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Theory and Methodology

# Generating pseudo-random time series with specified marginal distributions

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**Abstract**

Our goal is to generate a target time series with a specified marginal distribution and a specified lag-one autocorrelation. We consider an existing approach: first transform a known autocorrelated reference series into the corresponding uniform autocorrelated series and then apply the specified inverse transformation to each observation producing the target series. This approach is simple, except that the lag-one reference-series autocorrelation must be determined in a set-up step. We propose a method for determining this autocorrelation.

*Keywords:* Time series; Marginal distribution; Autocorrelation

## 1. Introduction

Simulation is a very powerful tool for the analysis of a wide variety of problems. Estimation of a system's performance by means of simulation often requires autocorrelated random variables as input. Examples of queueing models with correlated arrival- and service-times processes have been given by Heffes (1973, 1980), Heffes and Lucantoni (1986), and Lee et al. (1991).

In many applications, the most prominent properties of a stationary time-series model can be described by the marginal distribution and the autocorrelation structure. That such models can be summarized with the marginal distribution and low-order autocorrelations is important because complete specification of the joint distributions is usually not possible, because of limited time, data, or practitioner expertise.

We consider a simple case: generation of a stationary series $\{Y_i\}_{i=1}^{n}$, which we refer to as a *target series*, with a marginal cumulative distribution function (c.d.f.) $F_Y$ and a lag-one correlation $\rho_Y(1)$. For simplicity $\rho_Y$ will represent $\rho_Y(1)$. This paper considers only lag-one autocorrelation because it is a first step to including autocorrelation. The concepts in this paper can be directly extended to higher-order autocorrelations, but specific methods for such extensions are beyond the scope of this paper.

Most existing methods for generating autocorrelated series fall into three classes:

1) *Correlation-oriented approach:* This approach is exemplified in papers by Lewis (1980, 1985) and Lawrance and Lewis (1981, 1987). This approach develops a recursive algorithm for $Y_i$ given $Y_{i-1}$. For each marginal distribution $F_Y$, $\rho_Y$ is provided as a parameter to the algorithm. The advantage of this

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approach is that the autocorrelation function is tractable (hence, we call this the correlation-oriented approach). The limitation is that separate algorithms must be devised for different $F_Y$.

2) *Marginal-oriented approach:* This approach first transforms a known autocorrelated series, which is referred to as a *reference series*, into its corresponding uniform autocorrelated random variables, and then applies an inverse transformation method to generate the target series. The marginal distribution is easily preserved (hence we call this the marginal-oriented approach), while the autocorrelation sometimes becomes intractable through the two stages of transformation. Finding the autocorrelation of the reference series such that the autocorrelation of the target series meets a specific value is a root finding problem (see Chen and Schmeiser, 1994). Among the earliest to discuss the marginal-oriented approach were Gujar and Kavanagh (1968), Broste (1971), and Li and Hammond (1975). In particular, Li and Hammond (1975) used Newton–Raphton numerical root finding to determine the autocorrelation of the reference series. The autocorrelation $\rho_Y$ is a function of the reference-series autocorrelation, which involves a double integral; Li and Hammond did not explain how the upper and lower limits of the double integral are to be determined for any specified target series. Lakhan (1981) also followed this approach and empirically derived the relationship between the autocorrelation of the uniform, Rayleigh, and exponential target distributions corresponding to a given reference-series autocorrelation. Lakhan's results are incorrect because of an error in the autocorrelation of his reference series. Schmeiser (1990), in his review of simulation experiments, discussed this approach for generating a random vector, but does not discuss how to obtain a specified autocorrelation for the random vector. Melamed et al. (1992) presented a special case, called the TES methodology, which uses a correlated uniform series as a reference series, thus avoiding the first transformation in the marginal-oriented approach. A heuristic search is used in the TES methodology to obtain the approximate autocorrelation function of the target series.

3) *Conditional-distribution-oriented approach:* In this approach, the conditional probability of $Y_i$ given $Y_{i-1}$ is derived by dividing the joint probability density function of $Y_i$ and $Y_{i-1}$ by the marginal distribution of $Y_{i-1}$. This approach shifts the problem to one of generating bivariate distributions. Johnson and Tenenbein (1981) described a general scheme for generating a continuous bivariate distribution with specified marginals and several dependence measures. Schmeiser and Lal (1982) showed how to generate bivariate gamma random vectors with any correlation.

