3.σ^2置信区间

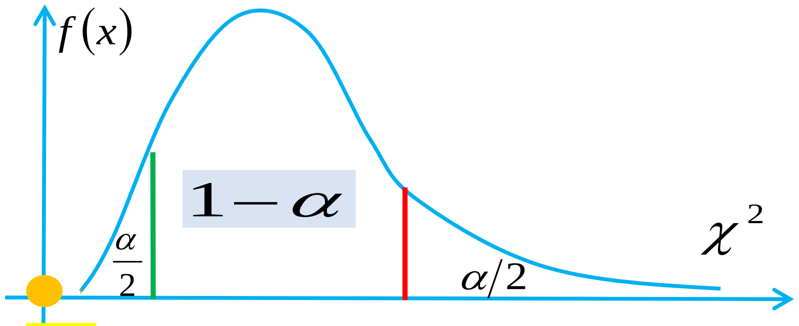
(1)双侧

$$P\left(T_1 \le \sigma^2 \le T_2\right) = 1-\alpha$$

$\le \sigma^2 <$ 但是

i） σ^2的点估计量为 S^2

ii）由抽样分布定理二有： $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$



s^2

iii）由小概率事件原理： 若$P(A)\le\alpha\ (0<\alpha<1)$，则$A$在一次抽样下不会发生。

一次抽样得到的样本方差值 s^2不是小概率事件， 须满足：

$$P\left(\boxed{\chi^2_{1-\frac{\alpha}{2}}(n-1)} \le \frac{(n-1)S^2}{\sigma^2} \le \boxed{\chi^2_{\frac{\alpha}{2}}(n-1)}\right) = 1-\alpha$$

$$P\left(\chi^2_{1-\frac{\alpha}{2}}(n-1) \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\frac{\alpha}{2}}(n-1)\right)=1-\alpha$$

iv）由上式解得： $$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}\right)=1-\alpha$$

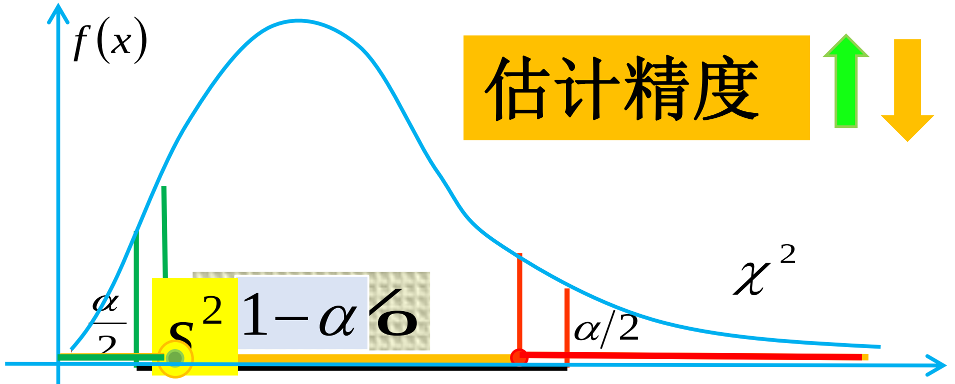
$$P\left(T_1 \le \sigma^2 \le T_2\right)=1-\alpha$$

