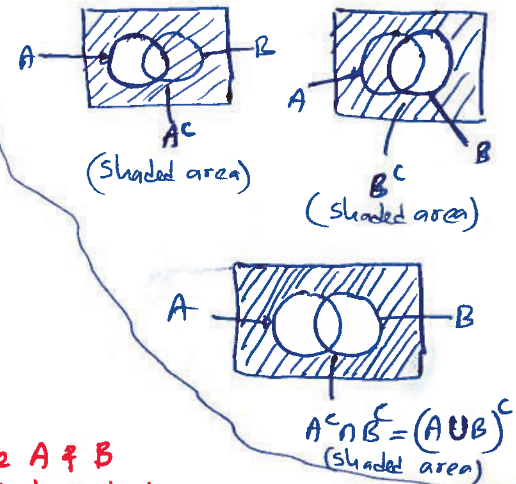
1. (7 marks)

(a) (3 marks) If A and B are independent events, show that A^C and B^C are also independent.

Since A and B are independent

$P(A \cap B) = P(A) \cdot P(B)$ ← ½



$$
\begin{aligned}
P(A^c \cap B^c) &= P[(A \cup B)^c] \\
&= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(A \cap B)] \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= [1 - P(A)] - P(B) + P(A) \cdot P(B) \quad \leftarrow \text{since } A \,\&\, B \text{ are independent} \\
&= [1 - P(A)] - P(B)[1 - P(A)] \\
&= [1 - P(A)][1 - P(B)]
\end{aligned}
$$

method 1.5 marks

$P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$. Therefore A^c & B^c are indepedent. ← 1

(b) (4 marks) A system consists of two components. The probability that the second component functions in a satisfactory manner during its design life is 0.9, the probability that at least one of the two components does so is 0.96, and the probability that both components do so is 0.75. What is the probability that the second component functions in a satisfactory manner if it is given that the first component functions in a satisfactory manner throughout its design life?

Let A : First component functions satisfactorily

B : second component functions satisfactorily

$P(B) = 0.9$

$P(A \cup B) = 0.96$

$P(A \cap B) = 0.75$

1

1

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.75}{0.81} = \underline{\underline{0.9259}} \leftarrow 1$$

$$
\left.
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
0.96 &= P(A) + 0.9 - 0.75
\end{aligned}
\right\} \ ½
$$

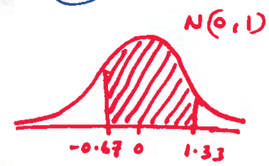
$\Rightarrow P(A) = 0.81$ ← ½

2. (9 marks) The breakdown voltage of a randomly chosen diode of a certain type is known to be normally distributed with mean value 40 V and standard deviation 1.5 V.

(a) What is the probability that the voltage of a single diode is between 39 and 42?

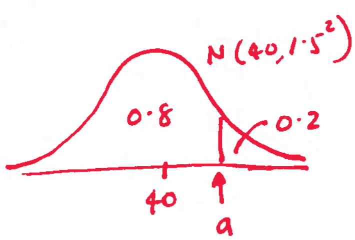
$X \equiv$ breakdown voltage

$\mu = 40, \ \sigma = 1.5$



$$
\begin{aligned}
P(39 < X < 42) &= P\left(\frac{39-40}{1.5} < Z < \frac{42-40}{1.5}\right) \leftarrow (1) \\
&= P(-0.67 < Z < 1.33) \\
&= P(Z < 1.33) - P(Z < -0.67) \leftarrow (1) \\
&= 0.9082 - (1 - 0.7486) \\
&= \underline{\underline{0.6568}} \leftarrow (1)
\end{aligned}
$$

(b) What value is such that only 20% of all diodes have voltage exceeding that value?



we desire the 80th percentile.

$$P\left(Z \le \frac{a-40}{1.5}\right) = 0.8 \leftarrow (1)$$

$$\Rightarrow \frac{a-40}{1.5} = 0.84 \leftarrow (1) \text{ table value}$$

$$\Rightarrow \underline{\underline{a = 41.26}} \leftarrow (1)$$

(c) If four diodes are independently selected, what is the probability that at least one has a voltage between 39 and 42?

