# Classical Electrodynamics

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# Chapter 1

# Vector and Tensor Analysis

A physical observable can be described by a geometric object. We shall describe the observables in classical electrodynamics mathematically in terms of scalars, pseudoscalars, vectors, pseudovectors, tensors or pseudotensors.

*Convention.*–For convenience, in this chapter we adopt the convention that Latin indices $i, j, k, l, ...$ run over the range $1, 2, 3$ to denote vector or tensor components in the real Euclidean three-dimensional (3D) configuration space $\mathbb{R}^3$, and Greek indices $\mu, \nu, \lambda, ...$, which are used in four-dimensional (4D), run over the range $0, 1, 2, 3$. Finally, in a few cases where the generic $M$-dimensional space is considered, we may use either Latin indices $i, j, k, l, ...$ or Greek indices $\mu, \nu, \lambda, ...$ to denote the components, running over the range $1, 2, ..., M$.

## 1.1 Definitions

Let us consider an $M$-dimensional vector space which is defined through the $M$ orthogonal bases $\hat{e}_1, \hat{e}_2, ..., \hat{e}_M$. For example, our real space is a three-dimensional (3D) Euclidean space with the bases usually denoted by $\hat{e}_x, \hat{e}_y, \hat{e}_z$. A vector in the the $M$-dimensional space can be defined by

$$\mathbf{A} = \sum_j^M A^j \hat{e}_j, \tag{1.1}$$

where $A^j$ is the $j$th component of the vector. For an intuitive definition, a vector is a quantity which has both magnitude and direction, and $A_j$th represents the component of the vector long the direction $\hat{e}_j$. In the Euclidian space, the magnitude, namely, the norm of a vector is defined by

$$|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + ... + A_M^2}. \tag{1.2}$$

The 3D position and velocity vectors in Cartesian coordinate system are given by

$$\mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z. \tag{1.3}$$

$$\mathbf{v} = v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z. \tag{1.4}$$

In contrast, a scalar $a$ is a quantity which does not have a direction (no components) but only has a magnitude. Actually, the norm of a vector is a scalar. For example,

in 3D real space the mass $m$, mass density $\rho$, light intensity $I$, etc, are scalar physical quantities.

The definitions of vector and scalar can be generalized to tensor, for which we extend the vector bases $\{\hat{e}_j; j = 1, ..., M\}$ to the tensor bases

$$\{\hat{e}_{j_1}\hat{e}_{j_2}...\hat{e}_{j_N}; j_1, j_2, ..., j_N = 1, ..., M\},$$

which is a group of $M^N$ elements. For the 3D real space, the bases with $N = 2$ includes the following 9 elements (dyad)

$$\begin{array}{c} \hat{e}_1\hat{e}_1,\ \hat{e}_1\hat{e}_2,\ \hat{e}_1\hat{e}_3; \\ \hat{e}_2\hat{e}_1,\ \hat{e}_2\hat{e}_2,\ \hat{e}_2\hat{e}_3; \\ \hat{e}_3\hat{e}_1,\ \hat{e}_3\hat{e}_2,\ \hat{e}_3\hat{e}_3; \end{array}$$

A rank-$N$ (contravariant) tensor in the $M$-dimensional space can be generically defined in the following form

$$T = \sum_{j_1, j_2, ..., j_N} T^{j_1 j_2, ..., j_N} \hat{e}_{j_1}\hat{e}_{j_2}...\hat{e}_{j_N}, \tag{1.5}$$

where $T^{j_1 j_2, ..., j_N}$ is a real element of the tensor corresponding to the base $\hat{e}_{j_1}\hat{e}_{j_2}...\hat{e}_{j_N}$. For convenience, we also define the covariant tensor based on the dual space with the orthogonal bases denoted as $\hat{e}^1, \hat{e}^2, ..., \hat{e}^M$. The rank-$N$ covariant tensor is defined by

$$T = \sum_{j_1, j_2, ..., j_N} T_{j_1 j_2, ..., j_N} \hat{e}^{j_1}\hat{e}^{j_2}...\hat{e}^{j_N}. \tag{1.6}$$

It is easy to see that the vector defined above corresponds to the case with $N = 1$, and the scalar corresponds to $N = 0$. A rank-2 (covariant) tensor can be described in the matrix form

$$T = \begin{bmatrix} T_{11}, & T_{12}, & ..., & T_{1M} \\ T_{21}, & T_{22}, & ..., & T_{2M} \\ .... & & & \\ T_{M1}, & T_{M2}, & ..., & T_{MM} \end{bmatrix}.$$

A sophisticated example is the Cauchy stress tensor $\sigma$, which is of rank 2. The physical meaning of the Cauchy stress tensor is that it takes a direction $\mathbf{n}$ as input and produces the stress $T^{(\mathbf{n})}$ on the surface normal to this vector for output, thus expressing a relationship between these two vectors that

$$T_j^{(\mathbf{n})} = \sum_k \sigma_{kj} n^k. \tag{1.7}$$

The Kronecker delta symbol, which is a very useful rank-2 tensor, is defined by

$$\delta_{ij} = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases} \tag{1.8}$$

Another common and useful tensor is the fully antisymmetric tensor of rank-3, also known as the Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ is an even permutation of 1,2,3} \\ 0, & \text{if at least two of } i, j, k \text{ are equal} \\ -1, & \text{if } i, j, k \text{ is an odd permutation of 1,2,3} \end{cases} \tag{1.9}$$

which satisfies the following property

$$\sum_i \epsilon_{ijk}\epsilon_{iml} = \delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km}. \tag{1.10}$$

More generally, we can define the $(n_1, n_2)$-type tensor in the following way

$$T^{(n_1,n_2)} = \sum_{i_1,...,i_{n_1},j_1,...,j_{n_2}} T^{i_1 i_2,...,i_{n_1}}_{\quad\quad\quad\quad j_1 j_2,...,j_{n_2}} \hat{e}_{i_1}\hat{e}_{i_2}...\hat{e}_{i_{n_1}}\hat{e}^{j_1}\hat{e}^{j_2}...\hat{e}^{j_{n_2}}. \tag{1.11}$$

The lower and upper indices represent the covariant and contravariant indices, respectively. The contravariant forms of Kronecker delta symbol and Levi-Civita tensor are related to their covariant forms by $\delta_{ij} = \delta^i_j = \delta^{ij}$ and $\epsilon_{ijk} = \epsilon^{ijk}$. It is noteworthy that in the most generic case a different relative order between the upper $(i_1, ..., i_{n_1})$ and lower indices $(j_1, ..., j_{n_2})$ can give a different $(n_1, n_2)$-type tensor. For example, $\epsilon^i{}_{jk} = -\epsilon_j{}^i{}_k$. For simplicity, we shall not discuss in more detail on this issue, but will specify it when necessary.

The metric tensor (fundamental tensor) is a particularly important rank-2 tensor. In covariant component form we shall denote it as $g_{\mu\nu}$. This metric tensor determines the relation between an arbitrary covariant vector $A_\mu$ and its contravariant counterpart $A^\mu$ according to the following rule:

$$A_\mu \equiv \sum_\nu g_{\mu\nu}A^\nu. \tag{1.12}$$

This rule is often called lowering of index. Accordingly, we have $A^\mu \equiv \sum_\nu g^{\mu\nu}A_\nu$, which is called raising of index. Similarly, for a rank-2 tensor, we can have

$$T_{\mu\nu} = \sum_{\lambda k} g_{\mu\lambda}g_{k\nu}T^{\lambda k}. \tag{1.13}$$

This process is also known as a tensor contraction.

For the 3D Euclidian space, the metric tensor is $g_{ij} = \delta_{ij}$ in the Cartesian coordinates, from which we know that the corresponding contravariant form and covariant form of a tensor in the 3D are simply the same. In this case, it is not essential to distinguish the contravariant and covariant indices. On the other hand, in the 4D Minkowski space, the metric tensor is $g_{\mu\nu} = \eta_{\mu\nu} = [-1, 1, 1, 1]^{\text{diag}}$. Thus whenever a zeroth component (regarding the time dimension) of a 4D vector/tensor is considered in the index lowering or raising, a minus sign should be included. Instead, for other components (regarding the 3D spatial dimension), there is no minus sign in the index lowering or raising.

For the electrodynamics, the above results also tell that for electrostatics and magnetostatics, it is not essential to distinguish the contravariant and covariant indices, while for the relativistic electrodynamics, we only need to distinguish the contravariant and covariant indices for the zeroth component.

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## 1.2 Algebra

**Scalar product.**– The scalar product of two vectors is defined by

$$\mathbf{A}\cdot\mathbf{B} \equiv \sum_\mu A^\mu B_\mu = \sum_{\mu\nu} \delta^\nu_\mu A^\mu B_\nu == \sum_{\mu\nu} g_{\mu\nu}A^\mu B^\nu. \tag{1.14}$$

Note that the norm of a vector is given by

$$\begin{aligned} |\mathbf{A}|^2 &= \mathbf{A}\cdot\mathbf{A} = \sum_{\mu} A^{\mu}A_{\mu} = \sum_{\mu\nu} g_{\mu\nu}A^{\mu}A^{\nu} \\ &= \sum_{\mu\nu\lambda} g_{\mu\nu}g^{\mu\lambda}A_{\lambda}A^{\nu} = \sum_{\nu\lambda}\delta^{\lambda}_{\nu}A_{\lambda}A^{\nu}. \end{aligned} \tag{1.15}$$

From the last equality we have that

$$\sum_{\mu} g_{\mu\nu}g^{\mu\lambda} = \delta^{\lambda}_{\nu}. \tag{1.16}$$

**Vector product.**–The vector product can be defined for two vectors in a 3D space. In the 3D Euclidean space $\mathbb{R}^3$, this takes the form

$$\mathbf{A}\times\mathbf{B} \equiv \sum_{ijk}\epsilon^{ijk}A_jB_k\hat{e}_i. \tag{1.17}$$

In the more generic case, the vector product of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A}\wedge\mathbf{B}$, which includes the basis $\hat{e}_i\wedge\hat{e}_j = -\hat{e}_j\wedge\hat{e}_i$, with the definition valid beyond 3D space. The wedge $\mathbf{A}\wedge\mathbf{B}$ indeed gives a rank-2 tensor. However, in the 3D space, due to the anti-symmetric nature $\mathbf{A}\wedge\mathbf{B}$ only has 3 independent components, which can then be rewritten as a vector through the formula (1.17). From the Eqs. (1.14) and (1.17) one can show the following results

$$\begin{aligned} &\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \\ &= -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = -\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}), \end{aligned} \tag{1.18}$$

and

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \tag{1.19}$$

We show the details of proof for Eq. (1.19) that

$$\begin{aligned} \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \sum_{jlkmi}\epsilon^{ijk}\epsilon^{klm}A_jB_lC_m\hat{e}_i = \sum_{jlkmi}\epsilon^{kij}\epsilon^{klm}A_jB_lC_m\hat{e}_i \\ &= \sum_{jlmi}(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})A_jB_lC_m\hat{e}_i \\ &= \sum_{jli}(\delta^{jm}A_jC_m)B^i\hat{e}_i - \sum_{jli}(\delta^{jl}A_jB_l)C^i\hat{e}_i \\ &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \end{aligned} \tag{1.20}$$

**Tensor product.**–The simplest case of the tensor product is a dyadic product between two vectors $\mathbf{A}$ and $\mathbf{B}$. Let the two vectors be covariant, the dyadic product is defined by

$$\mathbf{A}\otimes\mathbf{B} \equiv \sum_{ij}A_iB_j\hat{e}^i\otimes\hat{e}^j, \tag{1.21}$$

which is a $(0,2)$-type tensor of rank 2. The above formula can also be described as

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix} A_1B_1, & A_1B_2, & ..., & A_1B_M \\ A_2B_1, & A_2B_2, & ..., & A_2B_M \\ .... & & & \\ A_MB_1, & A_MB_2, & ..., & A_MB_M \end{bmatrix}.$$

Operating on this dyad from the right and from the left with an inner product of an vector $\mathbf{C}$ one obtains

$$(\mathbf{A}\otimes\mathbf{B})\cdot\mathbf{C} \equiv \mathbf{A}(\mathbf{B}\cdot\mathbf{C}), \tag{1.22}$$
$$\mathbf{C}\cdot(\mathbf{A}\otimes\mathbf{B}) \equiv (\mathbf{C}\cdot\mathbf{A})\mathbf{B}. \tag{1.23}$$

In more general, the tensor product of two tensors $T^{(n_1,n_2)}$ and $T^{(m_1,m_2)}$ is given by $T^{(n_1,n_2)}\otimes T^{(m_1,m_2)}=T^{(n_1+m_1,n_2+m_2)}$, namely, yielding a $(n_1+m_1,n_2+m_2)$-type tensor.

## 1.3 Differential Calculations

The concepts of the scalar, vector, and tensor are not restricted to be constants. We can define the scalar, vector, and tensor fields which are functions of coordinates of the $M$-dimensional space, defined as

$$a(\mathbf{x}) = a(x^1,x^2,...,x^M), \tag{1.24}$$
$$\mathbf{A}(\mathbf{x}) = \sum_i A_i(x^1,x^2,...,x^M)\hat{e}^i, \tag{1.25}$$
$$T^{(n_1,n_2)}(\mathbf{x}) = \sum_{i_1,...,i_{n_1},j_1,...,j_{n_2}} T^{i_1i_2,...,i_{n_1}}_{j_1j_2,...,j_{n_2}}(x^1,x^2,...,x^M)\hat{e}_{i_1}\hat{e}_{i_2}...\hat{e}_{i_{n_1}}\hat{e}^{j_1}\hat{e}^{j_2}...\hat{e}^{j_{n_2}}. \tag{1.26}$$

In the 3D Euclidian space, we take that $x^i=(x,y,z)$. In the 4D Minkowski space we take $x^\mu=(ct,x,y,z)$ and $x_\mu=(-ct,x,y,z)$.

**The *del* operator.**–In $\mathbb{R}^3$ the del operator is a differential vector operator, denoted in Gibbs notation by $\nabla$ and defined as

$$\nabla\equiv\sum_i^3\hat{e}^i\frac{\partial}{\partial x^i}. \tag{1.27}$$

In "component" notation we can write down it as

$$\partial_i\equiv(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}). \tag{1.28}$$

This operator can be generalized to generic $M$-dimensional Euclidean space. In the Cartesian coordinate system, the *del* operator for the $M$-dimensional flat space is defined by

$$\partial_\mu\equiv(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},...,\frac{\partial}{\partial x^M}). \tag{1.29}$$

The contravariant component representation of the $M$-dimensional *del* operator is given by

$$\partial^\mu = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_M}). \tag{1.30}$$

In 4D Minkowski space the contravariant form of the four-del operator can be represented as

$$\partial^\mu = (-\frac{1}{c}\frac{\partial}{\partial t}, \nabla). \tag{1.31}$$

and the covariant form as

$$\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla). \tag{1.32}$$

Taking the scalar product of these two, one obtains

$$\partial_\mu \partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = \Box^2, \tag{1.33}$$

which is the dAlembert operator, sometimes denoted as $\Box$, and sometimes defined with an opposite sign convention. With the help of the del operator we can define the gradient, divergence and curl of a tensor (in the generalised sense).

