y\right) + v_y\left(E_y + v_z B_x - v_x B_z\right) +\right.$$

$$\left. v_z\left(E_z + v_x B_y - v_y B_x\right)\right] = 0$$

which gives

$$v_x E_x + v_y Ey + v_z E_z = 0$$

which can also be written as

$$\mathbf{v} \cdot \mathbf{E} = 0 \tag{5}$$

which says that the particles move perpendicular to the electric field which is characteristic of the Lorentz force. For the parabolic functions,

$$\hat{f}_1 = \begin{cases} \frac{1}{4v_a^3}\left(1 - \frac{v^2}{2v_a^2}\right) & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$$\hat{f}_2 = \begin{cases} \frac{1}{12v_a^3}\left(1 + \frac{v^2}{2v_a^2}\right) & v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_1}{\partial v_x} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_x}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_1}{\partial v_y} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_y}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_1}{\partial v_z} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_z}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_2}{\partial v_x} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_x}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

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$$\frac{\partial \hat{f}_2}{\partial v_y} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_y}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_2}{\partial v_z} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_z}{2v_a^2}\right) & -v_a < v < v_a \\ \text{undefined} & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

In the range $v \in \mathbb{R}\setminus[-v_a, v_a]$ the equation becomes $0 = 0$ which is trivial. The equation becomes undefined for $v = v_a$ and $v = -v_a$ but we can ignore that for now. In the interesting range of $v \in (-v_a, v_a)$, the equation becomes

$$\frac{1}{4v_a^3}\left(\frac{-2}{2v_a^2}\right)[v_x(E_x + v_yB_z - v_zB_y) + v_y(E_y + v_zB_x - v_xB_z) + v_z(E_z + v_xB_y - v_yB_x)] = 0$$

for $\hat{f}_1$ and

$$\frac{1}{12v_a^3}\left(\frac{2}{2v_a^2}\right)[v_x(E_x + v_yB_z - v_zB_y) + v_y(E_y + v_zB_x - v_xB_z) + v_z(E_z + v_xB_y - v_yB_x)] = 0$$

for $\hat{f}_2$ which both give the same equation as $\widehat{f_M}$

$$v_x E_x + v_y Ey + v_z E_z = 0 \qquad \text{or} \qquad \mathbf{v} \cdot \mathbf{E} = 0$$

$\hat{f}_1$ and $\hat{f}_2$ behave similar to $\hat{f}_M$ when plugged into the Vlasov equation.

3.3 Plugging into the expression for Magnetic Mirror

Since the quantity $\frac{sin^2\theta}{B_z}$ is conserved in a magnetic mirror, it is a good exercise to calculate the average value or expectation of $sin^2\theta$, $\langle sin^2\theta \rangle = \left\langle \frac{v_\perp^2}{v^2} \right\rangle = \left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle$ For the Maxwellian $\widehat{f_M}$,

$$\left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}$$

$$\int_{v_x=-\infty}^{v_x=\infty} dv_x \int_{v_y=-\infty}^{v_y=\infty} dv_y \int_{v_z=-\infty}^{v_z=\infty} dv_z \left(\frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2}\right) \exp\left(-\frac{v_x^2}{v_{th}^2}\right) \exp\left(-\frac{v_y^2}{v_{th}^2}\right) \exp\left(-\frac{v_z^2}{v_{th}^2}\right)$$

Likewise for the parabolic functions,

$$\left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \frac{1}{4v_a^3}$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \left(\frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2}\right)\left(1 - \frac{v_x^2}{2v_a^2} - \frac{v_y^2}{2v_a^2} - \frac{v_z^2}{2v_a^2}\right)$$

for $\hat{f}_1$ and

$$\left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \frac{1}{12v_a^3}$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \left(\frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right) \left(1 + \frac{v_x^2}{2v_a^2} + \frac{v_y^2}{2v_a^2} + \frac{v_z^2}{2v_a^2} \right)$$

for $\hat{f}_2$ which are all very difficult to evaluate because of the denominators.

So an approximation $\left\langle \dfrac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \dfrac{\left\langle v_x^2 + v_y^2 \right\rangle}{\left\langle v_x^2 + v_y^2 + v_z^2 \right\rangle} + c_0$ can be done where c_0 is an error term.

For $\widehat{f_M}$, $\left\langle v_x^2 + v_y^2 \right\rangle = \dfrac{2KT}{m}$ and $\left\langle v_x^2 + v_y^2 + v_z^2 \right\rangle = \dfrac{3KT}{m}$ so

$$\left\langle \sin^2\theta \right\rangle = \left\langle \frac{v_\perp^2}{v^2} \right\rangle = \left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \frac{\left\langle v_x^2 + v_y^2 \right\rangle}{\left\langle v_x^2 + v_y^2 + v_z^2 \right\rangle} + c_0 = \frac{2}{3} + c_0$$

For $\hat{f}_1$, $\left\langle v_x^2 + v_y^2 \right\rangle = \dfrac{22}{45}v_a^2$ and $\left\langle v_x^2 + v_y^2 + v_z^2 \right\rangle = \dfrac{11}{15}v_a^2$ so $\left\langle \sin^2\theta \right\rangle = \dfrac{2}{3} + c_0$ and

For $\hat{f}_2$, $\left\langle v_x^2 + v_y^2 \right\rangle = \dfrac{98}{135}v_a^2$ and $\left\langle v_x^2 + v_y^2 + v_z^2 \right\rangle = \dfrac{49}{45}v_a^2$ so $\left\langle \sin^2\theta \right\rangle = \dfrac{2}{3} + c_0$

$\hat{f}_1$ and $\hat{f}_2$ behave similar to $\widehat{f_M}$ when plugged into the expression for $\left\langle \sin^2\theta \right\rangle$.

Since $\hat{f}_1$ and $\hat{f}_2$ behave similar to $\widehat{f_M}$ when plugged into the Vlasov equation and the expression for $\left\langle \sin^2\theta \right\rangle$, it seems that $\hat{f}_1$ and $\hat{f}_2$ are nice distribution functions to work with.

4   Plasma simulation with particles

5   Vlasov - Poisson and Particle in a cell method

References

[1] Chen, F. F. (1984). *Introduction to plasma physics and controlled fusion* (Vol. 1, pp. 8-11). New York: Plenum press.

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