置信度

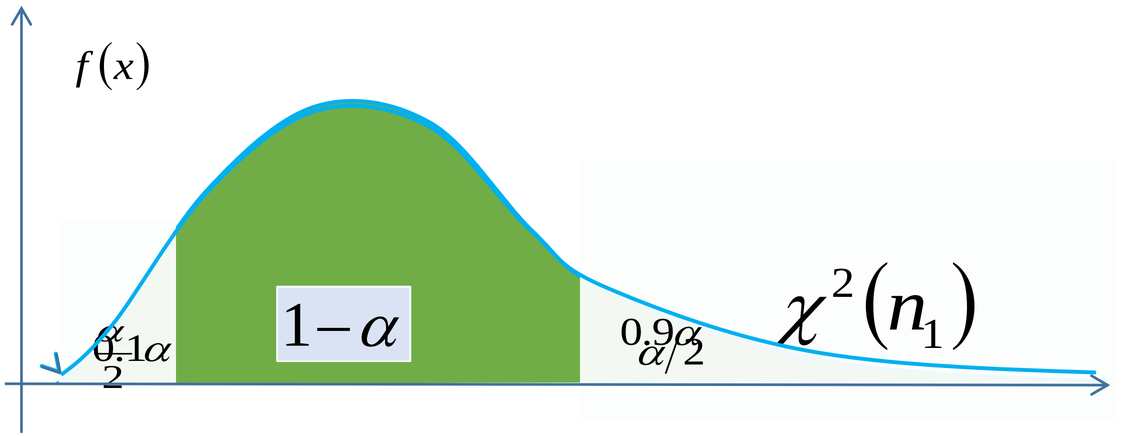
估计精度

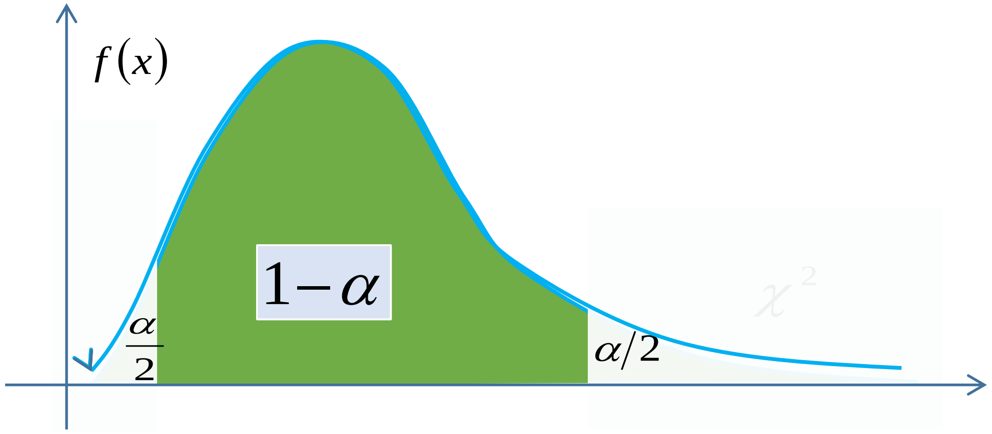
$$\sigma^2:\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)},\ \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$$

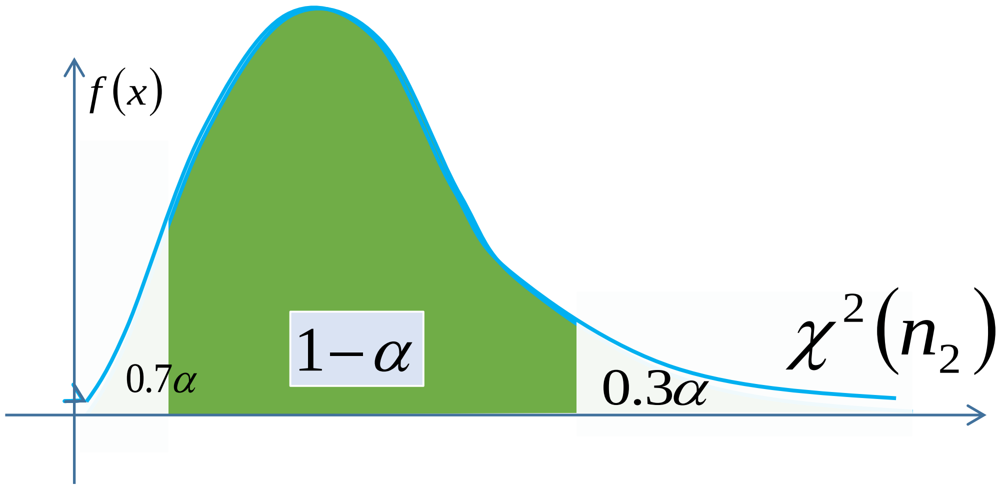
所有s^2



所有s^2







χ^2 不是对称分布，α仍**均分**两侧，

得双侧置信区间未必最短，但对

所有自由度来说总体较好。

例4.设某种金属丝长度 $X \sim N(\mu,\sigma^2)$ **，现从这批金属丝中随机抽取9根，测得其长度 数据如下1532,1297,1647,1356,1435,1483,1574,1517,1463，试估计该批金属丝长度方差** σ^2 **的0.95的置信区间。**

