4

Lecture 3
Equivalence for Repeated Cash Flows

Mrs Dashwood's Annuity

Mrs Dashwood’s Annuity, cont’d

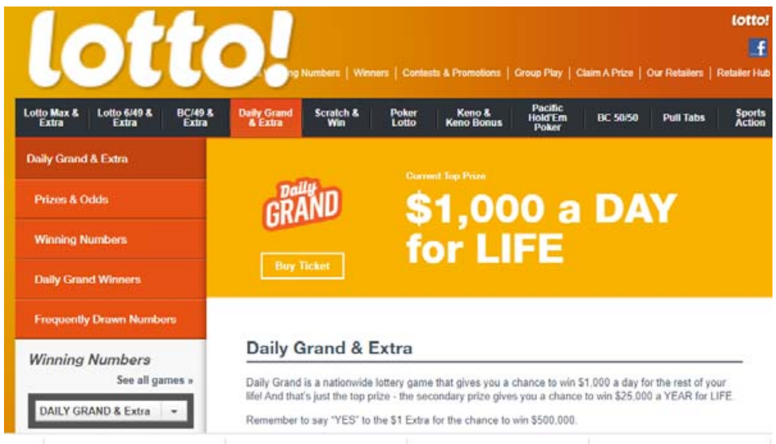
- Mr John Dashwood and his avaricious wife, Fanny, are debating how he can most cheaply discharge his obligations toward his half-sister, Mrs Henry Dashwood, and her three daughters
- He first considers giving her a lump sum of £1,500; then, since £1,500 seems a lot to part with in one lump, he considers paying her an annual sum of £100 – an *annuity* – for as long as she lives

Mrs Dashwood’s Annuity, cont’d

- But, objects Fanny, although Mrs Dashwood is old, such an arrangement will encourage her to cling to life for an unreasonable time; and, should she survive for more than 15 years, her half-brother will lose money by the arrangement
- It is clear from this discussion that Fanny Dashwood cannot have studied “engineering” economics; for in fact, Mr Dashwood *would* save money by the latter alternative, even if Mrs Henry Dashwood were to live considerably longer than 15 years

Mrs Dashwood's Annuity, cont'd

- This is like a Lottery For Life problem (except the lottery corporation gets to choose the amounts instead of you)
- Which would you choose: $1,000 a day for life, or $7 million once?



Learning Objectives

- Understand opportunity cost when comparing alternatives
- Convert uniform series of cash flows to their equivalent present or future values
- Use arithmetic and geometric gradients to solve correctly modelled problems
- Use *continuously compounded interest* with uniform payment series

Key Summary: Course to date and coming soon

- Variables and parameters (puzzle pieces):
  - Different kinds of interest rates
  - Discount rates
  - **Costs and cost savings or revenues, now and in the future**
  - Different expected lives of the possible project/purchases
  - Salvage value
  - Taxes and tax savings
  - How these escalate
- Analysis methods (ways to put the pieces together):
  - Present worth analysis (Net Present Value)
  - Equivalent uniform annual cost analysis
  - Rate of return analysis
  - Benefit-cost ratio analysis
  - Payback period
  - Cost-effectiveness analysis

Opportunity Cost

Opportunity cost is the net benefit (the textbook sometimes uses 'return') of your next best alternative.

Example 1:
If I go and get a waffle cone at Rain or Shine, and I pick London Fog (because it's amazing) over Strawberry, the monetary cost is the same, but my opportunity cost is the enjoyment of Strawberry ice cream, which I don't get by eating London Fog.

London Fog has a higher net benefit, because it has a higher benefit and the same cost. In this case, we can't monetize the net benefit because the benefit isn't measurable: we just know it's higher for London Fog (as evidenced by the choice made).

Example 2:
You need to install a new server. You have two options: modify and install an old server that you have in inventory for $5,000. Alternatively, you could buy and install a new server for $8,000. The old server could be sold off for $2,000.

