3. 常用分布的数学期望

(1) $\boxed{X \sim B(1, p) \quad E(X) = p}$

$$E(X) = 0 \times (1-p) + 1 \times p = p$$

$$E(X^2) = 0^2 \times (1-p) + 1^2 \times p = p$$

(2) $\boxed{X \sim B(n, p) \quad E(X) = np}$

$$P(X = k) = C_n^k p^k q^{n-k} \quad k = 0,1,2\ldots n$$

$$E(X) = \sum_{k=0}^{n} k C_n^k p^k q^{n-k} = \sum_{k=0}^{n} \cancel{k} \frac{n!}{\cancel{k}!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!\,p}{(k-1)!((n-1)-(k-1))!} p^{k-1} q^{(n-1)-(k-1)}$$

$$= np \boxed{\sum_{k-1=0}^{n-1} C_{n-1}^{k-1} p^{k-1} q^{(n-1)-(k-1)}}$$

$$= (p+q)^{n-1} np = np$$

(2)证：$X = \sum_{k=1}^{n} X_k$; $X_k \sim B(1, p), k = 0,1\cdots n$

$$E\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} EX_k = nEX_k = np$$

$$E\left(X^2\right)=\sum_{k=0}^{n} k^2 C_n^k p^k q^{n-k}=\sum_{k=0}^{n}\left(k^2-k+k\right)C_n^k p^k q^{n-k}$$

$$=\sum_{k=2}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k q^{n-k}+\boxed{\sum_{k=0}^{n} k C_n^k p^k q^{n-k}}$$

$$=\sum_{k-2=0}^{n-2}\frac{p^2 n(n-1)(n-2)!}{(k-2)!\left((n-2)-(k-2)\right)!}p^{k-2}q^{(n-2)-(k-2)}+np$$

$$\underset{(t=k-2)}{=}p^2 n(n-1)\sum_{t=0}^{n-2}\frac{(n-2)!}{t!\left((n-2)-t\right)!}p^t q^{(n-2)-t}+np$$

$$=p^2 n(n-1)(p+q)^{n-2}+np=p^2 n(n-1)+np$$

(3) $\boxed{X \sim P(\lambda) \quad E(X)=\lambda}$

$$\left(\sum_{n=0}^{+\infty} \frac{x^n}{n\,!} = e^x \right)$$

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda} \quad k=0,1,2....$$

$$E(X)=\sum_{k=0}^{+\infty} k\frac{\lambda^k}{k!}e^{-\lambda}=\sum_{k=1}^{+\infty}\frac{\lambda^{k-1}\lambda}{(k-1)!}e^{-\lambda} \underset{t=k-1}{=} \lambda\sum_{t=0}^{+\infty}\frac{\lambda^t}{t\,!}e^{-\lambda} = \lambda$$

$$E(X^2)=\sum_{k=0}^{+\infty} k^2\frac{\lambda^k}{k!}e^{-\lambda}=\sum_{k=0}^{+\infty}(k^2-k+k)\frac{\lambda^k}{k!}e^{-\lambda}=\lambda^2\sum_{k=2}^{+\infty}k(k-1)\frac{\lambda^{k-2}}{k\,!}e^{-\lambda}+\lambda$$

$$=\lambda^2\sum_{k-2=0}^{+\infty}\frac{\lambda^{k-2}}{(k-2)!}e^{-\lambda}+\lambda \; = \lambda^2+\lambda$$

(4) $\boxed{X \sim G(p) \quad E(X) = 1/p}$

$$P(X = k) = p(1-p)^{k\text{-}1} \quad k = 1,2\ldots$$

$$E(X) = \sum_{k=1}^{+\infty} kp(1-p)^{k\text{-}1} = \sum_{k=1}^{+\infty} kpq^{k\text{-}1} = p\boxed{\sum_{k=1}^{+\infty} kq^{k\text{-}1}}$$

$$\frac{(1-q)+q}{(1-q)^2} = \frac{1}{p^2}$$

设 $f(q) = \sum_{k=1}^{+\infty} q^k = \dfrac{q}{1-q}$， 则 $f'(q) = \left(\sum_{k=1}^{+\infty} q^k\right)' = \left(\dfrac{q}{1-q}\right)'$

$$E(X) = p\sum_{k=1}^{+\infty} kq^{k\text{-}1} = p\frac{1}{p^2} = \frac{1}{p}$$

$$E\left(X^2\right)=\sum_{k=1}^{+\infty}k^2p(1-p)^{k\text{-}1}=\sum_{k=1}^{+\infty}k^2pq^{k\text{-}1}=p\sum_{k=1}^{+\infty}k(k-1)q^{k\text{-}1}+\frac{1}{p}$$