解：由 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$， 可得 $P\left(\chi^2_{0.975}(8) < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{0.025}(8)\right) = 0.95$

解得 $P\left(\frac{(n-1)S^2}{\chi^2_{0.025}(8)} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{0.975}(8)}\right) = 0.95$ $\qquad n = 9 \quad s^2 = 11494$

$$\left(\frac{(n-1)S^2}{\chi^2_{0.025}(8)}, \frac{(n-1)S^2}{\chi^2_{0.975}(8)}\right) = \left(\frac{8\times 11494}{17.535}, \frac{8\times 11494}{2.18}\right) = (5244, 42179)$$

$\left(\chi^2_{0.025}(8) = 17.535,\ \chi^2_{0.975}(8) = 2.18\right)$

(2) σ^2单侧上限　　$P\left(\sigma^2 \le T\right) = 1-\alpha$

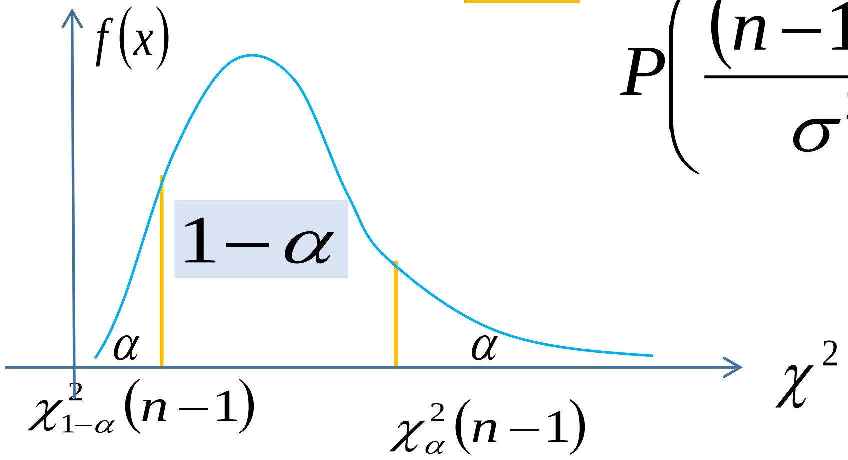
有些实际问题方差越小越好，　如：袋装大米质量　$X \sim N\left(50, \sigma^2\right)$

此时求σ^2的置信区间只需考虑单侧上限：

$$P\left(\frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha}(n-1)\right) = 1-\alpha$$

因为σ^2越小越好，$\dfrac{(n-1)S^2}{\sigma^2}$越大越好

错



$$P\left(\frac{(n-1)S^2}{\sigma^2} \ge \chi^2_{1-\alpha}(n-1)\right) = 1-\alpha \qquad \sigma^2 : \left(0, \frac{(n-1)S^2}{\chi^2_{1-\alpha}(n-1)}\right)$$

$$\sigma^2 : \left(\boxed{0}\ \frac{-1)S^2}{\chi_{\alpha/2}(n-1)}, \frac{(n-1)S^2}{\chi^2_{1-\boxed{\alpha}}(n-1)}\right)$$

(3) σ^2 单侧下限　　$P\left(\sigma^2 \geq T\right)=1-\alpha$

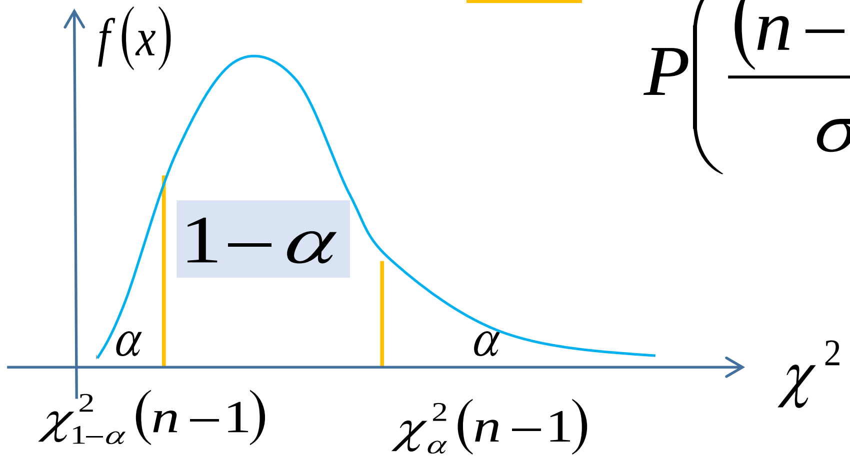
还有些实际问题方差大些较好，如：考试成绩　　$X \sim N\left(80, \sigma^2\right)$