**The gradient.**–The gradient of a scalar field $a(\mathbf{x})$ is defined by the *del* operator acting on the scalar field. In an $M$-dimensional Euclidean space it is defined as

$$\sum_\mu^M \hat{e}^\mu \partial_\mu a(x^\nu) = \sum_\mu^M \frac{\partial a(x^\nu)}{\partial x^\mu}\hat{e}^\mu, \tag{1.34}$$

which yields a vector field. In the 4D Minkowski space this is defined by

$$\sum_\mu \hat{e}^\mu \partial_\mu a(x^\nu) = \frac{1}{c}\frac{\partial a(x^\nu)}{\partial t}\hat{e}^0 + \sum_{j=1}^3 \frac{\partial a(x^\nu)}{\partial x^j}\hat{e}^j, \tag{1.35}$$

$$\sum_\mu \hat{e}_\mu \partial^\mu a(x^\nu) = -\frac{1}{c}\frac{\partial a(x^\nu)}{\partial t}\hat{e}_0 + \sum_{j=1}^3 \frac{\partial a(x^\nu)}{\partial x^j}\hat{e}_j. \tag{1.36}$$

In the 3D space, the gradient is

$$\nabla a(\mathbf{x}) = \sum_j^3 \frac{\partial a(\mathbf{x})}{\partial x^j}\hat{e}^j. \tag{1.37}$$

It can be verified that

$$\nabla(|\mathbf{x}-\mathbf{x}'|) = \sum_j^3 \frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial x^j}\hat{e}^j = -\sum_j^3 \frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial x'^j}\hat{e}^j = -\nabla'(|\mathbf{x}-\mathbf{x}'|) = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{1.38}$$

Similarly, we have for $\mathbf{x} \neq \mathbf{x}'$ that

$$\nabla(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) = -\nabla'(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{1.39}$$

**The divergence.**–The divergence of a vector is defined by the *del* operator acting on the vector from left side by the scalar product. In an $M$-dimensional Euclidean space it is defined as

$$\sum_{\mu}^{M} \partial_\mu A^\mu(x^\nu) = \sum_{\mu}^{M} \frac{\partial A^\mu(x^\nu)}{\partial x^\mu}, \tag{1.40}$$

which yields a scalar field. In the 4D Minkowski space this is defined by

$$\sum_{\mu} \partial_\mu A^\mu(x^\nu) = \frac{1}{c}\frac{\partial A^0(x^\nu)}{\partial t} + \sum_{j=1}^{3} \frac{\partial A^j(x^\nu)}{\partial x^j}, \tag{1.41}$$

$$\sum_{\mu} \partial^\mu A_\mu(x^\nu) = -\frac{1}{c}\frac{\partial A_0(x^\nu)}{\partial t} + \sum_{j=1}^{3} \frac{\partial A_j(x^\nu)}{\partial x^j}. \tag{1.42}$$

In the 3D space, the divergence is

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \sum_{j}^{3} \frac{\partial A_j(\mathbf{x})}{\partial x_j}. \tag{1.43}$$

It can be verified that

$$\sum_{\mu}^{M} \partial_\mu A^\mu(x^\nu) = \sum_{\mu}^{M} \partial^\mu A_\mu(x^\nu). \tag{1.44}$$

**The Laplacian** The 3D Laplace operator or Laplacian can be described as the divergence of the gradient operator:

$$\nabla^2 = \Delta = \nabla \cdot \nabla = \sum_{i,j} \hat{e}_i \cdot \hat{e}^j \partial^i \partial_j = \sum_{ij} \delta_i^j \partial^i \partial_j = \sum_{j} \partial_j^2. \tag{1.45}$$

A very useful formula in 3D $\mathbb{R}^3$ space is

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{1.46}$$

where $\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ is a 3D *Dirac delta* function.

**The curl.**–The curl of a 3D vector is defined by the *del* operator acting on the vector from left side by the vector product (for $\mathbf{A} = \sum_i A^i e_i$)

$$\nabla \times \mathbf{A}(\mathbf{x}) = \sum_{i,j,k}^{3} \epsilon_{ijk} \partial_j A^k(x^\nu) \hat{e}^i, \tag{1.47}$$

which yields a vector field. In the 4D Minkowski space the covariant 4D generalisation of the curl of a four-vector field $A_\mu(x^k)$ is the antisymmetric four-tensor field

$$T_{\mu\nu}(x^k) = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.48}$$

A vector with vanishing curl is said to be *irrotational*. In 3D Euclidean space, the curl of a gradient of a scalar is zero, and accordingly, the divergence of the curl of a vector is zero

$$\nabla \times [\nabla a(\mathbf{x})] = 0, \tag{1.49}$$
$$\nabla \cdot [\nabla \times \mathbf{A}(\mathbf{x})] = 0. \tag{1.50}$$

We have assumed that the scalar field $a(\mathbf{x})$ and the vector field $\mathbf{A}(\mathbf{x})$ are smooth functions in the space.

The four curl of four gradient of a smooth scalar field can be shown to be zero, similar as the case in the 3D $\mathbb{R}^3$ space. Actually, we have

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) a(x^\mu) = 0. \tag{1.51}$$

However, the four divergence of four curl of a vector field is generically *not* zero

$$\sum_{\nu=0}^{3} \partial^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \sum_{\nu=0}^{3} \partial_\mu \partial^\nu A_\nu - \Box^2 A_\mu \neq 0. \tag{1.52}$$

**Discussion**.–We should keep in mind that the difference in the results of Eqs. (1.50) and (1.52) is essentially because of the different definitions in the curl, rather than the different dimensions. As having been pointed out previously, in general the curl is a tensor product, but for the 3D curl we have redefined it into a vector by multiplying the antisymmetric tensor $\epsilon_{ijk}$. This is the reason why the divergence of the 3D curl vanishes. Instead, if we do not redefine the 3D curl into a vector through $\epsilon_{ijk}$, the divergence is generically not zero. On the other hand, if we redefine the 4D curl through $\tilde{T}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} T_{\rho\sigma}$, with $\epsilon^{\mu\nu\rho\sigma}$ being the 4D antisymmetric tensor, we can check straightforwardly that $\partial_\mu \tilde{T}^{\mu\nu} = 0$, say the 4D divergence of $\tilde{T}^{\mu\nu}$ indeed vanishes.

**Einstein summation convention.**–For convenience, in the following chapters we adopt the Einstein summation convention which states that a repeated index in a term implies summation over the range of the index in question. For example, with the Einstein summation convention the following formulas can be written down in simpler forms

$$\sum_{\mu}^{M} \hat{e}^\mu \partial_\mu a(x^\nu) \equiv \hat{e}^\mu \partial_\mu a(x^\nu) \tag{1.53}$$
$$\sum_{\mu}^{M} \partial_\mu A^\mu(x^\nu) \equiv \partial_\mu A^\mu(x^\nu), \tag{1.54}$$
$$\sum_{i,j,k}^{3} \epsilon_{ijk} \partial_j A^k(x^\nu) \hat{e}^i \equiv \epsilon_{ijk} \partial_j A^k(x^\nu) \hat{e}^i. \tag{1.55}$$
$$\sum_{j} \partial_j^2 \equiv \partial_j^2 = \Delta. \tag{1.56}$$

## 1.4 Further reading of the vector space

In the above sections we have adopted covariant and contravariant vectors which are vectors in the dual spaces defined with the bases $\{\hat{e}^1, \hat{e}^2, ..., \hat{e}^M\}$ and $\{\hat{e}_1, \hat{e}_2, ..., \hat{e}_M\}$,

respectively. For the Euclidean space, the bases satisfy $\hat{e}_i \cdot \hat{e}^j = \delta^i_j$. In the following we present a more rigorous description in mathematics.

In the 3D $\mathbb{R}^3$ space, the position vector can be described by $\mathbf{r} = x^1\hat{e}_1 + x^2\hat{e}_2 + x^3\hat{e}_3$. It is easy to know that when we do transformation on the bases, to make sure that the vector is unchanged, the components $\{x^i\}$ of the position vector should be transformed oppositely compared with the bases (or reference axes). Due to this property we call that $\mathbf{r}$ is a contravariant vector in the $\mathbb{R}^3$ space. This concept can be extended to a general definition: if the **components** of a vector $\mathbf{A}$ transforms oppositely to the transformation of the bases $\{\hat{e}_i\}$ (or equivalently, it transforms similar as the transformation of **components** of the coordinate vector $\mathbf{r}$), we call $\mathbf{A}$ contravariant, and it can be described by

$$\mathbf{A} = A^i\hat{e}_i. \tag{1.57}$$

Accordingly, if the **components** of a vector $\mathbf{A}$ transforms similar as the transformation of the bases $\{\hat{e}_i\}$ (or equivalently, oppositely to the transformation of **components** of the coordinate vector $\mathbf{r}$), the vector is covariant, and can be described by

$$\mathbf{A} = A_i\hat{e}^i. \tag{1.58}$$

We emphasize that in the above definitions, *both the covariant and contravariant features are defined by comparing with the transformation of the coordinator components $x^i$ in the space spanned by $\{\hat{e}_i\}$. Please do not wrongly interpret that for a covariant vector $A_i$ its components transform the same as the transformation of the bases $\{\hat{e}^i\}$.* The above definitions can be directly applied to generic $M$-dimensional space.

According to the above definition, when we proceed a position transformation with the coordinates $\{x^\mu\} \to \{\tilde{x}^\nu\}$, we must have the following result for the contravariant vector

$$\tilde{A}^\mu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}A^\nu, \tag{1.59}$$

so that it transforms similar as the coordinat components $x^\mu$. By contrast, the covariant vector satisfies

$$\tilde{A}_\mu = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}A_\nu. \tag{1.60}$$

This property can be applied to generic tensors. For example, for rank-2 tensors, we have

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\lambda}{\partial\tilde{x}^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}g_{\lambda\sigma}, \quad \tilde{g}^{\mu\nu} = \frac{\partial\tilde{x}^\mu}{\partial x^\lambda}\frac{\partial\tilde{x}^\nu}{\partial x^\sigma}g^{\lambda\sigma}. \tag{1.61}$$

With the above results, we can show a few useful results, which shall enable a deeper understanding. First, we can show easily that $dx^\mu$ is a contravariant vector. Actually, in the transformation $\{x^\mu\} \to \{\tilde{x}^\nu\}$, we have

$$d\tilde{x}^\mu = \frac{\partial\tilde{x}^\mu}{\partial x^\lambda}dx^\lambda. \tag{1.62}$$

On the other hand, the gradient of a scalar function $\partial f/\partial x^\mu$ yields a covariant vector. The reason is because

$$\frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\lambda}. \tag{1.63}$$

Finally, how to prove that the quantity $\partial f/\partial x_\mu$ is a contravariant vector? Note that using the metric tensor we have the following relation

$$\frac{\partial}{\partial x_\mu} = g^{\mu\nu}\frac{\partial}{\partial x^\nu}. \tag{1.64}$$

Furthermore, under the coordinator transformation we have

$$\tilde{g}^{\mu\nu}\frac{\partial}{\partial \tilde{x}^\nu} = \tilde{g}^{\mu\nu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\frac{\partial}{\partial x^\rho}. \tag{1.65}$$

Using the second equation in (1.61) we get that

$$\begin{aligned}
\tilde{g}^{\mu\nu}\frac{\partial}{\partial \tilde{x}^\nu} &= g^{\sigma\lambda}\frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial \tilde{x}^\nu}{\partial x^\lambda}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\frac{\partial}{\partial x^\rho} \\
&= g^{\sigma\lambda}\frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x^\lambda}\frac{\partial}{\partial x^\rho} \\
&= g^{\sigma\lambda}\frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\delta^\rho_\lambda\frac{\partial}{\partial x^\rho} \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\sigma}g^{\sigma\lambda}\frac{\partial}{\partial x^\lambda} \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial}{\partial x_\sigma}.
\end{aligned} \tag{1.66}$$

This result shows that the quantity $\partial/\partial x_\mu$ is a contravariant vector. In the above derivative we have used the result $\frac{\partial x^\rho}{\partial x^\lambda} = \delta^\rho_\lambda$, which is valid for orthogonal coordinate systems. With the above results we can easily derive more generic relations satisfied by the generic contravariant and covariant tensors.

# Chapter 2

# Maxwell Equations

## 2.1 Maxwell equations in vacuum

### 2.1.1 Maxwell equations

The equations now known as Maxwells equations were obtained over an extended period, principally during the early nineteenth century. Here, we shall take as our starting point the set of four differential equations as they were presented by Maxwell in about 1861. It was Maxwell who completed the process of constructing the equations, thereby achieving the first unification of fundamental theories in physics. Prior to Maxwell, there were two essentially independent theories, one describing electricity and the other describing magnetism, and it was he who brought about the synthesis that unified them into a single theory of electromagnetism. It was only later, after Einstein developed the theory of Special Relativity in 1905, that the magnitude of Maxwells achievement really became clear. Especially, a quite remarkable feature of Maxells 1861 equations is that they are already completely compatible with special relativity, with no need for modification of any kind. Aside from changes in notation and units, Maxwells equations have remained otherwise unaltered since 1861.

Let us begin by considering Maxwells equations in free space, by which is meant that the space outside of any conducting surfaces is assumed to be a vacuum. Using the SI (systeme international units) system of units, Maxwells equations are:

$$\begin{aligned} \nabla \cdot \vec{E}' &= \frac{\rho_0'}{\epsilon_0},\ \nabla \times \vec{B}' - \mu_0\epsilon_0 \frac{\partial \vec{E}'}{\partial t} = \mu_0 \vec{J}', \\ \nabla \cdot \vec{B}' &= 0,\ \nabla \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t} = 0. \end{aligned} \tag{2.1}$$

Here we have written these equations with a "prime" on the electric field $\vec{E}$, the magnetic field $\vec{B}$, the electric charge density , and the electric current density $J$. This is to indicate that these quantities are all expressed in the SI system of units. The remaining quantities appearing in the above equation are the constants $\epsilon_0$ and $\mu_0$, which are, respectively, the permitivity of free space and the permeability of free space. They have the values

$$\epsilon_0 \approx 8.85419 \times 10^{-12} \text{Farads/metre}, \tag{2.2}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{Henries/metre} \tag{2.3}$$

From the Maxwell equations we can easily derive the dynamical equations of $\vec{E}'$ and $\vec{B}'$. Using the vector identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$, we have that

$$\begin{aligned} \nabla \times \nabla \times \vec{E}' + \partial_t(\nabla \times \vec{B}') &= \nabla(\nabla \cdot E') - \nabla^2 \vec{E}' + \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}'}{\partial t^2} + \mu_0 \frac{\partial \vec{J}'}{\partial t} \\ &= 0. \end{aligned} \tag{2.4}$$

This equation can be simplified as

$$\nabla^2 \vec{E}' - \frac{1}{c^2} \frac{\partial^2 \vec{E}'}{\partial t^2} = \nabla \left( \frac{\rho'}{\epsilon_0} \right) + \mu_0 \frac{\partial \vec{J}'}{\partial t}, \tag{2.5}$$

where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 2.99792 \times 10^8$metres/second is the speed of light in vacuum. Similarly, we have that

$$\nabla^2 \vec{B}' - \frac{1}{c^2} \frac{\partial^2 \vec{B}'}{\partial t^2} = -\mu_0 \nabla \times \vec{J}', \tag{2.6}$$

In the vacuum, there are no charge and no current. We thus have $\rho' = \vec{J}' = 0$, and then

$$\nabla^2 \vec{E}' - \frac{1}{c^2} \frac{\partial^2 \vec{E}'}{\partial t^2} = 0, \tag{2.7}$$

$$\nabla^2 \vec{B}' - \frac{1}{c^2} \frac{\partial^2 \vec{B}'}{\partial t^2} = 0, \tag{2.8}$$

which are invariant under the transformation

$$\vec{E}' \to \vec{B}', \quad \vec{B}' \to -\epsilon_0 \mu_0 \vec{E}', \quad \text{or} \quad \vec{E}' \to -\vec{B}', \quad \vec{B}' \to \epsilon_0 \mu_0 \vec{E}'. \tag{2.9}$$

Indeed we can also verify that the whole Maxwell equations in the vacuum are invariant under the above transformation. This property corresponds to the electromagnetic duality. The wave equations of $\vec{E}'$ and $\vec{B}'$ imply that the electromagnetic waves, which are solutions to the above equations, travel in vacuum with the speed of $c$.

### 2.1.2 Gaussian units

SI units have their virtues for some purposes, but they can also be quite inconvenient in practice. This seems to be especially true in electromagnetism, and for this reason it is often more convenient to stick with an earlier system, known as Gaussian units, in which system the units of $\vec{E}$ and $\vec{B}$ fields are the same. In the system of Gaussian units, Maxwells equations in free space take the form

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi\rho, \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J}, \\ \nabla \cdot \vec{B} &= 0, \qquad \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \end{aligned} \tag{2.10}$$

Note that here, we are writing the equations using the unprimed quantities $\vec{E}$, $\vec{B}$, and $\vec{J}$, and it will probably therefore come as no surprise that it is this Gaussian system of

units that we prefer to use. It should already be evident upon comparing Eqs. (2.1) and (2.10) that Gaussian system is somewhat simpler, in that one needs only one "fundamental constant" $c$ (i.e. the speed of light) rather than two (the permittivity and the permeability of free space). The introduction of the $4\pi$ factors in the Gaussian units may perhaps seem tiresome, but the advantage of doing so will become apparent in due course. (Indeed, this is because of the fact that a unit sphere has area $4\pi$.)

