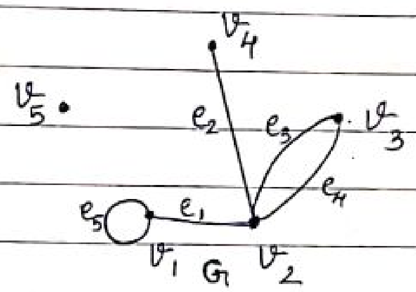
Graph Theory.

- A graph is an ordered pair (V, E)
  where V = non-empty set of vertices.
  E = set of edges.
- A vertex is a point / node. An edge is a curve / line joining two vertices. A loop is an edge joining a vertex to itself. Parallel / Multiple edges means more than one edges joining the same pair of vertices.
  eg: $G = (V, E)$
  where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$



Here, $e_1 = v_1v_2$ (edge e_1 joins vertices v_1 & v_2)
$e_2 = v_2v_4$
$e_3 = e_4 = v_2v_3$ → multiple edges.
$e_5 = v_1v_1$ → loop

- A graph without loops and multiple edges is called a simple graph. ~~A graph with loop~~
  A graph which is not simple is called multigraph.

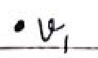
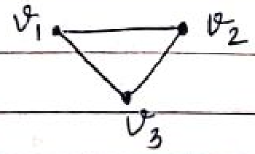
- If edge e is formed by vertices v_1 and v_2 then v_1, v_2 are called adjacent vertices.
  If vertex v is common between ~~vert~~ edges e_1 and e_2 then e_1, e_2 are called adjacent edges.
  Eg: In above example,
  v_1, v_2 are adjacent vertices while v_1, v_4 are not adjacent vertices.
  e_1, e_2 are adjacent edges but e_2, e_5 are not.

- Degree of a vertex v is the number of edges starting or ending at the vertex v, written as $d(v)$.
  Eg: In graph G of page 1,
  $d(v_1) = 3$ (as loop is counted 2 times)
  $d(v_2) = 4$
  $d(v_3) = 2$
  $d(v_4) = 1$ .... $\therefore v_4$ is called pendant vertex.
  $d(v_5) = 0$ .... $\therefore v_5$ is called isolated vertex.

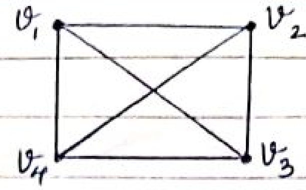
- Handshaking Theorem:
  The sum of degrees of all vertices in a graph is twice the number of edges in that graph
  i.e. $\Sigma d(v) = 2E$
  In graph G of page 1:
  $$\Sigma d(v) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$$
  $$= 3 + 4 + 2 + 1 + 0$$
  $$= 10.$$
  E = no. of edges in G = 5
  $\therefore \Sigma d(v) = 10 = 2 \times 5 = 2E$.

- Types of graph:

1. Complete Graph
   A simple graph in which every vertex is adjacent with every other vertex.
   ie. A simple graph on n vertices in which degree of every vertex is $n-1$.
   It is denoted by K_n.

Eg:		no. of edges
K_1		0
K_2		1
K_3		3

K_4



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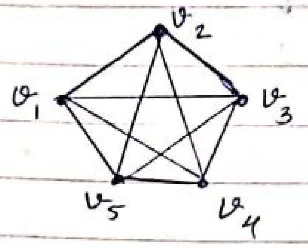
K_5



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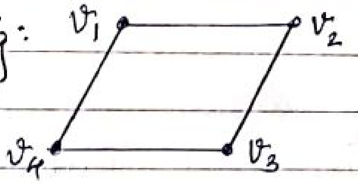
Note: The total no. of edges in $K_n = \frac{n(n-1)}{2}$

2. Regular Graph

A graph in which every evertex has same degree k is called a 'k-regular graph'.