出题者只需考虑 σ^2 的置信区间单侧下限：

$$P\left(\frac{(n-1)S^2}{\sigma^2} \geq \chi^2_{1-\alpha}(n-1)\right)=1-\alpha$$

因为 σ^2 越大越好，$\dfrac{(n-1)S^2}{\sigma^2}$ 越小越好

错



$$P\left(\frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha}(n-1)\right)=1-\alpha \quad \sigma^2 : \left(\frac{(n-1)S^2}{\chi^2_{\alpha}(n-1)}, \infty\right)$$

$$\sigma^2 : \left(\frac{(n-1)S^2}{\chi^2_{\alpha}(n-1)}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$$

$\rightarrow$

例5. 为估计制造某种产品所需的单位平均工作时间（h），现制造5件，所需工作时间如下：10.5,11,11.2,12.5,12.8. 假设所需工作时间服从正态分布 $X \sim N(\mu,\sigma^2)$ 试求方差 σ^2 的单侧置信上限。

$$n=5 \quad \bar{x}=11.6 \quad s^2=0.995 \quad \chi^2_{0.95}(4)=0.711$$

解：$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \Longrightarrow P\left(\sigma^2 \le \frac{(n-1)S^2}{\chi^2_{0.975}(4)}\right)=1-\alpha$$

$$\left(0, \frac{(n-1)S^2}{\chi^2_{0.95}(4)}\right)=\left(0, \frac{(5-1)0.995}{0.711}\right)$$

$$=(0, 5.597)$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{0.025}(4)} \le \boxed{\sigma^2 \le \frac{(n-1)S^2}{\chi^2_{0.975}(4)}}\right)=1-\alpha$$

4. σ_1^2/σ_2^2 的置信区间

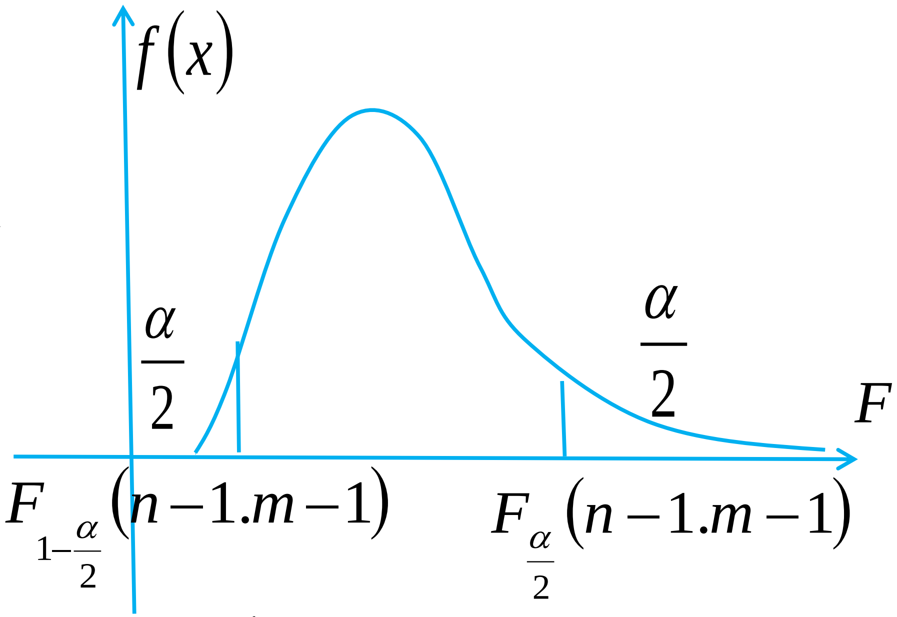
(1)双侧

$$\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n-1,m-1)$$

$$P\left(\frac{1}{F_{\frac{\alpha}{2}}(m-1,n-1)} \le \frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \le F_{\frac{\alpha}{2}}(n-1,m-1)\right) = 1-\alpha$$

