Logistic Function

$$P(x) = P = \frac{e^z}{1+e^z}$$

$$\Leftrightarrow \quad P(1+e^z) = e^z$$

$$\Leftrightarrow \quad P + Pe^z = e^z$$

$$\Leftrightarrow \quad P = e^z - e^z \cdot P$$

$$\Leftrightarrow \quad P = e^z(1-P)$$

$$e^z = \frac{P}{1-P}$$

$$P(x) = \frac{e^z}{e^z+1} \quad \Big| \text{ kürzen mit } e^z = \frac{:e^z}{:e^z}$$

$$= \frac{e^z : e^z}{(e^z+1):e^z} = \frac{1}{\underbrace{(e^z : e^z)}_{=1} + 1 : e^z} = \frac{1}{1+\left(\frac{e^z}{e^z}\right):e^z}$$

$$= \frac{1}{1+\frac{e^z}{e^{2z}}}$$

$$= \frac{1}{1+e^{z-2z}}$$

$$= \frac{1}{1+e^{-z}} = P(x)$$

Parametrisierung Weibullverteilung für gegebene $\underset{=10}{E(x)}$ und $\underset{=2}{SD(x)}$ 25.04.18

$\Rightarrow Var(x) = 4$

$$E(x) = \lambda\, \Gamma\left(1+\frac{1}{k}\right)$$

$$\lambda = \frac{E(x)}{\Gamma\left(1+\frac{1}{k}\right)}$$

$$Var(x) = \lambda^2 \left(\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right)$$

$$= \left(\frac{E(x)}{\Gamma\left(1+\frac{1}{k}\right)}\right)^2 \left(\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right)$$

$$= \frac{[E(x)]^2}{\left[\Gamma\left(1+\frac{1}{k}\right)\right]^2} \left(\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right)$$

$$\Leftrightarrow \quad Var(x) \cdot \left[\Gamma\left(1+\frac{1}{k}\right)\right]^2 = [E(x)]^2 \left(\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right)$$

$$\frac{Var(x)}{[E(x)]^2} \cdot \left[\Gamma\left(1+\frac{1}{k}\right)\right]^2 = \Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2$$

$$\frac{Var(x)}{[E(x)]^2} \cdot 1 = \frac{\Gamma\left(1+\frac{2}{k}\right)}{\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2} - 1$$

$$\frac{Var(x)}{[E(x)]^2} - \frac{\Gamma\left(1+\frac{2}{k}\right)}{\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2} + 1 = 0$$

→ Uniroot - Verfahren → Root - Finding

λ = scale
k = shape

Alternative

$$\text{Minimize} \quad \left(E(x) - \overset{\text{given } E(x)}{10}\right)^2 + \left(Var(x) - \overset{\text{given } Var(x) = [SD(x)]^2}{4}\right)^2$$

$$= \text{Min} \left(\lambda\, \Gamma\left(1+\frac{1}{k}\right) - 10\right)^2 + \left(\lambda^2 \cdot \Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2 - 4\right)^2$$

↳ use "optim" function in R
or "nlminb"

Quantile Function Gompertz

$$F(x) = 1 - e^{-\frac{b}{a} \cdot e^{ax} - 1}$$

$$\Leftrightarrow \quad u = 1 - e^{-\frac{b}{a} \cdot e^{ax} - 1}$$

$$1 - u = e^{-b/a \cdot e^{ax} - 1}$$

$$\ln(1-u) = -\frac{b}{a} \cdot e^{ax} - 1$$

$$\ln(1-u) + 1 = -\frac{b}{a} e^{ax}$$

$$-\frac{a}{b}\left[\ln(1-u) + 1\right] = e^{ax}$$

$$\ln\left[\ln\left[-\frac{a}{b}(\ln(1-u) + 1\right]\right] = ax$$

$$\frac{1}{a}\left[\ln\left[-\frac{a}{b}[\ln(1-u) + 1\right]\right] = x$$

$$= \frac{1}{a}\left[\ln\left[1 - \frac{a}{b}(\ln(1-u))\right]\right] = x = F^{-1}(u)$$

