三. 点估计优良性的评选标准

1.无偏性

2.有效性

3.一致性（相合性）

1.无偏性 若θ的估计量$\hat{\theta}=\hat{\theta}(X_1, X_2\cdots X_n)$，满足$E(\hat{\theta})=\theta$，

则称$\hat{\theta}$是θ的无偏估计量。

例1.X_1, X_2, X_3是总体X的一个样本，问下列估计量哪一个是总体均数μ的无偏估计量？$\hat{\mu}_1=\frac{1}{6}X_1+\frac{1}{3}X_2+\frac{1}{2}X_3$，

$\hat{\mu}_2=\frac{2}{5}X_1+\frac{2}{5}X_2+\frac{1}{5}X_3$，$\hat{\mu}_3=\frac{1}{3}X_1+\frac{2}{9}X_2+\frac{1}{7}X$

解：$E(\hat{\mu}_1)=\frac{1}{6}EX_1+\frac{1}{3}EX_2+\frac{1}{2}EX_3=\mu$ 无偏

$E(\hat{\mu}_2)=\frac{2}{5}EX_1+\frac{2}{5}EX_2+\frac{1}{5}EX_3=\mu$ 无偏

$$E(\hat{\mu}_3) = \frac{1}{3}EX_1 + \frac{2}{9}EX_2 + \frac{1}{7}EX_3 = \frac{44}{63}\mu \quad \text{有偏}$$

例2. 证明 $\boxed{\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ 是 } EX = \mu \text{ 的无偏估计量。}}$

证：$E\left(\overline{X}_n\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} EX_i = EX = \mu,$

则$\overline{X}_n$是EX的无偏估计量。

例3. 证明S^2是σ^2的无偏估计量， B_2是σ^2的有偏估计量。

$$\text{证明：}\quad B_2=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2=\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2-2\overline{X}_nX_i+\overline{X}_n^2\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}X_i^2-2\overline{X}_n\boxed{\frac{1}{n}\sum_{i=1}^{n}X_i}+\overline{X}_n^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\overline{X}_n^2\left(=A_2-A_1^2\right)$$

$$E(B_2)=E\left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2\right\}=E\left\{\frac{1}{n}\sum_{i=1}^{n}X_i^2-\overline{X}_n^2\right\}$$

$$=\left\{\frac{1}{n}\sum_{i=1}^{n}EX_i^2-E\overline{X}_n^2\right\}=EX^2-E\overline{X}_n^2$$

$$EB_2 = EX^2 - E\overline{X}_n^{\,2}$$

$$= \left\{DX + (EX)^2\right\} - \left\{D\overline{X}_n + \left(E\overline{X}_n\right)^2\right\}$$

$$= \sigma^2 + \mu^2 - \left(\frac{\sigma^2}{n} + \mu^2\right) = \frac{(n-1)\sigma^2}{n}$$

B_2是σ^2的有偏估计量，修正B_2，$E\left(\frac{n}{n-1}B_2\right) = \sigma^2$

令$\boxed{S^2 = \frac{n}{n-1}B_2}$，有$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$，$ES^2 = \sigma^2$无偏

2.有效性

定义：设$\hat{\theta}_1$，$\hat{\theta}_2$是θ无偏估计量，如果$D(\hat{\theta}_1) \le D(\hat{\theta}_2)$，

则称$\hat{\theta}_1$比$\hat{\theta}_2$更有效。

3.一致性（相合性）

定义：设$\hat{\theta}_n = g(X_1, X_2, \ldots X_n)$是$\theta$的一个估计量，若对$\forall \varepsilon > 0$有

$$\lim_{n \to \infty} P\left(\left|\hat{\theta}_n - \theta\right| < \varepsilon\right) = 1 \qquad \left(\hat{\theta}_n \xrightarrow{p} \theta\right)$$