This paper applies the marginal-oriented approach. As mentioned above, autocorrelation is not invariant through the necessary conversions. We give an iterative procedure for determining a lag-one autocorrelation of a reference series $\{X_n\}$ such that the resulting series $\{Y_n\}$ exhibits a specific lag-one correlation $\rho_Y = \rho_Y^*$. This paper is an updated and expanded version of Song and Hsiao (1993a). Pursuing the basic idea proposed in Song and Hsiao (1993a), we more thoroughly analyze the selection of the reference series and discuss the monotonicity between $\rho_Y$ and $\rho_X$. Also this paper presents more complete Monte Carlo results. A Fortran implementation, which allows very general choices as to the marginal distribution of the target series, is available from the authors.

The organization of the remainder of this paper is as follows. In Section 2, we present the procedure for generating a target series. Specifically, we discuss the choice of a reference series and the monotonicity relationship between the lag-one autocorrelation of the reference series and target series. Section 3 applies the procedure of Section 2 to a variety of target series. Section 4 summarizes the results of the paper.

## 2. Procedure

To generate a stationary series $Y_1, Y_2, \ldots$, with a marginal c.d.f. $F_Y$ and lag-one autocorrelation $\rho_Y^*$, we propose the iterative procedure shown in Fig. 1. Each iteration goes through Steps 1–5. Step 0 is an initialization step.

Step 1 is the marginal-oriented approach, which includes three sub-steps.

*Step 1(i).* Generate a selected reference series $\{x_i\}$ for $i = 1, 2, \ldots, n$ with autocorrelation $\rho_X$. For convenience, we assume that the variance of $X$ is 1.

*Step 1(ii).* Transform $x_i$ into a correlated uniform series

$$u_i = F_X(x_i) \equiv \mathbf{P}(X \leqslant x_i)$$

for $i = 1, 2, \ldots, n$.

*Step 1(iii).* Transform $u_i$ into the target series

$$y_i = F_Y^{-1}(u_i),$$

for $i = 1, 2, \ldots, n$.

In Step 1(i) we choose the first-order autoregressive (AR(1)) (Box and Jenkins, 1976) as a reference series; that is,

$$x_i = \phi x_{i-1} + \epsilon_i,$$

where

$$\epsilon_i \sim \text{iid Normal}(0, \sigma_\epsilon^2),$$

where $\sigma_\epsilon^2 = 1 - \phi^2$, $\phi = \rho_X$, and $-1 \leqslant \phi \leqslant 1$. The selection of the reference series is discussed in Sec-