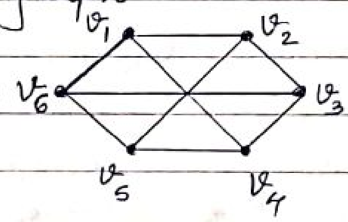
Eg:

(1)



2-regular

(2)

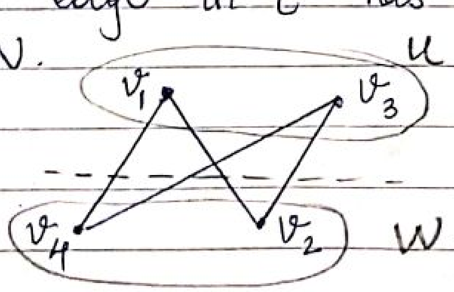


3-regular.

3. Bipartite Graph

A graph $G=(V,E)$ is called bipartite if the vertex set V is partitioned into two sets U and W such that every edge in E has one vertex in U and other in W.

Eg:



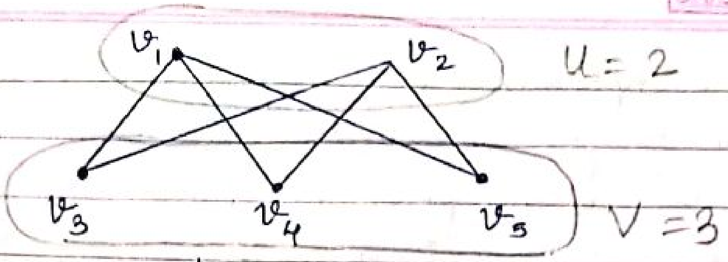
Redrawing the graph (1) above. (2-regular)

4. Complete Bipartite Graph

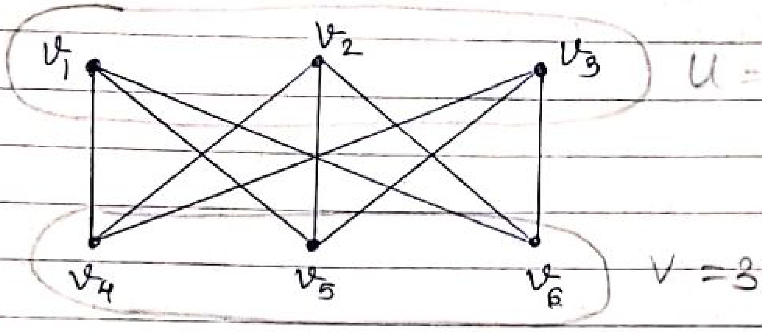
A simple bipartite graph which is complete.

It is denoted by $K_{m,n}$.

Eg :



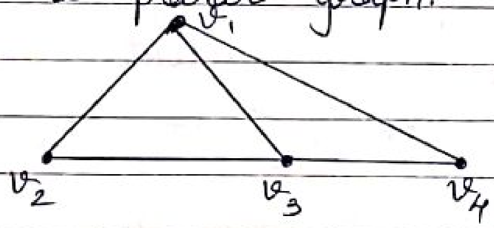
$K_{2,3}$.



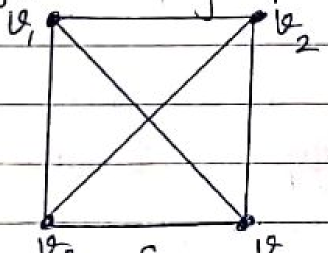
5. Planar Graph

A graph in which no two edges cross each other is called a planar graph.

Eg :

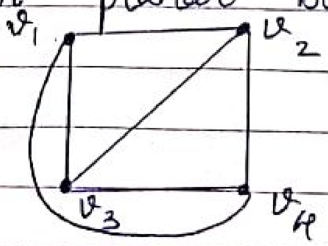


Sometimes, the given graph is not planar, but upon drawing the same graph in a little different way, the graph becomes planar.



G

Here, G is not planar but if the same graph is drawn like



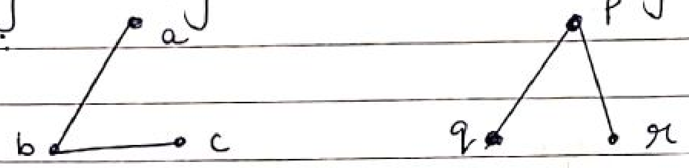
Now G is planar $\therefore$ Given graph G is planar.

