Algorithms for numerical solution of initial value problems

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Question 1

$y_{n+3} + (2b-3)(y_{n+2} - y_{n+1}) - y_n = hb(f_{n+2} + f_{n+1}), b \in \Re$
Method stability determined by roots of first characteristic polynomial of magnitude ≤ 1 with values of b.
$y_{n+3} + (2\,b-3)\;y_{n+2} + (3-2\,b)\,y_{n+1} - y_n$
Giving alphas: $\alpha_3 = 1$ $\alpha_2 = 2b-3$ $\alpha_1 = 3-2b$ $\alpha_0 = -1$
Which gives us the first characteristic polynomial:
$\rho(\xi) = \xi^3 + (2\,b-3)\;\xi^2 + (3-2\,b)\;\xi - 1$
Factorizing gives: $(\xi - 1)\;\left(\xi^2 + 2\,b\,\xi - 2\,\xi + 1\right)$
Root at $\xi = 1$. Quadratic formula gives the roots:
$\left[\xi = -\sqrt{b^2 - 2\,b} - b + 1, \xi = \sqrt{b^2 - 2\,b} - b + 1, \xi = 1\right]$
The interval of values for the real constant b for which the method is stable is $0 < b < 2$

Question 2

A linear multistep method is of order p if:

$$C_0 = C_1 = \ldots = C_p = 0, C_{p+1} \neq 0$$

Constants to check for equality to 0:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$
$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 - (\beta_0 + \beta_1 + \beta_2 + \beta_3)$$
$$C_j = \frac{1}{j!}(\alpha_1 + 2^j\alpha_2 + 3^j\alpha_3) - \frac{1}{(j-1)!}(\beta_1 + 2^{j-1}\beta_2 + 3^{j-1}\beta_3)$$

Question 2

Using:

$$\Sigma_{i=0}^{k}\alpha_i y_{n+i} = h\Sigma_{i=0}^{k}\beta_i f_{n+i}, n = 0, 1, \ldots$$

Gives alphas:

$$\begin{aligned}
\alpha_3 &= 1 \\
\alpha_2 &= 2b - 3 \\
\alpha_1 &= -(2b - 3) \\
\alpha_0 &= -1
\end{aligned}$$

And betas:

$$\begin{aligned}
\beta_3 &= 0 \\
\beta_2 &= b \\
\beta_1 &= b \\
\beta_0 &= 0
\end{aligned}$$

Question 2

Calculate the constants

$$
\begin{aligned}
C_0 &= -1 - (2b-3) + (2b-3) + 1 = 0 \\
C_1 &= -(2b-3) + 2(2b-3) + 3 - 2b = 0 \\
C_2 &= \frac{1}{2}(-(2b-3) + 4(2b-3) + 9) - 3b = \frac{1}{2}(6b - 9 + 9) - 3b = 0 \\
C_3 &= \frac{1}{6}[-(2b-3) + 8(2b-3) + 27] - \frac{1}{2}(b + 4b) \\
&= \frac{1}{6}[7(2b-3) + 27] - \frac{1}{2}5b = \frac{14b}{6} + 1 - \frac{5b}{2} \\
&= 1 - \frac{b}{6} \neq 0, \texttt{for} \;\; b \in (0,2)
\end{aligned}
$$

So,

$$p = 2$$

Question 3

For t_1:

$$\begin{aligned}
t_1 &= t_0 + h_E = h_E \\
y_{1,1} &= h_E(\mu - \beta(h_E) y_{1,0} y_{3,0}) + y_{1,0} \\
y_{2,1} &= h_E \left(\beta(h_E) y_{1,0} y_{3,0} - \frac{1}{\lambda} y_{2,0} \right) + y_{2,0} \\
y_{3,1} &= h_E \left(\frac{1}{\lambda} y_{2,0} - \frac{1}{\eta} y_{3,0} \right) + y_{3,0}
\end{aligned}$$

If we had numbers, we should now solve these three equations to get numerical answer for $y_{1,1}$ $y_{2,1}$ and $y_{3,1}$