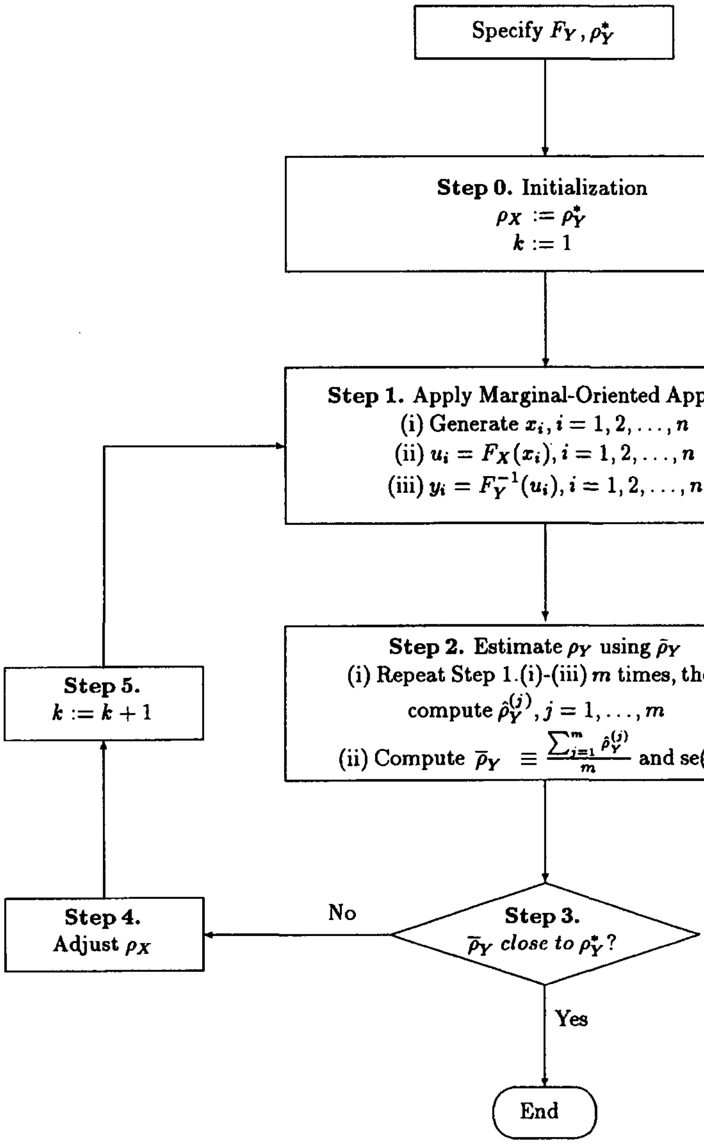



Fig. 1. Flowchart of the iterative procedure.

tion 2.1. As an initial approximation, we choose the autocorrelation $\rho_X$ equal to the target autocorrelation $\rho_Y^*$. Step 1(ii) generates correlated random numbers for use in the inverse transformation in Step 1(iii). Since Steps 1(ii) and 1(iii) are both transformations, the autocorrelation $\rho_Y$ will usually be different from $\rho_X$. Thomson (1954) and Barrett and Coals (1955) have shown that $\rho_Y$ is a monotonically increasing function of $\rho_X$ when both $\rho_X$ and $\rho_Y$ are positive. Thus, if the value of $\rho_X$ used in Step 1(i) does not produce the target $\rho_Y^*$, we can adjust $\rho_X$ in the same direction as the necessary change in $\rho_Y$. The relationship between $\rho_Y$ and $\rho_X$ is discussed in detail in Subsection 2.2.

In Step 2 we estimate $\rho_Y$. We first repeat Step 1(i)–(iii) $m$ times, each with a sample of size $n$. Each replication generates one estimate of $\rho_Y$. We then have $m$ estimates, which are denoted by $\hat{\rho}_Y^{(1)}, \hat{\rho}_Y^{(2)}, \ldots, \hat{\rho}_Y^{(m)}$. The estimator of $\rho_Y$ and its standard error are defined as $\bar{\rho}_Y$ and se($\bar{\rho}_Y$), where

$$\bar{\rho}_Y = \sum_{j=1}^{m} m^{-1} \hat{\rho}_Y^{(j)}$$

and

$$\mathrm{se}(\bar{\rho}_Y) = \sqrt{m^{-1}(m-1)^{-1} \sum_{j=1}^{m} \left(\hat{\rho}_Y^{(j)} - \bar{\rho}_Y\right)^2}.$$

When necessary, we use $\bar{\rho}_Y(k)$ and se($\bar{\rho}_Y(k)$) to specifically indicate $\bar{\rho}_Y$ and se($\bar{\rho}_Y$) computed at iteration $k$.

In Step 3, we accept the current value of $\rho_X$ and terminate the iterative procedure if $|\bar{\rho}_Y - \rho_Y^*| \leqslant \mathrm{se}(\bar{\rho}_Y)$. We need not be concerned with the issues of convergence or the efficiency of the termination criteria, which seem crude and heuristic, since empirical results show that the procedure works well for a variety of examples, such as those described in Section 3 of this paper.

In Step 4, we adjust $\rho_X$ if necessary. For the first iteration (i.e., $k=1$), if $\bar{\rho}_Y(1) > \rho_Y^*$, we adjust $\rho_X$ by taking

$$\Delta = |\bar{\rho}_Y(1) - \rho_Y^*|$$

as the decrement. If $\bar{\rho}_Y(1) < \rho_Y^*$, we take $\Delta$ as the increment. In subsequent iterations ($k = 2, 3, \ldots$), we keep taking $\Delta = |\bar{\rho}_Y(k) - \rho_Y^*|$ as the decrement if $\bar{\rho}_Y(j) > \rho_Y^*$ for each iteration $j \leqslant k$. Similarly, we keep taking $\Delta$ as the increment if $\bar{\rho}_Y(j) < \rho_Y^*$ for each iteration $j \leqslant k$. Otherwise, we use interpolation to adjust $\rho_X$.

In Step 5, we update the iteration number $k$ and reapply Steps 1 through 5.