Isomorphism of graphs:

Two graphs $G=(V,E)$ and $G'=(V',E')$ are said to be isomorphic if there is a one-to-one correspondence $f: V \to V'$ such that for $v_1, v_2 \in V$, $v_1', v_2' \in V'$, $f(v_1)=v_1'$, $f(v_2)=v_2'$ and v_1 is adjacent to v_2 then v_1' is adjacent to v_2'.

ie. [If we can find a bijective function from vertices to G to vertices of G' s.t. two vertices adjacent in G have their images adjacent in G' i.e. adjacency preserving.

Eg:



Here a and b are adjacent

- If $f(a)=q$, $f(b)=p$
  then p, q are adjacent
  ∴ proper mapping - can be considered.
- If $f(a)=q$, $f(b)=r$.
  then q and r are not adjacent
  ∴ improper mapping - cannot be considered.]

Note: If $f(a)=b$ then $\deg a = \deg b$.

How to check whether given graphs are isomorphic or not?

Step 1: Check whether no. of vertices in G = no of vertices in G'.
If Yes then proceed to step 2, otherwise conclude 'G and G' are not isomorphic'.

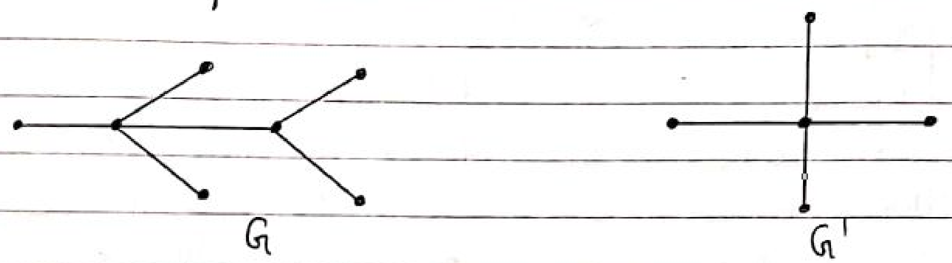
Step 2: Check whether no. of edges in G = no. of edges in G'.
If Yes then proceed to step 3, otherwise conclude 'G and G' are not isomorphic'.

Step 3: Check whether degree of vertices in G and G' are same or not.
If Yes then, proceed to step 4, otherwise conclude 'G and G' are not isomorphic'.

Step4: Define a function $f: G \to G'$ on vertices satisfying adjacency of vertices i.e. adjacency preserving map. If finding such f is possible then G and G' are isomorphic otherwise they are not isomorphic.

Eg: Check whether the following pair of graphs are isomorphic or not.

1.

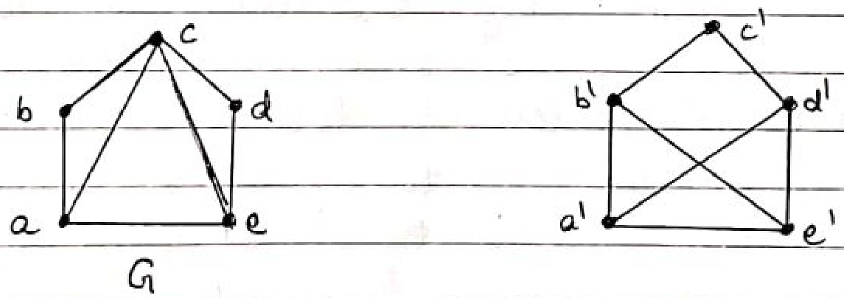


Step1: no. of vertices in $G = 7$

no. of vertices in $G' = 5$

$\therefore$ G and G' are not isomorphic.

2.



Step 1: no. of vertices in G = no. of vertices in $G' = 5$

Step 2: no. of edges in G = no. of edges in $G' = 7$.