Calculating Inverse Function

$$f(x) = 3 + 4x$$

$$f(x) := u$$

$$u = 3 + 4x$$

$$\frac{u-3}{4} = x = f^{-1}(u)$$

$$f(x) = 3 + g(x) \qquad \text{with } g(x) = 2x$$

$$g(x) := y$$

$$y = 2x$$

$$\frac{y}{2} = x = f^{-1}(y)$$

$$f(x) := u$$

$$u = 3 + g(x)$$

$$u - 3 = g(x)$$

$$g^{-1}(u-3) = x$$

Test: for $x = 2$

$$u = 3 + 4 \cdot 2 = 11$$

$$g^{-1}(11-3) = g^{-1}(8) = \frac{8}{2} = 4 = x$$

$$F(x) = 1 - \frac{\Gamma(\alpha, \rho + \theta x)}{\Gamma(\alpha, \rho)}$$

$$F(x) := u \quad ; \quad \underbrace{x := \rho + \theta x}_{\text{substitution}}$$

$$(1-u) \cdot \Gamma(\alpha, \rho) = \Gamma(\alpha, x)$$

$$\Gamma^{-1}(\alpha, (1-u)\Gamma(\alpha, \rho) = x \quad | \text{ Re-substitution}$$

$$\Gamma^{-1}(\alpha, (1-u)\Gamma(\alpha, \rho) = \rho + \theta x \quad | -\rho \quad | :\theta$$

$$\frac{\Gamma^{-1}(\alpha, (1-u)\Gamma(\alpha, \rho)) - \rho}{\theta} = x$$

Inverse Function of FTG - CDF (= Quantile Function)

1. Pareto:
Wie nach $E(x) = 10$ auflösen
wenn $E(x)_{Am} = E(x)_{EU} - \lambda_{EU}$

2. Kurtosis / Schiefe Parametrisierung
Nicht möglich für Stoppa?, Lognormal, Pareto
siehe Formel für Momente
FTG Ableitung Momente
in Sage funktioniert nicht, ~~Alpha ≥ 50 geht~~

3. Problem AD-Tests

4. FTG Parametrisierung $E(X) = 10 / SD = 2$
nicht machbar für $\alpha = 2$

4. FTG: $E(x) = 10$ + $Var(x) = 4$ für
Alpha ≥ 50 möglich
Formel funktioniert nicht

A: Tail-Index

B: * durch \cdot ersetzen

Parametrization Full-Tail-Gamma Distribution (FTG)

(Castillo et. al. 2012)

PDF: $f(x; \alpha, \theta, \rho) = \theta (\rho + \theta x)^{\alpha - 1} e^{-(\rho + \theta x)} / \Gamma(\alpha, \rho)$

- with incomplete Gamma function $\Gamma(\alpha, \rho) = \int_{\rho}^{\infty} t^{\alpha-1} e^{-t} dt$
- support on $(0, \infty)$
- defined by $\alpha \in \mathbb{R}$, $\theta > 0$, $\rho > 0$
- $\alpha > 0$: FTG is left truncated gamma distribution relocated to the origin (Tail of Gamma function)

  $\alpha \leq 0$: FTG is full exponential model from a canonical statistic
- $c = \frac{\rho}{\theta} > 0$, $\alpha < 0$: if $\rho \to 0$, then

  $PDF_{FTG} \to PDF_{Pareto}$ for fixed c using

  $\theta = \frac{\rho}{c}$; $PDF_{Pareto} = f(x; \alpha, c) = -\alpha c^{-1} (1 + \frac{x}{c})^{\alpha - 1}$

  CDF_{Pareto} $F(x; \alpha, c) = 1 - (1 + \frac{x}{c})^{\alpha}$

  s.t. $\lim_{\rho \to 0} \rho^{\alpha} / \Gamma(\alpha, \rho) = -\alpha$

  $\Rightarrow$ boundary parameter sets

  (a) $\{\alpha < 0, \theta = 0, c > 0\}$ → Pareto Distribution

  (b) $\{\alpha > 0, \theta > 0, \rho = 0\}$ → Gamma Distribution

Moment-Generating Function of FTG

$$M(t; \alpha, \theta, \rho) = (1 - t/\theta)^{-\alpha} \exp(-\rho t/\theta) \Gamma(\alpha, \rho(1 - t/\theta)) / \Gamma(\alpha, \rho)$$

for $t < 0$

$$K(t) = \log(M(t)) = -t\rho/\theta - \alpha \log(1 - t/\theta) - \log \Gamma(\alpha, (1 - t/\theta)\rho)$$

