Problem 1. **(8 points)**

The wait time (after a scheduled arrival time) in minutes for a train to arrive is Uniformly distributed over the interval $[0,12]$. You observe the wait time for the next 95 trains to arrive. Assume wait times are independent.

Part a) What is the approximate probability (to 2 decimal places) that the sum of the 95 wait times you observed is between 536 and 637? ___

Part b) What is the approximate probability (to 2 decimal places) that the average of the 95 wait times exceeds 6 minutes? ___

Part c) Find the probability (to 2 decimal places) that 92 or more of the 95 wait times exceed 1 minute. Please carry answers to at least 6 decimal places in intermediate steps. ___

Part d) Use the Normal approximation to the Binomial distribution (with continuity correction) to find the probability (to 2 decimal places) that 56 or more of the 95 wait times recorded exceed 5 minutes. ___

Solution: Let W be the wait time for a train to arrive. Then $W \sim \text{Uniform}[0,12]$ and we have:

$E(W) = \frac{0+12}{2} = 6 = \mu$
and
$\text{Var}(W) = \frac{(12-0)^2}{12} = 12 = \sigma^2$.

Part a)

By the Central Limit Theorem, $T = \sum_{i=1}^{100} W_i$ is approximately $\sim N(n\mu, n\sigma^2)$ where $n = 95$. Therefore,

$T \sim N(570, 1140)$ (approximately) and

$P(536 < T < 637) = P(\frac{536-570}{\sqrt{1140}} < Z < \frac{637-570}{\sqrt{1140}})$

$= P(-1.0070 < Z < 1.9844)$

$\approx 0.82.$

Part b)

By CLT, $\bar{W}$ approximately $\sim N(\mu, \frac{\sigma^2}{n})$ so we approximate

$W \sim N(6, 0.126315789473684)$ and compute

$P(\bar{W} > 6) = P(Z > \frac{6-6}{\sqrt{0.126315789473684}})$

$= P(Z > 0.0000)$

$\approx 0.5.$

Part c)

Let X be the count of the number of wait times (out of 100) that exceed 1 minute.

$[Aside : F_W(w) = \frac{w-0}{12} = w/12$

$P(W > 1) = 1 - F_W(1) = 1 - 1/12 = 11/12.$]

Then $X \sim \text{Binomial}(95, p = 11/12)$ so

$P(X \geq 92) = P(X = 92) + P(X = 93) + P(X = 94) + P(X = 95)$

$=^{95} C_{92}(\frac{11}{12})^{92}(\frac{1}{12})^3 +^{95} C_{93}(\frac{11}{12})^{93}(\frac{1}{12})^2 +^{95} C_{94}(\frac{11}{12})^{94}(\frac{1}{12})^1 + (\frac{11}{12})^{95}$

$\approx 0.04.$

Alternatively, it is possible to use the Normal approximation. Here, $np = 95 \times 11/12 \geq 5$ and $n(1-p) = 95 \times 1/12 \geq 5$ so it is okay to use the Normal approximation.
We take $X \sim N(np, npq)$ so

$X \sim N(87.0833, 7.2569)$

$P(X \geq 92) = P(X \geq 91.5)$ [continuity correction]

$= P(Z \geq \frac{91.5-87.0833}{\sqrt{7.2569}})$

$= P(Z \geq 1.6395)$

$\approx 0.05.$

Part d)

Use the Normal approximation to Binomial distribution.

$P(W > 5) = 1 - F_W(5) = 1 - 5/12 = 7/12.$

Let X be the count of the number of wait times (out of 95) that exceed 5 minutes.

$X \sim \text{Binomial}(n = 95, p = 7/12)$

$X \sim N(\underbrace{55.4167}_{np}, \underbrace{23.0903}_{npq})$

$P(X \geq 56) = P(X \geq 55.5)$ [continuity correction]

$= P(Z \geq \frac{55.5-55.4167}{\sqrt{23.0903}})$

$= P(Z \geq 0.0173)$

$\approx 0.49.$

Answer(s) submitted:

- `0.82`
- `0.5`
- `0.05`
- `0.49`

(correct)

Correct Answers:

- `0.82`
- `0.5`
- `0.045`
- `0.49`

Problem 2. **(2 points)**

An exam consists of 41 multiple-choice questions. Each question has a choice of five answers, only one of which is correct. For each correct answer, a candidate gets 1 mark, and no penalty is applied for getting an incorrect answer. A particular candidate answers each question purely by guess-work.

Using Normal approximation to Binomial distribution with continuity correction, what is the estimated probability this student obtains a score greater than or equal to 10? Please use R to obtain probabilities and keep at least 6 decimal places in intermediate steps.

- A. 0.5785
- B. 0.3059
- C. 0.6941
- D. 0.1846
- E. 0.3919

Solution: We have $n = 41$ and $p = 0.2$, and $X \sim \text{Bin}(41, 0.2)$.