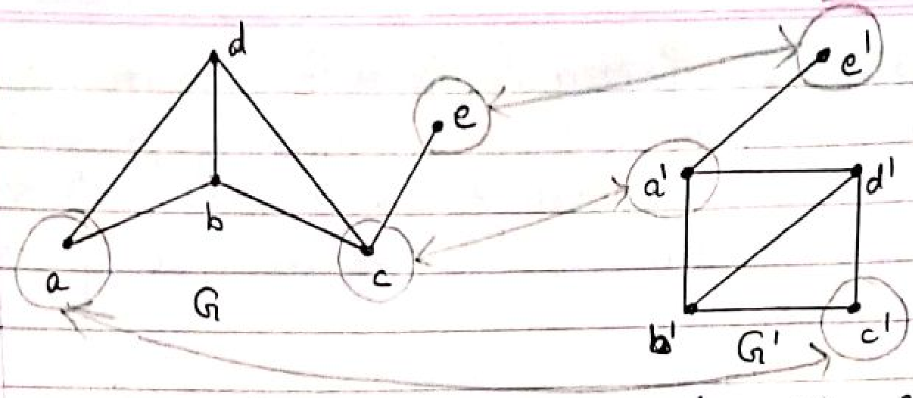
Step 3:

Degree	2	3	4
no. of vertices in G	2	2	1
no. of vertices in G'	1	4	–

Degree of vertices is not same

$\therefore$ G and G' are not isomorphic.

3.



Step 1: no. of vertices in G = no. of vertices in G' = 5
Step 2: no. of edges in G = no. of edges in G' = 6.
Step 3:

Degree	1	2	3
No. of vertices in G	1	1	3
No. of vertices in G'	1	1	3

Step 4: Mapping vertex to vertex with same degree.

- From step 3, there is only one vertex of degree 1 in G and G'
  ∴ mapping e to e' ie. f(e) = e'
- Similarly, only one vertex of degree ~~3~~ 2
  ∴ mapping a to c' ie. f(a) = c'
- Now, only left to find f(b), f(c), f(d).
- e is adjacent to c in G
  e' is adjacent to a' in G'
  f(e) = e'
  ∴ mapping c to a' ie. f(c) = a'.

  a → c'
  b → ?
  c → a'
  d → ?
  e → e'

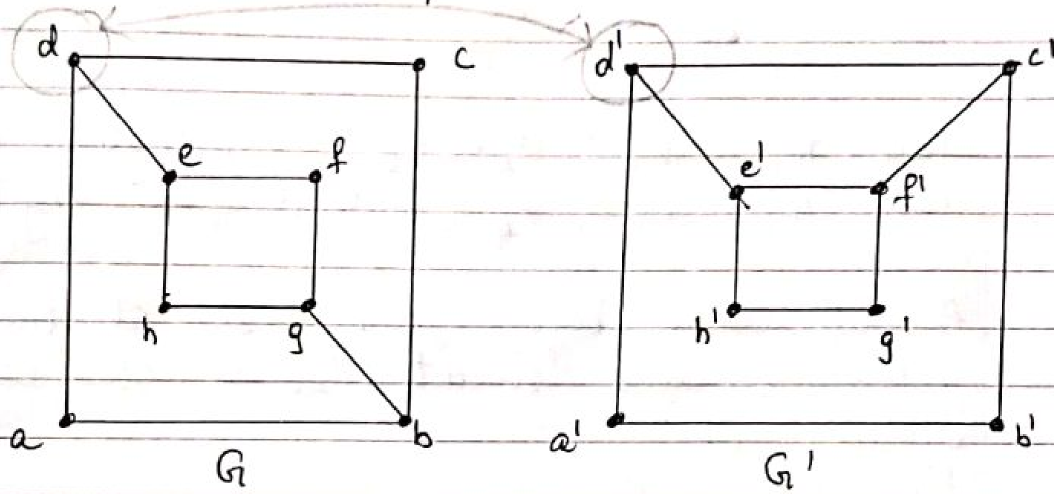
- Now, only left to find f(b), f(d).
  Here both are valid and free selection is possible.
  ∴ let f(b) = b', f(d) = d'
  OR f(b) = d', f(d) = b'.

