Tutorial 12 solutions

1. Vertex Cover

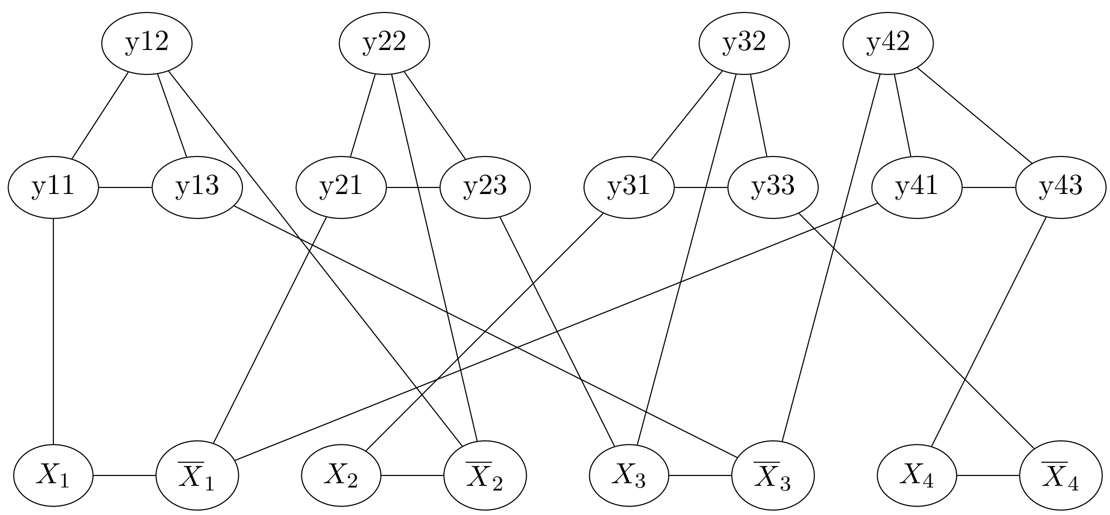
   (a) If someone gives us a collection of at most K vertices that they claim is a vertex cover for G, we need to verify that:

   - It contains at most K vertices.
   - Every edge of the graph has an endpoint in the set.

   This can be done (almost brute-force) in $O(mK)$ time, where m is the number of edges of the graph.

   (b) We need to reduce a known NP-complete problem (say, 3SAT) to the vertex cover problem. Then an arbitrary instance of a problem in NP can be solved by first reducing it to 3SAT in polynomial time (doable because 3SAT is NP-complete), then reducing the instance of 3SAT to an instance of vertex cover (which we will show how to do in polynomial time), and finally solving the instance of vertex cover. If we can solve the instance of vertex cover in polynomial time, then that gives a polynomial time solution to the arbitrary instance of a problem in NP.

   (c)



   (d) We choose $K = n + 2m$.

   (e) Suppose the instance of 3SAT is satisfiable. We obtain a vertex cover with K vertices by looking an an assignment of truth values to the variables that satisfies all the clauses.

   - For each variable X_i, if X_i was assigned true then we add X_i to the vertex cover. Otherwise we assign $\overline{X}_i$.
   - For each clause C_j, suppose the ith literal l of C_j is true. Then we add the vertices $Y_{j,k}$ to the vertex cover for both values of k that aren't equal to i

(for instance, if the first literal of C_j is true, then we add $Y_{j,2}$ and $Y_{j,3}$ to the vertex cover).

It's relatively easy to see that every edge of the graph will have at least one endpoint in the vertex cover.

(f) Suppose now that the graph constructed by the reduction has a vertex cover with at most K elements. At least one of the pair of vertices that correspond to each variable in the instance of 3SAT must be in the vertex cover. Morever, because every one of the triangle corresponding to clauses ($\{Y_{j,1}, Y_{j,2}\}$, $\{Y_{j,1}, Y_{j,3}\}$ and $\{Y_{j,2}, Y_{j,3}\}$) needs at least two vertices in the vertex cover, at most one of the pair of vertices that correspond to each variable in the instance of 3SAT must be in the vertex cover.

Let us assign the value True to X_i if X_i is in the vertex cover, and let us assign the value False to X_i if $\overline{X}_i$ is in the vertex cover. For each clause C_j, one of the three vertices $Y_{j,1}, Y_{j,2}, Y_{j,3}$ is not in the vertex cover (otherwise it would have more than $n+2m$ elements). This means the vertex for the corresponding literal **is** in the vertex cover, which means the literal is True.

2. First we need to prove that *largest common subgraph* belongs to NP. Our certificate will be the lists E'_1 and E'_2 of edges of G_1 and G_2, ordered so that each edge of E'_2 is in the same position in this list as the corresponding edge of E'_1 (so the first edges of both lists would be mapped to each other in the isomorphism, the second edges of both lists would be mapped to each other, etc). Given this certificate, we can verify that

   - $|E'_1| = |E'_2| \geq K$;
   - Each edge of E'_1 is an edge of E_1;
   - Each edge of E'_2 is an edge of E_2;
   - For every vertex of V_1 that appears in one or more edges of E'_1, there is one (and only one) vertex of V_2 that appears at the same place in the corresponding edges of E'_2.

   The last condition ensures that (V_1, E'_1) and (V_2, E'_2) are isomorphic.

   Next, we need to show how to reduce each instance of *hamiltonian path* to an instance of *largest common subgraph.* An instance of *hamiltonian path* is simply a graph $G = (V, E)$. Give this graph, we construct the following instance of *largest common subgraph*:

   - $G_1 = G$;
   - G_2 is a graph with $|V|$ vertices $w_1, w_2, \ldots, w_n$ and the edges $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\ldots \{w_{n-1}, w_n\}$.
   - $K = |V| - 1$.

If the graph G contains a hamiltonian path, then this hamiltonian path will be isomorphic to the graph G_2 by definition. So G_1 and G_2 have a common subgraph with $|V| - 1$ edges. Conversely, if G_1 and G_2 have a common subgraph with $|V| - 1$ edges, then this common subgraph must be isomorphic to G_2 (because G_2 also has exactly $|V| - 1$ edges). Therefore G_1 has a subgraph isomorphic to G_2, which means G contains a hamiltonian path.