Magnetic Mirror Effect in Magnetron Plasma:

Modeling of Plasma Parameters

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1.1 Density Function

$f(x, y, z, v_x, v_y, v_z, t)$:
number of particles at position (x, y, z) at time t with velocities between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, v_z and $v_z + dv_z$
Notation:
$f(\mathbf{r}, \mathbf{v}, t) := f(x, y, z, v_x, v_y, v_z, t)$

$$f(\mathbf{r}, t) = \int_{all \quad v_x} dv_x \int_{all \quad v_y} dv_y \int_{all \quad v_z} dv_z f(\mathbf{r}, \mathbf{v}, t)$$

Also written as

$$\int_{all \quad \mathbf{v}} d^3 v f(\mathbf{r}, \mathbf{v}, t) \quad \text{or} \quad \int_{all \quad \mathbf{v}} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t)$$

Density function

Normalized density function denoted $\hat{f}(\mathbf{r}, \mathbf{v}, t)$

$$\int_{all \quad \mathbf{v}} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) = 1$$

Average velocity

$$\bar{v} = \int_{all \quad \mathbf{v}} d\mathbf{v} \;\; v \;\; f(\mathbf{r}, \mathbf{v}, t)$$

Other parameters

Root Mean Square velocity v_{rms}

average absolute velocity $|\bar{v}|$

average velocity in z direction $\bar{v_z}$

1.2 Maxwell Boltzmann Distribution

Density Function

$$\widehat{f_M}(\mathbf{r}, \mathbf{v}, t) = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(-\frac{v^2}{v_{th}^2}\right) \tag{1}$$

$$v_{th}^2 = \frac{2KT}{m}$$

Features of the Maxwell Boltzmann Distribution (Maxwellian)

$$v_{rms} = \sqrt{\frac{3KT}{m}}, |\bar{v}| = 2\sqrt{\frac{2KT}{\pi m}}, |\bar{v_z}| = \sqrt{\frac{2KT}{\pi m}}, \bar{v_z} = 0$$

2. Equation of Motion

Derivative in 1 dimension

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x$$

If there are no external disturbances and interactions

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x = 0$$

which simply means

$$\frac{df}{dt} = 0 \tag{2}$$

Equation of Motion

Derivative in 3 dimensions

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial v_x}a_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial v_y}a_y + \frac{\partial f}{\partial z}v_z + \frac{\partial f}{\partial v_z}a_z$$

which is often written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_{\mathbf{v}} f$$

If there are no external disturbances and interactions

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_{\mathbf{v}} f = 0$$

which is again just equation (2)

2.2 Vlasov and Boltzmann equations

Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f = 0 \tag{3}$$

Lorentz force is substituted for the acceleration term in the 3 dimensional equation of motion without external disturbances and interactions

Boltzmann equation:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t} \right)_c \tag{4}$$

Collision terms are included to describe the change in the density function

Examples: Coulomb force, Collision with neutral particles

3.1 Magnetic Mirror - Conserved quantities

Magnetic Moment : Adiabatic Invariant

$$\mu = \frac{mv_{\perp}^2}{2B_z}$$

Kinetic energy

$$KE = \frac{1}{2}mv^2$$

Writing $v_{\perp} = v\ sin\theta$

$$\frac{\mu}{KE} = \frac{sin^2\theta}{B_z}$$

is conserved

I am still a bit confused on how to use this for a dynamic velocity distribution to get an expression for reflected and lost flux.

TO DO 1

So things after here in Magnetic mirror are not entirely clear.

$$\frac{sin^2\theta}{B_z} = \frac{sin^2\theta_0}{B_{z_0}}$$

a

$$sin^2\theta = \frac{B_z}{B_{z_0}}\ sin^2\theta_0$$

Since

$$sin^2\theta \le 1 \qquad \frac{B_z}{B_{z_0}}\ sin^2\theta_0 \le 1$$

The reflected fraction is given by

$$f_{loss} = \frac{\displaystyle\int_{loss\ cone} f(\mathbf{r}, \mathbf{v}, t)\ \ d\mathbf{v}}{\displaystyle\int_{all\ \ \mathbf{v}} f(\mathbf{r}, \mathbf{v}, t)\ \ d\mathbf{v}}$$