For convenience, we work out the explicit relation between SI units and Gaussian units, with which one can transfer the quantities in one set of units to another whenever desired. We introduce the constant factors $\alpha, \beta, \gamma$ and $\delta$ that relate the primed to the unprimed quantities by

$$\vec{E}' = \alpha\vec{E},\ \vec{B}' = \beta\vec{B},\ \rho' = \gamma\rho,\ \vec{J}' = \delta\vec{J}, \tag{2.11}$$

and plug these formulas into Eq. (2.1) to make sure that the resulted equations have the form of Eq. (2.10). Simple algebra gives that

$$\alpha = \frac{\beta}{\sqrt{\epsilon_0\mu_0}},\ \gamma = \delta = 4\pi\beta\sqrt{\frac{\epsilon_0}{\mu_0}}. \tag{2.12}$$

Note that we still have one undetermined constant, say, the value of the constant $\beta$ has not yet been determined. This constant can be determined by considering Coulombs law, giving the force between two electric charges separated by a distance R. We again need to distinguish between the charges $q_1'$ and $q_2'$ expressed in SI units, and the charges $q_1$ and $q_2$ expressed in Gaussian units. Since we have the relation $\rho' = \gamma\rho$ between charge densities in the two systems, and since the unit of volume is the same in the two systems, it follows that the charges will also be related by the same factor of $\gamma$:

$$q' = \gamma q. \tag{2.13}$$

In the SI system the Coulomb's law is expressed by

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{R^2} = \frac{\gamma^2}{4\pi\epsilon_0}\frac{q_1 q_2}{R^2}. \tag{2.14}$$

On the other hand, from the Eq. in Gaussian units $\nabla\cdot\vec{E} = 4\pi\rho$, we have the solution by assuming $\rho = q_1\delta^{(3)}(\mathbf{r})$ that

$$\vec{E} = \frac{q_1}{R^2}\hat{e}_R. \tag{2.15}$$

Then a charge $q_2$ staying with a distance $R$ away from $q_1$ experiences a force of the magnitude

$$F = q_2 E = \frac{q_1 q_2}{R^2}. \tag{2.16}$$

Thus comparing the Eq. (2.14) and (2.16) we reach that

$$\gamma = \sqrt{4\pi\epsilon_0}. \tag{2.17}$$

With the above solution we soon get all the other constants. Finally we reach the transformation between the two unit systems

$$\vec{E}' = \frac{1}{\sqrt{4\pi\epsilon_0}}\vec{E}, \quad \vec{B}' = \sqrt{\frac{\mu_0}{4\pi}}\vec{B}. \tag{2.18}$$

$$\rho' = \sqrt{4\pi\epsilon_0}\rho, \quad \vec{J}' = \sqrt{4\pi\epsilon_0}\vec{J}, \quad q' = \sqrt{4\pi\epsilon_0}q. \tag{2.19}$$

With these relations established, we can happily proceed by using the more convenient Gaussian units in this course, and anyone who wishes to re-express things in SI units can do so using the above transformation.

In the Gaussian units we can see that the unit of charge $[q] = [F]^{1/2}[r]$ can be characterized in terms of the units of length, mass, and time. On the other hand, the units of the fields $[E]$ and $[B]$ can be directly related to the charge and current. Thus in the Gaussian units all the quantities in electrodynamics can be characterized in units of length, mass, and time. This is in sharp contrast to the SI units, in which the charge has a fundamental unit independent of those of the length, mass, and time.

### 2.1.3 Macroscopic media

In principle, every problem in classical electromagnetism can be viewed as a problem formulated in free space, together with a number of electric point charges carried by electrons, protons, etc. In practice, however, it is often the case that the number of individual point charges is so large that it would not be convenient to consider them all separately, and instead, it is preferable to make a "macroscopic approximation". One obvious example is the notion of a conductor: It would be very clumsy and unwieldy to treat every electrostatics problem involving a conducting surface as a problem involving $10^{23}$ or so positive and negative point charges that are bound together to form a sheet of metal. Instead, we can typically just forget about the microscopic explanation of why the protons, neutrons and electrons have formed themselves into a metal, and instead simply abstract from this the macroscopic notion of a surface on which the electrostatic potential is constant. Another example where a macroscopic viewpoint is very useful is when one considers materials (such as glass) that exhibit a dielectric permittivity, or else materials that exhibit a magnetic permeability. One certainly can give a microscopic understanding of why these materials behave as they do, but it is convenient not to consider these details every time we want to work out the effect of a slab of glass in an electrostatics problem.

In order to give a macroscopic formulation of Maxwells theory in the presence of media, we now interpret $\vec{E}$ and $\vec{B}$ as averaged values of the electric and magnetic fields, where the averaging is performed over the distance scale of order the interatomic spacing in the medium. The point here is that we don't want to get involved in looking at the (enormous) microscopic variations in the fields that occur on the atomic length scale as one moves around close to individual electrons and protons. Having performed this averaging, the meanings of $\vec{E}$ and $\vec{B}$ are the same as they are in free space. For example, $\vec{E}$ still measures the potential difference between neighbouring points divided by their spatial separation. We must also introduce two new quantities, called $\vec{D}$ and $\vec{H}$ , which are related to $\vec{E}$ and $\vec{B}$, respectively. The standard names for all four fields

are:

$$\begin{array}{llll} \vec{E}: & \text{Electric field,} & \vec{D}: & \text{Electric displacement,} \\ \vec{B}: & \text{Magnetic induction,} & \vec{H}: & \text{Magnetic field.} \end{array}$$

In free space, we have

$$\vec{D} = \vec{E}, \quad \vec{B} = \vec{H}. \tag{2.20}$$

In a medium, on the other hand, $\vec{D}$ represents a "back-reacted" version of $\vec{E}$, which takes into account the fact that the positive and negative charges in the medium are displaced because of the presence of the externally-applied $\vec{E}$ field, and thus they feed back into the system. To leading order, the system of positive and negative charges in the medium (which is neutral on balance) distorts so that there is an effective electric dipole, or polarisation $\vec{P}$, and

$$\vec{D} = \vec{E} + 4\pi\vec{P}. \tag{2.21}$$

In a similar way, if the medium has magnetic properties there will be a similar relation

$$\vec{H} = \vec{B} - 4\pi\vec{M}, \tag{2.22}$$

where $\vec{M}$ is a dipole magnetisation term. We note that in the SI units the corresponding formulas are given by $\vec{D}' = \epsilon_0\vec{E}' + \vec{P}'$ and $\vec{B}' = \mu_0(\vec{H}' + \vec{M}')$. This can be obtained by introducing the transformation that

$$\vec{D}' = \sqrt{\frac{\epsilon_0}{4\pi}}\vec{D}, \quad \vec{P}' = \sqrt{4\pi\epsilon_0}P, \tag{2.23}$$

and

$$\vec{H}' = \sqrt{\frac{1}{4\pi\mu_0}}\vec{H}, \quad \vec{M}' = \sqrt{\frac{4\pi}{\mu_0}}M. \tag{2.24}$$

The effect of all this is that the Maxwell equations are modified in the presence of the medium. Instead of the free-space equations we shall now have

$$\begin{aligned} \nabla\cdot\vec{D} &= 4\pi\rho, \ \nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}, \\ \nabla\cdot\vec{B} &= 0, \quad \nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0. \end{aligned} \tag{2.25}$$

Notice that it is the first two equations, the ones that have the $\rho$ and $J$ source-terms on the right-hand side, that are modified. The remaining two equations are completely unchanged from their free-space forms.

We note that the charge density and electric current density in the Eqs. (2.10) are different from those in the Eqs. (2.25). For the former they are *total charge density* and *total electric current density*, which shall be denoted as $\rho_t$ and $\vec{J}_t$ in future, respectively, while for the latter they are *free charge density* and *free electric current density*, denoted as $\rho_f$ and $\vec{J}_f$, respectively. The word "free" means that the charge is not bounded to

atoms or molecules in the dielectric media, but can move freely in e.g. a metal. Being more precise, the free charges are not originated from the media. Any charge polarized from the media due to external field is not free according to this definition (without the polarization induced by external field, the media should be neutral and such charge disappears), and a free charge should indeed be an "extra" charge which resides in the media but can exist without the media. For example, for a metal put in an external electric field, even the charge polarized due to screening can move freely, it is not a free charge. In comparison, if a charge is transferred from one material to another by e.g. triboelectric effect, the transferred charge is a free charge for the latter material. The free current means that the current is carried by the free charges. Comparing the Eqs. (2.10) and Eqs. (2.25), we can show that

$$\begin{aligned} \nabla \cdot \vec{P} &= -(\rho_t - \rho_f) \\ &= -\rho_b, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \nabla \times \vec{M} &= \frac{1}{c}(\vec{J}_t - \vec{J}_f) - \frac{1}{c}\frac{\partial \vec{P}}{\partial t} \\ &= \frac{1}{c}(\vec{J}_t - \vec{J}_f - \vec{J}_b). \\ &= \frac{1}{c}\vec{J}_m \end{aligned} \tag{2.27}$$

Here $\rho_b$ and $\vec{J}_b = \partial \vec{P}/\partial t$ are the bound charge density and bound charge current density, which emerge due to the polarization and magnetization of the media. $\vec{J}_m$ is called molecular current density which excludes the current carried by bound charges.

### 2.1.4 Uniform media

The relations between $\vec{D}, \vec{H}$ and $\vec{E}, \vec{B}$ characterize the properties of a medium. A common situation is that the medium is uniform and isotropic (meaning that it is the same in all directions), such as air, water, and simple glass, with the polarization and magnetization being linearly proportional to external driving field. However, the situations can be complicated in the nonlinar media, such as ferroelectric and ferromagnetic materials, where such relation are generically nonlinear and spatially anisotropic. In this course, the linear media will be mostly considered.

**Static media**.–A simplest situation is that we assume that the medium responses to the external driving field "instantaneously". In this case the quantities $\vec{D}$ and $\vec{H}$ are simply constant multiples of $\vec{E}$ and $\vec{B}$ respectively:

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H} \tag{2.28}$$

The constant $\epsilon$ is called the relative permittivity of the medium, and the constant $\mu$ is called the relative permeability of the medium. In free space, we clearly have $\epsilon = 1$ and $\mu = 1$. Then from the relations (2.21) and (2.22) we have

$$\vec{P} = \chi_e \vec{E}, \quad \vec{M} = \chi_m \vec{H}, \tag{2.29}$$

where $\chi_e$ and $\chi_m$ are called electric and magnetic susceptibilities, respectively, and are given by

$$\chi_e = \frac{\epsilon - 1}{4\pi}, \quad \chi_m = \frac{\mu - 1}{4\pi}. \tag{2.30}$$

Assume that in a static medium there is no free charge, nor free electric current. From the Maxwell equations we can derive the wave equations for the electric and magnetic fields and obtain

$$\nabla^2\vec{E} - \frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0, \tag{2.31}$$

$$\nabla^2\vec{B} - \frac{n^2}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0, \tag{2.32}$$

where

$$n = \sqrt{\mu\epsilon}$$

is the refractive index of the medium. The above equations imply that the electromagnetic waves travel in the medium with a velocity $v_p = c/n$ different from that the in the vacuum. We call $v_p$ the phase velocity of light. In terms of susceptibilities, it is also given by

$$n = \sqrt{1 + 4\pi(\chi_e + \chi_m) + 16\pi^2\chi_e\chi_m}. \tag{2.33}$$

**Dispersive media**.–In general, a material cannot polarize instantaneously in response to an applied field. For a *linear* media, whose susceptibilities are independent of applied fields, the general formulations of the polarization and magnetization in terms of external field can be described by convolutions of time

$$\vec{P}(t) = \int_{-\infty}^{t} dt'\chi_e(t-t')\vec{E}(t'), \quad \vec{M}(t) = \int_{-\infty}^{t} dt'\chi_m(t-t')\vec{H}(t'). \tag{2.34}$$

The above formulas tell that the polarization and magnetization at time $t$ are generically determined by the fields in all the time before $t$. We can transform the above equations into frequency space by Fourier transformation, and obtain

$$\vec{P}(\omega) = \chi_e(\omega)\vec{E}(\omega), \quad \vec{M}(\omega) = \chi_m(\omega)\vec{H}(\omega). \tag{2.35}$$

From the above formulae we emphasize that the Eqs. (2.28) and (2.29) are essentially written in the frequency domain, while it was not specified previously for simplicity. The dependence of $\chi_{e,m}(\omega)$ gives the dispersion relations of the medium material. In a dispersive medium, the phase velocity $v_s$ of light is generically function of its frequency

$$v_p = \frac{c}{n(\omega)} = \frac{c}{\sqrt{1 + 4\pi[\chi_e(\omega) + \chi_m(\omega)] + 16\pi^2\chi_e(\omega)\chi_m(\omega)}}. \tag{2.36}$$

The wave vector of light in the medium is then given by $k = \omega/v_p$, yielding

$$k = \frac{\omega}{c}n(\omega). \tag{2.37}$$

On the other hand, we consider a wave packet which is a superposition of monochromatic plan-waves with a narrow distribution of wave vector and frequency, given by

$$\vec{E}(\mathbf{r}, t) = \sum_{\mathbf{k}} \vec{E}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{2.38}$$

Here $\mathbf{k} \in [\mathbf{k}_0 - \Delta\mathbf{k}, \mathbf{k}_0 + \Delta\mathbf{k}]$ and $\omega \in [\omega_0 - \Delta\omega, \omega_0 + \Delta\omega]$, with $|\Delta\mathbf{k}|$ and $\Delta\omega$ denoting small variations around the central wave-vector $\mathbf{k}_0$ and frequency $\omega_0$ $(= v_p(\mathbf{k}_0)k_0)$. We can rewrite the above formula as

$$\begin{aligned} \vec{E}(\mathbf{r}, t) &= \sum_{\delta\mathbf{k}} \vec{E}_{\delta\mathbf{k}} e^{i(\delta\mathbf{k}\cdot\mathbf{r} - \delta\omega t)} e^{i(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t)} \\ &= \vec{E}^{(0)}(\Delta\omega, \Delta\mathbf{k}) e^{i(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t)}, \end{aligned} \tag{2.39}$$

where $\vec{E}^{(0)}(\Delta\omega, \Delta\mathbf{k})$ represents the amplitude of the wave packet. Note that this amplitude is a function of the frequency and wave-vector distributions, characterizing how the wave-packet center travels in the space. Let that $\Delta\omega \to 0$ and $\Delta\mathbf{k} \to 0$, we can define the derivative of the frequency with respect to the wave vector as the "group velocity" of the light, which takes the form

$$v_g \equiv \frac{\Delta\omega}{\Delta k}|_{\Delta k \to 0} \equiv \frac{d\omega}{dk} = \frac{c}{n(\omega) + \omega\frac{dn}{d\omega}}. \tag{2.40}$$

It can be seen that the group velocity characterizes the propagation of the wave-packet.

The group velocity crucially depends on the dispersion relation of light in the medium. When $dn/d\omega > 0$, one has that group velocity is less than the phase velocity $v_g < v_p$. Materials with this property are called *normal dispersion medium*. On the other hand, if $dn/d\omega < 0$, one has $v_g > v_p$, namely, the group velocity is greater than the phase velocity. The materials with this property are called *anomalous dispersion medium*. Note that in the strongly dispersive medium with $dn/d\omega \gg 1$, one can have that the group velocity is greatly suppressed with respect to the vacuum (slow light): $v_g \ll c$. In particular, if

$$\frac{dn}{d\omega} \to +\infty. \tag{2.41}$$

The group velocity vanishes (light stop)

$$v_g \approx \frac{c}{\omega}\frac{d\omega}{dn} \to 0. \tag{2.42}$$

Usually, in the classical medium, the strong dispersion is associated with strong dissipation (decay). Thus the slow light or light stop is very difficult to be observed in the typical classical media. To observe such exceptional phenomena, one has to on one hand greatly enhance the dispersion of the medium, on the other hand suppress the dissipation of such medium. Such an exotic situation has been achieved in the recent 20 years using a mechanism called *electromagnetically induced transparency* (EIT), which is a hot research topic of quantum optics in the past two decades. Through the EIT mechanism the slow light with $V_g = 17$m/s was observed by L. V. Hau etal in 1999 [see Nature, **397**, 594(1999)] and also the light stop (with $v_g \to 0$) was successfully achieved by D. F. Phillips etal. in 2001 [See Phys. Rev. Lett. **86**, 783 (2001)]. The light stop may have novel applications to the light storage and even quantum memories [see Theory of dark-state polaritons, M. Fleischhauer and M. D. Lukin, Phys. Rev. Lett. **84**, 5094 (2000); Phys. Rev. A **65**,022314(2002)].

For the anomalous dispersive medium, if $n + \omega dn/d\omega < 1$, one has $v_g > c$, which is called the *superluminal* regime. In this regime the group velocity exceeds the speed of

light. However, we emphasize that this does not mean the breaking down of causality: the group velocity in the anomalous dispersion regime does not represent the propagation velocity of the information or energy, but only reflects the complicated dispersive properties of the medium. In general, by their definitions, both the phase and group velocities only characterize the dispersive properties of a medium, and we should not equalize the group velocity and the information or energy velocity, while in the normal dispersive medium they are indeed equivalent in many cases.

