准备工作：

一.定义基本概念（1.总体，2.样本）

构造统计量（4.统计量）

3. 样本的联合分布

$$\mu \xleftarrow{P} \overline{X}$$

$$\sigma^2 \longleftarrow S^2$$

$$\mu_1 - \mu_2 \leftarrow \overline{X} - \overline{Y}$$

$$\sigma_1^2 / \sigma_2^2 \leftarrow S_1^2 / S_2^2$$

$\bar{x} = 7.9$

μ在7.9左右

$$\overline{X} \sim N\left(\mu, \sigma^2/n\right)$$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

二.定义统计量的分布

$N(0,1)$分布（σ已知）

$\chi^2(n)$分布

$t(n)$分布（σ未知）

$F(n,m)$分布

三.推导抽样分布

$\overline{X}$的分布

S^2的分布

$\overline{X} - \overline{Y}$的分布

S_1^2 / S_2^2的分布

三．正态总体的抽样分布

1.抽样分布定理一 $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ （σ已知）

2.抽样分布定理二 $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

3.抽样分布定理三 $\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n)$ （σ未知）

4.抽样分布定理四 $\dfrac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n_1,n_2)$

1.抽样分布定理一 $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ （σ已知）

设总体$X \sim N(\mu, \sigma^2)$，$X_1, X_2 \cdots X_n$为简单随机样本，相互独立

$\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$为样本均数，则有 $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ （σ已知）

证明：因$X \sim N(\mu, \sigma^2)$，由正态分布的性质有 $\overline{X_n} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$

$$\frac{\overline{X_n}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \qquad (\sigma\text{已知})$$

2.抽样分布定理二 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

设总体$X \sim N(\mu, \sigma^2)$, $X_1, X_2 \cdots X_n$为简单随机样本, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$,

为样本均数，$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ 为样本方差，则有

(1) $\overline{X}$与S^2相互独立。 (2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$。

$\overline{X}$

说明：总体$X \sim N(\mu,\sigma^2)$, $X_1, X_2 \cdots X_n$为样本， $X_i \sim N(\mu,\sigma^2)$

$i = 1,2,\cdots n$

$$\sum_{i=1}^{n}\left(\frac{X_i - \boxed{\mu}}{\sigma}\right)^2 \sim \chi^2(n) \quad \xrightarrow[\text{已被证明}]{\text{需要证明}} \quad \sum_{i=1}^{n}\left(\frac{X_i - \boxed{\overline{X}}}{\sigma}\right)^2 \sim \chi^2(n-1)$$

$$S^2 = \frac{1}{n\text{-}1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \longrightarrow \sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = (n-1)S^2$$

$$\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2 = \frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{\sigma^2} = \boxed{\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)}$$

3.抽样分布定理三 $\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$ σ未知

设总体$X \sim N(\mu,\sigma^2)$, $X_1, X_2 \cdots X_n$为简单随机样本，相互独立

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ 为样本均数， $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ 为样本方差，

则有： $$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

证明：总体$X \sim N(\mu, \sigma^2)$，$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ 为样本均数，则

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1), \text{ 又 } \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\left(t = \frac{X}{\sqrt{Y / n}} \sim t(n) \right)$$

且$\overline{X}$与S^2相互独立，则

$$t = \frac{\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2} \Big/ (n-1)}} = \boxed{\frac{\overline{X} - \mu}{S / \sqrt{n}} \sim t(n-1)}$$

4.抽样分布定理四

总体$X \sim N(\mu,\sigma^2)$,样本$X_1, X_2 \cdots X_n$，样本均数 $\overline{X}_n$ 样本方差 S_1^2

总体$Y \sim N(\mu,\sigma^2)$,样本$Y_1, Y_2 \cdots Y_m$ 样本均数$\overline{Y}_m$ 样本方差 S_2^2

X与Y独立，则

(1) $$\frac{(\overline{X}_n - \overline{Y}_m) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n} + \dfrac{\sigma_2^2}{m}}} \sim N(0,1), \quad (\sigma_1^2 \text{，} \sigma_2^2 \text{已知})$$

(2) $$\frac{(\overline{X}_n - \overline{Y}_m) - (\mu_1 - \mu_2)}{s_w\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}} \sim t(n+m-2) \quad (\sigma_1^2 = \sigma_2^2 \text{未知})$$

(3) $$\frac{S_1^2 / S_2^2}{\sigma_1^2 / \sigma_2^2} \sim F(n-1, m-1)$$

$$\left(\text{其中} S_\omega^2 = \frac{(n-1)S_1^2 - (m-1)S_2^2}{n+m-2}\right)$$

(1)证明：$X \sim N(\mu_1, \sigma_1^2)$，$\overline{X}_n \sim N\left(\mu_1, \dfrac{\sigma_1^2}{n}\right)$，$\dfrac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2(n-1)$

$Y \sim N(\mu_2, \sigma_2^2)$，$\overline{Y}_m \sim N\left(\mu_2, \dfrac{\sigma_2^2}{m}\right)$，$\dfrac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2(m-1)$ X与Y独立

则 $\overline{X}_n - \overline{Y}_m \sim N\left(\mu_1 - \mu_2, \dfrac{\sigma_1^2}{n} + \dfrac{\sigma_2^2}{m}\right)$，

$$\boxed{\frac{(\overline{X}_n - \overline{Y}_m) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n} + \dfrac{\sigma_2^2}{m}}} \sim N(0,1) \quad (\sigma_1^2，\sigma_2^2\text{已知})}$$

