二. 方差

1.方差的定义

2.方差的计算公式

3.方差的性质

4.常用分布的方差

甲厂电视机寿命值 $X \sim N\left(\mu_1, \sigma_1^2\right)$ $\mu_1 = 8$(年)

乙厂电视机寿命值 $Y \sim N\left(\mu_2, \sigma_2^2\right)$ $\mu_2 = 8$(年)

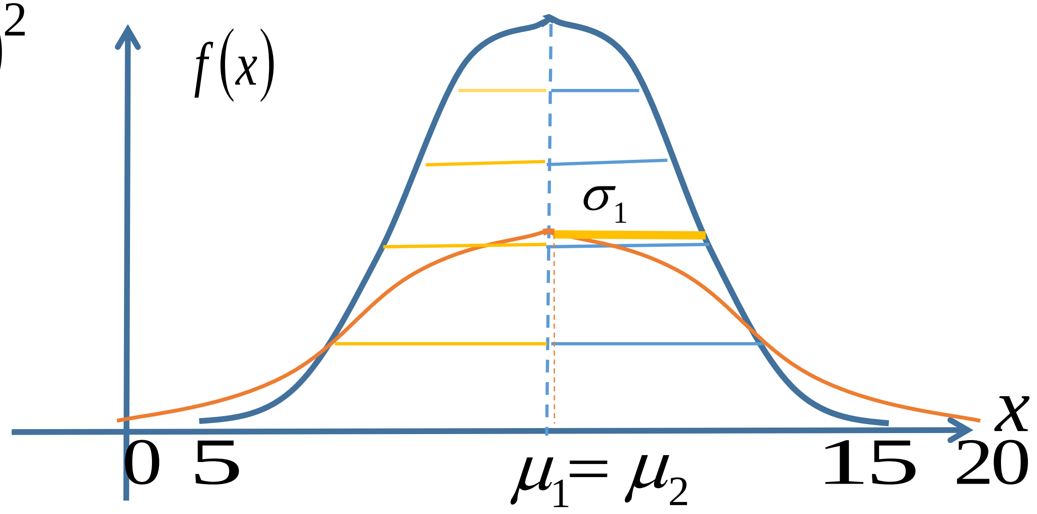
1.极差=最大-最小 $\begin{cases} 15-5=10 \\ 20-0=20 \end{cases}$ $\begin{pmatrix} \text{优点：简　单} \\ \text{缺点：太简单} \end{pmatrix}$

从图上看会买蓝色甲厂，下面设法描述两厂差别

2. $E(X-\mu) = EX - \mu \equiv 0$

$$E|X-\mu| \xrightarrow{\text{讨厌绝对值}} E(X-\mu)^2$$

$$\sqrt{E(X-\mu)^2} \neq E|X-\mu|$$



（重点可比，具体长度次之）

1.定义：（常数-描述变量的离散程度）

X 是随机变量，若期望 $E(X-E(X))^2$存在，则称其为X 的方差，记为 $D(X)=E(X-E(X))^2$。

设X 的分布列为 $P(X=x_i)=p_i, i=1,2\ldots,$

$$D(X)=E(X-E(X))^2=\sum_{i=1}^{n}(x_i-E(X))^2 p_i;$$

设X 的分布密度为$f(x)$,

$$D(X)=E(X-E(X))^2=\int_{-\infty}^{+\infty}(x-E(X))^2 f(x)dx$$

标准差：$\sigma_X=\sqrt{D(X)}=\sqrt{E(X-E(X))^2}$

2.计算公式： $D(X)=EX^2-(EX)^2$

证明：$D(X)=E(X-EX)^2=E\left(X^2-2XEX+(EX)^2\right)$

$$=EX^2-E(2XEX)+E(EX)^2$$

$$=EX^2-2EXEX+(EX)^2=EX^2-(EX)^2$$

$Ec=c$

$E(cX)=cEX$

$E(aX+b)$

$=aE(X)+b$

3.性质：

(1) $D(c)=0$，c为常数

(2) $D(cX)=c^2D(X)$

(3) $D(aX+b)=a^2D(X)$

$D(X)=E(c-E(c))^2=0$

$D(aX+b)=E\{(aX+b)-E(aX+b)\}^2$

$$=E\{aX+b-aEX-b\}^2$$

$$=a^2E(X-EX)^2=a^2D(X)$$

$$D(X \pm Y) = E\{(X \pm Y) - E(X \pm Y)\}^2 = E\{X \pm Y - EX \mp EY\}^2$$

$$= E\{(X - EX) \pm (Y - EY)\}^2$$

$$E(X \pm Y) = EX \pm EY$$

$$= E\left\{(X - EX)^2 \pm 2(X - EX)(Y - EY) + (Y - EY)^2\right\}$$

$$= E(X - EX)^2 \pm 2E(X - EX)(Y - EY) + E(Y - EY)^2$$

$$= DX + DY \pm 2E(X - EX)(Y - EY)$$

$$E(X - EX)(Y - EY) = E\{XY - YEX - XEY + EXEY\}$$

$$= EXY - EXEY - EYEX + EXEY$$

$$= EXY - EXEY \quad = 0$$

$\{X$与Y独立 $EXY = EXEY\}$

$$D(X \pm Y) = \begin{cases} DX + DY & X与Y独立 \\ DX + DY \pm 2E(X-EX)(Y-EY) \end{cases}$$