**Kramers-Kronig relations**.– In general, the susceptibility $\chi(\omega)$ is a complex function with $\chi(\omega) = \chi_1 + i\chi_2$. The real and imaginary parts, namely $\chi_1$ and $\chi_2$, are not independent. Their relation can be derived as follow. From the relation $\epsilon(\omega) = 1 + 4\pi\chi(\omega)$ we have

$$\begin{aligned}\epsilon(\omega) &= 1 + 4\pi \int_0^\infty \chi(\tau) e^{i\omega\tau} d\tau \\ &= 1 + 4\pi \int_0^\infty d\tau \frac{1}{2\pi} \int_{-\infty}^\infty \chi(\omega') e^{-i\omega'\tau} e^{i\omega\tau} d\omega' \\ &= 1 + 2\int_0^\infty d\tau \int_{-\infty}^\infty \frac{\epsilon(\omega') - 1}{4\pi} e^{-i(\omega'-\omega)\tau} d\omega'. \end{aligned} \tag{2.43}$$

It is reasonable to require that $\epsilon(\omega)$ is analytic in the upper closed half complex plane. For this to ensure that the above integral converges, we need to replace $\omega$ with $\omega + i\delta^+$, where $\delta^+$ is a positive infinitesimal. With this the contribution from upper bound $\tau \to +\infty$ of the integral vanishes. The above formula then reads

$$\epsilon(\omega) = 1 + \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\epsilon(\omega') - 1}{\omega' - \omega - i\delta^+} d\omega'. \tag{2.44}$$

Since $\epsilon(\omega)$ is analytic in the upper closed half infinite plane, we can rewrite the above formula as a contour integral enclosing the upper half complex plane

$$\epsilon(\omega) = 1 + \frac{1}{2\pi i} \oint \frac{\epsilon(\omega') - 1}{\omega' - \omega - i\delta^+} d\omega'. \tag{2.45}$$

Note that we have the following identity

$$\frac{1}{\omega' - \omega - i\delta^+} = \mathcal{P}(\frac{1}{\omega' - \omega}) + i\pi\delta(\omega' - \omega), \tag{2.46}$$

where $\mathcal{P}$ denotes the principle integral. Substituting the above result into the former formula yields that (note that some cancellation between the left and right hand sides of the formula is applied)

$$\epsilon(\omega) = 1 + \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^\infty \frac{\epsilon(\omega') - 1}{\omega' - \omega} d\omega'. \tag{2.47}$$

This gives that

$$\Re\big[\epsilon(\omega)\big] = 1 + \frac{1}{\pi} \mathcal{P} \int_{-\infty}^\infty \frac{\Im\big[\epsilon(\omega')\big]}{\omega' - \omega} d\omega', \tag{2.48}$$

$$\Im\big[\epsilon(\omega)\big] = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^\infty \frac{\Re\big[\epsilon(\omega') - 1\big]}{\omega' - \omega} d\omega'. \tag{2.49}$$

The above formulas show that the real and imaginary parts of $\epsilon(\omega)$ are related to each other, and one of them can be determined from the other. Accordingly, the real and imaginary parts of the susceptibility reads

$$\chi_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi_2(\omega')}{\omega'-\omega}d\omega', \tag{2.50}$$

$$\chi_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi_1(\omega')}{\omega'-\omega}d\omega'. \tag{2.51}$$

### 2.1.5 Boundary conditions

A situation that one encounters frequently when studying physical problems in electromagnetism is where there is a boundary or interface between two different materials or media. The simplest such situation in electrostatics is the case where there is a conducting surface in otherwise free space. Another example would be an interface between two materials with different dialectric constants. In fact the conductor in free space can just be viewed as a special case of the interface between two dielectric materials, with one of them (free space) having $\epsilon = 1$ and the other (the conductor) having $\epsilon = \infty$.

The boundary conditions on the electric and magnetic fields at an interface between two media can be determined by performing appropriate integrals of the Maxwell equations. Let us label the media by "1" and "2", and likewise place "1" and "2" subscripts on the various electric and magnetic fields on the two sides of the interface. Beginning with $\vec{D}$, we can integrate $\nabla\cdot\vec{D} = 4\pi\rho_f$ over a so-called "Gaussian pillbox" that straddles the interface. The pillbox is a like a very short length of circular cylinder, with the ends capped off so as to form a closed surface. One should imagine that the size of the whole pillbox is very small, and in fact eventually one takes the limit where the size tends to zero. At all stages in the limiting process, the height of the box (i.e. the length of the cylinder) is very small compared with its radius. The caps of the cylinder are taken to be parallel to the interface, with the interface slicing through the box; one cap on each side.

The divergence theorem states that for any vector field $\vec{v}$ we have

$$\int_V \nabla\cdot\vec{v}dV = \int_S \vec{v}\cdot d\vec{S}, \tag{2.52}$$

where $S$ is a closed surface enclosing a volume $V$. Integrating $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ over the pillbox and using the above formula, we therefore find

$$\int_S \vec{D}\cdot d\vec{S} = 4\pi q_f, \tag{2.53}$$

where $q_f$ is the charge inside the pillbox. Because the height of the pillbox is taken to be very small compared to its diameter, we can neglect the contributions to the $\vec{D}$ integral coming from the sides. Since the pillbox itself will be taken to have infinitesimal size we can think of the interface where the pillbox is placed as being planar. Let $\vec{n}$ be the unit normal vector pointing from medium 1 into medium 2. If the cross-sectional area of the pillbox is $\Delta A$, then Eq. (2.53) gives

$$\vec{n}\cdot(\vec{D}_2 - \vec{D}_1)\Delta A = 4\pi\sigma_f\Delta A, \tag{2.54}$$

where $\sigma_f$ is the surface free charge density at the interface. Thus we have

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 4\pi\sigma_f. \tag{2.55}$$

By the same procedure, the integration of the Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$ over the same pillbox gives

$$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0. \tag{2.56}$$

The zero on the right-hand side reflects the fact that there are no magnetic charges.

Further boundary conditions follow by appropriately integrating the remaining Maxwell equations across the interface. This time, we consider a rectangular loop formed by two infinitesimally-separated parallel line elements that straddle the interface, joined by adding connecting lines at the two ends. Eventually the size of this loop is scaled to zero. We now make use of Stokes' theorem, which states that for any vector field $\vec{v}$ we have

$$\int_\Sigma (\nabla \times \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{l}, \tag{2.57}$$

where $\Sigma$ denotes an (open) surface whose boundary is the closed loop $C$. Suppose again the unit normal from medium 1 to medium 2 is $\vec{n}$ at the chosen point on the interface that we are considering. We also choose a unit vector $\vec{m}$ that is tangent to the the interface at the selected point. Integrating the Maxwell equation $\nabla \times \vec{E} = -(1/c)\partial\vec{B}/\partial t$ over the area $\Sigma$ of the loop and applying (2.25) gives

$$\oint \vec{E} \cdot d\vec{l} = \frac{1}{c}\frac{\partial}{\partial t}\int_\Sigma \vec{B} \cdot d\vec{S}. \tag{2.58}$$

Since $\vec{B}$ is assumed to be finite, as also is $\partial\vec{B}/\partial t$, it follows that the right-hand side is infinitesimal since the separation between the two line elements of the loop is infinitesimal, implying that the area $\Sigma$ is infinitesimal. We finally reach the boundary condition

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0. \tag{2.59}$$

Finally, we perform an analogous integral over the last Maxwell equation, $\nabla \times \vec{H} = (1/c)\partial\vec{D}/dt + (4\pi/c)\vec{J}_f$. The finiteness of $\partial\vec{D}/\partial t$ means that its area integral over the loop gives zero, but $\vec{J}$ will have a non-zero area integral in general, since there can be a surface free current density $\vec{K}_f$ (analogous to the surface charge density $\sigma_f$). Thus we find

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c}\vec{K}_f. \tag{2.60}$$

To summarize, the boundary conditions we have derived above are given by

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 4\pi\sigma_f, \quad \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0, \tag{2.61}$$

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0, \quad \vec{n} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c}\vec{K}_f. \tag{2.62}$$

These give the junction conditions at the interface between medium 1 and medium 2, where $\vec{n}$ is the unit normal vector pointing from 1 to 2 at the interface, $\sigma$ is the surface charge density and $\vec{K}_f$ is the surface current density.

**Metallic surface**.–A special case of frequent interest arises for an electric field in free space, in the presence of a conducting surface. In the free-space region we have $\vec{D} = \vec{E}$, and the conductor can be viewed as the surface of a medium having infinite dielectric constant, which means that $\vec{E} = 0$ there. Thus the pillbox integration of $\nabla \cdot \vec{D} = 4\pi\rho_f$ becomes just the integral of $\nabla \cdot \vec{E} = 4\pi\rho_t$, with $\vec{E} = 0$ in "medium 1". We then have the boundary conditions

$$\vec{n} \cdot \vec{E} = 4\pi\sigma_t, \qquad \vec{n} \times E = 0. \tag{2.63}$$

The second equation says that there is no component of $\vec{E}$ tangent to the conducting surface, and the first equation says that the normal component of the electric field at the conductor is equal to $4\pi\sigma_t$. It is noteworthy that the electric displacement $\vec{D}$ can be generically nonzero in the conductor (see problems in homework No.2).

## 2.2 Conservation laws in electromagnetic theory

**Charge conservation**.–The charge conservation can be derived straightforwardly from the first two equations in (2.10), given by (we use the notations of total charge and total charge current)

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot \vec{J}_t = 0. \tag{2.64}$$

**Energy conservation**.–The energy can be transferred between matter and electromagnetic field through the work done by electric field on charged particles in the matter. Consider a particle with charge $q$ moving with velocity $\vec{v}$ in the electric field $\vec{E}$. The work done per unit time on the charge is given by $q\vec{v} \cdot \vec{E}$. Then, consider a charge current density $\vec{J}_t$ in a matter, the work done by electric field per unit time is given by

$$W = \int d^3\mathbf{x} \vec{J}_t \cdot \vec{E}. \tag{2.65}$$

Our goal is to rewrite the above equation in a form similar as the continuous equation for charge conservation. Using the Maxwell equation we can rewrite the above formula as

$$W = \int d^3\mathbf{x} \left[ \frac{c}{4\pi} \vec{E} \cdot (\nabla \times \vec{B}) - \frac{1}{4\pi} \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right]. \tag{2.66}$$

We apply the following identity for vectors

$$\nabla \cdot (\vec{E} \times \vec{B}) = B \cdot (\nabla \times E) - \vec{E} \cdot (\nabla \times \vec{B}), \tag{2.67}$$

and also the Faraday's law of electromagnetic induction: $\nabla \times \vec{E} = -(1/c)\partial \vec{B}/\partial t$. We then obtain that

$$W = -\int d^3\mathbf{x} \left[ \frac{c}{4\pi} \nabla \cdot (\vec{E} \times \vec{B}) + \frac{1}{4\pi} \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{4\pi} \vec{B} \cdot \frac{\partial B}{\partial t} \right]. \tag{2.68}$$

Since the left side of the above formula represents the work per unit time done by the electromagnetic field, it follows that the integral in the right hand side must represent the change of the total energy of the electromagnetic field (we assume that there is no dissipation in the medium). Similar as the continuous equation of charge conservation (2.64), we introduce the energy density of the electromagnetic field as

$$u = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2), \tag{2.69}$$

and the energy flux density, which is called **Poynting vector**:

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}.$$

We can show that for an electromagnetic wave the magnitude of Poynting vector simply equals the product of the energy density and speed of light. Then the equation (2.68) becomes

$$-\int d^3\mathbf{x}\vec{J}_t \cdot \vec{E} = \int d^3\mathbf{x}\left(\frac{\partial u}{\partial t} + \nabla \cdot \vec{S}\right). \tag{2.70}$$

This formula tells clearly that the change of the energy of the electromagnetic field can be resulted from two facts. The first is due to the work done by the field on charged particles in the medium, and the second is the energy flowing through the boundary. Accordingly, if there is no charge current in the space, the left side of the above equation is zero, and

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0. \tag{2.71}$$

In this case the total energy of the electromagnetic field is conserved.

**Momentum conservation**.–We can derive the momentum conservation in the similar way. The force exerting on a moving charged particle by electromagnetic field is called Lorentz force, given by (in Gaussian units)

$$\vec{F} = q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}), \tag{2.72}$$

where $\vec{v}$ is the velocity of the particle. Similarly, let us consider a system of charged particles (charge density $\rho$ and current density $\vec{J}_t$) with total momentum denoted as $\vec{P}^{(\mathrm{src})}$, we have

$$\frac{d\vec{P}^{(\mathrm{src})}}{dt} = \int d^3\mathbf{x}(\rho\vec{E} + \frac{1}{c}\vec{J}_t \times \vec{B}). \tag{2.73}$$

Again we apply the Maxwell equation and replace $\rho$ and $\vec{J}$ with differential forms of electromagnetic field. We have

$$\rho\vec{E} + \frac{1}{c}\vec{J}_t \times \vec{B} = \frac{1}{4\pi}\vec{E}(\nabla \cdot \vec{E}) + \frac{1}{4\pi c}\vec{B} \times \frac{\partial \vec{E}}{\partial t} - \frac{1}{4\pi}\vec{B} \times (\nabla \times \vec{B}). \tag{2.74}$$

The second term in the right hand side of the above equation can be replaced with

$$\vec{B} \times \frac{\partial \vec{E}}{\partial t} = \vec{E} \times \frac{\partial \vec{B}}{\partial t} - \frac{\partial}{\partial t}(\vec{E} \times \vec{B}). \tag{2.75}$$

Substituting this result into the former equation and applying the Maxwell equations $\partial\vec{B}/\partial t = -c\nabla\times\vec{E}$ and $\nabla\cdot\vec{B}=0$ yields

$$\frac{d\vec{P}^{(\mathrm{src})}}{dt} + \frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3\mathbf{x}(\vec{E}\times\vec{B}) = \frac{1}{4\pi}\int d^3\mathbf{x}\big[\vec{E}(\nabla\cdot\vec{E}) + \vec{B}(\nabla\cdot\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - \vec{B}\times(\nabla\times\vec{B})\big]. \quad (2.76)$$

Finally, we notice the following identity:

$$\vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E}) = \partial_j\big[(\vec{E}\otimes\vec{E})_{ij} - \frac{1}{2}|\vec{E}|^2\delta_{ij}\big]\hat{e}^i. \quad (2.77)$$

For the magnetic field, we have similar identity. With these identities, we can simplify the Eq. (2.76) into

$$\frac{d\vec{P}^{(\mathrm{src})}}{dt} + \frac{\partial\vec{P}^{(\mathrm{field})}}{\partial t} = -\int d^3\mathbf{x}\partial_j T_{ij}\hat{e}^i, \quad (2.78)$$

where $T_{ij}=T_{ji}$ is the **Maxwell's stress tensor**, defined by

$$T_{ij} = -\frac{1}{4\pi}\big[(\vec{E}\otimes\vec{E})_{ij} + (\vec{B}\otimes\vec{B})_{ij} - \frac{1}{2}(|\vec{E}|^2+|\vec{B}|^2)\delta_{ij}\big]. \quad (2.79)$$

Note that the appearance of tensor $T_{ij}$ in the Eq. (2.78) is by no means surprising. This equation describes the conservation of momentum, which is a vector field. On the other hand, the right hand side must be the divergence of some quantity, and the divergence yields a vector field. Thus in the right hand side it must be divergence of a rank-2 tensor (in the 3D $\mathbb{R}^3$ space). The total momentum of electromagnetic field has been defined by

$$\vec{P}^{(\mathrm{field})} = \frac{1}{4\pi}\frac{1}{c}\int d^3\mathbf{x}(\vec{E}\times\vec{B}) = \int d^3\mathbf{x}\vec{g}(\mathbf{x}). \quad (2.80)$$

In the last equation we have defined the momentum density $\vec{g}(\mathbf{x}) = \vec{E}\times\vec{B}/(4\pi c)$, which is related to the Poynting vector by

$$\vec{g} = \vec{S}/c^2. \quad (2.81)$$

This relation has an intuitive understanding: the unit of the Poynting vector is $[E]\cdot[v] = [P]\cdot[v^2]$, where $[P]$ and $[v]$ represent the units of momentum density and velocity, respectively. Note that the velocity of electromagnetic wave is $c$. The momentum density $\vec{g}$ of the electromagnetic field is thus given by $\vec{S}$ divided by the light speed. We shall see in the relativistic formalism of electrodynamics that the momentum density, energy flux density (Poynting vector), and Maxwell's stress tensor consist of the four-dimensional energy-momentum tensor.

## 2.3 Symmetries of Maxwell equations

In the final section of this chapter, we discuss the symmetries of the Maxwell equations. Symmetries are most important fundamental properties of a physical system, and they

determine the conserved quantities of such system. The important symmetries of the Maxwell's theory include the linearity, some discrete symmetries including inversion symmetry and time-reversal symmetry, Lorentz invariance, gauge invariance, and so on.