(2)证明：$(\sigma_1^2 = \sigma_2^2$记为σ^2未知$)$

$$\overline{X}_n - \overline{Y}_m \sim N\left(\mu_1 - \mu_2, \left(\frac{1}{n} + \frac{1}{m}\right)\sigma^2\right),$$

$$\frac{\left(\overline{X}_n-\overline{Y}_m\right)-\left(\mu_1-\mu_2\right)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}\sim N(0,1)$$

$$\frac{(m-1)S_2^2}{\sigma_2^2}=\frac{(m-1)S_2^2}{\sigma^2}\sim\chi^2(m-1)$$

$$\frac{(n-1)S_1^2}{\sigma_1^2}=\frac{(n-1)S_1^2}{\sigma^2}\sim\chi^2(n-1),$$

$$\frac{(n-1)S_1^2}{\sigma^2}+\frac{(m-1)S_2^2}{\sigma^2}=\frac{(n-1)S_1^2+(m-1)S_2^2}{\sigma^2}\sim\chi^2(n+m-2)$$

$\overline{X}_n-\overline{Y}_m$与$S_1^2$及$S_2^2$的函数独立。

$$t=\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)\Big/\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}{\sqrt{\frac{(n-1)S_1^2+(m-1)S_2^2}{\sigma^2}\Big/n+m-2}}$$

$$\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}\sim N(0,1)$$

$$\frac{(n-1)S_1^2+(m-1)S_2^2}{\sigma^2}\sim\chi^2(n+m-2)$$

$$=\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)}{\sqrt{\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\sqrt{\frac{1}{n}+\frac{1}{m}}}\sim t(n+m-2)$$

$$\left(t=\frac{X}{\sqrt{Y/n}}\sim t(n)\right)$$

$$=\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{n}+\frac{1}{m}}}$$

记 $S_w=\sqrt{\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}$

$$S_w=\sqrt{\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}$$

称 $S_w^2=\dfrac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}$ 为合并方差

$$\boxed{t=\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{n}+\frac{1}{m}}}\sim t(n+m-2)}$$

$$\left(z=\frac{\left(\overline{X}_n-\overline{Y}_m\right)-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}\right)$$

关于合并方差：

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2=\frac{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2}{n-1};$$

$$(n-1)S^2=\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2$$

$$S_w^2=\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}=\frac{\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2+\sum_{j=1}^{m}\left(Y_j-\overline{Y}_m\right)^2}{n-1+m-1}$$

3) S_1^2/S_2^2 的分布

证明：由于 $\dfrac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2(n-1)$ $\dfrac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2(m-1)$

S_1^2与S_2^2独立

$$F = \frac{\dfrac{(n-1)S_1^2}{\sigma_1^2}\Big/(n-1)}{\dfrac{(m-1)S_2^2}{\sigma_2^2}\Big/(m-1)} = \boxed{\frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n-1, m-1)}$$

参数μ：$\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$（$\sigma$已知）；$\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$（$\sigma$未知）

参数σ^2：$\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$； 参数$\dfrac{\sigma_1^2}{\sigma_2^2}$：$\dfrac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} \sim F(n-1, m-1)$

参数$\mu_1-\mu_2$：$\dfrac{(\overline{X}_n-\overline{Y}_m)-(\mu_1-\mu_2)}{\sqrt{(\sigma_1^2/n)+(\sigma_2^2/m)}} \sim N(0,1)$（$\sigma_1^2$，$\sigma_2^2$已知）

$$\frac{(\overline{X}_n-\overline{Y}_m)-(\mu_1-\mu_2)}{s_w\sqrt{(1/n)+(1/m)}} \sim t(n+m-2)\ (\sigma_1^2=\sigma_2^2\text{未知})$$

例1.$X_1, X_2 \cdots X_n$， X_{n+1}是正太总体$X \sim N\left(\mu, \sigma^2\right)$的样本，

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$

试证明统计量 $U = \sqrt{\frac{n-1}{n+1}}\frac{\boxed{X_{n+1} - \overline{X}_n}}{\boxed{S_n}} \sim t(n-1)$

证明：

$$\left.\begin{aligned} X_{n+1} &\sim N\left(\mu, \sigma^2\right) \\ \overline{X}_n &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \end{aligned}\right\} \Rightarrow X_{n+1} - \overline{X}_n \sim N\left(0, \frac{n+1}{n}\sigma^2\right),$$

$$\frac{X_{n+1} - \overline{X}_n}{\sigma\sqrt{n+1/n}} \sim N(0,1), \quad X_{n+1} - \overline{X}_n\text{与}S_n^2\text{独立}$$

$$又S^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2,\quad 有\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1),$$

$$(n-1)S^2=\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2$$

$$\frac{\boxed{\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2}}{\sigma^2}\sim\chi^2(n-1),$$

$$S_n^2=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2,\ nS_n^2=\boxed{\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2,}$$

$$则\boxed{\frac{nS_n^2}{\sigma^2}\sim\chi^2(n-1);}\qquad\boxed{\frac{X_{n+1}-\overline{X}_n}{\sigma\sqrt{n+1/n}}\sim N(0,1),}$$

$$U=\frac{X_{n+1}-\overline{X}_n}{\sigma\sqrt{n+1/n}}\Bigg/\sqrt{\frac{nS_n^2}{\sigma^2}\Big/n-1}=\sqrt{\frac{n-1}{n+1}}\frac{X_{n+1}-\overline{X}_n}{S_n}\sim t(n-1)$$