$E(X \pm Y) = EX \pm EY$

$D(X \pm Y) = DX + DY$

X与Y独立

一般地，$X_1, X_2, \ldots X_n$为n个相互独立的随机变量，则

$$D\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} D(X_i);$$

$$D\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} D(X_i); \quad \left(D(\overline{X}) = \frac{D(X)}{n} \quad X_i独立同分布 \right)$$

标准化随机变量：$X^* = \dfrac{X - E(X)}{\sqrt{D(X)}}$，有$E(X^*) = 0$；$D(X^*) = 1$。

总结： $D(X)=E(X-E(X))^2=EX^2-(EX)^2$

$D(c)=0$，c为常数

$D(cX)=c^2D(X)$

$D(aX+b)=a^2D(X)$

$$D(X\pm Y)=\begin{cases} D(X)+D(Y)\pm 2\{E(XY)-EXEY\} \\ D(X)+D(Y), X\text{与}Y\text{独立} \end{cases}$$

$D(\overline{X})=D(X)/n$ （$\overline{X}$为样本均数）

例1. 设随机变量 X 的分布列为

X	-1	0	$1/2$	1	2
P	$1/3$	$1/6$	$1/6$	$1/12$	$1/4$

求：$E(X), E(X^2), E(2X-3)$,

$D(X), D(X^2), D(2X-3)$.

解：$$E(X) = -\frac{1}{3} + \frac{1}{2}\times\frac{1}{6} + \frac{1}{12} + 2\times\frac{1}{4} = \frac{1}{3}$$

$$E(2X-3) = 2EX - 3 = 2\times\frac{1}{3} - 3 = -\frac{7}{3}$$

$$E(X^2) = \frac{1}{3} + \frac{1}{2^2}\times\frac{1}{6} + \frac{1}{12} + 2^2\times\frac{1}{4} = \frac{35}{24}$$

$$D(X) = EX^2 - (EX)^2 = \frac{35}{24} - \frac{1}{3^2} = \frac{97}{72}$$

$$D(2X-3) = 4D(X) = \frac{97}{18}$$

$$D(X^2) = EX^4 - (EX^2)^2$$

$$E(X^4) = \frac{1}{3} + \frac{1}{2^4}\times\frac{1}{6} + \frac{1}{12} + 2^4\times\frac{1}{4} = \frac{425}{96}$$

例2. 设随机变量 *X* 的密度函数为 $f(x)=\begin{cases}\dfrac{3}{(x+1)^4}, & x>0\\ 0, & x\le 0\end{cases}$**，求*D*(*X*).**

解：$E(X)=\int_{-\infty}^{+\infty} xf(x)dx=\int_0^{+\infty} x\dfrac{3}{(x+1)^4}dx$

$$=3\int_0^{+\infty}\frac{x+1}{(x+1)^4}-\frac{1}{(x+1)^4}dx=\left[\frac{-3}{2(x+1)^2}+\frac{1}{(x+1)^3}\right]_0^{+\infty}=\frac{1}{2}$$

$$E(X^2)=\int_0^{+\infty} x^2\frac{3}{(x+1)^4}dx=3\int_0^{+\infty}\frac{(x+1)^2}{(x+1)^4}-\frac{2x}{(x+1)^4}-\frac{1}{(x+1)^4}dx=1$$

$$D(X)=EX^2-(EX)^2=1-\frac{1}{2^2}=\frac{3}{4}$$

例3.二维随机变量分布列如表，求 $E(2X+3Y)$, $D(2X-3Y)$。

解：$E(2X+3Y)=2E(X)+3E(Y)=2\times0.6+3\times0.7=3.3$

$$D(2X-3Y)=4D(X)+9D(Y)-12(EXY-E(X)E(Y))$$

$$=4\times0.6\times0.4+9\times0.7\times0.3-12(0.4-0.6\times0.7)$$

XY	0	1
p	0.6	0.4

$Y \backslash X$	0	1	
0	0.1	0.2	0.3
1	0.3	0.4	0.7
	0.4	0.6	

例4. 设二维随即变量(X,Y)的密度函数为：

$$f(x,y)=\begin{cases}6e^{-2x-3y}, x>0, y>0; \\ 0, \qquad\qquad 其他。\end{cases}$$ 求 $E(2X-3Y), D(2X-3Y).$