Parametrization: with known α

$E(x) = (\alpha - \rho + e^{-\rho} \cdot \rho^{\alpha} / \Gamma(\alpha, \rho)) / \theta$ with $\mu = e^{-\rho} \cdot \rho^{\alpha} / \Gamma(\alpha, \rho)$

$$\theta = \frac{\alpha - \rho + e^{-\rho} \cdot \rho^{\alpha} / \Gamma(\alpha, \rho)}{E(x)}$$ substitute

$$Var(x) = (\alpha + (1 + \rho - \alpha)\mu - \mu^2) / \theta^2$$

$$\Leftrightarrow \quad 0 = (\alpha + (1 + \rho - \alpha) \cdot \mu - \mu^2) / \theta^2 - Var(x)$$

American Stoppa II

CDF:

$$F(x) = \left[1 - \left(\frac{\lambda}{\lambda + x} \right)^{k_1} \right]^{k_2} \qquad \text{with} \quad x \geq 0$$

PDF:

$$F'(x) = f(x) = k_2 \cdot \left[1 - \left(\frac{\lambda}{\lambda+x} \right)^{k_1} \right]^{k_2 - 1} \cdot - \lambda^{k_1} \cdot (-k_1) \cdot (\lambda + x)^{-k_1 - 1} \cdot 1$$

$$= k_2 \cdot \left[1 - \left(\frac{\lambda}{\lambda+x} \right)^{k_1} \right]^{k_2 - 1} \cdot k_1 \cdot \frac{\lambda^{k_1}}{(\lambda + x)^{k_1 + 1}}$$

$$= \frac{k_2 \cdot k_1}{\lambda + x} \cdot \left(\frac{\lambda}{\lambda + x} \right)^{k_1} \cdot \left[1 - \left(\frac{\lambda}{\lambda + x} \right)^{k_1} \right]^{k_2 - 1}$$

Quantile Function (inverting CDF $F(x)$)

$$F(x) = y = \left[1 - \left(\frac{\lambda}{\lambda+x} \right)^{k_1} \right]^{k_2}$$

$$\sqrt[k_2]{y} = y^{\frac{1}{k_2}} = 1 - \left(\frac{\lambda}{\lambda + x} \right)^{k_1} = 1 - \frac{\lambda^{k_1}}{(\lambda + x)^{k_1}}$$

$$\frac{-\sqrt[k_2]{y} + 1}{\lambda^{k_1}} = \frac{1}{(\lambda + x)^{k_1}}$$

$$(\lambda + x)^{k_1} = \frac{\lambda^{k_1}}{-\sqrt[k_2]{y} + 1}$$

$$x = \sqrt[k_1]{\frac{\lambda^{k_1}}{-\sqrt[k_2]{y} + 1}} - \lambda$$

$$= \lambda \cdot \left(1 - y^{1/k_2} \right)^{-1/k_1} - \lambda$$

Parametrization Pareto Distribution (EUROPEAN)

PDF: $f(x) = \dfrac{\alpha \cdot \lambda^{\alpha}}{x^{\alpha+1}}$ | CDF: $F(x) = 1 - \left(\dfrac{\lambda}{x}\right)^{\alpha} \quad \forall x > \lambda$

$\lambda > 0$ (scale)

$\alpha > 0$ (shape)

$x \in [\lambda, +\infty]$

Mean $E(x) \begin{cases} \infty & \text{for } \alpha \leq 1 \\ \dfrac{\alpha \lambda}{\alpha - 1} & \text{for } \alpha > 1 \end{cases}$

Variance $Var(x) \begin{cases} \infty & \text{for } \alpha \leq 2 \\ \dfrac{\lambda^2 \alpha}{(\alpha-1)^2(\alpha-2)} & \text{for } \alpha > 2 \end{cases}$

$$E(x) = \frac{\alpha \lambda}{\alpha - 1} \quad \Leftrightarrow \quad \lambda = \frac{E(x)(\alpha-1)}{\alpha}$$

$$\Rightarrow \quad Var(x) = \frac{\left(\dfrac{E(x)(\alpha-1)}{\alpha}\right)^2 \alpha}{(\alpha-1)^2(\alpha-2)}$$

$$\Leftrightarrow \quad 0 = \frac{\left(\dfrac{E(x)(\alpha-1)}{\alpha}\right)^2 \alpha}{(\alpha-1)^2(\alpha-2)} - Var(x)$$ for $\alpha > 1$ as lower bound

or $\quad | \cdot (\alpha-1)^2(\alpha-2)$

$$0 = \left(\frac{E(x)(\alpha-1)}{\alpha}\right)^2 \alpha - Var(x)(\alpha-1)^2(\alpha-2) \quad \forall \alpha$$
$\rightarrow \alpha > 0$

für n-tes Moment:

$$E(X^n) = \begin{cases} \lambda^n \dfrac{\alpha}{\alpha - n}, & \alpha > n \\ \infty & \alpha \leq n \end{cases}$$