If you install the new server, it will cost $8,000 minus $2,000 for selling the old server = $6,000 cost.
If you install the old server, it will cost $5,000.
Installing the old server is the best option. The opportunity cost is the net benefit of the next best alternative, which is a cost of $6,000.

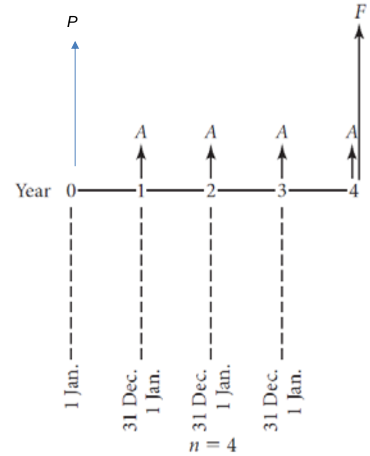
Notice that you need to include all the benefits and costs of the next best alternative.

Net benefit: Consider costs and benefits.
Net cost: choose which one is bigger.

Comparing and converting cash flows over time

The next several slides will explore how to convert and compare cash flows that occur right now, the present (P), regularly over time annually (A), and occur once at some future time (F).

- We rely on the concept of equivalency that we learned last time.
- The interest rate *i* is a crucial parameter to these conversions.



Uniform Series Compound Interest Formulas

- Uniform series (A) is defined as:
  - An end-of-period cash receipt or disbursement in a uniform series, continuing for n periods
- Let's start with this: What is the equivalent amount of money if we want to convert it to (compare it with) one point in time in the future (F)?
- Each amount A will earn interest until time F.
- The earlier deposits A earn more, because they have longer to earn interest.

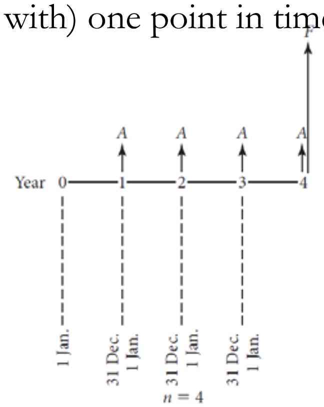


FIGURE 4-1A The general relationship between *A* and *F*.

Uniform Series Compound Interest Formulas, cont'd

- In the general case:

$$F = A(1+i)^{n-1} + \ldots + A(1+i)^2 + A(1+i) + A \qquad (1)$$

- Multiplying by $(1+i)$:

$$(1+i)F = A(1+i)^n + \ldots + A(1+i)^3 + A(1+i)^2 + A(1+i) \qquad (2)$$

- Factoring out A and subtracting (1) from (2):

$$\begin{aligned} (1+i)F &= A[(1+i)^n + \ldots + (1+i)^3 + (1+i)^2 + (1+i)] \\ \underline{-F} &\underline{= A[(1+i)^{n-1} + \ldots + (1+i)^2 + (1+i) + 1]} \\ iF &= A[(1+i)^n - 1] \end{aligned}$$

Uniform Series Compound Interest Formulas, cont'd

- Rearranging the previous equation

$$F = A\boxed{\left[\frac{(1+i)^n - 1}{i}\right]}$$

- The notation is $F = A\,\boxed{(F/A, i\%, n)}$

- The part in brackets is called the *uniform series compound amount factor.*

- F = A * $\boxed{\textit{uniform series compound factor}}$

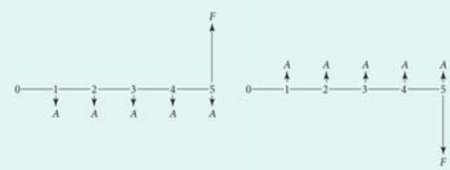
Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-1

You deposit \$500 in a credit union at the end of each year for five years. The credit union pays 5% interest, compounded annually. How much do you have in your account immediately after the fifth deposit?