解：$E(2X-3Y)=\int_0^{+\infty}\int_0^{+\infty}(2x-3y)6e^{-2x-3y}dxdy=0$

$$E(2X-3Y)^2=\int_0^{+\infty}\int_0^{+\infty}(2x-3y)^2 6e^{-2x-3y}dxdy=2$$

$D(2X-3Y)=2 \quad DX=EX^2-(EX)^2=EX^2(当 EX=0)$

4.常用分布的期望和方差

$$(1)\ X\sim B(1,p),\ E(X)=p,\ \ D(X)=p(1-p)$$

$$\left(E\left(X^{2}\right)=p\right)$$

$$D(X)=EX^{2}-(EX)^{2}=p-p^{2}=p(1-p)$$

$$(2)\ X\sim B(n,p),\ E(X)=np,\ D(X)=np(1-p)$$

$$\left(E\left(X^{2}\right)=p^{2}n(n-1)+np\right)$$

$$D(X)=EX^{2}-(EX)^{2}=p^{2}n(n-1)+np-n^{2}p^{2}\ =np(1-p)$$

$$D(X)=D\left(\sum_{i=1}^{n}X_{i}\right)=\sum_{i=1}^{n}D(X_{i})=np(1-p)$$

$$X=\sum_{i=1}^{n}X_{i},\ X_{i}\sim B(1,p)\text{相互独立}$$

$$(3)\ X\sim P(\lambda),\ E(X)=\lambda,\ \ D(X)=\lambda$$

$$D(X)=EX^{2}-(EX)^{2}=\lambda^{2}+\lambda-\lambda^{2}=\lambda$$

$$\left(E\left(X^{2}\right)=\lambda^{2}+\lambda\right)$$

$$(4)\ X \sim G(p),\ \ E(X)=\frac{1}{p},\ D(X)=\frac{q}{p^2}$$

$$\left(E(X^2)=\frac{1+q}{p^2}\right)$$

$$D(X)=EX^2-(EX)^2=\frac{1+q}{p^2}-\frac{1}{p^2}=\frac{q}{p^2}$$

$$(5)\ X \sim U(a,b),\ E(X)=\frac{a+b}{2},\ D(X)=\frac{(b-a)^2}{12}$$

$$\left(E(X^2)=\frac{a^2+ab+b^2}{3}\right)$$

$$D(X)=EX^2-(EX)^2=\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2=\frac{(b-a)^2}{12}$$

$$(6)\ X \sim e(\lambda),\ E(X)=\frac{1}{\lambda},\ D(X)=\frac{1}{\lambda^2}$$

$$\left(E(X^2)=\frac{2}{\lambda^2}\right)$$

$$D(X)=EX^2-(EX)^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

$$(7)\ X \sim N(\mu,\sigma^2), E(X)=\mu,\ \ D(X)=\sigma^2$$

$$DX = \int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{+\infty}(x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\overset{t=\frac{x-\mu}{\sigma}}{\underset{dx=\sigma dt}{=}} \int_{-\infty}^{+\infty}(t\sigma)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{+\infty} t^2 e^{-\frac{t^2}{2}} dt$$

$$= -\frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{+\infty} t de^{-\frac{t^2}{2}} = -\frac{2\sigma^2}{\sqrt{2\pi}} \left(te^{-\frac{t^2}{2}} \Big|_0^{+\infty} - \int_0^{+\infty} e^{-\frac{t^2}{2}} dt \right)$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = \sigma^2$$

$$D(X) = EX^2 - (EX)^2$$

$$EX^2 = D(X) + (EX)^2$$

例**5.** $(P_{118}-2)$ 设随机变量 X 与 Y 独立，且都服从均数为**0**，方差为**1/2** 的正态分布，求随机变量 $|X-Y|$ 的方差。

解：$Z=X-Y \qquad Z\sim N(0,1)$

$$D|X-Y|=D|Z|=E|Z|^2-(E|Z|)^2=EZ^2-(E|Z|)^2=1-(E|Z|)^2$$

$$E|Z|=\int_{-\infty}^{+\infty}|z|\varphi(z)dz=\int_{-\infty}^{+\infty}|z|\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=2\int_0^{+\infty}z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$=\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz^2 \quad =-\frac{2}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\Big|_0^{+\infty}=\sqrt{\frac{2}{\pi}}$$

$$D|X-Y|=1-(E|Z|)^2=1-2/\pi$$