设 $f(q)=\sum_{k=1}^{+\infty}q^k=\frac{q}{1-q}$，则 $f''(q)=\left(\sum_{k=1}^{+\infty}q^k\right)''=\left(\frac{q}{1-q}\right)''$

有 $\sum_{k=1}^{+\infty}k(k-1)q^{k-2}=2\frac{1}{p^3}$

$$E\left(X^2\right)=p\sum_{k=1}^{+\infty}k(k-1)q^{k\text{-}1}+\frac{1}{p}$$

$$=pq\sum_{k=1}^{+\infty}k(k-1)q^{k\text{-}2}+\frac{1}{p}\quad=pq\left(2\frac{1}{p^3}\right)+\frac{1}{p}\quad=\frac{1+q}{p^2}$$

(5) $\boxed{X \sim U(a,b) \quad E(X)=\frac{a+b}{2}}$

$$f(x)=\begin{cases}\frac{1}{b-a}, & a<x<b; \\ 0, & \text{其他}\end{cases}$$

$$EX=\int_{-\infty}^{+\infty} x f(x) dx=\int_a^b x \frac{1}{b-a} dx=\frac{a+b}{2}$$

$$E\left(X^2\right)=\int_{-\infty}^{+\infty} x^2 f(x) dx=\int_a^b x^2 \frac{1}{b-a} dx=\frac{a^2+ab+b^2}{3}$$

(6) $\boxed{X \sim e(\lambda) \quad E(X) = \frac{1}{\lambda}}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0; \\ 0, & x \le 0. \end{cases}$$

$$EX = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^{+\infty} x \lambda e^{-\lambda x} dx = -\int_0^{+\infty} x de^{-\lambda x}$$

$$= -xe^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$EX^2 = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx = -\int_0^{+\infty} x^2 de^{-\lambda x}$$

$$= -x^2 e^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} 2xe^{-\lambda x} dx = \int_0^{+\infty} 2xe^{-\lambda x} dx = \frac{2}{\lambda}\int_0^{+\infty} x\lambda e^{-\lambda x} dx$$

$$= -\frac{2}{\lambda} xe^{-\lambda x}\Big|_0^{\infty} + \frac{2}{\lambda}\int_0^{+\infty} e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$(7)\ \boxed{X \sim N\left(\mu,\sigma^2\right) \quad E\left(X\right)=\mu}$$

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty<x<+\infty$$

$$EX=\int_{-\infty}^{+\infty}xf\left(x\right)dx=\int_{-\infty}^{+\infty}x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$\underset{dx=\sigma dt}{\overset{t=\frac{x-\mu}{\sigma}}{=}}\int_{-\infty}^{+\infty}\left(t\sigma+\mu\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

$$=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}te^{-\frac{t^2}{2}}dt+\frac{\mu}{\sqrt{2\pi}}\boxed{\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2}}dt}=\mu$$

$$\boxed{\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}}$$

1.期望的定义

$$P(X=x_i)=p_i\ i=1,2\ldots,$$

$$EX=\sum_{i=1}^{\infty}x_ip_i$$

$$Y=g(X),$$

$$E(Y)=E(g(X))=\sum_{i=1}^{\infty}g(x_i)p_i$$

X的密度函数为$f(x)$,

$$EX=\int_{-\infty}^{+\infty}xf(x)dx$$

$$EY=E(g(X))=\int_{-\infty}^{+\infty}g(x)f(x)dx$$

分布列为$P(X=x_i,Y=y_j)=p_{ij}$

$$Z=g(X,Y)$$

$$EZ=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}g(x_i,y_j)p_{ij}\text{。}$$

联合密度为$f(x,y)$

$$EZ=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}g(x,y)f(x,y)dxdy.$$

2.期望的性质

(1) $Ec = c$,

(2) $E(cX) = cEX$.

(3) $E(aX+b) = aE(X)+b$

(4) $E(X \pm Y) = EX \pm EY$

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = EX$$

(5) $E(XY) = E(X)E(Y)$.

(X与Y独立)

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(3) $X \sim P(\lambda) \quad E(X) = \lambda$

(4) $X \sim G(p) \quad E(X) = 1/p$

(5) $X \sim U(a, b) \quad E(X) = \dfrac{a+b}{2}$

(6) $X \sim e(\lambda) \; E(X) = \dfrac{1}{\lambda}$

(7) $X \sim N(\mu, \sigma^2) \quad E(X) = \mu$