Consider a map $f: G \to G'$ defined by

$f(a) = c'$		$f(a) = c'$
$f(b) = b'$	OR.	$f(b) = d'$
$f(c) = a'$		$f(c) = a'$
$f(d) = d'$		$f(d) = b'$
$f(e) = e'$		$f(e) = e'$.

This is an adjacency preserving map.
$\therefore$ G is isomorphic to G'.

4.

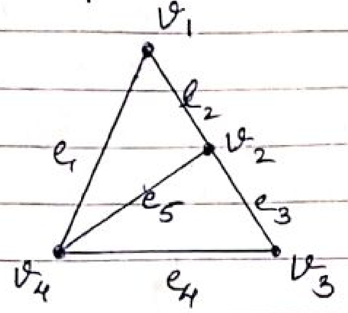


Step 1: no. of vertices in G = no. of vertices in G'. = 8
Step 2: no. of edges in G = no. of edges in G' = 10
Step 3:

Degree	2	3.
No. of vertices in G	4	4
No. of vertices in G'	4	4

Step 4:
Let $f(d) = d'$
Now, d is (connected) ~~to~~ adjacent to a, e, c of which deg a = deg c = 2, deg e = 3.
And, d' is adjacent to a', e', c' of which deg e' = deg c' = 3, deg a' = 2.
$\therefore$ d cannot be mapped to d'
Similarly,
d cannot be mapped to e', f', c'.
$\therefore$ Find $f: G \to G'$ is not possible.
$\therefore$ G is not isomorphic to G'.

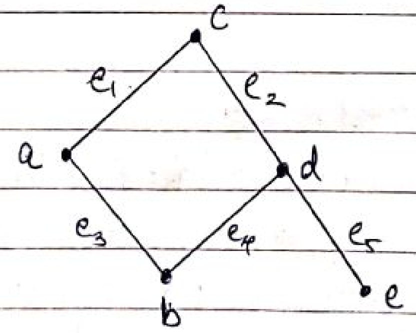
Path is an alternating sequence of vertices and edges of a graph in which neither vertex nor edge is repeated.
Length of path = no. of edges in the path
Eg:



Path $\pi_1 = v_1 e_1 v_4 e_5 v_2$, length = 2.
Path $\pi_2 = v_1 e_1 v_4 e_4 v_3 e_3 v_2 e_2 v_1$, length = 4.

A path that begins and ends at the same vertex is called 'circuit'. It is also called a 'closed path'.
Eg: path π_2 is a circuit.
path π_2 is not a circuit.

Connected graph
A graph is called connected if there is a path between every pair of vertices in that graph.
Eg:



From a – b is a direct path $a\, e_3\, b$
a – c ——— " ——— $a\, e_1\, c$
a – d ⇒ $\pi_1 : a\, e_1\, c\, e_2\, d$
$\pi_2 : a\, e_3\, b\, e_4\, d.$

$a - e \Rightarrow \pi_1 : a e_1 c\, c_2 d e_5 e$

$\pi_2 : a\, e_3 b e_4 d e_5 e$

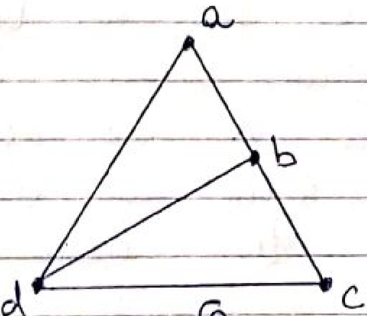
Similarly, we can find a path ~~two~~ between every other pair.

NOTE: Here,

- $\pi : a e_1 c e_2 d$ can also be written as $a c d$ as there is only one edge e_1 between 'a' and 'c'
- Writing acd as a path is easier than writing $ae_1 ce_2 d$.
- Length of path π = 2 = no. of edges.

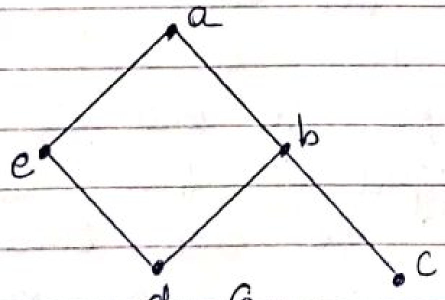
Eulerian Graph.