Parametrization Pareto Distribution (AMERICAN)

PDF : $f(x) = \frac{\alpha \lambda^{\alpha}}{(\lambda + x)^{\alpha+1}}$ | $F(x) = 1 - \left(\frac{\lambda}{\lambda+x}\right)^{\alpha}$ $\forall x > 0$

$\lambda > 0$ (scale)

$\alpha > 0$ (shape)

$0 < x < \infty$

$\frac{\lambda\alpha}{\alpha-1} - \frac{\lambda}{\alpha-1} = \frac{\lambda\alpha - \lambda}{\alpha-1}$

$= \frac{\lambda(\alpha-1)}{\alpha-1} = \lambda$

Mean European Pareto | Mean American Pareto

Mean $E(X) = \frac{\lambda}{\alpha-1}$ for $\alpha > 1$ | $\frac{\lambda \cdot \alpha}{\alpha-1} : \alpha = \frac{\lambda}{\alpha-1}$

Variance $Var(X) = \frac{\lambda^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$ same as European Pareto

$E(X) = \frac{\lambda}{\alpha-1} \iff \lambda = E(X)\cdot(\alpha-1)$

$\Rightarrow Var(X) = \frac{(E(X)(\alpha-1))^2 \alpha}{(\alpha-1)^2(\alpha-2)}$ $\left| - Var(X) \right| \cdot \left[(\alpha-1)^2(\alpha-2)\right]$

$\iff 0 = (E(X)(\alpha-1))^2 \alpha - Var(X)\cdot(\alpha-1)^2(\alpha-2)$

$\iff 0 = [E(X)]^2 \cdot (\alpha-1)^2 \cdot \alpha - Var(X)\cdot(\alpha-1)^2(\alpha-2)$ $| : (\alpha-1)^2$

$\iff 0 = [E(X)]^2 \cdot \alpha - Var(X)\cdot(\alpha-2)$

Quantile Function of CDF $F(x) = F^{-1}(x)$ (inverse)

$F(X) = y = 1 - \left(\frac{\lambda}{\lambda+x}\right)^{\alpha} = 1 - \frac{\lambda^{\alpha}}{(\lambda+x)^{\alpha}}$

$\iff \sqrt[\alpha]{y-1} = \frac{\lambda}{\lambda+x}$

$\iff \sqrt[\alpha]{y-1}\cdot(\lambda+x) = \lambda$

$\iff x = \frac{\lambda}{\sqrt[\alpha]{y-1}} - \lambda = F^{-1}(X)$

EUROPEAN STOPPA $\underline{II}$

CDF:

$$F(x) = \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2} = \left[1 - \lambda^{k_1} \cdot x^{-k_1} \right]^{k_2}$$

PDF:

$$F'(x) = f(x) = k_2 \cdot \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2 - 1} \cdot \left(-\lambda^{k_1} \cdot (-k_1) \cdot x^{-k_1 - 1} \right)$$

$$= k_2 \cdot \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2 - 1} \cdot k_1 \cdot \frac{\lambda^{k_1}}{x^{k+1}}$$

$$= \frac{k_2 \cdot k_1}{x} \cdot \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2 - 1} \cdot \frac{\lambda^{k_1}}{x^{k_1}}$$

$$= \frac{k_2 \cdot k_1}{x} \cdot \left(\frac{\lambda}{x}\right)^{k_1} \cdot \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2 - 1}$$

Moments

$$m_n = k_2 \cdot \text{Beta}\left(1 - \frac{n}{k_1}, k_2 \right) \cdot \lambda^n$$

