Tutorial Problem 5

Question- Number of runs of the basic algorithm of randomized max cut to get a fixed value of the probability that the expected cut size is at least $|E|/2$.
Answer- Simple randomized algorithm outputs a cut of size atleast $|E|/2$ with probabilty atleast $\frac{1}{|E|+2}$.
Now, let q be the fixed probability that we intend to achieve. We define $p \geqslant \frac{1}{m+2}$, where $|E| = m$. With p probability, we get the size of cut atleast $m/2$. Thus, with $(1-p)$, we get the size of the cut smaller than $m/2$. Let k be the number of run by the algorithm, where $k \geqslant 1$. Thus,

$$
\begin{aligned}
& p + (1-p)p + \ldots + (1-p)^{k-1}p = q \\
\implies & p[1 + (1-p) + \ldots + (1-p)^{k-1}] = q \\
\implies & p\frac{1-(1-p)^k}{1-(1-p)} = q \\
\implies & 1-(1-p)^k = q \\
\implies & k = \frac{\log(1-q)}{\log(1-p)} \\
\implies & k \geqslant \frac{\log(1-q)}{\log(1-\frac{1}{m+2})} \\
\implies & k \geqslant \frac{\log(1-q)}{\log(\frac{m+1}{m+2})} \\
\implies & k = \left\lceil \frac{\log(1-q)}{\log(\frac{m+1}{m+2})} \right\rceil
\end{aligned}
$$

For example, let $q = \frac{1}{2}$ and [redacted] $= 1022$. The required value of k is given bellow.

$$
\begin{aligned}
& k = \left\lceil \frac{\log(1-\frac{1}{2})}{\log(\frac{1022+1}{1022+2})} \right\rceil \\
\implies & k = \left\lceil \frac{\log(\frac{1}{2})}{\log(\frac{1023}{1024})} \right\rceil \\
\implies & k = \lceil 709.4 \rceil \\
\implies & k = 710
\end{aligned}
$$