Tutorial 3 solutions

1. a. We construct a new graph $G' = (V, E')$ where E' contains every pair $\{x, y\}$ of vertices of G that is NOT an edge of G (that is, every $\{x, y\}$ such that $\{x, y\} \notin E$), and find the largest clique in G'. Because every subset W of V that is a clique in G' is an independent set in G, and every subset W of V that is an independent set in G is a clique in G', the largest clique in G' **is** the largest independent set in G.

   b. Given G, we contruct a graph G' by adding one vertex v_e for each edge $e \in E$, and connecting this vertex to the endpoints of e. That is, if $e = \{x, y\}$ then we add the edges $\{v_e, x\}$ and $\{v_e, y\}$.

   Suppose that G has a vertex cover W with at most K vertices. The same vertices will be a dominating set in G': a vertex of G' that is not in W is either (1) a vertex from G that's the endpoint of an edge of G, whose other endpoint must be in W, or (2) one of the new vertices v_e, which is connected to both endpoints of the edge e of G, and hence to a member of W.

   Finally suppose that G' has a dominating set W with at most K vertices. We construct a vertex cover W' of G by choosing

   - every element of W that is a vertex of G.
   - one of the endpoints of the edge e, for every element v_e of W.

   Consider an edge e of G: the vertex v_e of G' was dominated, which means that either it was in W (and so W' will contain one of the endpoints of e), or one of its two neighbours (an endpoint of e) is in W'. This means W' is a vertex cover of G.

2. a. We introduce a boolean variable X_i that corresponds to each vertex v_i of the graph. A True value assigned to this variable will mean that v_i belongs to V_1; a False value will mean that v_i belongs to V_2.

   We then add the following clauses to the instance of satisfiability: for every pair v_i, v_j of vertices of the graph:

   - If the edge $\{v_i, v_j\}$ belongs to the graph, then we add the clause $X_i \vee X_j$ (the intended meaning is that it's not possible for both vertices to belong to V_2).
   - If the edge $\{v_i, v_j\}$ does not belong to the graph, then we add the clause $\overline{X_i} \vee \overline{X_j}$ (the intended meaning is that it's not possible for both vertices to belong to V_1).

b.

$$\overline{X_0} \vee \overline{X_1}$$
$$\overline{X_1} \vee \overline{X_2}$$
$$\overline{X_1} \vee \overline{X_4}$$
$$\overline{X_2} \vee \overline{X_3}$$

$$X_0 \vee X_2$$
$$X_0 \vee X_3$$
$$X_0 \vee X_4$$
$$X_1 \vee X_3$$
$$X_2 \vee X_4$$
$$X_3 \vee X_4$$

c. Suppose that G is a split graph. We assign the value True to every vertex of V_1 and the value False to every vertex of V_2. Consider a clause in the instance of SAT generated by our algorithm:

- If the clause is of the form $X_i \vee X_j$, then we know the edge $\{v_i, v_j\}$ belongs to the graph. At least one endpoint of this edge must belong to V_1 because no two vertices of V_2 are connected by an edge. Thus either X_1 or X_2 was assigned the value True, which means the clause is satisfied.
- If the clause is of the form $\overline{X_i} \vee \overline{X_j}$, then we know the edge $\{v_i, v_j\}$ does not belong to the graph. At least one endpoint of this edge must belong to V_2 because every two vertices of V_1 are connected by an edge. Thus either X_1 or X_2 was assigned the value False, which means that either $\overline{X_1}$ or $\overline{X_2}$ is True. Therefore the clause is also satisfied.

d. Suppose that there is a way to assign values to the variables in the instance of SAT that makes every clause TRUE. We define

- V_1 as the set of all vertices v_i for which X_i is True.
- V_2 as the set of all vertices v_i for which X_i is False.

Because every variable has a truth value, every vertex of the graph belongs to exactly one of V_1, V_2. Consider now two vertices v_i, v_j of the graph.

- If v_i and v_j both belong to V_1, then X_i and X_j were both assigned the value True. This means that the clause $\overline{X_i} \vee \overline{X_j}$, which evaluates to False, is **not** in the instance of SAT. Thus is it not the case that v_i and v_j are not connected by an edge. That is, v_i and v_j **are** connected by an edge in the graph.
- If v_i and v_j both belong to V_2, then X_i and X_j were both assigned the value False. This means that the clause $X_i \vee X_j$, which evaluates to False, is **not** in the instance of SAT. Thus v_i and v_j are not connected by an edge in the graph.