Quantile Function (inverting CDF $F(x)$)

$$F(x) = y = \left[1 - \left(\frac{\lambda}{x}\right)^{k_1} \right]^{k_2}$$

$$\sqrt[k_2]{y} = y^{\frac{1}{k_2}} = 1 - \frac{\lambda^{k_1}}{x^{k_1}} \Leftrightarrow -y^{\frac{1}{k_2}} = -1 + \frac{\lambda^{k_1}}{x^{k_1}}$$

$$\frac{-\sqrt[k_2]{y} + 1}{\lambda^{k_1}} = \frac{1}{x^{k_1}} \Leftrightarrow \sqrt[k_1]{x^{k_1}} = x^{k_1 \cdot \frac{1}{k_1}} = x = \sqrt[k_1]{\frac{\lambda^{k_1}}{1 - \sqrt[k_2]{y}}}$$

$$x = \left(\lambda^{k_1} \cdot \left(1 - y^{\frac{1}{k_2}} \right) \right)^{1/k_1}$$

$$= \lambda \cdot \left(1 - y^{1/k_2} \right)^{-1/k_1}$$

$x \geq \lambda$; $\lambda, \underbrace{k_1, k_2}$ > 0

scale shape parameters

23.04.18

Parametrisierung Lognormalverteilung mit $\underbrace{E(X) = 10 \text{ und } SD(X) = 2}_{\text{für gegebene } E(X) \text{ und } SD(X).}$

$$E(X) = e^{\mu + \frac{1}{2} \cdot \sigma^2}$$

$$10 = e^{\mu + 0{,}5\,\sigma^2}$$

$$\ln(10) = \mu + 0{,}5\,\sigma^2$$

$$\ln(10) - 0{,}5\,\sigma^2 = \mu \quad \Big| \quad 2\ln(10) - 2\mu = \sigma^2$$

$$SD(X) = E(X)\sqrt{e^{\sigma^2} - 1}$$

$$2 = 10\sqrt{e^{\sigma^2} - 1}$$

$$4 = \left(10\sqrt{e^{\sigma^2} - 1}\right)^2$$

$$4 = 100 \cdot \left(e^{\sigma^2} - 1\right)$$

$$4 = 100\,e^{\sigma^2} - 100$$

$$104 = 100 \cdot e^{\sigma^2}$$

$$\frac{SD^2 + E^2}{E^2} \leftarrow 1{,}04 = e^{\sigma^2}$$

$$\ln(1{,}04) = \sigma^2$$

$$0{,}03922 = \sigma^2 \checkmark$$

$$E(X) = e^{\mu + 0{,}5\,\sigma^2}$$

$$10 = e^{\mu + 0{,}5 \cdot \ln(1{,}04)}$$

$$SD(X) = e^{\overbrace{\ln(10) - 0{,}5\,\sigma^2}^{\mu} + 0{,}5\,\sigma^2} \cdot \sqrt{e^{\sigma^2} - 1}$$

$$2 = e^{\ln(10) - 0{,}5\,\sigma^2 + 0{,}5\,\sigma^2} \cdot \sqrt{e^{\sigma^2} - 1}$$

$$2 = e^{\ln(10)} \cdot \sqrt{e^{\sigma^2} - 1}$$

$$4 = \left(e^{\ln(10)} \cdot \sqrt{e^{\sigma^2} - 1}\right)^2 = e^{2\ln(10)} \cdot \left(e^{\sigma^2} - 1\right)$$

$$4 = e^{2\ln(10) + \sigma^2} - e^{2\ln(10)}$$

$$4 + e^{2\ln(10)} = e^{2\ln(10) + \sigma^2}$$

$$\ln\left(4 + e^{2\ln(10)}\right) = 2\ln(10) + \sigma^2$$

$$\ln\left(4 + e^{2\ln(10)}\right) - 2\ln(10) = \sigma^2 = 0{,}03922$$

$$\ln(10) = \mu + 0{,}5 \cdot \ln(1{,}04)$$

$$\ln(10) - 0{,}5 \cdot \ln(1{,}04) = \mu = 2{,}282975$$

Allgemein

$$\ln(E(X)) - 0{,}5 \cdot \ln\left(\frac{[E(X)]^2 + (SD(X))^2}{[E(X)]^2}\right) = \mu$$

$$\ln\left(\underbrace{(SD(X))^2}_{Var(X)} + e^{2\ln(E(X))}\right) - 2\ln(E(X)) = \sigma^2$$

$$= \ln\left(\frac{(SD(X))^2 + [E(X)]^2}{[E(X)]^2}\right) = \sigma^2 = \ln\left(\frac{(SD(X))^2}{[E(X)]^2} + 1\right)$$

Proof: $\sigma^2 = \ln\left(\frac{(SD(X))^2}{(E(X))^2} + 1\right) = \ln\left(\frac{\left(E(X)\sqrt{e^{\sigma^2} - 1}\right)^2}{(E(X))^2} + 1\right)$

$$= \ln\left(e^{\sigma^2} - 1 + 1\right) = \ln\left(e^{\sigma^2}\right) = \sigma^2$$