Chapter 4 - Random Variables and Distributions

STAT 251

Lecture 11

Maximum & Minimum of Independent Random Variables

Activity 2

and

More Examples

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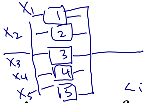
Chapter 4 - Learning Outcomes

- Notations
- Discrete Random Variables
- Continuous Random Variables
- Probability mass function (pmf)
- Probability density function (pdf)
- Cumulative Distribution Function (cdf)
- The Mean, the Variance and the Standard Deviation, Covariance
- Max and Min of Independent Random Variables

Maximum of Independent Random Variables

e.g.

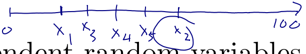


-until the maximum lifetime of 1 component

-At least 1 work

lifetime

<independent rand. var>



The maximum of a sequence of n independent random variables are of practical interest

We don't know when fail - random.

Max.

The **maximum** $V = max\{X_1, X_2, \cdots, X_n\}$ can be used to model

- The lifetime of a system on n components connected in Parallel

  X_i = lifetime of the i^{th} component

- The maximum flood level of a river in the next n years

  X_i = maximum flood level in the i^{th} year

Maximum of Independent Random Variables

Given that n independent random variables are $X_1, X_2, \cdots, X_n$. X_i's are identically distributed with the pdf $f_X(x)$ and cdf $F_X(x)$.

Find the pdf of the maximum

$$V = max\{X_1, X_2, \cdots, X_n\}$$

first, find the cdf of V, $F_V(v)$

$$F_V(v) = P(V \leq v)$$

integrate pdf.

$$= P(X_1 \leq v, X_2 \leq v, \cdots, X_n \leq v)$$; if v is the maximum, then each of $X_1, X_2, \cdots, X_n$ are $\leq v$

$$= P(X_1 \leq v)P(X_2 \leq v) \cdots P(X_n \leq v)$$; X_i's are independent

$$= F_{X_1}(v)F_{X_2}(v) \cdots F_{X_n}(v)$$

indicate reason.

$$F_V(v) = [F_X(v)]^n$$; X_i's are identically distributed

same.

therefore $F_{X_1}(v) = F_{X_2}(v) = \cdots = F_{X_n}(v) = F_X(v)$

Maximum of Independent Random Variables

$$V = max\{X_1, X_2, \cdots, X_n\}$$

pdf of V is $f_V(v)$

$$\begin{aligned}
f_V(v) &= F_V'(v) \\
&= \frac{d}{dv} F_V(v) \\
&= \frac{d}{dv} \left[F_X(v)\right]^n \qquad \underline{\text{by substitution.}} \\
&= n \left[F_X(v)\right]^{n-1} \frac{d}{dv} F_X(v) \\
f_V(v) &= n \left[F_X(v)\right]^{n-1} f_X(v)
\end{aligned}$$

Minimum of Independent Random Variables

The minimum of a sequence of n independent random variables are of practical interest

The **minimum** $U = min\{X_1, X_2, \cdots, X_n\}$ can be used to model

- The lifetime of a system on n components connected in series
  X_i = lifetime of the i^{th} component

- The minimum flood level of a river in the next n years
  X_i = minimum flood level in the i^{th} year

Minimum of Independent Random Variables

Given that n independent random variables are $X_1, X_2, \cdots, X_n$. X_i's are identically distributed with the pdf $f_X(x)$ and cdf $F_X(x)$.
Find the pdf of the minimum

$$U = min\{X_1, X_2, \cdots, X_n\}$$

first, find the cdf of U, $F_U(u)$

$$\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= 1 - P(U > u) \\
&= 1 - P(X_1 > u, X_2 > u, \cdots, X_n > u) \quad \text{; if } u \text{ is the minimum,} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{then each of } X_1, X_2, \cdots, X_n \text{ are } > u \\
&= 1 - P(X_1 > u)P(X_2 > u) \cdots P(X_n > u) \ \text{; } X_i\text{'s are independent} \\
&= 1 - [1 - F_{X_1}(u)][1 - F_{X_2}(u)] \cdots [1 - F_{X_n}(u)] \\
F_U(u) &= 1 - [1 - F_X(u)]^n \quad \text{; } X_i\text{'s are identically distributed}
\end{aligned}$$

