Chapter 6 - Some Probability Models

STAT 251

Lecture 17

Poisson Distribution

Examples

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Chapter 6 - Learning Outcomes

- Bernoulli Experiments
- Bernoulli and Binomial Random Variables
- Geometric Distribution
- Poisson process and associated random variables
- Poisson Approximation to the Binomial
- Heuristic Derivation of the Poisson and Exponential Distributions

Poisson Distribution

- Poisson distribution is a discrete distribution
- It expresses the probability of a given number of events occurring in a fixed interval of time/space if those events occur with a known constant rate and independently.
- A random Variable X: the number of occurences in a given interval/space has a poison distribution with $\lambda > 0$, if it has the pmf

$$X \sim Poisson(\lambda)$$

$$\underline{pmf} : P(X = x) = \frac{\lambda^x e^{-\lambda}}{x\,!} \qquad ; x = 0, 1, 2, 3, \cdots$$

where λ is the rate of occurrences

- The Poisson distribution is especially good at modeling rare events.

Poisson Distribution

Verify that $f(x) = P(X = x) = \dfrac{\lambda^x e^{-\lambda}}{x\,!} \quad ; x = 0, 1, 2, 3, \cdots$ is a pmf

(1) $f(x) \geq 0 for x = 0, 1, 2, 3, \cdots$

$$\frac{\lambda^x e^{-\lambda}}{x\,!} \geq 0 \text{ because } \lambda > 0$$

(2) showing $\sum_x f(x) = 1$

$$\sum_x f(x) = \sum_x P(X = x) = \sum_x \frac{\lambda^x e^{-\lambda}}{x\,!} = e^{-\lambda} \sum_x \frac{\lambda^x}{x\,!} = e^{-\lambda} e^{\lambda} = 1$$

** Check your calculus book for the result

$$e^k = 1 + k + \frac{k^2}{2\,!} + \frac{k^3}{3\,!} + \cdots = \sum_{x=0}^{\infty} \frac{k^x}{x\,!}$$

Mean and Variance of Poisson random variable

$$X \sim Poisson(\lambda)$$

$$\text{mean} = \mu = E(X) = \lambda$$

$$\text{variance} = \sigma^2 = Var(X) = \lambda$$

** derive the mean and the variance for Poisson random variable.

Poisson Process

Properties of a Poisson Process

- The number of occurrences of an event on non-overlapping intervals are independent.
- The number of occurrences of the event in an interval is proportional to the size of the interval.
- The probability of an event withing a certain interval does not change over different intervals.
- Events cannot occur simultaneously

Applications

- Failure of a machine in one month
- Number of typing errors on a page
- Number of phone calls arrive at a telephone switchboard in 3o minutes

Poisson process and associated random variables

Exponential Distribution:

- Let the random Variable X: the number of occurrences in a given interval has a Poisson distribution

$$X \sim Poisson(\lambda)$$

- Then let T be the time between two consecutive occurrences of events. We also can consider T as the waiting time until the first event. Then T is a continuous random variable and it has exponential density.

$$T \sim Exp(\lambda)$$

$$\text{pdf of } T\text{: } f(t) = \lambda e^{-\lambda t} \qquad ; t \geq 0$$

$$\text{cdf of } T\text{: } F(t) = 1 - e^{-\lambda t} \qquad ; t \geq 0$$

$$E(T) = \frac{1}{\lambda} \qquad Var(T) = \frac{1}{\lambda^2}$$

Example: 4

A switchboard receives calls of a rate at 3 per minute during a busy period. Let X_t denote the number of calls in t minutes during a busy period. Assuming the Poisson process assumptions are reasonable, calculated the probability of receiving more than 3 calls in 5 minutes interval during a busy period.

$$P(x \leq 3) = 1 - e^{-\lambda t}$$

$$= 1 - e^{-5/3}$$

$$P(x > 3) = e^{-5/3} = 0.188$$

Poisson Approximation to the Binomial

Let $X \sim Bin(n, p)$ be a Binomial random variable. If n is large ($n \geq 20$) and p is small ($np < 5$), then we can use a Poisson random variable with rate $\lambda = np$ to approximate the probabilistic behavior of the Binomial random variable X

In other words

$$Bin(n, p) \approx Poisson(np) \qquad \text{for all } x = 0, 1, 2, \cdots, n$$

Example: 5

If 1% of the output from a machine is defective. Then what is the probability that the more than 3 are defective in a random sample of 100?

(a) Calculate the exact probability using the Binomial distribution

(b) Use the Poisson approximation to the Binomial to calculate the probability.

Binom $= 1 - \binom{100}{3}(0.01)^3(0.99)^{97} + \binom{100}{2}(0.01)^2(0.99)^{98} \cdots \binom{100}{0}(0.01)^0(0.99)^{100}$

$= 1 - 0.9816$

$= 0.0184$ //

Poisson

$1 - P(X \leq 3)$

$1 - \{P(X=0) + P(X=1) \cdots P(X=3)\}$

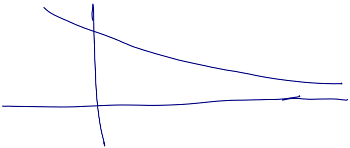
$= 1 - \left\{ \frac{1^0 e^{-1}}{0!} + \frac{1 e^{-1}}{1!} + \frac{1^2 e^{-1}}{2!} + \frac{1^3 e^{-1}}{3!} \right\}$

$= 0.0189$ //

Example: 6

$\mu = \frac{1}{\lambda}$

$\lambda = \frac{1}{3}$



The length of time to be served in a cafeteria is exponentially distributed with mean 3 minutes. What is the probability that a person is served in less than 1 minute at least five of the next six days?

pdf: $f(x) = \frac{1}{3} e^{-\frac{1}{3}t}$.

cdf ~ $1 - e^{-\frac{1}{3}t} \Big|_0^1 = \cancel{1} - e^{-\frac{1}{3}} - \cancel{1} + e^{-\frac{1}{3} \times 0}$

$= 1 - e^{-1/3} = 0.283$

$\binom{6}{5} 0.283^5 (0.716)^1$

$+$

$\binom{6}{6} 0.283^6 \times 1 = 0.008$ //

Before the next class ...

Visit the course website at canvas.ubc.ca

- Review Lecture 17 and related sections in the text book
- Topic of next class:
  - Chapter 6: Poisson Distribution, More Examples/Activity