$$= \frac{\int_0^{2\pi} d\phi \left[\int_0^{\theta_0} sin\theta d\theta + \int_{\pi-\theta_0}^{\pi} sin\theta d\theta\right] \int_{allv} \frac{f(\mathbf{r}, \mathbf{v}, t)\ \ dv}{4\pi v^2}}{\int_0^{2\pi} d\phi \int_0^{\pi} sin\theta d\theta \int_{allv} \frac{f(\mathbf{r}, \mathbf{v}, t)\ \ dv}{4\pi v^2}}$$

$$= 1 - cos\theta_0$$

The uniform distribution describes a collection of particles where a particle chosen at random is equally likely to have any velocity in the range.

$$f_0 = \begin{cases} \frac{1}{2v_a} & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

which is already normalized. The Dirac delta-like distribution would describe all the particles having the same velocity.

$$f_\delta = \begin{cases} 1 & v = v_a \\ 0 & else \end{cases}$$

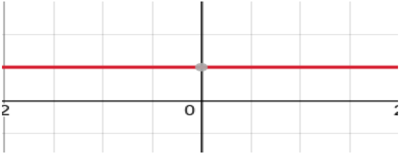

Figure 1: f_0

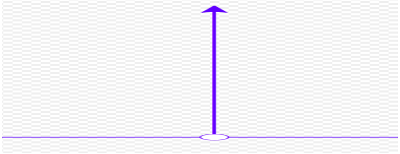

Figure 2: f_δ

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f_0 generated using desmos graphing calculator. f_δ snipped from the image in wikipedia for the Dirac delta function.

- Uniform and delta distributions are not very realistic in a system of large number of particles.
- Parabolic functions in a given range are better and yet simple to work with.

4.2 Parabolic density functions as approximation to Maxwellian

$$f_1 = \begin{cases} 1 - \frac{v^2}{2v_a^2} & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$$f_2 = \begin{cases} 1 + \frac{v^2}{2v_a^2} & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

These functions can be normalized as

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \;\; c_0 \left(1 - \frac{v_x^2 + v_y^2 + v_z^2}{2v_a^2}\right) = 1$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \;\; c_0 \left(1 + \frac{v_x^2 + v_y^2 + v_z^2}{2v_a^2}\right) = 1$$

Parabolic density functions - Normalized

$$\hat{f}_1 = \begin{cases} \frac{1}{4v_a^3}\left(1 - \frac{v^2}{2v_a^2}\right) & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$$\hat{f}_2 = \begin{cases} \frac{1}{12v_a^3}\left(1 + \frac{v^2}{2v_a^2}\right) & v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

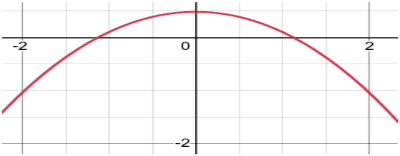

Figure 3: $\hat{f}_1$

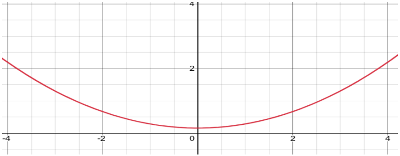

Figure 4: $\hat{f}_2$

4.3 $\hat{f}_1$, $\hat{f}_2$ and $\widehat{f_M}$ in the collisionless Vlasov equation

Plugging in $\hat{f}_1$, $\hat{f}_2$ and $\widehat{f_M}$ in the collisionless Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f = 0$$

For all three of the density functions

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial z} = 0$$

as all three have no explicit dependence on position or time. The Vlasov equation then becomes