Minimum of Independent Random Variables

$$U = min\{X_1, X_2, \cdots, X_n\}$$

pdf of U is $f_U(u)$

$$\begin{aligned}
f_U(u) &= F_U^{'}(u) \\
&= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} \left\{1 - \left[1 - F_X(u)\right]^n\right\} \\
&= 0 - n\left[1 - F_X(u)\right]^{n-1} \frac{d}{du}(-F_X(u)) \\
f_U(u) &= n\left[1 - F_X(u)\right]^{n-1} f_X(u) \qquad ; \frac{d}{du}(-F_X(u)) = -\frac{d}{du} F_X(u) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad = -f_X(u)
\end{aligned}$$

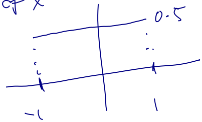
Example 9

Let X is uniformly distributed on the interval [-1, 1].

That is, $X \sim U[-1, 1]$.

Find the probability distribution of $Y = e^X$

pdf of x



pdf: $\begin{cases} 0.5 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

cdf : $\int_{-1}^{k} 0.5 \, dx = 0.5(x)\Big|_{-1}^{k}$

$= -0.5(-1) + 0.5k =$

$= 0.5(k+1)$

cdf of Y is

$P(Y \le y) = P(e^X \le y)$

$P(X \le \ln y)$ plug $\frac{1}{2}(\ln y + 1)$ //

Pdf of y

$\hookrightarrow \frac{d}{dy}$ cdf $\Rightarrow \frac{1}{2}\left(\frac{1}{y}\right) = \frac{1}{2y}$

Example 10

cdf of $x = \int_{-3}^{K} \frac{x^2}{18}\,dx$

$= \frac{x^3}{18 \times 3}\Big|_{-3}^{K} = \cdots$

Suppose pdf of X is

$$f(x) = \begin{cases} \dfrac{x^2}{18} & ; -3 \leq x \leq 3 \\ 0 & ; \text{otherwise} \end{cases}$$

The random variable Y is defined as $Y = X^2$

(i) Find the pdf of Y and

(ii) Calculate $P(Y > 4)$

Example 11

Suppose that $X_1,\ X_2,\ \ldots,\ X_n$ are independent random variables each having $Exp(\lambda)$ distribution. Obtain the probability density function of

(i) $Y = max\{X_1,\ X_2,\ \ldots,\ X_n\}$

(ii) $W = min\{X_1,\ X_2,\ \ldots,\ X_n\}$

Example 12

$\text{pdf } X_1 = \begin{cases} 1/20 & 20 \le x_1 \le 40 \\ 0 & \text{otherwise} \end{cases}$

$\text{cdf } X_1 = \int_{20}^{k} 1/20 \, dt = \frac{1}{20} t \Big|_{20}^{k} = \frac{k}{20} - 1 =$

$\text{pdf } X_2 = \begin{cases} 1/15 & 35 \le x_2 \le 50. \\ 0 & . \end{cases}$

$X_2 = \int_{35}^{k} 1/15 \, dt = 1/15 \, t \Big|_{35}^{k} = \frac{k}{5} - \frac{7}{3}.$

Suppose X_1 and X_2 are two independent random variables, where

$$X_1 \sim U(20, 40) \text{ and } X_2 \sim U(35, 50)$$

Let $Y = max\{X_1, \, X_2\}$. Find the pdf of Y

$Y = \max\{X_1, X_2\}$

cdf Y

$F_Y(y) = P(Y \le y) = P(X_1 \le y) \, P(X_2 \le y)$

$= \left(\frac{y-20}{20}\right)\left(\frac{y-35}{15}\right) \quad 35 \le y \le 40.$

PDF

$\begin{cases} 0 ; & y < 35 \\ \left(\frac{y-20}{20}\right)\left(\frac{y-35}{15}\right) ; & 35 \le 40 \\ 1/15 & 40 \le y \le 50 \\ 0 & y > 50. \end{cases}$

Activity 2

Please refer the Activity 2 (this will open one hour before the class)

Before the next class ...

Visit the course website at canvas.ubc.ca

- Review the lecture 11 and related sections in the text book
- Review the Activity 2 (Chapter 4) solutions and Examples and Answers
- Topic of next Lecture: Activities/Examples related to Chapter 4
- **Complete the Chapter-4 pre-activity worksheet before the next class. Also complete Activity 3**
  - You can find the pre-activity worksheet under the next lecture material (Lecture 12)
- **Topic of next class:** Wednesday, October 12 - Chapter 5: Normal Distribution