By a rule of thumb, since $np > 5$ and $np(1-p) > 5$, we can approximate $X \sim \text{N}(np, npq)$;
that is, $X \sim \text{N}(8.2, 6.56)$.

We want to compute $P(X \geq 10)$. We must apply a continuity correction, which makes the probability we want to compute $P(X \geq 9.5)$. Transforming this to a standard normal distribution, we find that the quantity we want to compute is

$$P(X \geq 9.5) = P\left(Z \geq \frac{9.5 - np}{\sqrt{npq}}\right) \tag{1}$$

$$= P(Z \geq (9.5 - 8.2)/2.5612) \tag{2}$$

$$= 0.3059 \tag{3}$$

Answer(s) submitted:

- `B`

(correct)

Correct Answers:

- `B`

Problem 3. **(10 points)**

70% of the employees in a specialized department of a large software firm are computer science graduates. A project team is made up of 8 employees.

Part a) What is the probability to 3 decimal digits that all the project team members are computer science graduates? ___

Part b) What is the probability to 3 decimal digits that exactly 3 of the project team members are computer science graduates? ___

Part c) What is the most likely number of computer science graduates among the 8 project team members? Your answer should be an integer. If there are two possible answers, please select the smaller of the two integers. ___

Part d) There are 46 such projects running at the same time and each project team consists of 8 employees as described. On how many of the 46 project teams do you expect there to be exactly 3 computer science graduates? Give your answer to 1 decimal place. ___

Part e) I meet 50 employees at random. What is the probability that the fourth employee I meet is the first one who is a computer science graduate? Give your answer to 3 decimal places. ___

Part f) I meet 28 employees at random on a daily basis. What is the mean number of computer science graduates among the 28 that I meet? Give your answer to one decimal place. ___

Solution: Part a) We have $X \sim \text{Binomial}(8, 0.7)$. Hence $P(X = 8) = p^8 = 0.7^8 = 0.0576$.

Part b) $P(X = 3) = \binom{8}{3} \cdot p^3 \cdot (1-p)^{(8-3)} = \binom{8}{3} \cdot 0.7^3 \cdot (0.3)^5 = 0.0467$.

Part c) The trick is to find a relationship between $P(X = x)$ and $P(X = x-1)$.

$$\begin{aligned}
\frac{P(X=x)}{P(X=x-1)} &= \frac{\binom{8}{x} p^x (1-p)^{8-x}}{\binom{8}{x-1} p^{x-1}(1-p)^{8-(x-1)}} \\
&= \frac{\frac{8!}{(8-x)!x!} p^x (1-p)^{8-x}}{\frac{8!}{(8-(x-1))!(x-1)!} p^{x-1}(1-p)^{8-(x-1)}} \\
&= \frac{\frac{8!}{(8-x)!x(x-1)!} p^x (1-p)^{8-x}}{\frac{8!}{(8-(x-1))!(x-1)!} p^{x-1}(1-p)(1-p)^{8-x)}} \\
&= \frac{x-8+1}{x} \cdot \frac{p}{1-p}
\end{aligned}$$

We want to solve for x such that $\frac{P(X=x)}{P(X=x-1)} > 1$. That is:

$$\begin{aligned}
\frac{P(X=x)}{P(X=x-1)} &> 1 \\
\frac{x-8+1}{x} \cdot \frac{p}{1-p} &> 1
\end{aligned}$$

So

$$\begin{aligned}
p \cdot (8+1-x) &> x(1-p) \\
p \cdot (8+1) &> x \\
0.7 \cdot 9 &> x
\end{aligned}$$

So $x < 6.3$. Since x must be an integer, we have $x = 6$.

Part d) Let W be the number of project teams with exactly 3 computer science graduates.

$$W \sim \text{Binomial}(46, p = 0.0467)$$

Then

$$\begin{aligned}
E(W) &= n \cdot p \\
&= 46 \cdot 0.0467 \\
&= 2.15
\end{aligned}$$

Part e) Let S be a random variable counting the number of people I meet until I meet the first computer science graduate.

$$S \sim \text{Geometric}(p = 0.7)$$

So

$$\begin{aligned}
P(X = 4) &= (1-p)^{4-1} \cdot p \\
&= (1-0.7)^3 \cdot 0.7 \\
&= 0.0189
\end{aligned}$$

Part f) Let T be a random variable counting the number of computer science graduates I meet daily.

$$T \sim \text{Binomial}(28, p = 0.7)$$

So

$$\begin{aligned}
E(T) &= n \cdot p \\
&= 28 \cdot 0.7 \\
&= 19.6
\end{aligned}$$

Answer(s) submitted:

- `0.058`
- `0.047`
- `6`
- `2.2`
- `0.019`
- `19.6`

(correct)
Correct Answers:

- `0.0576`
- `0.0467`
- `6`
- `2.15`
- `0.0189`
- `19.6`