### 2.1. Choice of a reference series

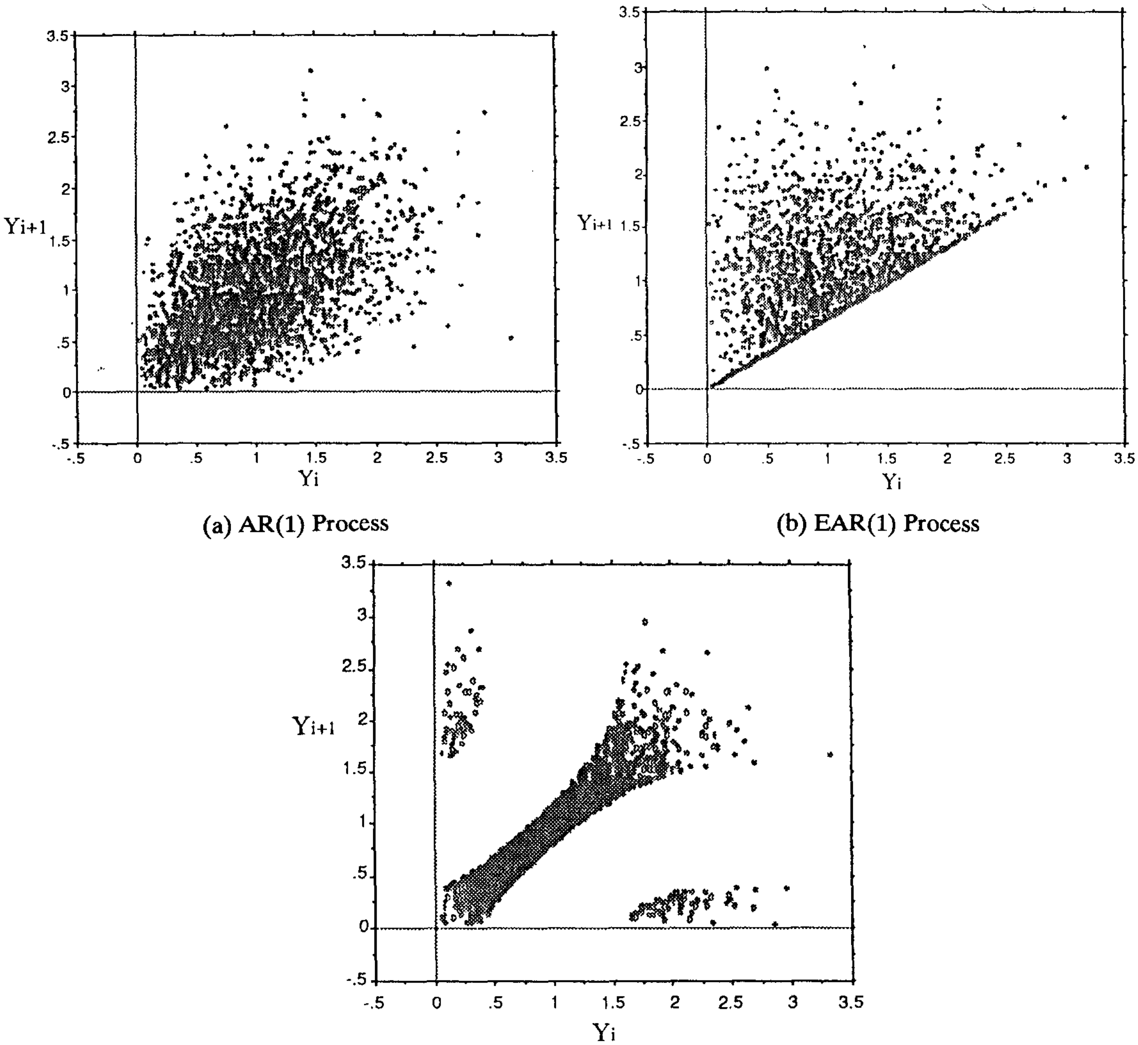
In the proposed procedure, we start with a selected autocorrelated series as a reference series in Step 1(i). A reference series is not fully determined by the specification of $F_Y$ and $\rho_Y^*$ for the target series; one may still choose among various autocorrelated series. For example, to generate a correlated series with marginal distribution $F_Y$ and lag-one correlation $\rho_Y^*$, the first-order autoregressive (AR(1)) process with a normal marginal distribution (Box and Jenkins, 1976), exponential AR(1) process with an exponential marginal distribution (Lewis, 1980), and various correlated $U(0, 1)$ processes (Melamed, 1991) have all been used with success. Inspection of the target bivariate joint distribution, shown in Fig. 2, motivates selection of the AR(1) process as a reference series for a broad range of systems. The unusual behavior in the end effects exhibited by the Correlated $U(0, 1)$ and the truncated characteristics exhibited by the exponential AR(1) recommend those series for certain other purposes (see Melamed, 1991).