**Linearity**.–The Maxwell equations are linear with respect to the physical quantities, including the electric and magnetic fields, the charge density, and charge current density. As a result, the solutions to the Maxwell equations satisfy the principle of linear superposition. In other words, if both the field $\vec{E}_1$ and $\vec{E}_2$ are solutions to the Maxwell equations, then $\vec{E}_1 + \vec{E}_2$ is also solution to the equations, similar for other quantities. The linearity also tells that if the fields in the space are contributed from multiple charges and multiple electric currents, the contributions from different charges and different currents are independent and can be linearly summed over. This feature is important when we solve the boundary problems.

**Discrete symmetry I: inversion symmetry**.–The Maxwell equations are invariant under the inversion transformation which sends $\mathbf{x} \to -\mathbf{x}$. Under this transformation we have the following properties for electric field and magnetic field

$$\vec{E} \to -\vec{E}, \quad \vec{B} \to \vec{B}. \tag{2.82}$$

Thus the magnetic field is unchanged under the inversion transformation. This property implies that the magnetic field is a pseudovector in the 3D space. *A pseudovector is defined that it transforms oppositely with respect to the vector under any reflection transformation.* For example, the angular momentum is a pseudovector, or called axial vector, similar as the magnetic field. Indeed, in the following section we shall introduce the vector potential for magnetic field $\vec{B} = \nabla \times \vec{A}$. Under the inversion transformation one has $\vec{A} \to -\vec{A}$, and thus $\vec{B}$ is unchanged.

**Discrete symmetry II: time-reversal symmetry**.–The Maxwell equations are invariant under the time-reversal transformation which sends $t \to -t$. Under this transformation we have the following properties for electric field and magnetic field

$$\vec{E} \to \vec{E}, \quad \vec{B} \to -\vec{B}, \tag{2.83}$$

so that the Maxwell equations are invariant. Indeed, in the following sections we shall see that $\vec{E} = -\nabla\varphi - (1/c)\partial\vec{A}/\partial t$, which requires that $\vec{A} \to -\vec{A}$ under time-reversal transformation. Thus from $\vec{B} = \nabla \times \vec{A}$ one gets the transformation in the above formula.

The time-reversal transformation is particularly important in quantum mechanics. The fact that the magnetic field changes sign under time-reversal transformation has explicit meaning: the magnetic field affects the phase of the wave function for a quantum system (more precisely, it enters the phase of the wave function through the path integral of vector potential $\vec{A}$). The time-reversal symmetry in quantum mechanics correspond to the complex conjugate transformation, which reverses the phase of a wave function. Thus the vector potential $A$ or magnetic field $\vec{B}$ reverses sign.

Finally, the time-reversal odd property of magnetic field also also tells that for a physical system, if we want to break its time-reversal symmetry, we can apply a magnetic field and introduce interactions between the magnetic field and the system. In this way, the time-reversal symmetry of the system is broken (note that the magnetic field is an external field). This is always considered in a quantum mechanic problem.

**Lorentz invariance**.–Lorentz invariance is one of the most important continuous symmetries of Maxwell equations. The principle of relativity states that the equations describing the laws of physics have the same form in all inertial reference frames. The relation between different inertial reference frames are given by Lorentz transformation. As a result, the Maxwell equations are also invariant under the Lorentz transformation. We shall study in detail this issue in the chapter of special relativity and electrodynamics. Note that Maxwell equations were introduced before the invention of special relativity and Lorentz transformation. Thus we can say the the special relativity was indeed birthed from Maxwell's theory.

**Gauge invariance**.–Considering the properties of the magnetic field that $\nabla\cdot\vec{B}=0$, we can introduce a vector field $\vec{A}$ and let

$$\vec{B}=\nabla\times\vec{A}. \tag{2.84}$$

Then from the Maxwell equation $\nabla\times\vec{E}+(1/c)\partial\vec{B}/\partial t=0$, we have

$$\nabla\times\big[\vec{E}+(1/c)\frac{\partial\vec{A}}{\partial t}\big]=0. \tag{2.85}$$

Therefore, the curl of $\vec{E}+(1/c)\partial\vec{A}/\partial t$ is zero, and we can introduce a gradient of a scalar field $\varphi(\mathbf{x})$ for it, rendering that

$$\vec{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}. \tag{2.86}$$

The quantities $\varphi$ and $\vec{A}$ are conventionally called scalar and vector potentials, respectively. However, we recall that the definition of a scalar or vector field depends on what space one considers. The potentials $\varphi$ and $\vec{A}$ are scalar and vector fields in the 3D real space, but they are not separately in the 4D Minkowski space. Instead, $\varphi$ and $\vec{A}$ together consist of the four-vector potential in the 4D Minkowski space.

It is clear that the choice of scalar and vector potential is not unique. The electric and magnetic fields are invariant under the following transformation

$$\varphi \;\rightarrow\; \varphi'=\varphi-\frac{1}{c}\frac{\partial\lambda}{\partial t}, \tag{2.87}$$

$$\vec{A} \;\rightarrow\; \vec{A}'=\vec{A}+\nabla\lambda. \tag{2.88}$$

This transformation is called gauge transformation, and Maxwell equations are then gauge invariant. In electrodynamics, to introduce scalar and vector potentials is to facilitate the solutions to Maxwell equations in different circumstances, but it is not necessary. However, when we go to quantum mechanics, it is required to introduce the gauge potentials $(\varphi,\vec{A})$, which gives rise to a phase factor for the wave function of a charged particle and can lead to very important physics. The most outstanding example is the Aharonov-Bohm effect, which cannot be explained solely by magnetic field without introducing the vector gauge potential $\vec{A}$. This implies that the electric and magnetic fields ($\vec{E}$ and $\vec{B}$) are not enough to fully describe electromagnetic theory, and they are *underdetermine*. On the other hand, when we introduce $\vec{A}$ and $\varphi$, all the different gauge potentials related by the above gauge transformation give the same

results, bringing about large redundance in the description. Thus the gauge potentials $(\varphi, \vec{A})$ are *overdetermine* (T.T. Wu and C.N. Yang, 1975).

To make the solution of $\vec{A}$ and $\varphi$ be determined, we can introduce gauge conditions or gauge fixing. For different situations different gauge conditions may be applied for convenience. One most popular gauge condition is the Lorentz gauge condition, described by

$$\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0. \tag{2.89}$$

This formula can also be written in the form of divergence of four-vector

$$\partial_\mu A^\mu = \partial^\mu A_\mu = 0, \tag{2.90}$$

where $A_\mu = (-\varphi, \vec{A})$. Under this condition, and substituting the vector and scalar potentials into the Maxwell equations we can obtain

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho_t. \tag{2.91}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}_t. \tag{2.92}$$

It can be seen that under the Lorentz gauge condition, the vector and scalar potentials obey independent equations. With proper boundary conditions one can solve the gauge potentials independently, and then the electric and magnetic fields. Different situations will be introduced in the following chapters. Another widely used gauge condition is the Coulomb gauge condition, given by $\nabla \cdot \vec{A} = 0$, or called transverse gauge.

# Chapter 3

# Electrostatics

In the electrostatics, we neglect the time-dependence of Maxwell equations, which follows then the Poisson equation for scalar potential $\nabla^2\varphi = -4\pi\rho$. All we need to do is to solve this equation under proper boundary conditions, and then find the electric field and charge distribution of the system.

## 3.1 Uniqueness theorem

Whenever one is solving a differential equation, such as the Poisson equation $\nabla^2\phi = -4\pi\rho$ that we encounter in electrostatics, the question arises as to what boundary conditions one must impose in order to obtain a unique solution. Expressed more physically, one may ask how much boundary information must be specified in order to pin down the physics of the problem completely. The answer for Poissons equation is that the solution for the potential $\phi$ inside a volume $V$ will be uniquely determined once its value at all points on the (closed) surface $S$ that bounds $V$ is specified. For example, if we are solving $\nabla^2\phi = -4\pi\rho$ inside a sphere, then the solution will be uniquely determined provided the value of $\phi$ at every point on the surface of the sphere is specified. This type of boundary condition, in which $\phi$ is specified on $S$, is known as a *Dirichlet boundary condition.* An alternative is to specify not $\phi$ itself, but its normal derivative $\partial\phi/\partial n$, on the boundary. This is known as a *Neumann boundary condition.* In this case the solution for $\phi$ is again unique, except for the (trivial) point that an arbitrary additive constant is undetermined. One can also consider mixed boundary conditions, which are Dirichlet on parts of the boundary and Neumann on the rest.

To prove these statements, we suppose that for given boundary conditions on S there exist two different solutions to $\nabla^2\phi = -4\pi\rho$. Let these solutions be $\phi_1$ and $\phi_2$. The idea will be to prove that actually $\phi_1 = \phi_2$, and so the solution is unique. With

$$\nabla^2\phi_1 = -4\pi\rho, \quad \nabla^2\phi_2 = -4\pi\rho, \tag{3.1}$$

it follows by subtraction that the function $\psi$ defined by $\psi = \phi_1 - \phi_2$ will satisfy Laplaces equation

$$\nabla^2\psi = 0, \tag{3.2}$$

in the volume $V$. Since $\phi_1$ and $\phi_2$ by definition satisfy identical boundary conditions on $S$, it follows that $\psi$ will satisfy either $\psi = 0$ (Dirichlet) or $\partial\psi/\partial n = 0$ (Neumann) on $S$.

We now multiply (3.2) by $\psi$, integrate over $V$, and then perform an integration by parts:

$$\begin{aligned} 0 &= \int_V \psi \nabla^2 \psi dV \\ &= \int_V \big[\nabla \cdot (\psi \nabla \psi) - \nabla \cdot \psi \cdot \nabla \psi\big] dV \\ &= \int_S \psi \nabla \psi \cdot d\vec{S} - \int_V |\nabla \psi|^2 dV. \end{aligned} \tag{3.3}$$

Note that the first term on the last line comes by using the divergence theorem. Now we see that since either $\psi$ or its normal derivative $\partial\psi/\partial n \equiv \vec{n} \cdot \nabla\psi$ vanishes at all points on $S$, we are left with

$$\int_V |\nabla \psi|^2 = 0. \tag{3.4}$$

The integrand is everywhere non-negative, and so the integral can be zero only if the integrand vanishes everywhere in $V$. But if $|\nabla\psi|^2 = 0$ it follows that

$$\nabla \psi = 0. \tag{3.5}$$

everywhere in $V$, and so we conclude that $\psi$ is constant everywhere in $V$. In other words, we have proved that

$$\phi_1 = \phi_2 + k \tag{3.6}$$

where $k$ is a constant. We have the following two cases.

I) In the case of Dirichlet boundary conditions we know that $\phi_1 = \phi_2$ on $S$, and so the constant $k$ must be zero. This proves that $\phi_1 = \phi_2$ everywhere in $V$, thus establishing that the solution is unique.

II) In the case of Neumann boundary conditions, where only the normal derivative is specified on S, it is clear that the constant $k$ can never be determined. This is of no consequence, since $\phi$ and $\phi + k$ give rise to the same physical $\vec{E}$ field in the studied region $V$ in any case. So in the Neumann case, the solution is also unique.

Note that the results above can apply not only to the problem of solving for $\phi$ inside a finite volume $V$ with finite-sized closed boundary $S$, but also in the case where volume $V$ is infinite. A typical example would be when there is a finite-sized surface $S_1$ (for example a spherical conductor) and the volume $V$ is taken to be the entire infinite space outside it. In this case there is no actual boundary at infinity, but we can treat the problem by imagining that we introduce a spherical boundary surface $S_2$ at some very large radius $R$, and eventually we send $R$ to infinity. When $R$ is large but finite, we have a finite volume $V$ bounded by the disconnected sum of the two surfaces $S_1$ (in the middle) and $S_2$ (at large distance).

The uniqueness arguments discussed above can then be applied to this situation, with the surface integral in (3.3) becoming the sum of two integrals, one over the component $S_1$ of the total boundary and the other over the component $S_2$. Dirichlet or Neumann boundary conditions are specified on $S_1$, and so that contribution to the surface integral will vanish. The surface integral over $S_2$ will become zero in the limit

that the radius $R$ is sent to infinity, provided that $\phi$ goes to zero sufficiently fast at infinity (for example, one can readily show that if the charges are distributed within a finite volume, the surface integral term always vanishes). Thus in practice we think of $S_2$ as "the spherical surface at infinity" and we impose the boundary condition that $\phi$ goes to zero at infinity, thereby ensuring that the $S_2$ component of the surface integral in (3.3) will vanish too. This ensures that again we are left with just the volume integral (3.4), and so the uniqueness proof goes through as before.

Note also that we can allow multiple disconnected surfaces at finite distance, provided that Dirichlet or Neumann boundary conditions are imposed on all of them. In summary, therefore, we have a uniqueness proof too in the case where the volume $V$ is infinite, provided that we not only impose Dirichlet or Neumann boundary conditions on any boundary surfaces at finite distance, but we also impose a fall-off condition on the potential at infinity.

Note that we have established uniqueness of the solution subject to the imposition of either Dirichlet or Neumann boundary conditions at each point on the boundary. It could be Dirichlet for some points, and Neumann for others, but at any given point one must specify only one of Dirichlet or Neumann. With such boundary conditions specified, the problem is said to be well posed. This means that these boundary conditions are neither too weak, leaving the problem underdetermined and not fully pinned down, nor are they too strong, leaving the problem overdetermined and therefore admitting no solution.

An example of an overdetermined problem would be if one tried to impose both Dirichlet and Neumann boundary conditions at each point on $S$. In other words, if one tried to specify both the potential and its normal derivative at each point on $S$. Specifying both $\phi$ and $\partial\phi/\partial n$ on $S$ is known as specifying Cauchy boundary conditions. That this would be an overdetermination is obvious from the fact that Dirichlet conditions alone are sufficient to give a unique solution. And, on the other hand, Neumann conditions alone are sufficient to give another unique solution. Except in the unlikely event that one picked precisely the matching set of Neumann conditions that would reproduce the solution with the Dirichlet conditions, there will be a conflict between the two, implying that no solution would exist.

## 3.2 Green's theorem

Consider the charge density distribution $\rho(\mathbf{r})$ in the space, we gave the expression for the electrostatic potential by:

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \tag{3.7}$$

This result assumes that the charge distribution exists in otherwise free space, with no conductors or other boundaries present. In practice, as we have already remarked, a typical realistic situation is one where there are other conductors, etc., on which boundary conditions are specified. To handle the case where there are boundaries, the following procedure can be useful. We first derive a simple result known as Greens theorem, and then apply it to the case of interest.

Let $\phi$ and $\psi$ be two scalar functions. We can then consider

$$\begin{aligned}\nabla\cdot(\phi\nabla\psi-\psi\nabla\phi) &= \nabla\phi\cdot\nabla\psi+\phi\nabla^2\psi-\nabla\psi\cdot\nabla\phi-\psi\nabla^2\phi,\\ &= \phi\nabla^2\psi-\psi\nabla^2\phi. \end{aligned}\tag{3.8}$$

Integrating this over a volume $V$ bounded by surface $S$, and using the divergence theorem, we therefore find

$$\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)dV=\int_S(\phi\nabla\psi-\psi\nabla\phi)\cdot d\vec{S}.\tag{3.9}$$

This is the Green's theorem.

We apply it to our electrostatics problem by taking $\phi$ to be the electrostatic potential satisfying Poissons equation, and taking

$$\psi=\frac{1}{|\mathbf{r}-\mathbf{r}'|}.\tag{3.10}$$

We shall take $\mathbf{r}'$ to be the integration variable in (3.9), and so the derivatives in (3.9) will also be with respect to $\mathbf{r}'$. We shall therefore denote these with primes also. Note that we shall have

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}').\tag{3.11}$$

This can be used on the left-hand side of (3.9), and we also use $\nabla'^2\phi(\mathbf{r}')=-4\pi\rho(r')$. Thus we obtain

$$\phi(\mathbf{r})=\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'+\frac{1}{4\pi}\int_S\big[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\phi(\mathbf{r}')-\phi(\mathbf{r}')\nabla'(\frac{1}{|\mathbf{r}-\mathbf{r}'|})\big]\cdot d\vec{S}'.\tag{3.12}$$

The first term on the right-hand side of (3.12) is of the same form as the expression (3.7) that held in free space. The surface integrals in (3.12) represent the contribution from charge distributions on the boundary $S$ that we are now including. We can interpret (3.12) as giving the expression for the potential everywhere in $V$ in terms of the charge density $\rho$ in $V$ and the potential (and its normal derivative) on $S$. However, we cannot view (3.12) in itself as providing the answer we are seeking for how to solve for the potential in a general electrostatics problem. It can be seen from (3.12) that we would need to feed in the information about $\phi$ on the boundary and also about $\partial\phi/\partial n$ on the boundary in order to obtain the expression for $\phi$ in $V$. But we saw in the discussion of the uniqueness theorem that we are not allowed to specify independently the values of  and also of its normal derivative on the boundary; that would give an overdetermined problem that admitted no solution. Thus we can only regard (3.12) as an integral equation which will tell us what  is everywhere, once we know what it and its normal derivative are on the boundary. To solve the general boundary-value problem we will need to introduce another tool, which is called the Green function.