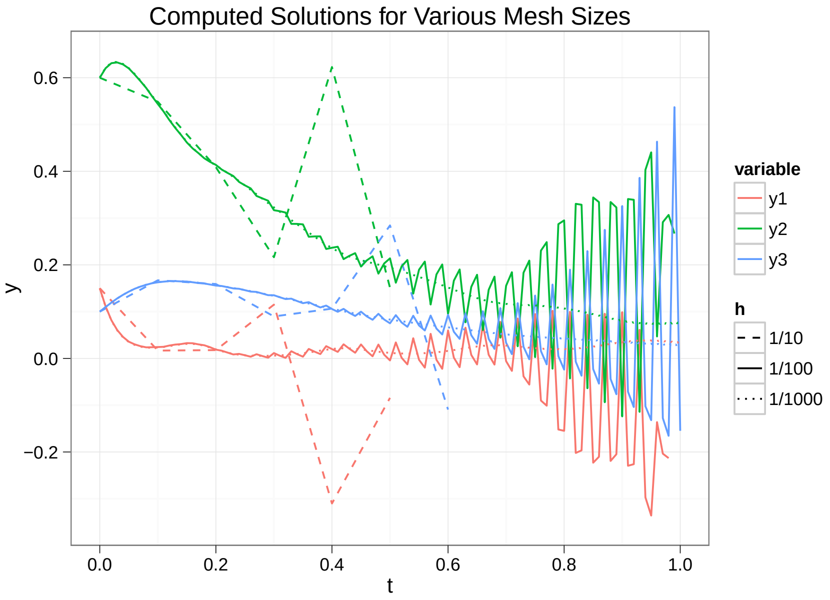
Question 3

For t_2:

$$
\begin{aligned}
t_2 &= 2h_E \\
y_{1,2} &= h_E(\mu - \beta(h_E)y_{1,0}y_{3,0}) + y_{1,0} + (\mu - \beta(2h_E)y_{1,0}y_{3,0})h_E \\
y_{2,2} &= h_E\left(\beta(h_E)y_{1,0}y_{3,0} - \frac{1}{\lambda}y_{2,0}\right) + y_{2,0} + h_E\left(\beta(2h_E)y_{1,0}y_{3,0} - \frac{1}{\lambda}y_{2,0}\right) \\
y_{3,2} &= h_E\left(\frac{1}{\lambda}y_{2,0} - \frac{1}{\eta}y_{3,0}\right) + y_{3,0} + h_E\left(\frac{1}{\lambda}y_{2,0} - \frac{1}{\eta}y_{3,0}\right)
\end{aligned}
$$

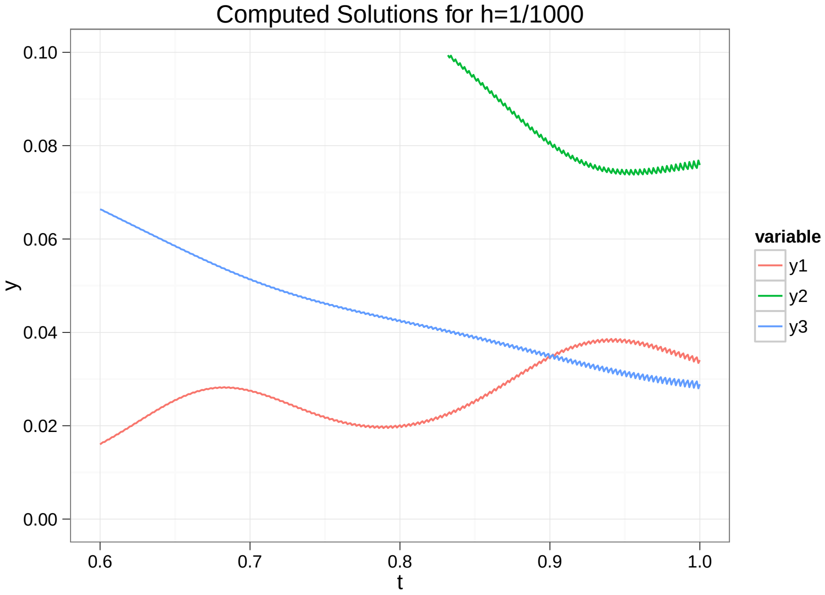
Again we should solve these equations simultaneously to $y_{1,2}$ $y_{2,2}$ and $y_{3,2}$.

Question 4



Computed Solutions for Various Mesh Sizes
0.6
0.4
0.2
0.0
−0.2
y
0.0
0.2
0.4
0.6
0.8
1.0
t
variable
y1
y2
y3
h
1/10
1/100
1/1000

Question 4


Computed Solutions for h=1/1000
y
t
0.00
0.02
0.04
0.06
0.08
0.10
0.6
0.7
0.8
0.9
1.0
variable
y1
y2
y3