Let $Y \equiv$ number of diodes with voltage between 39 and 42. out of 4 diodes

(0.5)

$\boxed{\text{Part(a)} \Rightarrow P(39 < X < 42) = 0.6568 = p}$

$Y \sim Bin(4, p) \leftarrow (0.5)$

$$
\left.
\begin{aligned}
P(Y \ge 1) &= 1 - P(Y = 0) \\
&= 1 - \binom{4}{0}(0.6568)^0 (1 - 0.6568)^4
\end{aligned}
\right\} \leftarrow (1) \text{ for procedure.}
$$

$$= \underline{\underline{0.9861}} \leftarrow (1)$$

[3] (9 marks) Suppose small aircraft arrive at a certain airport according to a Poisson process with rate $\alpha = 8$ per hour.

(a) (3 marks) What is the probability that at least 2 small aircraft arrive during a 1-hour period?

X: # of small aircraft arrive per hour.

$X \sim \text{Poisson}(8)$

$X \sim \text{Poisson}(\alpha)$

$\Rightarrow P(X=x) = \frac{\alpha^x e^{-\alpha}}{x!}$; $x = 0, 1, 2 \ldots$

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 1 - \frac{8^0 e^{-8}}{0!} - \frac{8^1 e^{-8}}{1!}$$
$$= 1 - e^{-8} - 8e^{-8} = 1 - e^{-8}(1+8)$$
$$= 0.99698 \approx 0.997$$

method – 2 marks

Answer – 1 mark.

(b) (3 marks) A small aircraft just arrived. What is the probability of waiting time greater than 20 minutes until the next small aircraft arrive at the airport?

Let Y = waiting time till next small aircraft (in hours)

$Y \sim \exp(8)$

1 mark

$Y \sim \exp(\lambda)$

$\Rightarrow f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$\Rightarrow F(x) = 1 - e^{-\lambda x}$

$F(x) = \int_0^x \lambda e^{-\lambda t}\, dt$

$= \lambda \left. \frac{e^{-\lambda t}}{-\lambda} \right|_0^x = -e^{-\lambda x} - (-1)$

$= 1 - e^{-\lambda x}$; $x > 0$

$$P(Y > 20 \text{ min}) = P(Y > \tfrac{1}{3} \text{ hours})$$
$$= 1 - P(Y < 1/3)$$
$$= 1 - F(1/3)$$
$$= 1 - [1 - e^{-8 \cdot \frac{1}{3}}]$$
$$= e^{-8/3}$$
$$= 0.0695$$

method 1 mark

Answer 1 mark

(c) (3 marks) What are the expected value and the standard deviation of the number of small aircraft that arrive during a 2.5-hour period?

W: # of aircraft arrive per 2.5 hour period.

$W \sim \text{Poisson}(8 \times 2.5) = \text{Poisson}(20)$

1 mark

$W \sim \text{poisson}(\lambda)$

$E[W] = \lambda$

$Var(W) = \lambda$

Expected value $= E[W] = 20$ ← 1 mark

$Var(W) = 20$

Standard deviation $= \sqrt{20} = 4.472$ ← 1 mark

4. A tall cup at Starbucks is designed to hold 355 mL of coffee. Suppose in reality, the amount of coffee poured into tall cups at Starbucks follow a normal distribution with mean 354 mL and standard deviation 1 mL. Assume the amount of coffee poured into tall cups is independent.

(a) Suppose you buy 3 randomly selected tall cups of coffee, what is the probability that exactly one of the 3 coffee cups is underfilled?

X = amount of coffee poured into tall cups $\sim N(354, 1^2)$

$P(\text{underfilled}) = P(X < 355) = P\left(Z < \frac{355 - 354}{1}\right) = P(Z < 1)$

≈ 0.84

(by 68-95-99.7 rule)

(Can also use $N(0,1)$ table)

Y = # of underfilled cups in 3

$Y \sim Bin(n = 3, p = 0.84)$

$P(Y=1) = \binom{3}{1} 0.84^1 (1-0.84)^{3-1} = 0.0645$

(b) Suppose you buy 40 randomly selected tall cups of coffee, what is the approximate probability that at most 26 are underfilled?

W = # underfilled cups in 40 $\sim Bin(n=40, p = 0.84)$

$np = 33.6 \geq 5, \quad n(1-p) = 6.4 \geq 5$

$W \overset{approx}{\sim} N(\mu = 33.6, \sigma^2 = np(1-p) = 5.376)$

$P(W \leq 26) = P\left(Z \leq \frac{26.5 - 33.6}{\sqrt{5.376}}\right)$ continuity correction

$= P(Z \leq -3.06)$

$= 1 - 0.9989 = 0.0011$