则称$\hat{\theta}_n$是θ的一致估计。

矩估计都是一致估计，极大似然估计是否是一致估计需要验证。

例1. X_1, X_2, X_3 是总体 X 的一个样本，问下列估计量哪一个是总体均数 μ 的无偏估计量？ $\hat{\mu}_1 = \frac{1}{6}X_1 + \frac{1}{3}X_2 + \frac{1}{2}X_3$,

$$\hat{\mu}_2 = \frac{2}{5}X_1 + \frac{2}{5}X_2 + \frac{1}{5}X_3,\quad \hat{\mu}_3 = \frac{1}{3}X_1 + \frac{2}{9}X_2 + \frac{1}{7}X$$

解： $E(\hat{\mu}_1) = \frac{1}{6}EX_1 + \frac{1}{3}EX_2 + \frac{1}{2}EX_3 = \mu$ 无偏 $D(\hat{\mu}_1) = \frac{14}{36}DX$

$E(\hat{\mu}_2) = \frac{2}{5}EX_1 + \frac{2}{5}EX_2 + \frac{1}{5}EX_3 = \mu$ 无偏 $D(\hat{\mu}_2) = \frac{9}{25}DX$

$E(\hat{\mu}_3) = \frac{1}{3}EX_1 + \frac{2}{9}EX_2 + \frac{1}{7}EX_3 = \frac{44}{63}\mu$ 有偏

$$D(\hat{\mu}_1) > D(\hat{\mu}_2)$$

例2.设总体X~$U(0，\theta)$，求 θ 矩估计，极大似然估计，验证无偏性，比较有效性。

解：$EX=\dfrac{\theta}{2}$，　令$\overline{X}=EX$，　得$\hat{\theta}_{矩}=2\overline{X}$。

$$L(\theta)=\prod_{i=1}^{n}\frac{1}{\theta}=\frac{1}{\theta^{n}},\quad\left(\frac{1}{\theta^{n}}\right)'\neq 0$$

若使$L(\theta)$极大，需$\theta\to 0$，但$0<x<\theta$

则$\hat{\theta}_{极大}=X_{(n)}$。

验证无偏性：　$E(\hat{\theta}_{矩})=E(2\overline{X})=2E\overline{X}=\theta$；　$\hat{\theta}_{矩}$无偏

$E\hat{\theta}_{极大}=EX_{(n)}$，令$Z=\hat{\theta}_{极大}=X_{(n)}$，　$EZ=\int_{-\infty}^{+\infty}zf_Z(z)dz$

$$F_z(z) = P(Z \le z) = P(\max(X_1, X_2, \cdots X_n) \le z) = P(X_1 \le z, X_2 \le z, \cdots X_n \le z)$$

$$= (P(X \le z))^n = \left(\frac{z}{\theta}\right)^n \qquad F(x) = \frac{x}{\theta}$$

$$f_z(x) = F_z'(z) = \left\{\left(\frac{z}{\theta}\right)^n\right\}' = n\frac{\mathrm{z}^{\mathrm{n-1}}}{\theta^{\mathrm{n}}}$$

$$f_z(x) = \begin{cases} n\dfrac{\mathrm{z}^{\mathrm{n-1}}}{\theta^{\mathrm{n}}}, & 0 < x < \theta \\ 0, & \text{其他。} \end{cases}$$

$$E\hat{\theta}_{\text{极大}} = EZ = \int_0^\theta zn\frac{\mathrm{z}^{\mathrm{n-1}}}{\theta^{\mathrm{n}}}dz = \frac{n}{n+1}\theta,$$

$$\text{修正}T = \frac{n+1}{n}\hat{\theta}_{\text{极大}},$$

$$ET = \theta$$

矩估计是无偏估计，极大似然估计是有偏估计， 修正后的T无偏。

$$\boxed{D(\hat{\theta}_{矩})} = D(2\overline{X}) = 4D\overline{X} = 4\frac{DX}{n} = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{\boxed{3n}}$$

$$EZ^2 = \int_0^{\theta} z^2 n\frac{z^{n-1}}{\theta^n}dz = \frac{n}{n+2}\theta^2$$

当$n>1$时，$n(n+2)>3n$，修正后的极大似然估计比矩估计更有效。

$$D\hat{\theta}_{极大} = DZ = EZ^2 - (EZ)^2$$

$$= \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$\boxed{DT} = D\left(\frac{n+1}{n}\hat{\theta}_{极大}\right) = \frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{\boxed{n(n+2)}}$$