1 Gödel's First Incompleteness Theorem

No consistent system of axioms for arithmetic is complete

Completeness is the property of a system of axioms which means for all sentences P, there is a proof for either P or ¬P. It is worth noting that complete systems may not be consistent, so they can have proofs for both P and ¬P. Godel's theorem states that for any axiom system capable of representing arithmetic, if it is consistent, it cannot be complete. If we assume a given system M to be complete, then we immediately know that it is not consistent. Conversely, if we assume M is consistent, then we know it cannot be complete. The proof for this theorem can be broken down into a few 'simple' steps as follows:

1. Build a formal system capable of representing arithmetic
2. Develop an identification system which assigns unique IDs for each sentence in the system. This identification system needs to be capable of self reference.
3. Create a syntactically correct formula F that represents the idea *"This sentence is unproveable."* This may be accomplished using the Diagonal lemma.
4. Consider the following truth values for the sentence F:
   (a) F is true. If F is true, then the formal system is known to be incomplete because is is unable to prove the true sentence F.
   (b) F is false. If F is false, then there exists a proof for F. If F can be proven, yet is false, then the system is not sound.

   By the law of excluded middle, we can conclude that any system capable of self reference and arithmetic is either incomplete or unsound.

2 Gödel's Second Incompleteness Theorem

No sound system of axioms for arithmetic
can prove its own consistency

In other words, this theorem states that there is no way for a consistent formal system of arithmetic to prove itself as being consistent. Let us consider a consistent formal system of arithmetic S and show why a proof for its consistency leads to a contradiction.

1. Let us create a syntactically correct sentence in S named g, which states *"the sentence g is unproveable"*.
2. The consistency of system S tells us that if there is a proof for a given sentence, then that sentence must be true. Therefore, if there is a proof for g, then g must be true. In other words, if ¬g is true (there is a proof for g), then g is true. Therefore, $\neg g \to g$. However, by nature of the consistency of S, we know that ¬ g ∧ g is not derivable.
3. Let us now assume that we have some proof for the consistency of S, represented by Γ. If we can prove that S is consistent, and we know ¬g leads to a contradiction, then we can conclude g by indirect proof.
4. We now have a proof for g, which due to the consistency of S, we know to be impossible.

Therefore, we can show by indirect proof that Γ cannot exist. If Γ cannot exist, then there can be no self-proof for the consistency of a consistent formal system of arithmetic.