## 3.3 Green functions and the boundary-value problem

The key point about the function $\psi = 1/|\mathbf{r} - \mathbf{r}'|$ that we needed in deriving the result (3.12) was that it satisfied (3.11). In fact there is a much broader class of functions that satisfy equation (3.11). This is an inhomogeneous equation with the delta function providing a source on the right-hand side, and so we can add to the solution (3.10) an arbitrary solution of the homogeneous equation. Thus we may take $\psi$ in (3.9) to be any function of the form

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}'), \tag{3.13}$$

where $F(\mathbf{r}, \mathbf{r}')$ is any solution of the homogeneous equation

$$\nabla'^2 F(\mathbf{r}, \mathbf{r}') = 0, \tag{3.14}$$

which implies that $F(\mathbf{r}, \mathbf{r}')$ is an harmonic function. Thus we have

$$\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{3.15}$$

The idea now is that we will choose $F(\mathbf{r}, \mathbf{r}')$ so that $G(\mathbf{r}, \mathbf{r}')$, which is called a Green function, satisfies appropriate boundary conditions. To see how this works, we first note that there is an analogous result to (3.12) where we replace $\psi$ with $G(\mathbf{r}, \mathbf{r}')$ rather than $1/|\mathbf{r} - \mathbf{r}'|$, namely

$$\phi(\mathbf{r}) = \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3\mathbf{r}' + \frac{1}{4\pi} \int_S \big[ G(\mathbf{r}, \mathbf{r}') \nabla' \phi(\mathbf{r}') - \phi(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') \big] \cdot d\vec{S}'. \tag{3.16}$$

Consider first the case where we wish to specify Dirichlet boundary conditions for the potential on the surface S. We achieve this by choosing the harmonic function $F(\mathbf{r}, \mathbf{r}')$ in so that $G(\mathbf{r}, \mathbf{r}')$ vanishes when $\mathbf{r}'$ lies in the surface $S$. Thus, denoting this Dirichlet Green function by $G_D(\mathbf{r}, \mathbf{r}')$, we have $G_D(\mathbf{r}, \mathbf{r}') = 0$ when $\mathbf{r}' \in S$. Using $G_D(\mathbf{r}, \mathbf{r}')$ in (3.16) we therefore obtain

$$\phi(\mathbf{r}) = \int_V \rho(\mathbf{r}') G_D(\mathbf{r}, \mathbf{r}') d^3\mathbf{r}' - \frac{1}{4\pi} \int_S \phi(\mathbf{r}') \nabla' G_D(\mathbf{r}, \mathbf{r}') \cdot d\vec{S}'. \tag{3.17}$$

This has achieved the goal of giving an expression for $\phi(\mathbf{r})$ everywhere in the volume $V$, expressed in terms of the given charge density $\rho(\mathbf{r})$ and the values of $\phi(\mathbf{r})$ on the boundary surface $S$. Thus, we may say that the Dirichlet boundary-value problem is solved, albeit somewhat formally.

One might worry that (3.17) has done little more than replace one difficult problem (solving $\nabla^2\phi = 4\pi\rho$ for $\phi$) by another equivalently difficult problem (solving for the harmonic function $F(\mathbf{r}, \mathbf{r}')$ that is needed in order to ensure the Green function satisfies (3.16). However, this is not quite true, and moreover, a very important advantage of adopting this Green-function approach is that solving just once for the Green function for the given geometry then allows us to solve many different boundary-value problems. The point is the following. Having once solved for $G_D(\mathbf{r}, \mathbf{r}')$ in the given geometry (i.e. for the specified boundary surface $S$), one can now construct the solution

for $\phi(\mathbf{r})$ for any choice of charge density $\rho(\mathbf{r})$ and for any choice of the boundary-value potential on the surface $S$. Thus finding the Dirichlet Green function just once for the chosen geometry allows us to solve any Dirichlet boundary-value problem for that geometry.

The solution to the Neumann problem goes rather similarly, although with a minor subtlety. One might think that now one should choose $F(\mathbf{r}, \mathbf{r}')$ so that the normal derivative of the Green function vanished on $S$,

$$\partial G_N(\mathbf{r}, \mathbf{r}')/\partial n' = 0, \quad \text{when } \mathbf{r}' \in S. \tag{3.18}$$

But this would lead to a contradiction, since from the divergence theorem we have

$$\int_S \nabla' G_N(\mathbf{r}, \mathbf{r}') \cdot d\vec{S}' = -4\pi, \tag{3.19}$$

and so we cannot impose the boundary condition (3.18) on $S$. The simplest choice is to impose

$$\partial G_N(\mathbf{r}, \mathbf{r}')/\partial n' = -4\pi/A, \quad \text{when } \mathbf{r}' \in S, \tag{3.20}$$

where $A$ is the area of the boundary $S$. Substituting (3.20) into (3.16), we therefore find

$$\phi(\mathbf{r}) = \langle\phi\rangle_S + \int_V \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') d^3\mathbf{r}' + \frac{1}{4\pi} \int_S G_N(\mathbf{r}, \mathbf{r}') \nabla' \phi(\mathbf{r}') \cdot d\vec{S}'. \tag{3.21}$$

where $\langle\phi\rangle_S$ denotes the average value of $\phi$ over the surface $S$,

$$\langle\phi\rangle_S = \frac{1}{A} \int_S \phi(\mathbf{r}') dS'. \tag{3.22}$$

Note that $\langle\phi\rangle_S$ is a constant, which is not important for the Neumann problem, as we already know that the solution to Neumann problem can have an arbitrary constant difference without affecting the physics of the system. The solutions (3.17) or (3.23) for $\phi$ in terms of the Dirichlet or Neumann Green function provide at least a formal solution to the boundary-value problem. How useful they are in practice depends upon the details of the geometry of the problem. It all comes down to the question of whether one can solve explicitly for the Green function $G_D(\mathbf{r}, \mathbf{r}')$ or $G_N(\mathbf{r}, \mathbf{r}')$. For a boundary $S$ of some generic type it will certainly be impossible. In certain special cases one can obtain closed-form expressions. We shall meet example later on where this can be done, in the case of an infinite planar boundary $S$, and in the case of a spherical boundary. It is worth making a few closing remarks about the physical interpretation of the Green function. The simplest example is when

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.23}$$

which is, as we have seen, the Green function for the Dirichlet problem where the only boundary is the "sphere at infinity". We can recognize (3.23) as being the electric potential at the point $\mathbf{r}$ due to a unit point charge at the point $\mathbf{r}'$. In fact (3.23) is

symmetrical under the exchange of $\mathbf{r}$ and $\mathbf{r}'$, and so it can be equivalently viewed as the potential at $\mathbf{r}'$ due to a unit charge at $\mathbf{r}$.

In the more general case, the Green function is of the form (3.13), where $F(\mathbf{r}, \mathbf{r}')$ is harmonic function. This means that $G(\mathbf{r}, \mathbf{r}')$ again has the interpretation of being the potential at $\mathbf{r}$ due to a unit charge at $\mathbf{r}'$, but now in the more complicated case where $G(\mathbf{r}, \mathbf{r}')$ (or its normal derivative, in the Neumann case) vanishes on $S$. One can show quite generally that in the case of Dirichlet boundary conditions, the Green function $G_D(\mathbf{r}, \mathbf{r}')$ is necessarily symmetrical under the exchange of $\mathbf{r}$ and $\mathbf{r}'$:

$$G_D(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}', \mathbf{r}). \tag{3.24}$$

(This can be done by using Greens theorem (3.9).) In the case of Neumann boundary conditions, symmetry under the exchange of $\mathbf{r}$ and $\mathbf{r}'$ is not automatic, but it can always be imposed.

In the following sections we shall see how the Green function method is applied to solve the problems in electrostatics.

## 3.4 Methods of images

Sometimes, if there is a boundary surface of particularly simple and symmetrical geometry, it is possible to solve the boundary-value problem by means of the Method of Images. Suppose, for example, the potential is specified to be zero on a surface $S$ in otherwise free space, and that one wishes to calculate the potential everywhere outside (in the volume $V$) due to a point charge located outside the surface. If $S$ is suitably symmetrical, it may be possible to "mock up" the same zero-potential surface by considering a totally free space, with no surfaces anywhere, but with one or more additional image charges judiciously introduced in the region of the total space that does not include $V$. (i.e., the additional image charges are on the "other side" of where the surface $S$ of the original problem was located.) Suppose that by introducing image charges in this way, one can arrange that the total potential due to the original charge plus the image charges is zero on the virtual surface $S$. It is then clear, by invoking the uniqueness theorem, that the potential at all points in $V$ must be the same in the image-charge "mock-up" and in the original problem with the actual physical conducting surface.

The challenge is to figure out how to achieve the "virtual" zero-potential surface by means of image charges. In practice, there are very few cases where it can be done. We shall discuss two of them now. Furthermore, together with the Green function method, we can generalize the solutions for those special cases to more generic situations.

### 3.4.1 Infinite planar conductor

The simplest example where the method of images can be employed is in the case of an infinite planar conductor. Let us suppose, for convenience, that Cartesian axes are chosen so that the conductor lies in the plane $z = 0$. We shall take the volume $V$ that lies outside the conductor to be the whole of the half-space $z > 0$. Suppose the conductor is fixed at zero potential.

If a point charge $q$ is located at some point in $V$, then it is obvious that if an image charge $-q$ is placed "behind" the conductor at precisely the mirror-image location, then by symmetry it must be the case that the total potential of original plus image charge, now taken to be in a completely free space with no conductor at all, will vanish on the plane $z = 0$. Therefore, the potential at any point in $V$ in the original problem with conductor will be given by the total potential in the image-charge "mock up".

To be more precise, let us suppose that the original charge $q$ is located at

$$\mathbf{r}_1 = (x_1, y_1, z_1), \quad \text{with} \quad z_1 > 0. \tag{3.25}$$

The image charge $-q$ is then located at

$$\mathbf{r}_2 = (x_2, y_2, z_2) = (x_1, y_1, -z_1). \tag{3.26}$$

Therefore, the total potential is given by

$$\phi(\mathbf{r}) = \frac{q}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} - \frac{q}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z+z_1)^2}}. \tag{3.27}$$

Clearly this potential indeed vanishes on the surface $z = 0$, and so therefore by the uniqueness theorem $\phi(\mathbf{r})$ describes the potential, at all points with $z > 0$, of the single charge q in the presence of the infinite conductor at $z = 0$.

### 3.4.2 Dirichlet and Neumann Green functions for infinite planar boundary

We are now in a position to construct the Dirichlet and Neumann Green functions for this case. Recall that the Dirichlet Green function $G_D(\mathbf{r}, \mathbf{r}')$ is defined to be the potential at $\mathbf{r}$ due to a unit strength charge at $\mathbf{r}'$, subject to the condition that $G_D(\mathbf{r}, \mathbf{r}')$ should vanish on the boundary $S$. Thus we can read off from Eq. (3.27) that the Dirichlet Green function in the case of the infinite planar boundary at $z = 0$ is given by

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}. \tag{3.28}$$

It can be seen that this is indeed symmetric under the exchange of $\mathbf{r}$ and $\mathbf{r}'$. To use the above formula in order to solve Dirichlet boundary-value problems in this geometry, we just plug it into the general expression (3.17). This requires that we evaluate the normal derivative of $G_D$ on the boundary, which in the present case means that we need

$$-\frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial z'}\Big|_{z'=0} = -\frac{2z}{\Big[(x-x')^2 + (y-y')^2 + (z-z')^2\Big]^{3/2}}. \tag{3.29}$$

We emphasize that the normal derivative in (3.17) is directed outwards from the volume $V$, which means in this case in the negative direction along the $z'$ axis. (Thus

$\partial G_D(\mathbf{r},\mathbf{r}')/\partial n' = -\partial G_D(\mathbf{r},\mathbf{r}')/\partial z'$, as given in the above formula.) Suppose we consider that there is charge distribution $\rho(\mathbf{r})$ in the above plane. Then we obtain the total potential in the studied space by

$$\begin{aligned}\phi(\mathbf{r}) &= \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &\quad - \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \\ &\quad + \frac{z}{2\pi}\int dx'dy' \frac{\phi(x',y',0)}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}} \\ &= \phi_1(\mathbf{r})+\phi_2(\mathbf{r})+\phi_3(\mathbf{r}), \end{aligned} \tag{3.30}$$

where $\phi_j(\mathbf{r})$ $(j=1,2,3)$ represents the $j$-th term in the right hand side of the above formula. The first term $\phi_1$ in the right hand side represents the contribution by the charge $\rho(\mathbf{r})$ above the infinite plane, the second term $\phi_2$ represents the contribution by the imagine charge, and the third term $\phi_3$ represents the contribution from the boundary value of $\phi$ on the plane at $z'=0$. The last term can be rewritten as

$$\phi_3(\mathbf{r}) = \int dx'dy'dz' \frac{\rho_S(x',y',z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}, \tag{3.31}$$

with

$$\rho_S(\mathbf{r}') = -\frac{1}{2\pi}\phi(x',y',0)\frac{d\delta(z')}{dz'}. \tag{3.32}$$

Thus the term $\phi_3$ can be understood as the contribution due to a charge density distribution $\rho_S(\mathbf{r}')$ at the infinite plane at $z'=0$. Now we have constructed the solution of the equation $\nabla^2\phi = -4\pi\rho$, in which the boundary value of $\phi$ on the infinite planar surface $z'=0$ is specified.

We can also easily construct the Neumann Green function $G_N(\mathbf{r},\mathbf{r}')$ for this geometry. In this case, it is defined to be the potential at $\mathbf{r}$ due to a unit strength charge at $\mathbf{r}'$, subject to the condition that the normal derivative of $G_N$ should vanish on the plane $z'=0$ [1]. This time, we can suspect that a small modification of the image-charge trick should give us the required result. Indeed this works, and all we need to do is to replace the minus sign in front of the second term in (3.28) by a plus sign, to give

$$G_N(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}. \tag{3.33}$$

It is easily to verify that this satisfies the required condition that

$$\frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial z'}|_{z'=0} = 0. \tag{3.34}$$

[1] In our general discussion for the Green function for Neumann boundary conditions, we had the requirement (3.20) that the normal derivative should equal $-4\pi/A$, where $A$ was the area of the boundary. In the present case this area is infinite, and so we can simply require that the normal derivative of GN should vanish.

We now plug the Neumann Green function into (1.94), in order to solve the general class of boundary-value problems in which the normal derivative of $\phi$ is specified on the infinite planar surface $z = 0$. Suppose the charge density $\rho(\mathbf{r})$ is nonzero above the plane. Plugging (3.33) into (3.21) then gives

$$\begin{aligned}\phi(\mathbf{r}) &= \langle\phi(\mathbf{r}')\rangle_S + \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &+ \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \\ &- \frac{1}{2\pi}\int dx'dy' \frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}} \left( \frac{\partial\phi(x',y',z')}{\partial z'}|_{z'=0} \right). \quad (3.35)\end{aligned}$$

The average value $\langle\phi(\mathbf{r}')\rangle_S$ at the boundary can be set as zero. Of course, since $\vec{E} = -\nabla\phi$, we may write the above formula as

$$\begin{aligned}\phi(\mathbf{r}) &= \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &+ \int dx'dy'dz' \frac{\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \\ &+ \frac{1}{2\pi}\int dx'dy' \frac{E_z(x',y',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}} \\ &= \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) + \phi_3(\mathbf{r}), \quad (3.36)\end{aligned}$$

where the last term can also be written as

$$\phi_3(\mathbf{r}) = \int dx'dy' \frac{\sigma(x',y',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}}, \quad (3.37)$$

with

$$\sigma(x',y',0) = E_z(x',y',0)/2\pi, \quad (3.38)$$

denoting an effective surface charge density distribution in the boundary.

### 3.4.3 Spherical conductor

A slightly more subtle example where the method of images can be employed is in the case of a spherical conductor. Suppose a conducting sphere of radius $a$ is held at zero potential, and that a charge $q$ is placed outside the sphere, at a distance $b$ from its origin. It turns out that this situation can be "mocked up" by considering instead entirely free space containing the original charge and also a certain charge $q'$ placed at a certain distance $c$ from the origin of the sphere, on the line joining the charge $q$ and the origin.