SOLUTION

The diagram on the left shows the situation from your point of view; the one on the right, from the credit union's point of view. Either way, the diagram of the five deposits and the desired computation of the future sum F matches the situation for the uniform series compound amount formula:



$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = (F/A, i, n)$$

where $A = \$500$, $n = 5$, $i = 0.05$, $F =$ unknown. Filling in the known variables gives

$$F = \$500(F/A, 5\%, 5) = \$500(5.526) = \$2{,}763$$

There will be \$2,763 in the account after the fifth deposit.

Uniform Series Compound Interest Formulas, cont'd

- We can look at the problem completely the other way around:
- Solving for A: $A = F\boxed{\left[\frac{i}{(1+i)^n - 1}\right]}$
- The notation is $A = F\,\boxed{(A/F, i\%, n)}$
- The part in brackets is called the *uniform series sinking fund factor.*
- A = F * $\boxed{\textit{uniform series sinking fund}}$

In this form, the equation answers the question *"How much would I need to regularly deposit if I want to end up with a certain amount of money at the end, and know how much interest can I earn?"*

Notice that we're working with the same formula: we just solved for a different variable.

The $\boxed{\textit{uniform series sinking fund}}$ = 1 / $\boxed{\textit{uniform series compound factor}}$

Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-2

Jim wants to buy some electronic equipment for \$1,000. He decided to save a uniform amount at the end of each month so that he would have the required \$1,000 at the end of one year. The local credit union pays 6% interest, compounded monthly. How much would Jim have to deposit each month?

SOLUTION

$$F = \$1{,}000 \qquad n = 12 \qquad i = 0.5\% \qquad A = \text{unknown}$$
$$A = 1{,}000(A/F, 0.5\%, 12) = 1{,}000(0.0811) = \$81.10$$

Jim would have to deposit \$81.10 each month.

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = 1000 \times \frac{0.05}{(1.05)^{11}} = 81.10$$

Uniform Series Compound Interest Formulas, cont'd

We can instead rework the same info into present day dollar terms:

- Remember that the *single payment compound amount* formula is
$$F = P(1+i)^n$$
- Substitute this in for F in the sinking fund formula:
$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = P(1+i)^n\left[\frac{i}{(1+i)^n - 1}\right]$$
- Therefore: $A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$
  - Notation: $A = P\,(A/P,\, i\%,\, n)$
- This is called the *uniform series capital recovery factor*

Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-3

An energy-efficient machine costs $5,000 and has a life of five years. If the interest rate is 8%, how much will it have to save every year in order for the initial capital amount to be recovered?

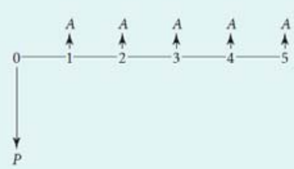
Switch to Excel example 4-3

Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-3

An energy-efficient machine costs $5,000 and has a life of five years. If the interest rate is 8%, how much will it have to save every year in order for the initial capital amount to be recovered?

SOLUTION



$P = \$5{,}000 \qquad n = 5 \qquad i = 8\% \qquad A = \text{unknown}$

$A = P(A/P, 8\%, 5) = 5{,}000(0.2505) = \$1{,}252$

The required annual saving to recover the capital is $1,252.

Uniform Series Compound Interest Formulas, cont'd

This formula can be flipped over too, to solve for how much money you would need to start with:

- Solving the capital recovery formula for P:

$$\boxed{P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]}$$

Note ERROR in textbook for this formula: p.100

  - Notation: $P = A(P/A, i\%, n)$
- This is called the *uniform series present worth formula*

Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-4

An investor holds a time-payment purchase contract on some machine tools. The contract calls for the payment of $140 at the end of each month for a five-year period. The first payment is due in one month. He offers to sell you the contract for $6,800 cash today. If you can otherwise make 1% per month on your money, would you accept or reject the investor's offer?

