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ASSIGNMENT 1

CS F214 Logic in Computer Science

<u>Group Members</u>

1. Dhruv Maheshwari - 2019A7PS0020U
2. Kshitiz Bansal - 2019A7PS0012U
3. Vignesh Bhatt - 2019A7PS0088U
4. Akshit Kacholiya -2019A7PS0067U
5. Aaradhya Chauhan -2019A7PS0023U

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An Introduction to Natural Deduction

Natural Deduction (ND) is a proof calculus in logic and proof theory. ND is a way to prove both real life and hypothetical problems. In ND, logical reasoning can be expressed by inference rules which is closely related to the natural way of judging and reasoning.

ND is composed of simple and self-evident inference rules based upon methods of proof and traditional ways of reasoning that have been applied since ages in deductive practice. An alternative to Hilbert-style axiomatic systems were challenged by the 1st formal ND systems which were developed by G. Gentzen and S. Jaśkowski in 1930s. Gentzen introduced a format of ND useful for theoretical investigations of the structure of proofs. A format of ND for practical purposes of proof search was developed by Jaśkowski.

In contrast to proofs in axiomatic systems, proofs in ND systems are based on the use of assumptions which are introduced freely and may be discarded as per situation and problem solving. It is easy to compose and decompose formulas in solutions as ND systems use many inference rules of simple language.

Finally, ND systems allow for the application of different proof-search strategies. Proofs in ND systems tend to be much shorter and easier to construct than in axiomatic or tableau systems.

ND is supposed to clarify the form and structure of our logical arguments, describe the appropriate means of justifying a conclusion with given set of premises and explain the sense in which the rules we use are valid.

Few rules:

The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\to$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\ \to\mathrm{i}$	$\dfrac{\phi \quad \phi \to \psi}{\psi}\ \to\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules:

$$\dfrac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{PBC} \qquad\qquad \dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Story

On a hot Sunday afternoon, the jailer decides to challenge his 3 wise prisoners. He keeps a condition that who so ever will win the challenge, he would be set free.

The challenge is that there are 2 black caps and 3 white caps. The jailor will randomly put the caps on each prisoner's head. The prisoners would be asked to stand in a line as per their heights and guess their own cap's colour. The first one to do so would be set free. They can neither discuss among themselves nor can they use any unfair means. However, they can listen to each other's answers and use it as a clue to guess their own hat's colour.

Is there a way that at least a prisoner is set free?

Solution using Natural Deduction

Let C be the tallest man followed by B and A. They are standing as shown in the figure. Therefore, C gets the chance to answer first.

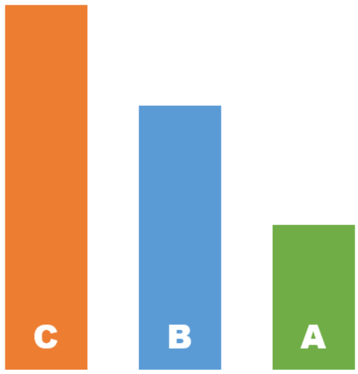

Fig 1: Prisoners standing

Let the atomic statements be as follows:

- p: A wears black
- q: B wears black
- r: C wears black

The table below summarizes the winning conditions of the 3 prisoners.

Statements	Explanation
From C's perspective	
$(p \land q) \rightarrow \neg r$	As soon as C sees 2 black caps, he can immediately tell that he's wearing a white cap as there are at most 2 black caps.
$(p \land \neg q) \rightarrow (r \lor \neg r)$	In all these conditions, if C sees at least one white cap, he is unable to determine the colour of his own cap. He can wear either black or white cap.
$(\neg p \land q) \rightarrow (r \lor \neg r)$	
$(\neg p \land \neg q) \rightarrow (r \lor \neg r)$	
From B's perspective	
$(r \lor \neg r) \land p \rightarrow \neg q$	As C is confused as which colour cap he's wearing, B would understand that A and B are

	either wearing same colour caps or different coloured caps (complement to each other). Seeing that A is wearing a black cap, B would infer he's wearing a white cap. If B would have been wearing a black instead, C would have answered in the previous case.
$(r \vee \neg r) \wedge \neg p \rightarrow (q \vee \neg q)$	As C is confused and B sees a white cap of A, he's unable to deduce anything.
From A's perspective	
$(r \vee \neg r) \wedge (q \vee \neg q) \rightarrow \neg p$	Upon lack of response from both C and B, A would understand that he's wearing a white cap because of which C and B are unable answer.

Let's apply these cases:

<u>Case 1: When A wears a black cap</u>

- B wears a black cap

  $p, q, (p \wedge q) \rightarrow \neg r \vdash \neg r$

  1) p — premise
  2) q — premise
  3) $(p \wedge q) \rightarrow \neg r$ — premise
  4) $(p \wedge q)$ — $\wedge_i$ 1,2
  5) $\neg r$ — $\rightarrow e$ 4,3

  C deduces that he is wearing a white cap and is set free.

