Chapter 4 - Radom Variables and Distributions

STAT 251

Lecture 8

Discrete Random Variables

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Chapter 4 - Learning Outcomes

- Notations
- Discrete Random Variables
- Continuous Random Variables
- Probability mass function (pmf)
- Probability density function (pdf)
- The Mean, the Variance and the Standard Deviation
- Cumulative Distribution Function (cdf)
- Max and Min of Independent Random Variables

Random Variable

- A random variable (rv) X is a function defined on the sample space S, assigning a number $x = X(w)$ to each outcome w in the sample space.

- Random variables are defined by uppercase letters such as $X, Y, \ldots$. If the random variable is X, then the lower case letter x represents a possible value of the random variable.

Example 1

A fair coin is flipped 3 times

equally likely, fair coin.

1/8 for any outcome.

- Let X be the number of heads in 3 flips
- Then the sample space S is

  $S = \{$ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$

Random Var: e.g. № of heads

$\hookrightarrow x = X(\omega)$ input.

$\omega = 0, 1, 2, 3$

Number of heads is limited to 3 possibilities

$x = 3/8$

$6 - 1 = 5$ 6, 1

$1 - 6 = -5$

$x = 0, 1, 2, 3.$

e.g. $X(HHH) = 3$ 3 head, $X(HTT) = 1$ 1 head

$5 \to -5$

Two types of random variables

- **Discrete random variable**
  Possible values are either finite set or an infinite sequence of values.

  infinite Nº of possible values, because continuous.

- **Continuous random variable**
  Possible values are in an interval on the number line or all numbers in a collection of intervals.
  For a continuous random variable X, $P(X = c) = 0$ for any possible value c.

  continuous

  any pt.

Any random variable whose possible values are 0 and 1 is called **Bernouli** random variable.

Probability Mass Function (pmf)

- Probability mass function (pmf) of a discrete random variable X, is defined as

  $f(x) = P(X = x)$ **for all possible values of** X

- $f(x)$ gives the probability of each possible value x of the rv X. It has the following properties.

  1. $f(x) \geq 0$ for all x in X $\longrightarrow$ can't have negative prob
  2. $\sum_x f(x) = 1$ → prob cannot be over 100%

d

Random var.

Example 2

Consider the example 1. Obtain the pmf of X, number of heads in 3 flips of a fair coin - $S = \{HHH, HTH, THH, HHT, TTH, THT, HTT, TTT\}$.

pmf of X can be given in a table

x	$f(x) = P(X = x)$
0	1/8
1	3/8
2	3/8
3	1/8.
	$\sum f(x) = 1$

or we can give the pmf of X as a function

y	$f(y) = P(Y = y)$
-10	0.3
-2	0.5
0	0.1
5	0.2
	$\sum f(x) = 1.1$

- Not PMF

Example 3

Y is a random variable with pmf $f(y)$ such that

y	-3	0	1	5
$f(y) = P(Y = y)$	k	0.4	$3k$	$2k$

Answer the following questions

$0.4 + 3k + 2k + k = 1$

$6k = 0.6$

$k = 0.1$

1. find k 0.1
2. $P(Y = 5)$ 0.2
3. $P(Y \geq 0)$ 0.9
4. $P(Y < 0)$ 0.4
5. $P(Y > 0.5)$ 0.5

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Cumulative distribution function (cdf) of a discrete rv

The cumulative distribution function (cdf) of a discrete random variable X with pmf $f(x)$ is defined as

$$F(x) = P(X \leq x) = \sum_{k \leq x} f(k), \qquad \text{for all real } x$$

The cumulative distribution function (cdf)) of a random variable X is denoted by $F(x)$ which is the probability that X is less than or equal to some x.

In many applications we work with $1 - F(x)$ instead of $F(x)$. Notice that $1 - F(x) = P(X > x)$ and therefore gives the probability that X will exceed the value x.