- A path or circuit which includes every 'edge' of the graph is said to be an Eulerian path / circuit.
- If a graph has an Eulerian circuit, it is called Eulerian graph.



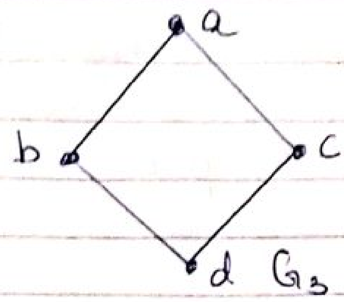
G_1

- a − b − c − d − a is a circuit but it leaves edge b−d so it is not Eulerian circuit.
- d − a − b − d − c − b is an Eulerian path.
- There is no possible Eulerian circuit so the given graph G_1 is not Eulerian.



G_2

- c − b − a − e − d − b is Eulerian path
- There is no possible Eulerian circuit, so graph G_2 is not Eulerian.



a - b - d - c - a is an Eulerian circuit
∴ The graph G_3 is Eulerian.

How to check whether the given graph is Eulerian or not?

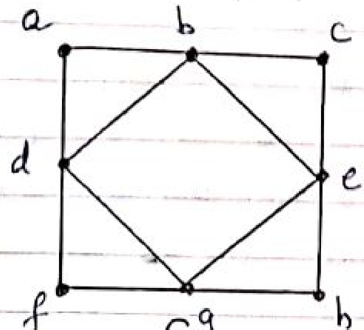
- Theorem:
  A connected graph is Eulerian ~~iff~~ if and only if degree of every vertex is even.
  Eg: In above example, there were odd degree vertices in G_1 and G_2 but in G_3 all vertices had even degree
  So, only G_3 is Eulerian.

- Theorem:
  A connected graph G has an Eulerian path but not an Eulerian circuit if there are exactly two vertices of odd degree
  i.e. If a graph has exactly two vertices of odd degree - u and v then there is an Eulerian path between u and v.
  NOTE: As there are two odd degree vertices, the graph will not have Eulerian circuit.
  Eg: In above example, G_1 and G_2 had 2 vertices of odd degree so they had Eulerian path but were not Eulerian. G_3 had all vertices of even degree so it had Eulerian circuit. Hence G_3 has / does not have Eulerian path.

[Q] Can a circuit be treated a path?]

Check whether the following graphs have Eulerian path and / or circuit.
[Whenever we say that the graph has Eulerian path / circuit, we need to specify that particular path / circuit. Just mentioning it has path / circuit is not proper].

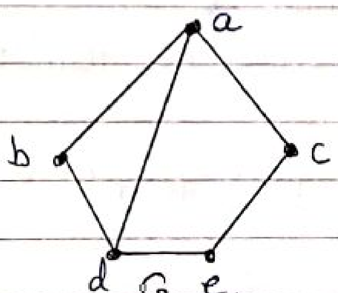
(1)



G_1

In graph G_1,
All vertices have even degree
∴ The graph is Eulerian.
π_1: a-b-c-e-h-g-f-d-g-e-b-d-a
is an Eulerian circuit.
Every circuit is a path.
∴ π_1 is an Eulerian path.

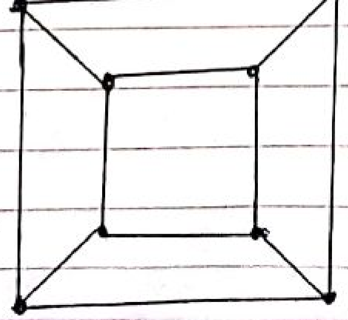
(2)



G_2

In graph G_2,
All vertices do not have even degree.
∴ The graph is not Eulerian.
But there are exactly two vertices a, d of odd degree so an Eulerian path between a and d exists. π_2: a-b-d-a-c-e-d.
NOTE: There is no Eulerian path between any other pair of vertices.

