Algorithms for numerical solution of initial value problems

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Question 1

$y_{n+3} + (2b-3)(y_{n+2} - y_{n+1}) - y_n = hb(f_{n+2} + f_{n+1}), b \in \Re$
Method stability determined by roots of first characteristic polynomial of magnitude ≤ 1 with values of b.
$y_{n+3} + (2\,b-3)\;y_{n+2} + (3-2\,b)\,y_{n+1} - y_n$
Giving alphas: $\alpha_3 = 1$ $\alpha_2 = 2b-3$ $\alpha_1 = 3-2b$ $\alpha_0 = -1$
Which gives us the first characteristic polynomial:
$\rho(\xi) = \xi^3 + (2\,b-3)\;\xi^2 + (3-2\,b)\;\xi - 1$
Factorizing gives: $(\xi-1)\;\left(\xi^2 + 2\,b\,\xi - 2\,\xi + 1\right)$
Root at $\xi = 1$. Quadratic formula gives the roots:
$\left[\xi = -\sqrt{b^2-2\,b} - b + 1, \xi = \sqrt{b^2-2\,b} - b + 1, \xi = 1\right]$
The interval of values for the real constant b for which the method is stable is $0 < b < 2$

Question 2

A linear multistep method is of order p if:
$C_0 = C_1 = ... = C_p = 0, C_{p+1} \neq 0$
Constants to check for equality to 0:
$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$
$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 - (\beta_0 + \beta_1 + \beta_2 + \beta_3)$
$C_j = \frac{1}{j!}(\alpha_1 + 2^j\alpha_2 + 3^j\alpha_3) - \frac{1}{(j-1)!}(\beta_1 + 2^{j-1}\beta_2 + 3^{j-1}\beta_3)$
Using: $\Sigma_{i=0}^{k}\alpha_i y_{n+i} = h\Sigma_{i=0}^{k}\beta_i f_{n+i}, n = 0, 1, \ldots$
Gives alphas: $\alpha_3 = 1$ $\alpha_2 = 2b - 3$ $\alpha_1 = -(2b-3)$ $\alpha_0 = -1$
And betas: $\beta_3 = 0$ $\beta_2 = b$ $\beta_1 = b$ $\beta_0 = 0$
$C_0 = -1 - (2b-3) + (2b-3) + 1 = 0$
$C_1 = -(2b-3) + 2(2b-3) + 3 - 2b = 0$
$C_2 = \frac{1}{2}(-(2b-3) + 4(2b-3) + 9) - 3b = \frac{1}{2}(6b - 9 + 9) - 3b = 0$
$C_3 = \frac{1}{6}[-(2b-3) + 8(2b-3) + 27] - \frac{1}{2}(b + 4b) =$
$\frac{1}{6}[7(2b-3) + 27] - \frac{1}{2}5b = \frac{14b}{6} + 1 - \frac{5b}{2} = 1 - \frac{b}{6} \neq 0, \texttt{for } b \in (0,2)$
So $p = 2$

Question 3

For t_1:

$t_1 = t_0 + h_E = h_E$

$y_{1,1} = h_E(\mu - \beta(h_E)y_{1,0}y_{3,0}) + y_{1,0}$

$y_{2,1} = h_E\left(\beta(h_E)y_{1,0}y_{3,0} - \frac{1}{\lambda}y_{2,0}\right) + y_{2,0}$

$y_{3,1} = h_E\left(\frac{1}{\lambda}y_{2,0} - \frac{1}{\eta}y_{3,0}\right) + y_{3,0}$

If we had numbers, we should now solve these three equations to get numerical answer for $y_{1,1}$ $y_{2,1}$ and $y_{3,1}$

For t_2:

$t_2 = 2h_E$

$y_{1,2} = h_E(\mu - \beta(h_E)y_{1,0}y_{3,0}) + y_{1,0} + (\mu - \beta(2h_E)y_{1,0}y_{3,0})h_E$