The primary advantage of the marginal-oriented approach using an AR(1) process as the reference series is simplicity. Other correlated $U(0, 1)$ time series, such as the TES series proposed recently by Jagerman and Melamed (1994), can also produce a natural appearance, an improvement over earlier and simpler versions of TES (Melamed, 1991).

### 2.2. Monotonicity between autocorrelations of X and Y

Steps 1(ii) and 1(iii) are both nonlinear transformations, so the lag-one autocorrelation $\rho_Y$ will not usually equal $\rho_X$. To motivate the adjustment of $\rho_X$ in the same direction as the necessary change in $\rho_Y$, we need to show that the relationship function between $\rho_X$ and $\rho_Y$ is monotonic.

Suppose that $X_t$ and $X_{t+h}$ are two bivariate normal random variables with autocorrelation func-




(a) AR(1) Process

(b) EAR(1) Process

(c) U(0,1) Process

Fig. 2. Target bivariate joint distributions with $\rho_Y^* = 0.5$.

tion $\rho_X(h)$. The relationship between $\rho_Y(h)$ and $\rho_X(h)$, where $Y = G(X)$, is as follows:

$$\rho_Y(h) = \sum_{n=0}^{\infty} \rho_X^n(h) C_n^2, \tag{1}$$

where

$$C_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x) Z_n(x) \mathrm{e}^{-x^2/2}\, \mathrm{d}x,$$

and $Z_n(x)$ is the $n$-th Hermite polynomial,

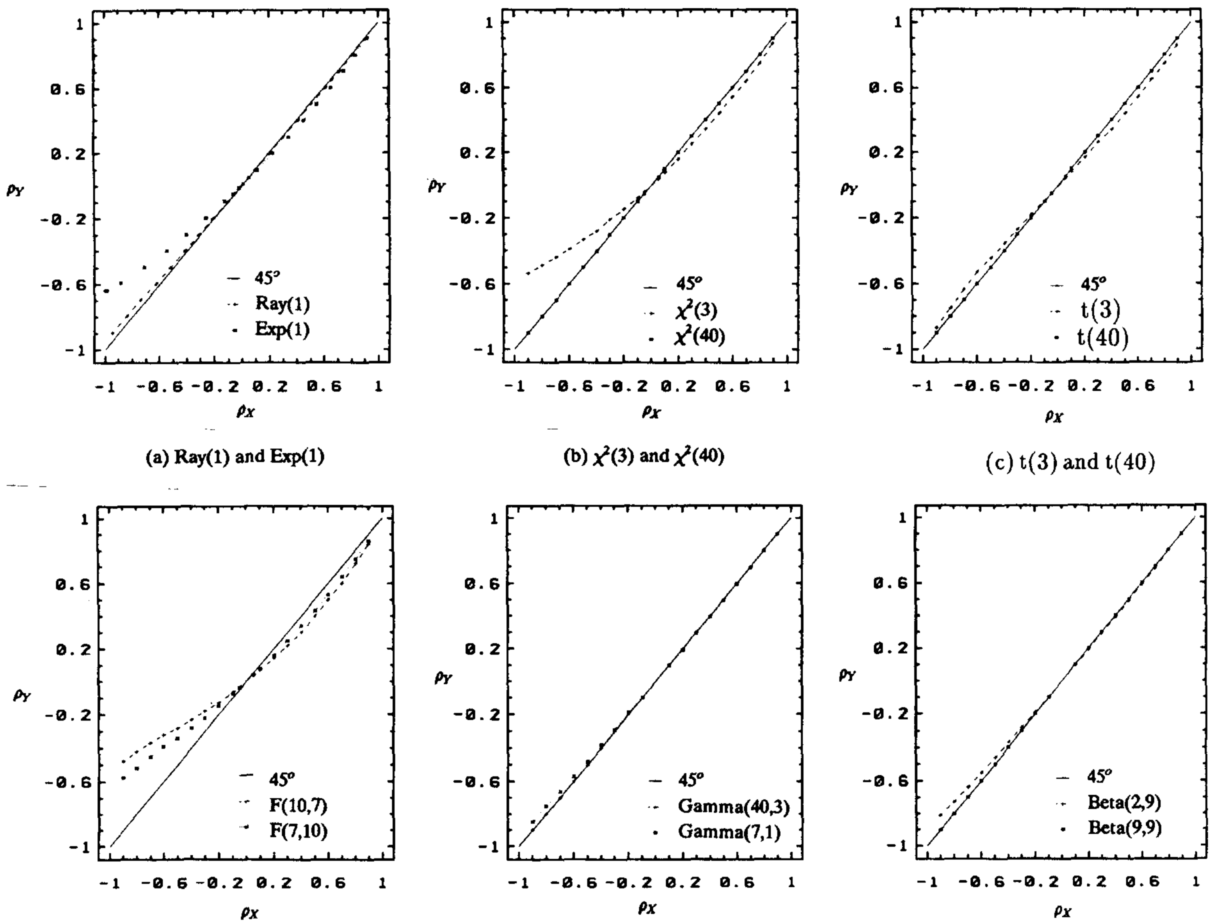
$$Z_n(x) = \frac{(-1)^n}{\sqrt{n!}} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2/2}.$$