The quickest way to derive this result is as follows. Imagine that the sphere is centred on the origin of Cartesian coordinates, and that the charge $q$ is placed at distance $b$ along the $z$ axis, i.e. at $(x, y, z) = (0, 0, b)$. The claim is that the image charge $q'$ should also lie on the z axis, at some point $(x, y, z) = (0, 0, c)$. If this does

indeed give rise to a spherical surface of radius a that has zero potential, then in particular it must be that the potential is zero at the two points $(0, 0, a)$ and $(0, 0, -a)$ on the sphere. Since these two points are aligned on the same axis as the charges, it is particularly easy to write down the conditions that the potential should be zero:

$$\frac{q}{b+a} + \frac{q'}{a+c} = 0, \quad \frac{q}{b-a} + \frac{q'}{a-c} = 0. \tag{3.39}$$

These two conditions determine $q'$ and $c$, giving

$$q' = -\frac{aq}{b}, \quad c = \frac{a^2}{b}. \tag{3.40}$$

Observe that since $b > a$, we have $c = a(a/b) < a$. Thus, as one would expect, the image charge is inside the spherical surface. It remains to verify that the potential then vanishes for an arbitrary point on the sphere. The problem has rotational symmetry around the $z$ axis, so it suffices to consider a point $P$ at angle $\theta$ from the $z$-axis. If the distance from $q$ to $P$ is $l_q$, and the distance from $q'$ to $P$ is $l_{q'}$, then the cosine rule gives

$$l_q^2 = a^2 + b^2 - 2ab\cos\theta, \quad l_{q'}^2 = a^2 + c^2 - 2ac\cos\theta. \tag{3.41}$$

We can see that $l_{q'} = (a/b)l_q$ and thus we have

$$\frac{q}{l_q} + \frac{q'}{l_{q'}} = 0. \tag{3.42}$$

Therefore the potential vanishes everywhere on the sphere $x^2 + y^2 + z^2 = a^2$.

It is useful also to give the result in a more general fashion, in which the original charge $q$ is placed at an arbitrary point $\mathbf{r}_1$ located outside the sphere, rather than lying specifically on the $z$ axis. Clearly, if the charge $q$ lies at $\mathbf{r}_1$ then the charge $q'$ must lie at a point $\mathbf{r}_2$ along the same direction, and since the second relation in Eq. (3.40) can be written as $c = (a^2/b^2)b$, we must have

$$\mathbf{r}_2 = \frac{a^2}{r_1^2}\mathbf{r}_1. \tag{3.43}$$

Thus, the potential at $\mathbf{r}$ outside the zero-potential sphere at $r = a$ due to a charge q located at $\mathbf{r}_1$ outside the sphere is given by

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}_1|} - \frac{qa/r_1}{|\mathbf{r} - (a^2/r_1^2)\mathbf{r}_1|}. \tag{3.44}$$

**Generalization**.–Because of the linearity of the Maxwell equations, it is straightforward to generalise the above result in a variety of ways. For example, instead of taking the conducting sphere to be at zero potential, we could consider a situation where it is held at a non-zero potential $V$ (relative to zero at infinity, still). All that need be done is to add another term to the potential (3.44), corresponding to the introduction of a point charge at the origin. Thus if we now take

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}_1|} - \frac{qa/r_1}{|\mathbf{r} - (a^2/r_1^2)\mathbf{r}_1|} + \frac{Q}{r}, \tag{3.45}$$

then the potential on the surface of the sphere becomes $\phi = Q/a$. Choosing $Q = Va$ therefore gives the required result.

As another generalisation, we can calculate the solution for a grounded sphere placed in a previously-uniform electric field. Without loss of generality, let us take the electric field to be directed along the $z$ axis. The uniform field can be achieved via a limiting process in which two point charges $\pm Q$ are placed at $z = \mp b$ respectively. Close to the origin, there will therefore be an approximately uniform electric field $E_0 \approx 2Q/b^2$ directed along z. Eventually, we take $b$ to infinity, while holding $E_0 = 2Q/b^2$ fixed, and the approximation becomes exact.

In the presence of the grounded sphere, each of the above charges will have its image charge, with $+Q$ at $-b$ having an image charge $-Qa/b$ at $z = -a^2/b$, and $-Q$ at $+b$ having an image charge $+Qa/b$ at $z = +a^2/b$. If we use spherical polar coordinates to write

$$\mathbf{r} = (x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \tag{3.46}$$

then from (3.45) we deduce that the total potential for the system we are considering will be

$$\begin{aligned}\phi(r,\theta,\varphi) &= \frac{Q}{\sqrt{r^2+b^2+2rb\cos\theta}} - \frac{Q}{\sqrt{r^2+b^2-2rb\cos\theta}} \\ &- \frac{Qa/b}{\sqrt{r^2+a^4/b^2+2a^2r/b\cos\theta}} + \frac{Qa/b}{\sqrt{r^2+a^4/b^2-2a^2r/b\cos\theta}},\end{aligned} \tag{3.47}$$

Expanding as a power series in $1/b$, we find

$$\phi(r,\theta,\varphi) = -\frac{2Q}{b^2}r\cos\theta + \frac{2Q}{b^2}\frac{a^3}{r^2}\cos\theta + ..., \tag{3.48}$$

where the higher-order terms involve higher inverse powers of b and therefore they will go to zero when $b$ is sent to infinity holding $E_0 = 2Q/b^2$ fixed. In this limit, we therefore find that

$$\phi(r,\theta,\varphi) = -E_0(r - \frac{a^3}{r^2})\cos\theta. \tag{3.49}$$

The first term in (3.49) can be written using Cartesian coordinates as $\phi = -E_0 z$, and so it just describes the purely uniform electric field $\vec{E} = -\nabla\phi = (0, 0, E_0)$ that would occur in the absence of the grounded sphere. The second term describes an electric dipole contribution to the potential, arising from the two pairs of charges plus images.

### 3.4.4 Dirichlet Green function for spherical boundary

We can use the results in section 3.4.3 to construct the Dirichlet Green function for the boundary-value problem where the potential is specified on the surface of a sphere. We just need to set $q = 1$ and $\mathbf{r}_1 = \mathbf{r}'$ in Eq. (3.44), leading to

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{a/r'}{|\mathbf{r} - (a^2/r'^2)\mathbf{r}'|}. \tag{3.50}$$

If we introduce $\gamma$ as the angle between $\mathbf{r}$ and $\mathbf{r}'$, then Eq. (3.50) can be written as

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{a^2+r^2r'^2/a^2-2rr'\cos\gamma}}. \tag{3.51}$$

Written in this form, it is manifest that $G_D(\mathbf{r},\mathbf{r}')$ is symmetric under the exchange of $\mathbf{r}$ and $\mathbf{r}'$. It is also manifest that $G_D(\mathbf{r},\mathbf{r}')$ vanishes, as it should, if $\mathbf{r}$ or $\mathbf{r}'$ lies on the surface of the sphere.

To use this expression in the general boundary-value integral in (3.17), we need to calculate the normal derivative with respect to $\mathbf{r}'$, evaluated on the sphere at $r' = a$. Bearing in mind that the outward normal from the volume $V$ (external to the sphere) is directed inwards towards the centre of the sphere, we therefore need

$$\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}|_{r'=a} = -\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial r'}|_{r'=a} = -\frac{r^2-a^2}{a[r^2+a^2-2ar\cos\gamma]^{3/2}}. \tag{3.52}$$

Substituting into (3.17) (and taking the charge density $\rho = 0$ for simplicity), we obtain

$$\phi(\mathbf{r}) = \frac{a(r^2-a^2)}{4\pi}\int \frac{\phi(a,\theta',\varphi')}{[r^2+a^2-2ar\cos\gamma]^{3/2}}d\Omega'. \tag{3.53}$$

where in the boundary integral we express the potential $\phi$ in terms of spherical polar coordinates $(r',\theta',\varphi')$. The area element on the sphere of radius $a$ is written as $\vec{n}\cdot d\vec{S}' = a^2 d\Omega'$, where $d\Omega' = \sin\theta' d\theta' d\varphi'$ is the area element on the unit sphere (i.e. the solid angle element). The expression (3.53) gives the result for the potential everywhere outside a spherical surface of radius $a$, on which the potential is specified to be $\phi(a,\theta',\varphi')$.

Note that the integration in (3.53) is actually rather complicated, even if $\phi(a,\theta',\varphi')$ itself is a simple function, because of the $\cos\gamma$ appearing in the denominator. Using spherical polar coordinates, the Cartesian components of $\mathbf{r}$ and $\mathbf{r}'$ are

$$\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \tag{3.54}$$
$$\mathbf{r}' = (r'\sin\theta'\cos\varphi', r'\sin\theta'\sin\varphi', r'\cos\theta'), \tag{3.55}$$

and so $\cos\gamma$, which is defined by $\mathbf{r}\cdot\mathbf{r}' = rr'\cos\gamma$, is given by

$$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'.$$

To have a concrete understanding of the applications of the above solution of $\phi$, we consider, as an example, the case where one hemisphere of the boundary surface is held at a constant potential $V$ , while the other hemisphere is held at potential $-V$ . Since we are using standard spherical polar coordinates, it is natural to orient things so that the two hemispheres correspond to the parts of the sphere with $z > 0$ and $z < 0$ respectively. In other words, we have

$$\phi(a,\theta,\varphi) = \begin{cases} +V, & \text{for} \quad 0 \le \theta < \pi/2, \\ -V, & \text{for} \quad \pi/2 \le \theta \le < \pi. \end{cases} \tag{3.56}$$

The solution of $\phi$ then reads

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{aV(r^2-a^2)}{4\pi}\int_0^{2\pi} d\varphi' \Big[\int_0^{\pi/2} \frac{\sin\theta'}{[r^2+a^2-2ar\cos\gamma]^{3/2}}d\theta' \\ &\quad - \int_{\pi/2}^{\pi} \frac{\sin\theta'}{[r^2+a^2-2ar\cos\gamma]^{3/2}}d\theta'\Big].\end{aligned} \tag{3.57}$$

By making the change of variables $\theta' \to \pi - \theta'$ and $\varphi' \to \varphi' + \pi$ in the second integral, this can be written as

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{aV(r^2-a^2)}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi/2} \Big[\frac{\sin\theta'}{[r^2+a^2-2ar\cos\gamma]^{3/2}}d\theta' \\ &\quad - \frac{\sin\theta'}{[r^2+a^2+2ar\cos\gamma]^{3/2}}d\theta'\Big]. \end{aligned} \tag{3.58}$$

Unfortunately, the integrations are still too complicated to admit a useful explicit closed-form result[2]. We can easily integrate (3.58) if we ask for the potential $\varphi$ only in the special case where we are on the $z$ axis, i.e. for $\theta = 0$. It then follows that $\cos\gamma = \cos\theta'$, and then elementary integration of (3.58) gives, for $z > a$,

$$\phi(\mathbf{r}) = V\left(1 - \frac{z^2-a^2}{z\sqrt{a^2+z^2}}\right). \tag{3.59}$$

In the absence of a closed-form expression for the general off-axis potential, one could resort to making a power-series expansion of the integrand in (3.58) in powers of $\cos\gamma$, and then performing the integrations term by term. This is a somewhat clumsy approach. In the future we shall develop an approach which will allow us to obtain the power series expression for the off-axis potential easily.

## 3.5 Separation of variables

### 3.5.1 Introduction

The boundary-value problem in electrostatics is formulated as the problem of solving Poissons equation $\nabla^2\phi = -4\pi\rho$ in a volume $V$ bounded by a surface $S$ on which appropriate boundary conditions are imposed. Quite commonly, we are interested in the situation where $\rho = 0$ in $V$, so that the potential $\phi$ in $V$ is governed entirely by the conditions that it, or its normal derivative, satisfies on $S$.

The geometry of the boundary surface $S$ typically dictates what type of coordinate system is best adapted to the problem. For example, if $S$ is formed by one or more planar surfaces, then Cartesian coordinates are likely to be the most convenient choice. If, on the other hand, the boundary $S$ is spherical, then spherical polar coordinates will probably be the best choice. For a boundary of cylindrical shape, cylindrical polar coordinates will be most convenient.

All three of these coordinate systems share the special property that when using them the Laplacian operator $\nabla^2$ is *separable*. This would not be true for some arbitrary choice of coordinate system. The defining property of a separable coordinate system is that Laplaces equation, which is itself a second-order partial differential equation, can be factored into a system of second-order ordinary differential equations. This is of enormous benefit when one tries to construct solutions.

[2]This illustrates an important point, that although we may say that the boundary-value problem for the spherical boundary has been solved once we obtained an explicit closed-form result for the Green function, it does not necessarily mean that we can present an explicit closed-form expression for the solution.

We shall describe the process of separation of variables in the three cases of Cartesian, spherical polar, and cylindrical polar, coordinates. In each case, the solution of the factored ordinary differential equations requires an understanding of certain classes of special functions. In the Cartesian case, the relevant special functions are just the familiar sine and cosine trigonometric functions. In the case of spherical polar coordinates, the Legendre and associated Legendre functions arise, whilst in the case of cylindrical polar coordinates it is Bessel functions that arise.

We begin in this section with the separation of variables in Cartesian coordinates.

### 3.5.2 Separation of variables in Cartesian coordinates

The Laplace equation in Cartesian coordinates is simply

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{3.60}$$

The separation of variables is achieved by considering a solution of the factorised form

$$\phi(x,y,z)=X(x)Y(y)Z(z). \tag{3.61}$$

Substituting this into (3.60) and dividing out by $\phi$, yields

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0. \tag{3.62}$$

The first term is independent of $y$ and $z$, the second is independent of $x$ and $z$, and the third is independent of $x$ and $y$. It therefore follows that each term must separately be a constant, with the three constants summing to zero. Therefore either two of the constants are positive with the third negative, or two are negative with the third positive. Let us take the constants in the first two terms to be negative, and the last to be positive, so that we may write

$$\frac{1}{X}\frac{d^2X}{dx^2}+\alpha^2=0,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}+\beta^2=0,\quad \frac{1}{Z}\frac{d^2Z}{dz^2}-\gamma^2=0, \tag{3.63}$$

with

$$\gamma^2=\alpha^2+\beta^2. \tag{3.64}$$

The solutions for $X$, $Y$ and $Z$ will therefore be of the forms

$$X\sim e^{\pm i\alpha x},\quad Y\sim e^{\pm i\beta y},\quad Z\sim e^{\pm\gamma z}. \tag{3.65}$$

Equivalently, the solutions for $X$ and $Y$ can be taken to be linear combinations of sine and cosine functions of their respective arguments, while $Z$ can be written in terms of hyperbolic functions.

The general solution to (3.60) can now be written as a sum over all the "basic" solutions of the form (3.61) that we have now constructed. Since $\alpha$ and $\beta$ are at this stage arbitrary constants, the general solution will have the form

$$\phi(x,y,z)=\int_{-\infty}^{+\infty}d\alpha\int_{-\infty}^{+\infty}d\beta e^{i(\alpha x+\beta y)}\big[a(\alpha,\beta)e^{\gamma z}+b(\alpha,\beta)e^{-\gamma z}\big], \tag{3.66}$$

where $a(\alpha, \beta)$ and $b(\alpha, \beta)$ are arbitrary functions, and $\gamma$ is given by (3.64). The general solution to (3.60) is expressed as an integral over a continuum of the basic solutions, as in (3.66). However, as soon as we also impose boundary conditions on the solution, the continuous integrals will be replaced by a discrete sum over basic solutions.

**Example: A rectangular hollow box:**

Suppose, for example, we wish to solve Laplace's equation inside a hollow rectangular box, with sides of length $a$, $b$ and $c$ in the $x$, $y$ and $z$ directions respectively. We may set up the axes so that the origin is at one corner of the box, so that the faces are located at $x = 0$ and $x = a$; at $y = 0$ and $y = b$; and at $z = 0$ and $z = c$. Suppose that the faces are all held at zero potential, except for the face at $z = c$, on which the potential is specified to be

$$\phi(x, y, z = c) = V(x, y), \tag{3.67}$$

for some specified voltage profile function $V(x, y)$.

