Chapter 8 - Statistical Modeling and Inference

STAT 251

Lecture 28

Hypothesis Testing about Mean - Examples
Type I and Type II Errors

Dr. Lasantha Premarathna

Chapter 8 - Learning Outcomes

- Point Estimation for μ and σ
- Bias of an estimator
- Confidence Interval for μ
- Testing of Hypotheses about μ
- One sample problems
- Two sample problems

Decisions and Types of errors in Hypothesis Testing

		Reality (population condition)	
		H_0 True	H_0 False
Decision	Reject H_0	Type I Error	Correct Decision
	Do Not Reject H_0	Correct Decision	Type II Error

Four possible outcomes

- Reject H_0
  - In reality null is false: we've made the **correct decision!**
  - In reality null is true: we've made an **error**
- Fail to reject H_0
  - In reality null is false: we've made an **error**
  - In reality null is true: we've made the **correct decision!**

Type I and Type II errors

- **Type I error** is rejecting H_0 when H_0 is true
- **Type II error** is not rejecting H_0 when H_0 is false

- What test is a good test

  A test that rarely makes type I and type II errors

- There are probabilities associated with each type of error

$$P(\text{Type I error}) = \alpha$$

$$P(\text{Type II error}) = \beta$$

Type I and Type II errors

- We can control the probability of type I error by our choice of the significance level, α
- It's difficult to control the probability of making type II error
- Statisticians avoid the risk of making a type II error by using "Do not reject H_0" and NOT "accept H_0"
- $1 - \beta$ referred to as the power of a test
$$1 - \beta = 1 - P(\text{Type II error}) = \text{power}$$
- We want the power to be large
- α, β are test properties, independent of data

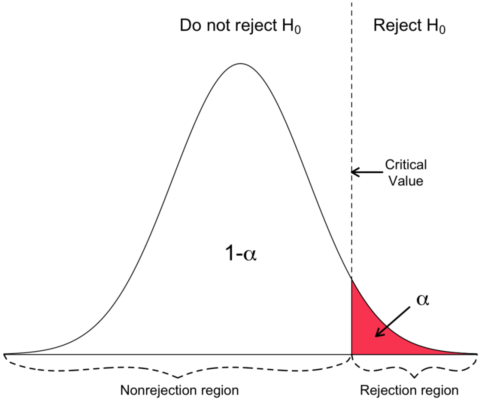
Power of a Test

Power is the probability of correctly rejecting the null hypothesis H_0, when H_0 is false

$$\begin{aligned} \text{power} &= P(\text{Reject } H_0 \text{ when } H_0 \text{ is false}) \\ &= 1 - \beta \end{aligned}$$

Type I error

Suppose $H_0 : \mu = 10$ vs. $H_1 : \mu > 10$ and $\alpha = 5\%$.
If our null hypothesis $H_0 : \mu = 10$ was actually true, what percent of the time would we wrongly reject H_0?



Type I and Type II errors

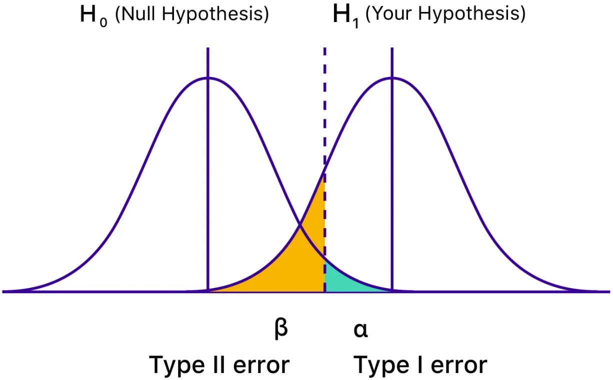
- The Type II error occurs when the null hypothesis is false, but we do not reject it.

- We control the Type I Error by specifying the significance level. However, the probability of Type II Error will depend on:
  - Effect size (i.e., the difference between the null hypothesis and reality)
  - The sample size
  - The probability of Type I Error

Type I and Type II errors

Suppose $H_0 : \mu = 10$ vs. $H_1 : \mu > 10$

The Type II error occurs when the null hypothesis is false, but we do not reject it.



Example: 3

A department store manager determines that the new billing system will be cost-effective only if the mean monthly account is more than \$170.

A random sample of 400 monthly accounts is drawn and the sample mean was found to be \$178. Assume that monthly accounts are approximately normally distributes with $\sigma = \$65$.

(a) Can we conclude that the new system will be cost-effective? Use $\alpha = 0.05$

(b) Describe type I and type II errors in the context of this problem

(c) Considering the test procedure, find the rejection region of $\bar{x}$.

(d) When $\mu = 180$, find the probability of type II error.