$$P\left(\frac{s_1^2/s_2^2}{F_{\frac{\alpha}{2}}(n-1,m-1)} < \sigma_1^2/\sigma_2^2 < \frac{s_1^2}{s_2^2}F_{\frac{\alpha}{2}}(m-1,n-1)\right) = 1-\alpha$$

$$\boxed{\left(\frac{s_1^2/s_2^2}{F_{\frac{\alpha}{2}}(n-1,m-1)},\ \frac{s_1^2}{s_2^2}F_{\frac{\alpha}{2}}(m-1,n-1)\right)}$$



$$=\frac{1}{F_{\frac{\alpha}{2}}(m-1,n-1)}$$

(2) σ_1^2/σ_2^2 单侧置信上限

$$\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n-1,m-1)$$

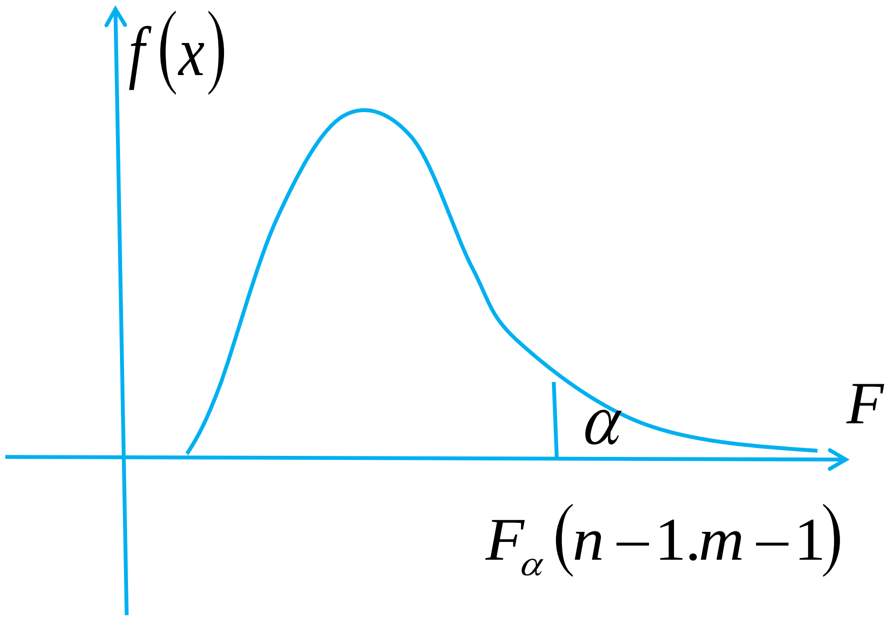
$$P\left(\sigma_1^2/\sigma_2^2 \le \frac{s_1^2}{s_2^2}F_{\alpha}(m-1,n-1)\right)=1-\alpha$$

$$\left(0\,,\ \frac{s_1^2}{s_2^2}F_{\alpha}(m-1,n-1)\right)$$

$$P(\theta \le T)=1-\alpha,$$

$$P\left(\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \le F_{\alpha}(n-1,m-1)\right)=1-\alpha$$