The proof of Eq. (1) can be found in Thomson (1954) and Barrett and Coals (1955).

Eq. (1) shows that for cases where $\rho_X \geqslant 0$, $\rho_Y$ is a monotonically increasing function of $\rho_X$. For cases where the target series $Y$ is symmetric (without the assumption of $\rho_X \geqslant 0$), application of the result that $C_n = 0$ when $n$ is an even number (see Gujar and Kavanagh, 1968) shows that the relationship between $\rho_X$ and $\rho_Y$ is monotonic.

For cases where $\rho_X$ is negative, we conjecture that monotonicity holds between $\rho_X$ and $\rho_Y$. Many Monte Carlo experiments, such as those described in the next paragraph, and the following intuitive explanation support this conjecture. Since both Pearson's correlation coefficient $\rho$ and Kendall's $\tau$ (Kruskall, 1958; Lehmann, 1966) are used to measure the degree of association or dependence between two random variables, it seems reasonable to assume monotonicity between $\rho_Y(h)$ and $\tau_Y(h)$ (denoted here as $\rho_Y(h) \propto \tau_Y(h)$). Unlike the correlation coefficient $\rho_Y(h)$, $\tau_Y(h)$ remains unchanged by monotonic functional transformations of the coordinates. The relations $\rho_Y(h) \propto \tau_Y(h)$, $\tau_Y(h) = \tau_X(h)$, and $\rho_X(h) \propto \tau_X(h)$ imply $\rho_Y(h) \propto \rho_X(h)$; that is, the relation between $\rho_X(h)$ and $\rho_Y(h)$ is monotonically increasing.

To find the relationship between $\rho_X$ and $\rho_Y$, we conduct a Monte Carlo experiment by first specifying $\rho_X = -0.9, -0.8, \ldots, 0.8, 0.9$ and then simply apply Steps 1 and 2 in our procedure to obtain the estimate $\bar{\rho}_Y$. In the experiment, the sample size $n = 10\,000$ is used to compute $\hat{\rho}_Y^{(j)}$, the $j$-th estimate of $\rho_Y$. As discussed in Schmeiser (1990), the stan-




(a) Ray(1) and Exp(1) (b) $\chi^2(3)$ and $\chi^2(40)$ (c) t(3) and t(40)

(d) F(10,7) and F(7,10) (e) Gamma(40,3) and Gamma(7,1) (f) Beta(2,9) and Beta(9,9)

Fig. 3. Functional relationships between $\rho_X$ and $\rho_Y$.

dard error of each $\hat{\rho}_Y^{(j)}$ for $j = 1, 2, \ldots, m$ is approximately

$$\left(1 - \rho_Y^2\right)/\sqrt{n},$$

so $n \geqslant 10\,000$ ensures almost two-decimal-place accuracy in $\hat{\rho}_Y^{(j)}$. We set the number of replications $m = 20$ to estimate the standard error of the estimates of $\rho_Y$.

The marginal distributions of the target marginals are Rayleigh with parameter 1, exponential with parameter 1, chi-square with parameters 3 and 40, $t$ with parameters 3 and 40, $F$ with degrees of freedom (10, 7) and (7, 10), gamma with shape parameter and scale parameter (7, 1) and (40, 3), and beta with parameters (2, 9) and (9, 9). The functions between $\rho_X$ and $\rho_Y$ for the above target distributions are shown in Fig. 3. Cases where the target distribution is Rayleigh or exponential are shown in Fig. 3a. Cases where the target distribution is chi-square, $t$, $F$, gamma, or beta distributions with two different parameters are shown in Figs. 3b–3f, respectively. For comparison, a 45° line, which represents the Gaussian case, is also shown in each figure. In all of the figures, for positive $\rho_X$, the value of the function is below the 45° line, and for negative $\rho_X$, the value is above the line. Notice that the relation between $\rho_X$ and $\rho_Y$ is monotonically increasing for both positive and negative autocorrelations.