Switch to Excel example 4-4

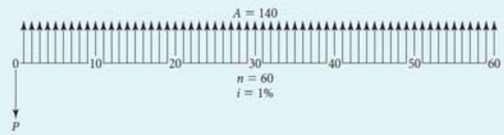
Uniform Series Compound Interest Formulas, cont'd

EXAMPLE 4-4

An investor holds a time-payment purchase contract on some machine tools. The contract calls for the payment of \$140 at the end of each month for a five-year period. The first payment is due in one month. He offers to sell you the contract for \$6,800 cash today. If you can otherwise make 1% per month on your money, would you accept or reject the investor's offer?

SOLUTION

Summarizing the data in a cash flow diagram, we have



Use the uniform series present worth formula to compute the present worth.

$$
\begin{aligned}
P = A(P/A, i, n) &= 140(P/A, 1\%, 60) \\
&= 140(44.955) \\
&= \$6{,}293.70
\end{aligned}
$$

If you are paying more than \$6,293.70, you will receive less than the required 1% per month interest. Reject the investor's offer.

Uniform Series Formulas: Problem 1

Joe wants to be able to purchase a dream car for about \$19,000 on 1 January 2004, just after he graduates from college. Joe has had a part time job and started making deposits of \$275 each month into an account that pays 9% compounded monthly beginning with the first deposit on 1 February 1999. The last deposit is to be made on 1 January 2004. Determine how much money he would have saved to buy the car.

Will he be able to buy his dream car?

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = \left[\frac{0.09}{(1.09)^{60} - 1}\right] \times 275$$

Uniform Series Formulas: Problem 1, cont'd

Solution

Monthly deposits, A = \$275. n = 2/1999 to 1/2004 = 60 periods.

Relevant formula: $F = A\left[\frac{(1+i)^n - 1}{i}\right]$

F = \$275 (F/A, 0.75%, 60) = \$275 (75.424) = \$20,741.64

Joe will have **\$20,741.64** available for his purchase.

Uniform Series Formulas: Problem 2

You are repaying a debt of \$10,000 with equal payments made at the end of four equal periods. If the interest rate is 10% per period, how much of the original \$10,000 principal will be paid in the second payment?

Uniform Series Formulas: Problem 2, cont'd

Principal = \$10,000
Number of equal payments (n) = 4. i = 10% per period.

Relevant formula: $A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$ (PMT formula in Excel)

$A = P(A/P, i, n) = 10{,}000(A/P, 10\%, 4) = 10{,}000 \times 0.3155 = \$3155$

→ Ch4 ppt ex2

Turn to Excel example:
[First payment interest on unpaid balance = 10,000 × 0.10 = \$1000.

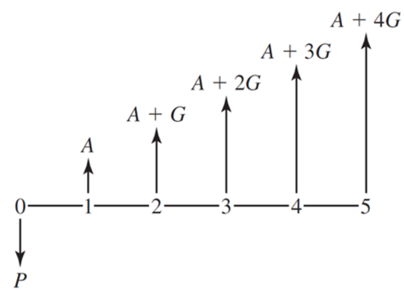
Principal repaid = A – 1,000 = \$2155.00
Second payment Principal due = 10,000 – 2155 = \$7845
Interest on unpaid balance = 7,845 × 0.10 = \$784.50.
Principal paid in the second payment = 3,155 – 784.50 = \$2370.50.]

For reference: Relationships between Compound Interest Factors

- Single Payment
  - Compound amount factor = 1/Present Worth Factor
  - $(F/P, i, n) = 1/(P/F, i, n)$
- Uniform Series
  - Capital recovery factor = 1/Present Worth Formula
  - $(A/P, i, n) = 1/(P/A, i, n)$
  - Compound amount factor = 1/Sinking Fund Factor
  - $(F/A, i, n) = 1/(A/F, i, n)$
  - Capital recovery factor = Sinking Fund Factor plus i
  - $(A/P, i, n) = (A/F, i, n) + i$

Arithmetic Gradient

- New wrinkle: what if the regularly occurring deposits are increasing linearly?
- A uniformly increasing series consists of two components:
  - Series component (*A*)
  - Gradient component (*G*)