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f = 0$$

which is

$$(E_x + v_y B_z - v_z B_y) \frac{\partial f}{\partial v_x} + (E_y + v_z B_x - v_x B_z) \frac{\partial f}{\partial v_y} +$$

$$(E_z + v_x B_y - v_y B_x) \frac{\partial f}{\partial v_z} = 0$$

$\widehat{f_M}$ in the collisionless Vlasov equation

For the Maxwellian,

$$\widehat{f_M} := \hat{f}(\mathbf{r}, \mathbf{v}, t) = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$\frac{\partial f_M}{\partial v_x} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_x}{v_{th}^2}\right)$$

$$\frac{\partial f_M}{\partial v_y} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_y}{v_{th}^2}\right)$$

$$\frac{\partial f_M}{\partial v_z} = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2v_z}{v_{th}^2}\right)$$

plugging in, the equation becomes

$$\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} exp\left(\frac{-v^2}{v_{th}^2}\right)\left(\frac{-2}{v_{th}^2}\right)\left[v_x\left(E_x + v_y B_z - v_z B_y\right) + v_y\left(E_y + v_z B_x - v_x B_z\right) + v_z\left(E_z + v_x B_y - v_y B_x\right)\right] = 0$$

$\widehat{f_M}$ in the collisionless Vlasov equation

which gives

$$v_x E_x + v_y Ey + v_z E_z = 0$$

which can also be written as

$$\mathbf{v} \cdot \mathbf{E} = 0 \tag{5}$$

which says that the particles move perpendicular to the electric field which is characteristic of the Lorentz force.

For the parabolic functions,

$$\hat{f}_1 = \begin{cases} \frac{1}{4v_a^3}\left(1 - \frac{v^2}{2v_a^2}\right) & -v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$$\hat{f}_2 = \begin{cases} \frac{1}{12v_a^3}\left(1 + \frac{v^2}{2v_a^2}\right) & v_a \leq v \leq v_a \\ 0 & else \end{cases}$$

$\hat{f}_1$ and $\hat{f}_2$ in the collisionless Vlasov equation

$$\frac{\partial \hat{f}_1}{\partial v_x} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_x}{2v_a^2}\right) & -v_a < v < v_a \\ \textit{undefined} & v = -v_a, v = v_a \\ 0 & \textit{else} \end{cases}$$

$$\frac{\partial \hat{f}_1}{\partial v_y} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_y}{2v_a^2}\right) & -v_a < v < v_a \\ \textit{undefined} & v = -v_a, v = v_a \\ 0 & \textit{else} \end{cases}$$

$$\frac{\partial \hat{f}_1}{\partial v_z} = \begin{cases} \frac{1}{4v_a^3}\left(\frac{-2v_z}{2v_a^2}\right) & -v_a < v < v_a \\ \textit{undefined} & v = -v_a, v = v_a \\ 0 & \textit{else} \end{cases}$$

$$\frac{\partial \hat{f}_2}{\partial v_x} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_x}{2v_a^2}\right) & -v_a < v < v_a \\ undefined & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_2}{\partial v_y} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_y}{2v_a^2}\right) & -v_a < v < v_a \\ undefined & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

$$\frac{\partial \hat{f}_2}{\partial v_z} = \begin{cases} \frac{1}{12v_a^3}\left(\frac{2v_z}{2v_a^2}\right) & -v_a < v < v_a \\ undefined & v = -v_a, v = v_a \\ 0 & else \end{cases}$$

In the range $v \in \mathbb{R}n\,[-v_a, v_a]$ the equation becomes $0 = 0$ which is trivial. The equation becomes undefined for $v = v_a$ and $v = -v_a$ but we can ignore that for now. In the interesting range of $v \in (-v_a, v_a)$, the equation becomes

$$\frac{1}{4v_a^3}\left(\frac{-2}{2v_a^2}\right)\left[v_x\left(E_x + v_yB_z - v_zB_y\right) + v_y\left(E_y + v_zB_x - v_xB_z\right) + v_z\left(E_z + v_xB_y - v_yB_x\right)\right] = 0$$

for $\hat{f}_1$ and

$$\frac{1}{12v_a^3}\left(\frac{2}{2v_a^2}\right)\left[v_x\left(E_x + v_yB_z - v_zB_y\right) + v_y\left(E_y + v_zB_x - v_xB_z\right) + v_z\left(E_z + v_xB_y - v_yB_x\right)\right] = 0$$

for $\hat{f}_2$ which both give

$$v_xE_x + v_yEy + v_zE_z = 0 \qquad \text{or} \qquad \mathbf{v}\cdot\mathbf{E} = 0$$

$\hat{f}_1$ and $\hat{f}_2$ behave similar to $\hat{f_M}$ when plugged into the Vlasov equation.