(3)

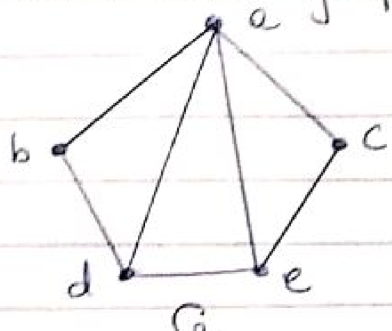

G_3

In Graph G_3,
All vertices have odd degree
∴ The graph does not have an Eulerian path ~~neither~~ nor Eulerian circuit.

Hamiltonian Graph

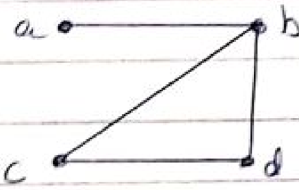
- A path or circuit which includes every 'vertex' of the graph is said to be an Hamiltonian path or circuit.
- If a graph has an Hamiltonian circuit, it is called Hamiltonian graph.

Eg.



G_1

In graph G_1,
$\pi_1 = a - b - d - e - c - a$ is a Hamiltonian circuit
Hence, G_1 is Hamiltonian.



G_2

In Graph G_2,
- $\pi_2 = a - b - d - c$ is a Hamiltonian path.
- There is no Hamiltonian circuit.

How to check whether the given graph is Hamiltonian or not?

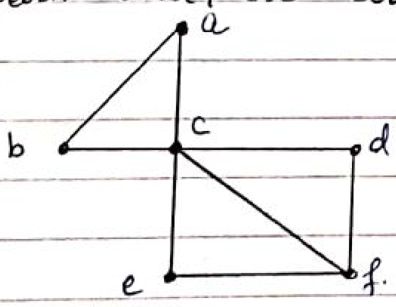
- Checking for Hamiltonian is not as easy as checking for Eulerian.
- ~~Finding Every connected graph~~
- Finding Hamiltonian path / circuit is not difficult.
- When concluding that the graph is not Hamiltonian, below stated theorems should be used.

Theorem

1. If G is connected graph with n vertices and if sum of degrees of each pair of vertices is greater than or equal to n-1 then there is Hamiltonian circuit.
2. In a simple connected graph with n vertices, if degree of each vertex is greater than or equal to $n/2$ then there is Hamiltonian Circuit.

Check whether the following graphs have Hamiltonian path and / or circuit.

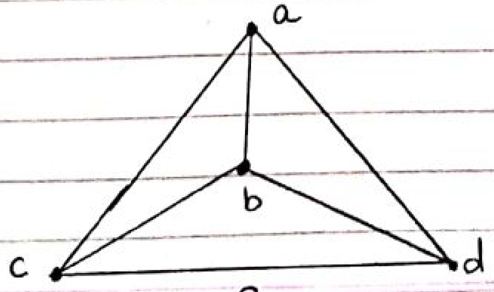
(1)



G_1

In Graph G_1, $n = 6$ $\therefore n/2 = 3$
$deg.(a) = 2$
$\therefore deg(a) \ngeq n/2$
$\therefore$ Hamiltonian circuit does not exist.
But Hamiltonian path exist -
$\pi_1 : a-b-c-d-f-e.$

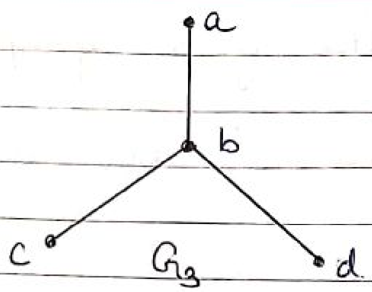
(2)



G_2

In graph G_2, $n=4$, $\therefore n/2 = 2$
~~deg (a)~~ degree of all vertices=3
which is greater than $n/2$
$\therefore$ Hamiltonian circuit exist
$\pi_2 : a-c-b-d-a.$

(3)



In graph G_3, $n = 4$, $n/2 = 2$
$\deg(a) \ngeq n/2$
$\therefore$ There is no Hamiltonian circuit.
Also, clearly there is no Hamiltonian path.