Arithmetic Gradient, cont'd

- Can calculate the conversion by splitting up into (1) Present Value of uniform payments (P') and (2) Present Value of increasing payments (P'')
- $P = P' + P'' = A\ (P/A, i, n) + G\ (P/G, i, n)$
- $= A\left[\frac{(1+i)^n-1}{i(1+i)^n}\right] + G\left[\frac{(1+i)^n-in-1}{i^2(1+i)^n}\right]$

Arithmetic Gradient, cont’d

- It is also possible to convert the linearly increasing series of payments (G) to a uniformly sized series of payments (A).
  - *Arithmetic gradient uniform series factor*:

$$A = G * (A/G, i, n) = G * \left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

Arithmetic Gradient, cont’d

EXAMPLE 4-9

On a certain piece of machinery, it is estimated that the maintenance expense will be as follows:

Year	Maintenance Cost
1	\$100
2	200
3	300
4	400

What is the equivalent uniform annual maintenance cost for the machinery if 6% interest is used?

SOLUTION

The first cash flow in the arithmetic gradient series is zero; hence the diagram is *not* in the proper form for the arithmetic gradient equation. As in Example 4-8, the cash flow must be resolved into two components:

$$A = 100 + 100(A/G, 6\%, 4) = 100 + 100(1.427) = \$242.70$$

The equivalent uniform annual maintenance cost is \$242.70.

Arithmetic Gradient, cont'd

EXAMPLE 4-9

On a certain piece of machinery, it is estimated that the maintenance expense will be as follows:

Year	Maintenance Cost
1	$100
2	200
3	300
4	400

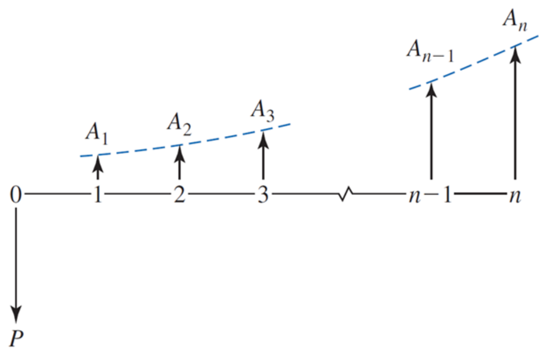
What is the equivalent uniform annual maintenance cost for the machinery if 6% interest is used?

- See Excel example

Potential break point

Geometric Gradient

- Next wrinkle: what if the regularly occurring deposits are increasing geometrically (non-linearly)?
- Period-by-period change is a uniform rate (*g*)
- Applicable for population growth or other levels of activity where changes over time are best modelled as a percentage of the previous year



Geometric Gradient, cont'd

- Two possible cases:
  - (1) Where: i ≠ g

$$P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{i-g}\right]$$

Note error in formula when repeated on p.123

  - Factor Notation: $(P/A, g, i, n)$

- Where the bracketed part is called the "*geometric series present worth factor*" (when $i \neq g$)

Geometric Gradient, cont'd

- Two possible cases:
  - (2) Where: i = g

$$P = A_1\ [n(1+i)^{-1}]$$

  - Factor Notation: $(P/A, g, i, n)$
- Where the bracketed part is called the "*geometric series present worth factor*" (where $i = g$)

Geometric Gradient, cont'd

EXAMPLE 4-12

The first-year maintenance cost for a new car is estimated to be $100, and it increases at a uniform rate of 10% a year. Using an 8% interest rate, calculate the present worth of cost of the first five years of maintenance.

- See Excel example

Geometric Gradient, cont'd

EXAMPLE 4-12

The first-year maintenance cost for a new car is estimated to be $100, and it increases at a uniform rate of 10% a year. Using an 8% interest rate, calculate the present worth of cost of the first five years of maintenance.

