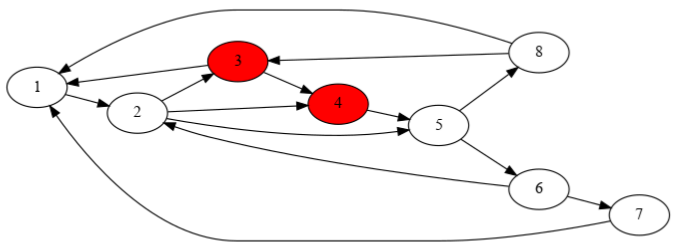
CPSC 320: NP-completeness and Feedback Vertex Set Solutions*

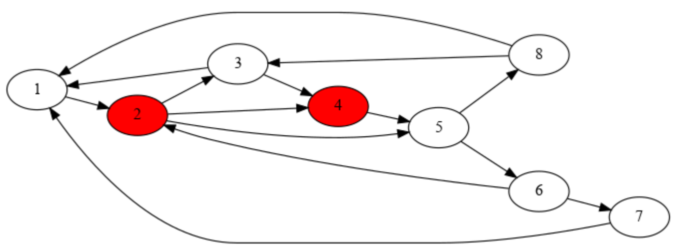
The *Feedback Vertex Set* problem is defined as follows:

> The input is a directed graph $G_{FVS} = (V_{FVS}, E_{FVS})$, and a positive integer $K_{FVS} \leq |V_{FVS}|$.
>
> The problem consists in answering the question "Is there a subset $V'_{FVS} \subseteq V_{FVS}$ such that $|V'_{FVS}| \leq K_{FVS}$ and every directed circuit in G_{FVS} includes at least one vertex from V'_{FVS}?"

For instance, in the following figure, vertices 3 and 4 are not a feedback vertex set, because neither vertex 3 nor vertex 4 are on the cycle 1, 2, 5, 6, 7, 1, or on the cycle 1, 2, 5, 8, 1.



However, vertices 2 and 4 are a feedback vertex set:



1. Prove that the *Feedback Vertex Set* problem belongs to NP by

   - Explaining what a "proof" that the answer to an instance of the problem is Yes might look like.
   - And describing an algorithm to verify such a proof.

   Hint: the algorithm from section 3.6 of your textbook may be helpful.

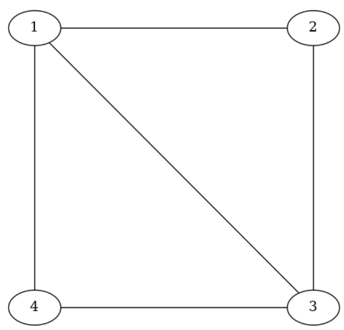
   SOLUTION: The "proof" will simply be the list of vertices in V'_{FVS}. If every directed circuit in G_{FVS} includes at least one vertex from V'_{FVS}, then G_{FVS} with the vertices of V'_{FVS} removed is a directed acyclic graph (DAG). So we can verify the proof by deleting the vertices in the list from G_{FVS}, and then attempting to find a topological ordering. If we find one, then V'_{FVS} was a feedback vertex set. This algorithm runs in $O(|V_{FVS}| + |E_{FVS}|)$, which is polynomial.

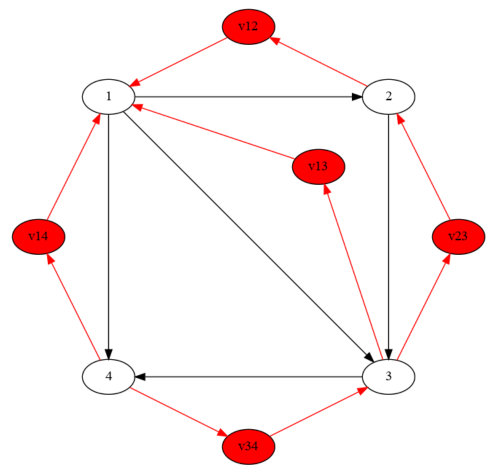
2. In this worksheet, you will prove that this problem is NP-complete using a reduction from the *Vertex Cover* problem. Recall that this problem, which you saw on assignment 1, is defined as follows:

   > The input is an undirected graph $G_{VC} = (V_{VC}, E_{VC})$ and an integer $K_{VC} \leq |V_{VC}|$.
   >
   > The problem consists in answering the question "Is there a subset $V'_{VC} \subseteq V_{VC}$ such that $|V'_{VC}| \leq K_{VC}$ and every edge in E_{VC} has at least one endpoint in V'_{VC}.

Describe a reduction from Vertex Cover to Feedback Vertex Set that you think might work. That is, show how to take an instance G_{VC}, K_{VC} of Vertex Cover and transform it into an instance G_{FVS}, K_{FVS} of Feedback Vertex Set. Hint: the reduction from Vertex Cover to Dominating Set you saw in question 3 of tutorial 3 might be a **very** good starting point.

SOLUTION: We construct G_{FVS} as follows: we start with G_{VC}, and then we add a direction arbitrarily to every edge of G_{VC}. Then, for every directed edge $e = (u, v)$, we add a new vertex $v_{u,v}$, and directed edges $(v, v_{u,v})$ and $(v_{u,v}, u)$ to form a directed triangle containing the three vertices. Finally we let $K_{FVS} = K_{VC}$. Here is an example: the instance of Vertex Cover is on the left, and the corresponding instance of Feedback Vertex Set is on the right.





3. Prove that if a graph G_{VC} has a vertex cover with at most K_{VC} elements, then the graph G_{FVS} constructed using your reduction has a feedback vertex set with at most K_{FVS} vertices.

   SOLUTION: Suppose the graph G_{VC} has a vertex cover V'_{VC} with at most K_{VC} elements. We claim the same subset V'_{FVS} of the vertices of G_{FVS} is a feedback vertex set. Indeed, consider a directed cycle inside G_{FVS}. If that cycle contains an edge that came from G_{VC}, then one of the endpoints of this edge belongs to V'_{VC}, which means the cycle contains a vertex in V'_{FVS}. If that cycle does not contain any edge from G_{VC}, then it must contain the pair of directed edges $(v, v_{u,v})$ and $(v_{u,v}, u)$ that were added based on an edge $e = \{u, v\}$ of G_{VC}. One of the endpoints of e belongs to V'_{VC}, which once again means the cycle contains a vertex in V'_{FVS}.

4. Finally, prove that if the graph G_{FVS} constructed using your reduction from an instance G_{VC}, K_{VC} of Vertex Cover has a feedback vertex set with at most K_{FVS} vertices, then G_{VC} has a vertex cover with at most K_{VC} elements.

   SOLUTION:

   Suppose the graph G_{FVS} constructed using our reduction from an instance G_{VC}, K_{VC} of Vertex Cover has a feedback vertex set V'_{FVS} with at most K_{FVS} vertices. We construct a vertex cover V'_{VC} of G_{VC} as follows:

   - Each element of V'_{FVS} that is a vertex of G_{VC} belongs to V'_{VC}.
   - For each element of V'_{FVS} that is the vertex $v_{u,v}$ added based on an edge $e = \{u, v\}$ of G_{VC}, we add u to the vertex cover.

   Now we need to show the set V'_{VC} we constructed is a vertex cover of G_{VC}. Consider an arbitrary edge $e = \{u, v\}$ of G_{VC}. The graph G_{FVS} contains a directed cycle $u, v, v_{u,v}, u$. Because every directed cycle of G_{FVS} contains an element of V'_{FVS}, this means one of u or v was added to V'_{VC}. This proves that every edge of G_{VC} has an endpoint in V'_{VC}, as required.