错



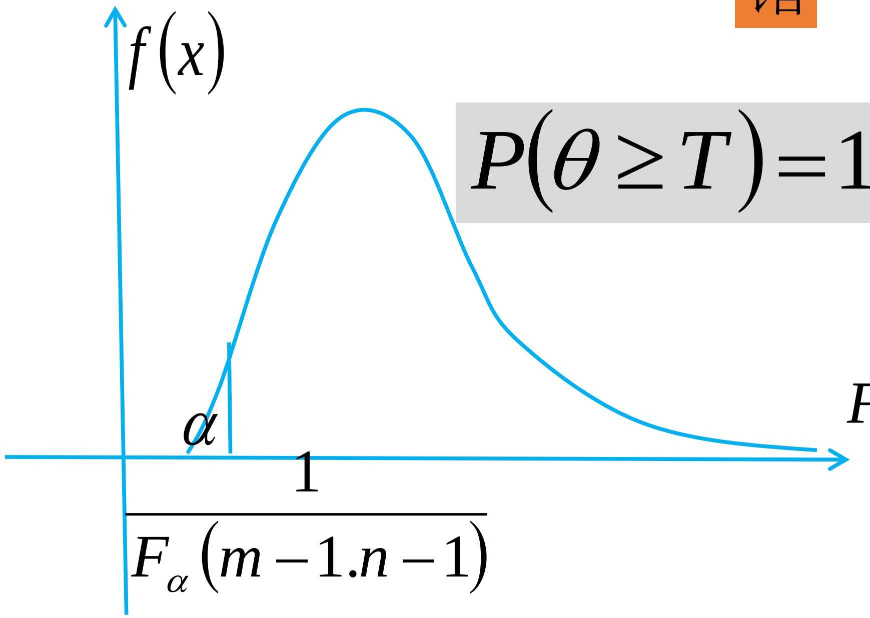
$$P\left(\frac{s_1^2/s_2^2}{F_{\frac{\alpha}{2}}(n-1,m-1)} \le \sigma_1^2/\sigma_2^2 \le \frac{s_1^2}{s_2^2}F_{\frac{\alpha}{2}}(m-1,n-1)\right)=1-\alpha$$

(3) σ_1^2/σ_2^2 单侧置信下限

$$P\left(\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \geq \frac{1}{F_\alpha(m-1,n-1)}\right)=1-\alpha$$

错

$$\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n-1,m-1)$$



$$P(\theta \geq T)=1-\alpha,$$

$$P\left(\sigma_1^2/\sigma_2^2 \geq \frac{s_1^2/s_2^2}{F_\alpha(n-1,m-1)}\right)=1-\alpha$$

$$\boxed{\left(\frac{s_1^2/s_2^2}{F_\alpha(n-1,m-1)}, +\infty\right)}$$

$$P\left(\frac{s_1^2/s_2^2}{F_{\frac{\alpha}{2}}(n-1,m-1)} \leq \sigma_1^2/\sigma_2^2 \leq \frac{s_1^2}{s_2^2}F_\alpha(m-1,n-1)\right)=1-\alpha$$

例6.设用两种不同的方法冶炼某种结束材料，分别抽样测试其杂质含量（单位：%）得到如下数据：

原始冶炼方法：26.9,22.3,27.2,25.1,22.8,24.2,30.2,25.7,26.1

新的冶炼方法：22.6,24.3,23.4,22.5,21.9,20.6,20.6,23.5

假设两种冶炼方法的杂质含量X,Y都服从正态分布，且方差σ_1^2，σ_2^2均未知，求方差比 $\boxed{\sigma_1^2/\sigma_2^2}$ 的置信度为0.9的置信下限。

$$\boxed{S_1^2=5.826,\ n=9;\quad S_2^2=1.799,\ m=8}$$

$S_1^2=5.826,\ n=9;\quad S_2^2=1.799,\ m=8\ \ 1-\alpha=0.9$

置信度为0.9的下限

$$\text{解：}\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2}\sim F(n-1,m-1)$$

$$P\left(\sigma_1^2/\sigma_2^2\ge\frac{s_1^2/s_2^2}{F_\alpha(n-1,m-1)}\right)=1-\alpha$$

$$P\left(\frac{1}{F_{\frac{\alpha}{2}}(m-1,n-1)}\le\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2}\le F_{\frac{\alpha}{2}}(n-1,m-1)\right)=1-\alpha$$

$$P\left(\frac{s_1^2/s_2^2}{F_{\frac{\alpha}{2}}(n-1,m-1)}<\sigma_1^2/\sigma_2^2<\frac{s_1^2}{s_2^2}F_{\frac{\alpha}{2}}(m-1,n-1)\right)=1-\alpha$$