STEP-BY-STEP SOLUTION

Year n	Maintenance Cost			$(P/F, 8\%, n)$	PW of Maintenance
1	100.00	= 100.00	×	0.9259 =	$ 92.59
2	100.00 + 10%(100.00)	= 110.00	×	0.8573 =	94.30
3	110.00 + 10%(110.00)	= 121.00	×	0.7938 =	96.05
4	121.00 + 10%(121.00)	= 133.10	×	0.7350 =	97.83
5	133.10 + 10%(133.10)	= 146.41		0.6806 =	99.65
					$480.42

SOLUTION USING GEOMETRIC SERIES PRESENT WORTH FACTOR

$$P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{i-g}\right] \quad \text{where } i \neq g$$

$$= 100.00\left[\frac{1-(1.10)^5(1.08)^{-5}}{-0.02}\right] = \$480.42$$

The present worth of cost of maintenance for the first five years is $480.42.

When Compounding Period and Payment Period Differ

- When compounding interest periods and payment periods differ, an adjustment is required in order to utilize the formulas.
- This is usually done by:
  1. Computing the equivalent payment amounts for each compounding period and applying the interest rate
  2. Computing an effective interest rate for the payment periods

Geometric Gradient, cont'd

Example 4-13:

On 1 Jan, a woman deposits $5,000 in a credit union that pays 8% nominal annual interest, compounded quarterly. She wishes to withdraw all the money in five equal yearly sums, beginning 31 Dec of the first year. How much should she withdraw each year?

20 interest periods, 5 withdrawals.

W W W W W

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$5,000

Geometric Gradient, cont'd

Example 4-13:

1. Determine what 20 withdrawals would look like (so that equivalent number of withdrawals and interest periods):

A A A A A A A A…

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

$5,000

$$using\ A = P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right] = \$306$$

Geometric Gradient, cont'd

Example 4-13:

2. Convert the series of 20 withdrawals (A=$306) into an equivalent set of 5 withdrawals. This means that n is now 4 (since each of the 5 larger withdrawals includes 4 quarters).

Using $F = A\left[\frac{(1+i)^n - 1}{i}\right] = \$1{,}260$ /yr.

4

306 -

- See Excel example for another method of solving this.

306 306 306 306 -

0 1 2 4

Reality and the Assumed Uniformity of *a*, *G*, and *g*

- We define and start with an *A* (uniform annual cost), a *G* (uniform annual gradient), and a *g* (uniform annual rate of increase) for three reasons:
  1. It is easier to start with simple models
  2. These model cash flows are convenient for bounding the problems often encountered in engineering economic analysis
  3. Not enough is known about the future and so it is approximated through uniform series and gradients

Continuous Compounding at Nominal Rate *r* per period

- Continuous Compounding Sinking Fund

$$[A/F\,,r,n]=\frac{e^{r}-1}{e^{rn}-1}$$

- Continuous Compounding Capital Recovery

$$[A/P,r,n]=\frac{e^{rn}(e^{r}-1)}{e^{rn}-1}$$

- Continuous Compounding Series Compound Amount

$$[F/A,r,n]=\frac{e^{rn}-1}{e^{r}-1}$$

- Continuous Compounding Series Present Worth

$$[P/A,r,n]=\frac{e^{rn}-1}{e^{rn}(e^{r}-1)}$$

Continuous Compounding at Nominal Rate *r* per period, cont'd

EXAMPLE 4-14

In Example 4-1, \$500 per year was deposited in a credit union that paid 5% interest, compounded annually. At the end of five years, \$2,763 was in the credit union account. How much would there have been if the institution paid 5% nominal interest, compounded continuously?

SOLUTION

$$A=\$500 \quad r=0.05 \quad n=5\text{ years}$$

$$F=A[F/A,r,n]=\left(\frac{e^{rn}-1)}{e^{r}-1}\right)=500\left(\frac{e^{0.05(5)}-1}{e^{0.05}-1}\right)$$

$$=\$2{,}769.84$$

- See Excel example