For certain types of target distributions the relationship between $\rho_X$ and $\rho_Y$ is independent of the corresponding distribution parameters. For example, in Fig. 3a for exponential(1) and Rayleigh(1), distributions, the value of the relation between $\rho_X$ and $\rho_Y$ is not dependent upon the selection of parameter 1. Therefore, the Rayleigh distribution with parameter 1 represents a general Rayleigh case with any other possible parameter value and the exponential distribution with parameter 1 represents a general exponential case with any other possible parameter value. For these examples, a linear function of the corresponding random variable is invariant with respect to the type of distribution. That is, if

$$Y \sim \text{Rayleigh}(1),$$

then

$$\alpha Y \sim \text{Rayleigh}(\alpha^{-2}),$$

and if

$$Y \sim \exp(1),$$

then

$$\alpha Y \sim \exp(\alpha^{-1}).$$

For the same reason, changing the mean and variance for the Normal distribution, the scale parameter for the gamma distribution, and exchanging the two parameters for the beta distribution do not affect the results.

It can be seen that in each of Figs. 3b–3f, the two plots (not including the 45° line) differ. This is because the relationship between $\rho_X$ and $\rho_Y$ depends on the distribution parameters, even for the same type of target distribution. Thus, no single plot represents the general case.

To generate a target distribution with a particular parameter and lag-one correlation, we can construct a plot similar to those in Fig. 3 to first find the corresponding value of $\rho_X$. We can then apply the marginal-oriented approach to generate the distribution. In this paper, however, we use a different approach: we present an algorithm to directly estimate the corresponding $\rho_X$ without plotting the full range of the relationship between $\rho_X$ and $\rho_Y$.

## 3. Examples

We apply the procedure of Section 2 to a variety of target series, including all target distributions shown in Fig. 3. In all examples the reference and target series have length $n = 1000$ and number of replications $m = 20$. For illustration, we arbitrarily choose three cases: Rayleigh(1), exponential(1), and $F(7, 10)$. See Song and Hsiao (1993b) for other cases.

In Tables 1–3, the first column is the target value $\rho_Y$. The second column gives the number of iterations required to reach a suitable $\rho_X$. The third column is the resulting value of $\rho_X$ (for which $\bar{\rho}_Y$ satisfies the termination criterion). Corresponding values of $\bar{\rho}_Y$ and se($\bar{\rho}_Y$) are given for reference in columns 4 and 5, respectively.

In these examples (including all examples discussed in Fig. 3) our procedure converges rapidly,

Table 1
Rayleigh (1)

| Target $\rho_Y^*$ | Iterations required | Resulting $\rho_X$ | $\bar{\rho}_Y$ given $\rho_X$ | se( $\bar{\rho}_Y$ ) |
|---|---|---|---|---|
| −0.900 | 2 | −0.946 | −0.898 | 0.002 |
| −0.800 | 2 | −0.840 | −0.797 | 0.004 |
| −0.700 | 2 | −0.744 | −0.698 | 0.006 |
| −0.600 | 2 | −0.629 | −0.598 | 0.007 |
| −0.500 | 2 | −0.523 | −0.499 | 0.007 |
| −0.400 | 2 | −0.418 | −0.399 | 0.008 |
| −0.300 | 2 | −0.314 | −0.299 | 0.008 |
| −0.200 | 2 | −0.209 | −0.200 | 0.008 |
| −0.100 | 1 | −0.100 | −0.095 | 0.008 |
| 0.100 | 1 | 0.100 | 0.098 | 0.008 |
| 0.200 | 1 | 0.200 | 0.195 | 0.008 |
| 0.300 | 1 | 0.300 | 0.292 | 0.008 |
| 0.400 | 2 | 0.409 | 0.400 | 0.008 |
| 0.500 | 2 | 0.510 | 0.500 | 0.007 |
| 0.600 | 2 | 0.610 | 0.600 | 0.007 |
| 0.700 | 2 | 0.709 | 0.700 | 0.006 |
| 0.800 | 2 | 0.806 | 0.800 | 0.004 |
| 0.900 | 2 | 0.904 | 0.900 | 0.003 |

never requiring more than four iterations to obtain the corresponding $\rho_X$ given any specific $\rho_Y^*$. The computation time per iteration is not more than five minutes on a 486-based personal computer. Most of the computation time is devoted to computing the inverse transformation in Step 1(iii). For the inverse