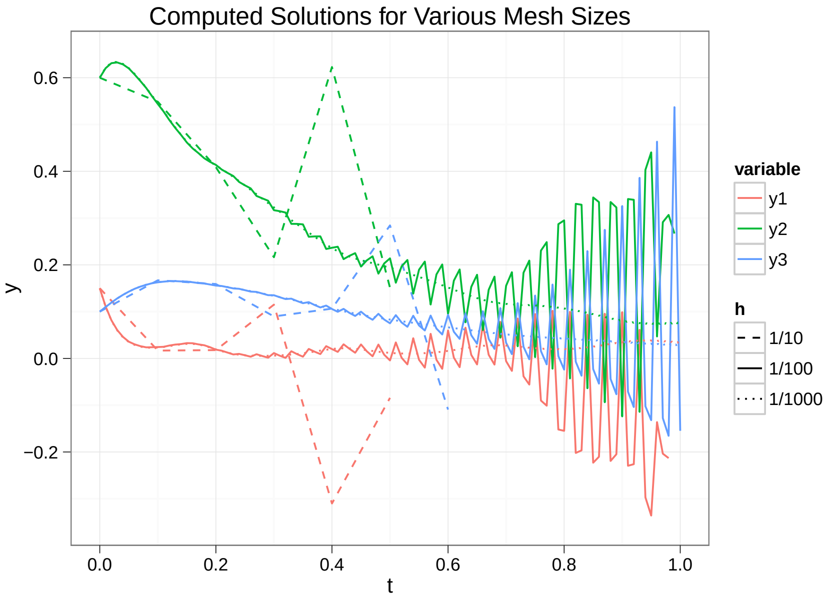
$y_{2,2} =$

$h_E\left(\beta(h_E)y_{1,0}y_{3,0} - \frac{1}{\lambda}y_{2,0}\right) + y_{2,0} + h_E\left(\beta(2h_E)y_{1,0}y_{3,0} - \frac{1}{\lambda}y_{2,0}\right)$

$y_{3,2} = h_E\left(\frac{1}{\lambda}y_{2,0} - \frac{1}{\eta}y_{3,0}\right) + y_{3,0} + h_E\left(\frac{1}{\lambda}y_{2,0} - \frac{1}{\eta}y_{3,0}\right)$

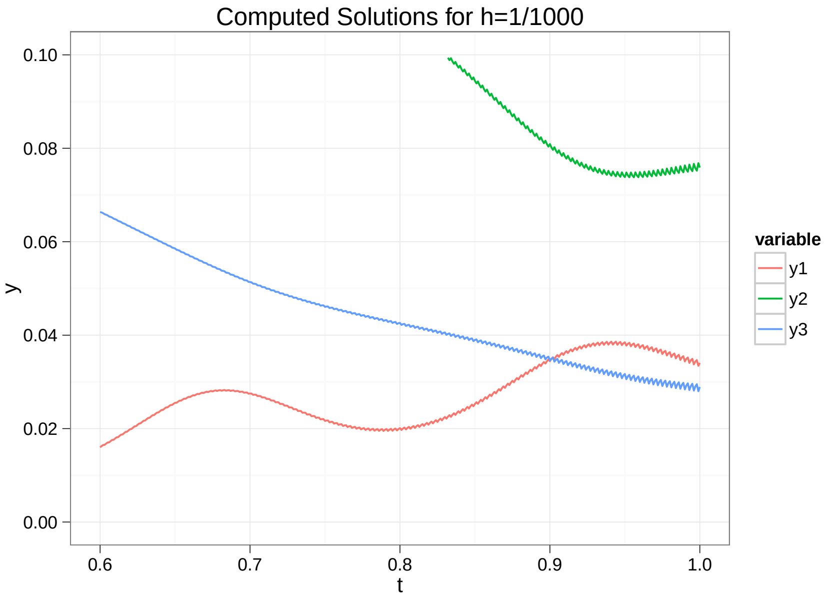
Again we should solve these equations simultaneously to $y_{1,2}$ $y_{2,2}$ and $y_{3,2}$.

Question 4



Computed Solutions for Various Mesh Sizes
0.6
0.4
0.2
0.0
−0.2
y
0.0
0.2
0.4
0.6
0.8
1.0
t
variable
y1
y2
y3
h
1/10
1/100
1/1000

Question 4


Computed Solutions for h=1/1000
0.10
0.08
0.06
0.04
0.02
0.00
y
0.6
0.7
0.8
0.9
1.0
t
variable
y1
y2
y3