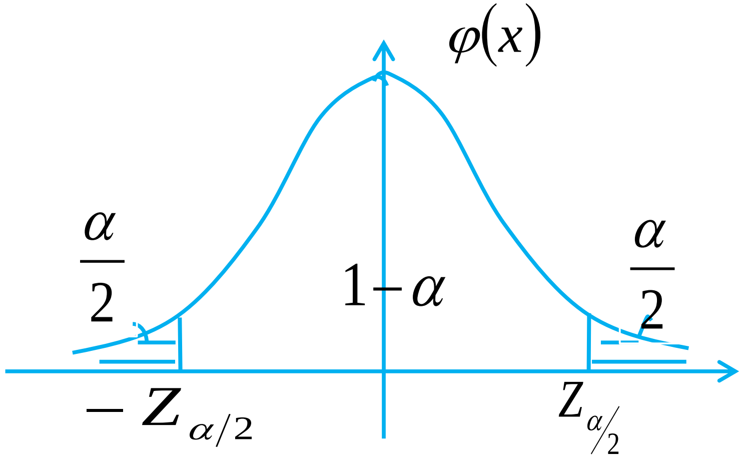
$$F_{0.1}(8,7)=2.75$$

$$\frac{\sigma_1^2}{\sigma_2^2}:\left(\frac{s_1^2/s_2^2}{F_\alpha(n-1,m-1)},+\infty\right)=\left(\frac{5.82/1.79}{2.75},+\infty\right)=(1.079,+\infty)$$

5.$\mu_1-\mu_2$置信区间 $(\sigma_1^2,\sigma_2^2$已知$)$

$$P(T_1\le\theta\le T_2)=1-\alpha$$

$$\frac{(\overline{X}_n-\overline{Y}_m)-(\mu_1-\mu_2)}{\sqrt{(\sigma_1^2/n)+(\sigma_2^2/m)}}\sim N(0,1)$$



$$P\left\{-Z_{\frac{\alpha}{2}}<\frac{(\overline{X}_n-\overline{Y}_m)-(\mu_1-\mu_2)}{\sqrt{(\sigma_1^2/n)+(\sigma_2^2/m)}}<Z_{\frac{\alpha}{2}}\right\}=1-\alpha$$

$\mu_1-\mu_2$的置信区间为: $\boxed{(\overline{X}_n-\overline{Y}_m)\pm Z_{\frac{\alpha}{2}}\sqrt{(\sigma_1^2/n)+(\sigma_2^2/m)}}$

例7.设用两种不同的方法冶炼某种结束材料，分别抽样测试其杂质含量（单位：%）得到如下数据：

原始冶炼方法：26.9,22.3,27.2,25.1,22.8,24.2,30.2,25.7,26.1

新的冶炼方法：22.6,24.3,23.4,22.5,21.9,20.6,20.6,23.5

假设两种冶炼方法的杂质含量*X*,*Y*都服从正态分布

$$X \sim N\left(\mu_1, 2.1^2\right), Y \sim N\left(\mu_2, 1.3^2\right).$$

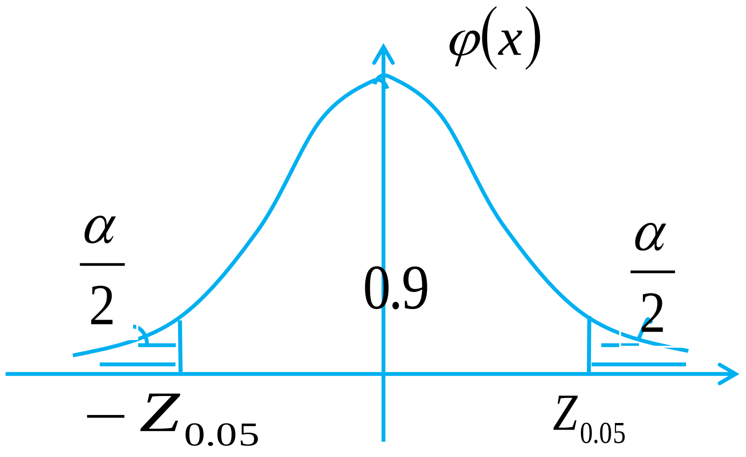
求两不同方法冶炼的材料，杂质含量均数差的置信度为0.9的置信区间。

$$\boxed{\bar{x} = 25.11 \quad n = 9; \quad \bar{y} = 22.42 \quad m = 8}$$

$\bar{x} = 25.11 \quad n = 9; \quad \bar{y} = 22.42, \quad m = 8 \qquad X \sim N\left(\mu_1, 2.1^2\right), Y \sim N\left(\mu_2, 1.3^2\right).$

$$1-\alpha = 0.9$$

解：$\dfrac{\left(\overline{X}_n - \overline{Y}_m\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\left(\sigma_1^2/n\right) + \left(\sigma_2^2/m\right)}} \sim N(0,1)$