4.4 Plugging into the expression for Magnetic Mirror

Since the quantity $\frac{sin^2\theta}{B_z}$ is conserved in a magnetic mirror, it is a good exercise to calculate the average value or expectation of $sin^2\theta$, $\langle sin^2\theta\rangle = \left\langle \frac{v_\perp^2}{v^2} \right\rangle = \left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle$

For the Maxwellian $\widehat{f_M}$,

$$\left\langle \frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2} \right\rangle = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}$$

$$\int_{v_x=-\infty}^{v_x=\infty} dv_x \int_{v_y=-\infty}^{v_y=\infty} dv_y \int_{v_z=-\infty}^{v_z=\infty} dv_z \left(\frac{v_x^2 + v_y^2}{v_x^2 + v_y^2 + v_z^2}\right) exp\left(-\frac{v_x^2}{v_{th}^2}\right)$$

$$exp\left(-\frac{v_y^2}{v_{th}^2}\right) exp\left(-\frac{v_z^2}{v_{th}^2}\right)$$

for $\hat{f}_1$

$$\left\langle \frac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right\rangle = \frac{1}{4v_a^3}$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \left(\frac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right)$$

$$\left(1 - \frac{v_x^2}{2v_a^2} - \frac{v_y^2}{2v_a^2} - \frac{v_z^2}{2v_a^2} \right)$$

for $\hat{f}_2$

$$\left\langle \frac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right\rangle = \frac{1}{12v_a^3}$$

$$\int_{v_x=-v_a}^{v_x=v_a} dv_x \int_{v_y=-v_a}^{v_y=v_a} dv_y \int_{v_z=-v_a}^{v_z=v_a} dv_z \left(\frac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right)$$

$$\left(1 + \frac{v_x^2}{2v_a^2} + \frac{v_y^2}{2v_a^2} + \frac{v_z^2}{2v_a^2} \right)$$

Integrals are **Very Difficult!** to evaluate because of the denominators.

So an approximation $\left\langle \dfrac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right\rangle = \dfrac{\left\langle v_x^2+v_y^2 \right\rangle}{\left\langle v_x^2+v_y^2+v_z^2 \right\rangle} + c_0$ can be done where c_0 is an error term.

For $\widehat{f_M}$, $\left\langle v_x^2+v_y^2 \right\rangle = \dfrac{2KT}{m}$ and $\left\langle v_x^2+v_y^2+v_z^2 \right\rangle = \dfrac{3KT}{m}$ so

$$\left\langle sin^2\theta \right\rangle = \left\langle \frac{v_\perp^2}{v^2} \right\rangle = \left\langle \frac{v_x^2+v_y^2}{v_x^2+v_y^2+v_z^2} \right\rangle = \frac{\left\langle v_x^2+v_y^2 \right\rangle}{\left\langle v_x^2+v_y^2+v_z^2 \right\rangle} + c_0 = \frac{2}{3} + c_0$$

For $\hat{f}_1$, $\left\langle v_x^2+v_y^2 \right\rangle = \dfrac{22}{45}v_a^2$ and $\left\langle v_x^2+v_y^2+v_z^2 \right\rangle = \dfrac{11}{15}v_a^2$ so $\left\langle sin^2\theta \right\rangle = \dfrac{2}{3} + c_0$ and

For $\hat{f}_2$, $\left\langle v_x^2+v_y^2 \right\rangle = \dfrac{98}{135}v_a^2$ and $\left\langle v_x^2+v_y^2+v_z^2 \right\rangle = \dfrac{49}{45}v_a^2$ so $\left\langle sin^2\theta \right\rangle = \dfrac{2}{3} + c_0$

- $\hat{f}_1$ and $\hat{f}_2$ behave similar to $\widehat{f_M}$ when plugged into the Vlasov equation
- $\hat{f}_1$ and $\hat{f}_2$ behave similar to $\widehat{f_M}$ when plugged into the expression for $\left\langle sin^2\theta \right\rangle$
- **Hence!** $\hat{f}_1$ and $\hat{f}_2$ are nice distribution functions to work with as an approximation to $\widehat{f_M}$.

6. Particle in a cell

References

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