Examples On Asymptotic Notation

We know that Big O is defined as:

$$O(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that } f(n) \le c\, g(n) \ \forall\, n \ge n_0 \}$$

- We will give here some examples on how to find the constants 'c' and 'n_0'

Example 1:

$100n + 5$:

Here we can easily see that

$100n + 5$ is $O(n^2)$

because

$$\begin{aligned} 100n + 5 &\le 100n + n \quad \text{for } n \ge 5 \\ &= 101n \\ &\le 101n^2 \end{aligned}$$

$\therefore$ $100n + 5$ is $O(n^2)$ for $n_0 = 5$, $c = 101$

Also,

$$\begin{aligned} 100n + 5 &\le 100n + 5n, \quad n \ge 1 \\ &= 105n \\ &\le 105n^2 \end{aligned}$$

$\therefore$ $100n + 5$ is $O(n^2)$ for $n_0 = 1$, $c = 105$

Here it is important to note that Big O gives the upper bound, so 100n+5 is $O(n^2)$ is correct, but we can tighten the upper bound, as:

$$100n + 5 \le 100n + n \quad \text{for } n \geq 5$$
$$= 101n$$

i.e. $100n + 5 \le 101n$ for $n \geq 5$

$\therefore$ $100n + 5$ is $O(n)$ for $n_0 = 5$, $c = 101$.

Example 2.

$100n^2 + 20n + 5$ is $O(n^2)$

Here

$$100n^2 + 20n + 5 \le 100n^2 + 20n^2 + 5n^2, \quad n \geq 1$$
$$= 125n^2$$

So $100n^2 + 20n + 5$ is $O(n^2)$ for $n_0 = 1$
$c = 125$

Alternatively,

$$100n^2 + 20n + 5 \le 100n^2 + n^2 + n^2 \quad \text{for } n \geq 20$$
$$= 102n^2$$

So, $100n^2 + 20n + 5$ is $O(n^2)$

for $n_0 = 20$, $c = 102$

Example 3

$3n^3 - 20n^2 + 5$

We see that

$$
\begin{aligned}
3n^3 - 20n^2 + 5 &\le 3n^3 + 5 \\
&\le 3n^3 + n^3 \quad \text{for } n \geq 5 \\
&= 4n^3
\end{aligned}
$$

So, $3n^3 - 20n^2 + 5$ is $O(n^3)$

for $n_0 = 5$

$c = 4$

Big Ω is defined as:

$$\Omega(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that } f(n) \leq c\,g(n) \ \forall\, n \geq n_0 \}$$

- We give here some examples on how to find the constants 'c' and 'n_0'

Example 1

$100n + 5$ is $\Omega(n)$

Here

$$100n + 5 \geq 100n \quad \text{for } n \geq 1$$

So, $100n + 5$ is $\Omega(n)$ for $n_0 = 1$
$c = 100$

Example 2

$100n^2 + 20n + 5$ is $\Omega(n^2)$

Here,

$$100n^2 + 20n + 5 \geq 100n^2 \text{ for } n \geq 1$$

So, $100n^2 + 20n + 5$ is $\Omega(n)$ for $n_0 = 1$
$c = 100$

Example 3

$3n^3 - 20n^2 + 5$

We see that

$$\begin{aligned} 3n^3 - 20n^2 + 5 &\geq 3n^3 - 20n^2 \\ &\geq 3n^3 - n^3 \quad \text{for } n \geq 20 \\ &= 2n^3 \end{aligned}$$

So, $3n^3 - 20n^2 + 5$ is $\Omega(n^3)$ for

$n_0 = 20$

$c = 2$

Big Θ is defined as

$$\Theta(g(n)) = \{ f(n) : \exists \text{ positive constants } c_1 \text{ and } c_2 \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \ \forall n \ge n_0 \}$$

- We give here some examples

Example 1.

$100n + 5$

$$100n+5 \le 101n \text{ for } n \ge 5$$

and $100n+5 \ge 100n$ for $n \ge 1$

So

$$100n \le 100n+5 \le 101n \text{ for } n \ge 5$$

i.e $c_1 = 100$, $c_2 = 101$, $n_0 = 5$

Example 2

$100n^2+20n+5$

$$100n^2+20n+5 \le 102n^2 \text{ for } n \ge 20$$

$$100n^2+20n+5 \ge 100n^2 \text{ for } n \ge 1$$

So,

$$100n^2 \le 100n^2+20n+5 \le 102n^2 \text{ for } n \ge 20$$

i.e. $c_1 = 100$, $c_2 = 102$, $n_0 = 20$

Example 3

$3n^3 - 20n^2 + 5$

Here

$$3n^3 - 20n^2 + 5 \leq 4n^3 \quad \text{for } n \geq 5$$

$$3n^3 - 20n^2 + 5 \geq 2n^3 \quad \text{for } n \geq 20$$

So

$$2n^3 \leq 3n^3 - 20n^2 + 5 \leq 4n^3 \quad \text{for } n \geq 20$$

i.e. $c_1 = 2$, $c_2 = 4$, $n_0 = 20$