Example 4

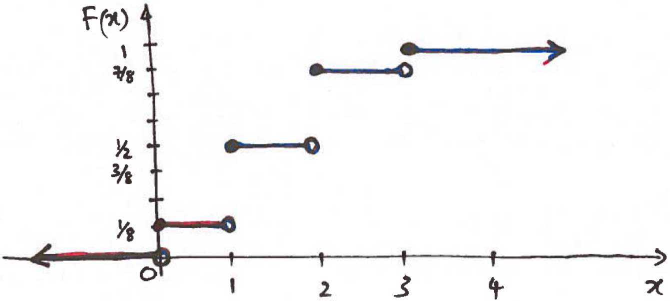
Consider example 1. The pmf of the rv X, number of heads in 3 flips of a fair coin

$$f(x) = \begin{cases} 1/8 & ; x = 0 \\ 3/8 & ; x = 1 \\ 3/8 & ; x = 2 \\ 1/8 & ; x = 3 \end{cases}$$

Then the cdf of X is

$$g(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ 1/8 & ; 0 \leq x < 1 \\ 4/8 & ; 1 \leq x < 2 \\ 7/8 & ; 2 \leq x < 3 \\ 1 & ; 3 \leq x. \end{cases}$$

Example 4 - Graph of cdf of X



Mean of a discrete random variable

- Mean/Expectation /Expected value of a discrete random variable X with pmf $f(x)$ is

$$\mu = E(X) = \sum_x x f(x)$$

- Expectation can be interpreted as the long run average of X over hypothetical repetitions of the experiment.

- Expected value of some function $g(X)$ corresponding to the random variable X with pmf $f(x)$ is

$$E[g(X)] = \sum_x g(x) f(x)$$

output probability

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Variance and Standard deviation of a discrete rv

Consider the discrete random variable X with pmf $f(x)$

- Variance of X is

$$\sigma^2 = \text{Var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

like $g(x)$

- standard deviation of X is

$$\sigma = \text{SD}(X) = \sqrt{\text{Var(X)}}$$

- Following formula is often used to calculate the variance

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$; will give the proof later

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Example 5

Calcule the mean & variance of the random variable X, the number of heads in 3 flips of a fair coin

From example 2, we know that the pmf of X can be given as follows

x	$f(x) = P(X = x)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$
	$\sum f(x) = 1$

Mean

$0 \times 1/8$
$+$
$1 \times 3/8 \quad = 3/8$
$2 \times 3/8 \quad = 6/8$
$+$
$3 \times 1/8 \quad 3/8$
$= 12/8$
$= 1.5$.

Variance.

$1/8\,(0-1.5)^2$
$+$
$3/8\,(1-1.5)^2$
$+$
$3/8\,(2-1.5)^2$
$+$
$1/8\,(3-1.5)^2$.
$= 0.75$.

Example 5 (contd.)

$$\text{Mean} = \mu = E(X) = \sum_x x f(x)$$

$$\text{Variance} = \sigma^2 = \text{Var}(X) = \sum_x (x-\mu)^2 f(x)$$

$= 3 -$

Example 5 (contd.)

Variance calculation : Method 2

$$
\begin{aligned}
E(X^2) &= \sum_x x^2 f(x) \\
&= (0 \times \tfrac{1}{8}) + (1^2 \times \tfrac{3}{8}) + (2^2 \times \tfrac{3}{8}) + (3^2 \times \tfrac{1}{8}) \\
&= 24/8 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
\text{Variance} = \sigma^2 = \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
&= 3 - (1.5)^2 \\
&= 3 - (1.5)^2 \\
&= 0.75
\end{aligned}
$$

It's easier to use the formula $\text{Var}(X) = E(X^2) - [E(X)]^2$ for hand calculation.

Before the next class ...

Visit the course website at canvas.ubc.ca

- Review the Lecture 8 and related sections in the text book
- Next class

  Chapter 4: Continuous Random Variables

$E(X)$ of fair die $= 3.5$.