(e) Evaluate the power of the test when $\mu = 180$.

Example: 3 - Solution

(a) Can we conclude that the new system will be cost-effective? Use $\alpha = 0.05$

We want to determine whether the mean monthly account (μ) is more than \$170

Hypotheses

$$H_0 : \mu \leq 170 \qquad \Rightarrow \mu_0 = 170$$
$$H_a : \mu > 170 \qquad \text{this is a right tail test}$$

We know that $n = 400, \bar{x} = 178, \sigma = 65$

Test statistic

$$Z_{obs} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{178 - 170}{65/\sqrt{400}}$$
$$= 2.46$$

Example: 3 - Solution

(a) contd...

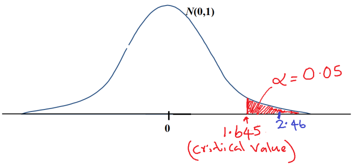
<u>Method 1: Critical value approach</u>

$\alpha = 0.05$, this is a right tail test

$Z_{obs} = 2.46 > Z_{0.05} = 1.645$

$\Rightarrow$ Reject H_0 at $\alpha = 0.05$

Conclusion: The new system will be cost effective at the significance level 0.05



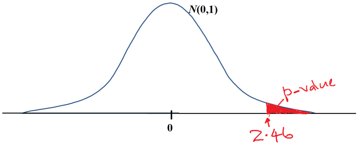
$Z_{obs} = 2.46$ is in the rejection region

Example: 3 - Solution

(a) contd...

Method 2: p-value approach

$$\begin{aligned}\text{p-value} &= P(\text{observing data as extreme or more extreme than what we observed, given } H_0 \text{ is true})\\ &= P(\bar{x} \geq 178 \text{ when } \mu = 170)\\ &= P(Z \geq 2.46) = 0.0069\end{aligned}$$



p-value $= 0.0069 < \alpha = 0.05$

$\Rightarrow$ Reject H_0 at $\alpha = 0.05$

Conclusion: The new system will be cost effective at the significance level 0.05

Example: 3 - Solution

(b) Describe type I and type II errors in the context of this problem

Type I error : Reject H_0 when H_0 is true

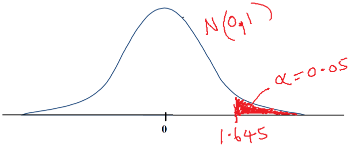
Conclude that the new billing system iscost effective (i.e. true mean > 170) when it really does not

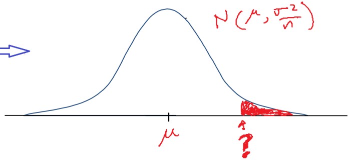
Type II error : Do not reject H_0 when H_0 false

Conclude that the new billing system is Not cost-effective when it really really cost-effective.

Example: 3 - Solution

(c) Considering the test procedure, find the rejection region of $\bar{x}$.





Reject when $Z > 1.645$

$$\Rightarrow Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > 1.645$$
$$\Rightarrow \bar{x} > 1.645\frac{\sigma}{\sqrt{n}} + \mu_0$$
$$\Rightarrow \bar{x} > 1.645\frac{65}{\sqrt{400}} + 170$$
$$\Rightarrow \bar{x} > 175.35$$

Therefore rejection region is $\bar{x} > 175.35$

Example: 3 - Solution

(d) When $\mu = 180$, find the probability of type II error.

$\mu = 180 \;\Leftarrow\;$ this belongs to H_a

When $\mu = \mu^* = 180$, $\bar{x}$ follows a normal distribution with mean 180 and standard deviation $65/\sqrt{400}$

$$\begin{aligned}
\beta &= P(\text{Type II error}) \\
&= P(\text{Do not reject } H_0 \text{ when } H_0 \text{ false}) \\
&= P(\bar{x} < 175.35 \text{ when } \mu = \mu^* = 180) \\
&= P\left(\frac{\bar{x} - \mu^*}{\sigma/\sqrt{n}} = \frac{175.35 - 180}{65/\sqrt{400}}\right) \\
&= P(Z < -1.43) \\
&= 0.0764
\end{aligned}$$

Example: 3 - Solution

(e) Evaluate the power of the test when $\mu = 180$

$$\begin{aligned} \text{Power} &= P(\text{Reject } H_0 \text{ when } H_0 \text{ false}) \\ &= 1 \text{ - } \beta \\ &= 1 \text{ - } 0.0764 \\ &= 0.9236 \end{aligned}$$

This is very useful test since it makes the correct decision 92.36% of the time when $\mu = 180$

Before the next class ...

Visit the course website at canvas.ubc.ca

- Review Lecture 28 and related sections in the text book
- Topic of next class: **Chapter 8: more on Hypothesis Testing about the Mean, Examples**