Table 2
Exponential (1)

| Target $\rho_Y^*$ | Iterations required | Resulting $\rho_X$ | $\bar{\rho}_Y$ given $\rho_X$ | se( $\bar{\rho}_Y$ ) |
|---|---|---|---|---|
| −0.640 | 4 | −0.950 | −0.632 | 0.008 |
| −0.600 | 4 | −0.885 | −0.592 | 0.008 |
| −0.500 | 4 | −0.714 | −0.495 | 0.005 |
| −0.400 | 3 | −0.550 | −0.396 | 0.006 |
| −0.300 | 3 | −0.403 | −0.298 | 0.006 |
| −0.200 | 3 | −0.261 | −0.197 | 0.007 |
| −0.100 | 2 | −0.126 | −0.094 | 0.008 |
| −0.050 | 2 | −0.065 | −0.050 | 0.008 |
| 0.050 | 1 | 0.050 | 0.050 | 0.009 |
| 0.100 | 2 | 0.113 | 0.098 | 0.009 |
| 0.200 | 2 | 0.228 | 0.196 | 0.009 |
| 0.300 | 2 | 0.339 | 0.297 | 0.009 |
| 0.400 | 2 | 0.447 | 0.397 | 0.009 |
| 0.500 | 2 | 0.551 | 0.500 | 0.008 |
| 0.600 | 2 | 0.650 | 0.602 | 0.007 |
| 0.700 | 2 | 0.745 | 0.704 | 0.006 |
| 0.800 | 3 | 0.831 | 0.800 | 0.005 |
| 0.900 | 2 | 0.920 | 0.903 | 0.003 |

Table 3
$F(7, 10)$

| Target $\rho_Y^*$ | Iterations required | Resulting $\rho_X$ | $\bar{\rho}_Y$ given $\rho_X$ | se( $\bar{\rho}_Y$ ) |
|---|---|---|---|---|
| −0.540 | 4 | −0.900 | −0.539 | 0.011 |
| −0.500 | 4 | −0.800 | −0.495 | 0.011 |
| −0.400 | 4 | −0.619 | −0.390 | 0.012 |
| −0.300 | 4 | −0.439 | −0.293 | 0.005 |
| −0.200 | 4 | −0.277 | −0.198 | 0.006 |
| −0.100 | 2 | −0.119 | −0.100 | 0.006 |
| −0.050 | 1 | −0.050 | −0.046 | 0.007 |
| −0.010 | 3 | −0.001 | −0.011 | 0.001 |
| 0.050 | 2 | 0.073 | 0.044 | 0.007 |
| 0.100 | 3 | 0.142 | 0.098 | 0.008 |
| 0.200 | 3 | 0.264 | 0.199 | 0.008 |
| 0.300 | 2 | 0.370 | 0.293 | 0.008 |
| 0.400 | 2 | 0.479 | 0.398 | 0.008 |
| 0.500 | 2 | 0.581 | 0.501 | 0.009 |
| 0.600 | 2 | 0.680 | 0.601 | 0.011 |
| 0.700 | 3 | 0.770 | 0.700 | 0.007 |
| 0.800 | 3 | 0.856 | 0.800 | 0.005 |
| 0.900 | 3 | 0.934 | 0.900 | 0.004 |

transformation in Step 1(iii) we use the Fortran code provided by Ding (1987).

In these examples, the estimates $\rho_X$, shown in the third column, are quite satisfactory, since the resulting estimates $\bar{\rho}_Y$ (shown in the fourth column) are very close to the target $\rho_Y^*$ (shown in the first column). Therefore, although the termination criteria in our procedure are crude, they are simple and effective.

## 4. Summary

Simulation modeling frequently calls for generating autocorrelated processes. This paper proposes a procedure to implement an existing approach – the marginal-oriented approach. We consider only lag-one autocorrelation because it is a first step to including autocorrelation. Our concepts can be directly extended to higher-order autocorrelations. For example, if the first lag-$p$ autocorrelations are considered, we suggest adopting the $p$-order autocorrelation (AR( $p$)) process as the reference series. The algorithm presented here can be efficiently applied to any distribution for which the inverse transformation of the cumulative distribution function can be calculated or approximated.

The computational efficiency of the current procedure could be improved by finding a more efficient way to estimate the process correlations and by finding a more efficient method of adjusting the lag-one autocorrelation of the AR process which exploits the monotone relationship. These topics will be left for future research.

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