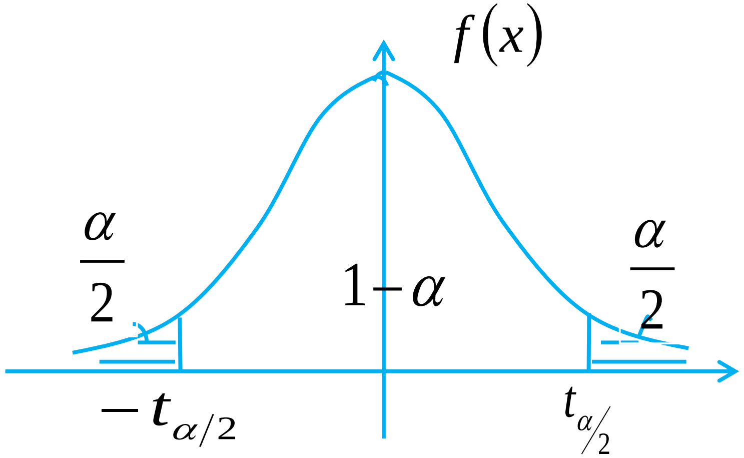
$$P\left\{-Z_{0.05} < \frac{\left(\overline{X}_n - \overline{Y}_m\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\left(\sigma_1^2/n\right) + \left(\sigma_2^2/m\right)}} < Z_{0.05}\right\} = 0.9$$

$\mu_1 - \mu_2$的置信区间为：$\left(\overline{X}_n - \overline{Y}_m\right) \pm Z_{0.05}\sqrt{\left(\sigma_1^2/n\right) + \left(\sigma_2^2/m\right)}$

$$(25.11 - 22.42) \pm 1.64\sqrt{\left(2.1^2/9\right) + \left(1.3^2/8\right)}$$

6. $\mu_1-\mu_2$置信区间 $\left(\sigma_1^2=\sigma_2^2\text{未知}\right)$

$$\frac{\left(\overline{X}_n-\overline{Y}_m\right)-\left(\mu_1-\mu_2\right)}{s_w\sqrt{\left(1/n\right)+\left(1/m\right)}}\sim t\left(n+m-2\right)$$



$$P\left\{-t_{\frac{\alpha}{2}}<\frac{\left(\overline{X}_n-\overline{Y}_m\right)-\left(\mu_1-\mu_2\right)}{s_w\sqrt{\left(1/n\right)+\left(1/m\right)}}<t_{\frac{\alpha}{2}}\right\}=1-\alpha$$

$\mu_1-\mu_2$置信区间为: $\boxed{\left(\overline{X}_n-\overline{Y}_m\right)\pm t_{\frac{\alpha}{2}}\left(n+m-2\right)s_w\sqrt{\left(1/n\right)+\left(1/m\right)}}$

$\mu_1-\mu_2$置信区间为:

$$\left(\overline{X}_n-\overline{Y}_m\right)\pm t_{\frac{\alpha}{2}}(n+m-2)s_w\sqrt{(1/n)+(1/m)}$$

$\mu_1-\mu_2$置信区间上限:

$$\left(\overline{X}_n-\overline{Y}_m\right)+t_{\alpha}(n+m-2)s_w\sqrt{(1/n)+(1/m)}$$

$\mu_1-\mu_2$置信区间下限:

$$\left(\overline{X}_n-\overline{Y}_m\right)-t_{\alpha}(n+m-2)s_w\sqrt{(1/n)+(1/m)}$$

$$s_w^2=\frac{(n-1)S_1^2+(m-1)S_2^2}{n-1+m-1}$$