- B wears a white cap

  $p, \neg q, (p \wedge \neg q) \rightarrow (r \vee \neg r) \vdash (r \vee \neg r)$

  1) p — premise
  2) $\neg q$ — premise
  3) $(p \wedge \neg q) \rightarrow (r \vee \neg r)$ — premise
  4) $(p \wedge \neg q)$ — $\wedge_i$ 1,2
  5) $(r \vee \neg r)$ — $\rightarrow e$ 4,3

  C is unsure of his cap colour. The chance passes on to B.

  $(r \vee \neg r), p, (r \vee \neg r) \wedge p \rightarrow \neg q \vdash \neg q$

  1) $(r \vee \neg r)$ — premise
  2) p — premise

3) $(r \lor \neg r) \land p \rightarrow \neg q$ premise
4) $(r \lor \neg r) \land p$ $\land_i$ 1,2
5) $\neg q$ $\rightarrow e$ 4,3

B is able to guess correctly that he is wearing a white colour hat and is set free.

Case 2: When A wears a white cap

- B wears a black cap

$\neg p, q, (\neg p \land q) \rightarrow (r \lor \neg r) \vdash (r \lor \neg r)$

1) $\neg p$ premise
2) q premise
3) $(\neg p \land q) \rightarrow (r \lor \neg r)$ premise
4) $(\neg p \land q)$ $\land_i$ 1,2
5) $(r \lor \neg r)$ $\rightarrow e$ 4,3

C is not able to deduce the colour of his cap as per the discussion in the table. The chance passes onto B.

$(r \lor \neg r), \neg p, (r \lor \neg r) \land \neg p \rightarrow (q \lor \neg q) \vdash (q \lor \neg q)$

1) $(r \lor \neg r)$ premise
2) $\neg p$ premise
3) $(r \lor \neg r) \land \neg p \rightarrow (q \lor \neg q)$ premise
4) $(r \lor \neg r) \land \neg p$ $\land_i$ 1,2
5) $(q \lor \neg q)$ $\rightarrow e$ 4,3

B is also not able to deduce the colour. The chance passes onto A.

$(r \lor \neg r), (q \lor \neg q), (r \lor \neg r) \land (q \lor \neg q) \rightarrow \neg p \vdash \neg p$

1) $(r \lor \neg r)$ premise
2) $(q \lor \neg q)$ premise
3) $(r \lor \neg r) \land (q \lor \neg q) \rightarrow \neg p$ premise
4) $(r \lor \neg r) \land (q \lor \neg q)$ $\land_i$ 1,2
5) $\neg p$ $\rightarrow e$ 4,3

A is able to guess the colour correctly and is set free.

- B wears a white cap

$\neg p, \neg q, (\neg p \land \neg q) \rightarrow (r \lor \neg r) \vdash (r \lor \neg r)$

1) $\neg p$ premise

2) $\neg q$ premise
3) $(\neg p \land \neg q) \rightarrow (r \lor \neg r)$ premise
4) $(\neg p \land \neg q)$ $\land_i$ 1,2
5) $(r \lor \neg r)$ $\rightarrow$e 4,3

C is not able to deduce the colour of his cap as per the discussion in the table. The chance passes onto B.

$(r \lor \neg r), \neg p, (r \lor \neg r) \land \neg p \rightarrow (q \lor \neg q) \vdash (q \lor \neg q)$

1) $(r \lor \neg r)$ premise
2) $\neg p$ premise
3) $(r \lor \neg r) \land \neg p \rightarrow (q \lor \neg q)$ premise
4) $(r \lor \neg r) \land \neg p$ $\land_i$ 1,2
5) $(q \lor \neg q)$ $\rightarrow$e 4,3

B is also not able to deduce the colour. The chance passes onto A.

$(r \lor \neg r), (q \lor \neg q), (r \lor \neg r) \land (q \lor \neg q) \rightarrow \neg p \vdash \neg p$

1) $(r \lor \neg r)$ premise
2) $(q \lor \neg q)$ premise
3) $(r \lor \neg r) \land (q \lor \neg q) \rightarrow \neg p$ premise
4) $(r \lor \neg r) \land (q \lor \neg q)$ $\land_i$ 1,2
5) $\neg p$ $\rightarrow$e 4,3

A is able to guess the colour correctly and is set free.

Limitations

- If there had been 3 caps instead of 2, the negation ¬ would have failed.
  Let's say we have 3 caps, white, black and orange.
  If p: white cap, ¬p can mean either black or orange cap.
  The decision is inconclusive.

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