Since the potential vanishes at $x = 0$ for all $y$ and $z$, it follows that we must arrange for $X(x)$ to vanish at $x = 0$. Since the general solution for $X(x)$ is

$$X(x) = a_1 e^{i\alpha x} + b_1 e^{-i\alpha x}, \tag{3.68}$$

we must have $b_1 = -a_1$, and thus $X(x) \sim \sin(\alpha x)$. The potential also vanishes at $x = a$ for all $y$ and $z$, and this means that we must have $X(a) = 0$. This implies that $\alpha$ must be restricted to take only a discrete (but infinite) set of values,

$$\alpha = \frac{m\pi}{a}, \quad m \in \mathbb{Z}. \tag{3.69}$$

Without loss of generality we may assume that m is a positive integer, since the negative values will just reproduce the same set of functions (multiplied by $-1$)[3]. In the same way, the vanishing of $\phi$ at $y = 0$ and $y = b$ implies that $Y(y)$ must be proportional to $\sin \beta y$, and that $\beta$ must be of the form

$$\beta = \frac{n\pi}{b}, \quad n \in \mathbb{Z}. \tag{3.70}$$

We also take $n$ to be positive integers.

The vanishing of $\phi$ at $z = 0$ implies that $Z(z)$ must be proportional to $\sinh \gamma z$. Since $\gamma$ is given in terms of $\alpha$ and $\beta$ through Eq. (3.64), it follows that the general solution for $\phi$ that satisfies all the boundary conditions except the one on the remaining face at $z = c$ can be written as

$$\phi(x, y, z) = \sum_{m\geq 1} \sum_{n\geq 1} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh\big(\pi z \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\big), \tag{3.71}$$

where $A_{mn}$ are arbitrary constants.

The constants $A_{mn}$ are determined by matching $\phi$ to the given boundary condition (3.67) at $z = c$. This amounts to constructing a two-dimensional Fourier series expansion for the function $V(x, y)$. To do this, it is useful to recall a few facts about expansions in terms of complete sets of orthogonal functions.

[3]Indeed, for the general solution, if we sum over the solutions with respect to $m$ and $-m$ ($m > 0$), the coefficients of the two summed terms can be simply combined and give a single coefficient with respect to $m$. The reason is because the two eigen-solutions for m and -m are actually identical and thus not independent. The general solution should be expanded over the independent eigen-solutions.

### 3.5.3 Generalised Fourier expansions

Suppose that the functions $u_n(x)$ for $n \geq 1$ form a *complete* set in the interval $a \leq x \leq b$. They may be either real, or complex, and so for generality we shall assume that they are complex, with complex conjugation denoted by a bar. We furthermore assume that the functions are *orthogonal*, meaning that

$$\int_a^b \bar{u}_m(x) u_n(x) dx = 0, \quad m \neq n. \tag{3.72}$$

The integral when $m = n$ is non-zero and finite, and so we may for convenience scale the functions so that when $m = n$ the integral gives unity. Thus we may assume orthonormal functions, satisfying

$$\int_a^b \bar{u}_m(x) u_n(x) dx = \delta_{mn}. \tag{3.73}$$

Suppose now we have an arbitrary function $f(x)$, which is assumed to be square integrable on the interval $a \leq x \leq b$. (i.e. $\int_a^b |f(x)|^2 dx$ is finite.) We may expand $f(x)$ in terms of the complete set of functions $u_n(x)$, by writing

$$f(x) = \sum_{n \geq 1} a_n u_n(x), \tag{3.74}$$

where $a_n$ are certain constants. These constants can be determined by multiplying (3.74) by $u_m(x)$, integrating over the interval $a \leq x \leq b$, and using the orthonormality relations (3.73) we shall have

$$a_n = \int_a^b \bar{u}_n(x) f(x) dx. \tag{3.75}$$

If we plug (3.75) into (3.74), being careful to distinguish between the argument $x$ in (3.74) and the integration variable $x$ in (3.75) (which we shall now call $x$), we obtain

$$f(x) = \int_a^b \Big[ \sum_{n \geq 1} \bar{u}_n(x') u_n(x) \Big] f(x') dx'. \tag{3.76}$$

Comparing this with the defining property of the delta function, we see that we may conclude that

$$\sum_{n \geq 1} \bar{u}_n(x') u_n(x) = \delta(x - x'). \tag{3.77}$$

The generalisation to expansions in more than one dimension is immediate. For example, suppose in two dimensions we have the orthonormal functions $u_m(x)$ as before, defined in $a \leq x \leq b$, and another orthonormal set $v_n(y)$ defined in the interval $c \leq y \leq d$. We can expand

$$f(x, y) = \sum_{m \geq 1} \sum_{n \geq 1} a_{mn} u_m(x) u_n(y), \tag{3.78}$$

and read off the coefficients $a_{mn}$ by first multiplying by $\bar{u}_p(x)\bar{v}_q(y)$ and integrating, to give

$$a_{mn} = \int_a^b dx \int_c^d dy \bar{u}_m(x)\bar{u}_n(y) f(x,y). \tag{3.79}$$

Now we can go back to our boundary-value problem in the rectangular box. Recall that we had obtained the expression (3.71) for $\phi(x,y,z)$ everywhere inside the box, expressed as a double summation. It remained for us to determine the expansion coefficients $A_{mn}$, by matching $\phi(x,y,z)$ to the given boundary potential $V(x,y)$ at $z=c$. In other words, we must find $A_{mn}$ such that

$$V(x,y) = \sum_{m\geq 1}\sum_{n\geq 1} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh\big(\pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\big), \tag{3.80}$$

This amounts to solving for the coefficients $a_{mn}$ such that

$$V(x,y) = \sum_{m\geq 1}\sum_{n\geq 1} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{3.81}$$

and then $A_{mn}$ will be given by

$$A_{mn} = \frac{a_{mn}}{\sinh\big(\pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\big)}. \tag{3.82}$$

To determine the coefficients $a_{mn}$ in (3.81), we recognise that this is an example of a two dimensional expansion in terms of the complete sets of functions $\sin\frac{m\pi x}{a}$ and $\sin\frac{n\pi y}{b}$, and so to read off $a_{mn}$ we just need to multiply by $\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}$ and integrate. It is straightforward to show the following results

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{p\pi x}{a}dx = \frac{1}{2}a\delta_{mp}. \tag{3.83}$$

and

$$\int_0^b \sin\frac{m\pi y}{b}\sin\frac{p\pi y}{b}dy = \frac{1}{2}b\delta_{mp}. \tag{3.84}$$

We thus obtain the coefficients $a_{mn}$ by

$$a_{mn} = \frac{4}{ab}\int_0^a dx \int_0^b dy V(x,y) \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{3.85}$$

## 3.6 Separation of variables in curvilinear coordinates

### 3.6.1 Separation of variables in spherical polar coordinates

The spherical polar coorinates $(r,\theta,\varphi)$ are related to Cartesian coordinates $(x,y,z)$ by

$$x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi,\ \ z = r\cos\theta. \tag{3.86}$$

Please refer to the handwriting notes for this section.

## 3.7 Multipole expansion

The multipole expansion provides a way of organising the expression for the electrostatic potential due to a localised distribution of charges, as a sum over terms proportional to the total charge, the dipole moment, the quadrupole moment, and so on. This topic will apply the techniques of vectors and tensors in three-dimensional Cartesian space.

### 3.7.1 Multipole expansion in Cartesian coordinates

Consider the electrostatic potential of $N$ point charges $q_a$, located at fixed positions $\mathbf{r}_a$. It is given by

$$\phi(\mathbf{r}) = \sum_{a=1}^{N} \frac{q_a}{|\mathbf{r} - \mathbf{r}_a|}. \tag{3.87}$$

In the continuum limit, the potential due to a charge distrubution characterised by the charge density $(\mathbf{r})$ is given by

$$\phi(\mathbf{r}) = \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.88}$$

Since we shall assume that the charges are confined to a finite region, it is useful to perform a multipole expansion of the potential far from the region where the charges are located. This amounts to an expansion in inverse powers of $r = |\mathbf{r}|$. This can be achieved by performing a Taylor expansion of $|\mathbf{r} - \mathbf{r}'|^{-1}$.

Recall that in one dimension, Taylors theorem gives

$$f(x + a) = f(x) + a f^{(1)}(x) + \frac{a^2}{2!} f^{(2)}(x) + \frac{a^3}{3!} f^{(3)}(x) + .... \tag{3.89}$$

Here $f^{(n)}(x)$ is the $n$-th order derivative of $f$ with respect to $x$. In three dimensions, the analogous expansion is

$$f(\mathbf{r} + \mathbf{a}) = f(\mathbf{r}) + a_i \partial_i f(\mathbf{r}) + \frac{1}{2!} a_i a_j \partial_i \partial_j f(\mathbf{r}) + \frac{1}{3!} a_i a_j a_k \partial_i \partial_j \partial_k f(\mathbf{r}) + .... \tag{3.90}$$

Note that the repeated indices in the above equation should be summed over $1, 2, 3$. We now apply this 3-dimensional Taylor expansion to the function $f(\mathbf{r}) = 1/|\mathbf{r}| = 1/r$, taking $\mathbf{a} = -\mathbf{r}'$. This gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} - x'_i \partial_i \frac{1}{r} + \frac{1}{2!} x'_i x'_j \partial_i \partial_j \frac{1}{r} - \frac{1}{3!} x'_i x'_j x'_k \partial_i \partial_j \partial_k \frac{1}{r} + .... \tag{3.91}$$

Now since $r^2 = x_j x_j$, it follows that $\partial_i r^2 = 2r \partial_i r = 2x_i$, and so

$$\partial_i r = \frac{x_i}{r}. \tag{3.92}$$

Note that we have (assuming $r \neq 0$) that

$$\nabla^2 \frac{1}{r} = \partial_i \partial_i \frac{1}{r} = -\partial_i \frac{x_i}{r^3} = -\frac{3}{r^3} + \frac{3x_i}{r^4} \frac{x_i}{r} = 0. \tag{3.93}$$

A consequence of this is that the multiple derivatives

$$\partial_i\partial_j\frac{1}{r},\quad \partial_i\partial_j\partial_k\frac{1}{r},\quad \partial_i\partial_j\partial_k\partial_l\frac{1}{r},\quad .... \tag{3.94}$$

are all *traceless* on any pair of indices, namely

$$\delta_{ij}\partial_i\partial_j\frac{1}{r}=0,\quad \delta_{ij}\partial_i\partial_j\partial_k\frac{1}{r}=0,\quad .... \tag{3.95}$$

We can use this property in order to replace the quantities

$$x'_ix'_j,\quad x'_ix'_jx'_k,\quad ... \tag{3.96}$$

that multiply the derivative terms in (3.91) by the totally tracefree quantities

$$(x'_ix'_j-\frac{1}{3}\delta_{ij}r'^2),\quad (x'_ix'_jx'_k-\frac{1}{5}[x'_i\delta_{jk}+x'_j\delta_{ik}+x'_k\delta_{ij}]r'^2),\quad ... \tag{3.97}$$

where $r'^2=x'_ix'_i$ and the factor $1/5$ in the second term is because the situation that all the indices are identical has an additional contribution $2r'^2$. Note that we can do this because the trace terms that we are subtracting out here give zero when they are contracted onto the multiple derivatives of $1/r$ in (3.91). It therefore follows from (3.88)and (3.91) that we have

$$\phi(\mathbf{r})=\frac{Q}{r}-p_i\partial_i\frac{1}{r}+\frac{1}{3}\frac{1}{2!}Q_{ij}\partial_i\partial_j\frac{1}{r}-\frac{1}{5}\frac{1}{3!}Q_{ijk}\partial_i\partial_j\partial_k\frac{1}{r}+.... \tag{3.98}$$

where $Q$ is the total charge, $\vec{p}$ is the electric dipole, $Q_{ij}$ is the electric quadrupole moment, and $Q_{ijk}, Q_{ijkl}$, etc., are the higher multipole moments. They are given by

$$\begin{aligned}
Q &= \int\rho(r')d^3\mathbf{r}', &(3.99)\\
p_i &= \int x'_i\rho(\mathbf{r}')d^3\mathbf{r}', &(3.100)\\
Q_{ij} &= \int(3x'_ix'_j-\delta_{ij}r'^2)\rho(\mathbf{r}')d^3\mathbf{r}', &(3.101)\\
Q_{ijk} &= \int\big[5x'_ix'_jx'_k-(x'_i\delta_{jk}+x'_j\delta_{ik}+x'_k\delta_{ij})r'^2\big]\rho(\mathbf{r}')d^3\mathbf{r}', &(3.102)
\end{aligned}$$

and so on. Note that by construction, all the multipole moments with two or more indices are symmetric and traceless on all indices.

Note that the terms in the multipole expansion (3.98) do indeed fall off with increasing inverse powers of $r$. For example, the dipole term is given by

$$\phi_{\rm Dipole}=-p_i\partial_i\frac{1}{r}=\frac{p_ix_i}{r^3}=\frac{p_in_i}{r^2}, \tag{3.103}$$

which falls off like $1/r^2$, since $n_i\equiv x_i/r$ is a unit-length vector. The quadrupole term is given by

$$\phi_{\rm Quadrupole}=\frac{1}{6}Q_{ij}\partial_i\partial_j\frac{1}{r}=\frac{1}{6}Q_{ij}\frac{3x_ix_j-r^2\delta_{ij}}{r^5}=\frac{1}{2}Q_{ij}\frac{x_ix_j}{r^5}=\frac{1}{2}Q_{ij}\frac{n_in_j}{r^3}, \tag{3.104}$$

which falls off like $1/r^3$. (The penultimate equality above follows because $Q_{ij}$ is traceless.)

In summary, we see that the multipole expansion of the potential due to a localised charge distribution takes the form

$$\phi(\mathbf{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{n}}{r^2} + \frac{1}{2}\frac{Q_{ij}n_in_j}{r^3} + \frac{1}{2}\frac{Q_{ijk}n_in_jn_k}{r^4} + .... \tag{3.105}$$

The electric field due to the monopole potential $\phi_{\text{Monopole}} = Q/r$ is the familiar one

$$\mathbf{E}_{\text{Monopole}} = \frac{Q\mathbf{r}}{r^3} = \frac{Q\mathbf{n}}{r^2}, \tag{3.106}$$

which falls off as the square of the distance. For the dipole potential, the electric field is easily calculated using index notation:

$$\partial_i\frac{p_jx_j}{r^3} = \frac{p_j\delta_{ij}}{r^3} - \frac{3p_jx_jx_i}{r^5} = -\frac{3n_in_jp_j - p_i}{r^3}, \tag{3.107}$$

and hence

$$\mathbf{E}_{\text{Dipole}} = \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}}{r^3}. \tag{3.108}$$

This falls off as the cube of the distance. The electric fields for the higher multipole terms can be calculated in a similar way.

The total charge $Q$ (the electric monopole moment) is of course a single quantity. The dipole moment pi is a 3-vector, so it has three independent components in general. The quadrupole moment $Q_{ij}$ is a symmetric 2-index tensor in three dimensions, which would mean $\mathrm{C}_4^2 = 3\times4/2 = 6$ independent components. But it is also traceless, $Q_{ii} = 0$, which is one condition. Thus there are $6-1 = 5$ independent components. The octopole moment $Q_{ijk}$ is a 3-index symmetric tensor, which would mean $\mathrm{C}_5^3 = 3\times4\times5/3! = 10$ independent components. But it is also traceless, $Q_{iij} = 0$, which has 3 conditions. Thus the octopole has in general $10-3 = 7$ independent components. In general, the $2^l$-pole has $C_{l+2}^l - C_l^2 = 2l+1$ $(l \geq 2)$ independent components. Here $C_{l+2}^l$ means that one selects $l$ numbers out of $3+(l-1) = l+2$ numbers, and $C_l^2$ denotes the number of traces. It is straightforward to see in the same way that the $2^l$-pole moment

$$Q_{i_1i_2\ldots i_l} = (2l-1)\int(x'_{i_1}x'_{i_2}\ldots x'_{i_l} - \text{traces})\rho(\mathbf{r}')d^3\mathbf{r}', \quad l > 1 \tag{3.109}$$

has $(2l+1)$ independent components.

### 3.7.2 Multipole expansion using spherical harmonics

In fact, the multipole expansion is equivalent to an expansion in spherical polar coordinates, using the spherical harmonics $Y_{lm}(\theta,\varphi)$:

$$\phi(r,\theta,\varphi) = \sum_{l\geq0}\sum_{m=-l}^{l} B_{lm}Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}. \tag{3.110}$$

At a given value of $l$ the terms fall off like $r^{-l-1}$, and there are $(2l+1)$ of them, with coefficients $B_{lm}$, since $m$ ranges over the integers $-l \le m \le l$. For each value of $l$, there is a linear relationship between the $(2l+1)$ components of $B_{lm}$ and the $(2l+1)$ components of the multipole moments $Q$, $p_i$, $Q_{ij}$, $Q_{ijk}$, etc. Likewise, for each $l$ there is a linear relationship between $r^{-l-1}Y_{lm}(\theta,\varphi)$ and the set of functions $\partial_{i_1}\partial_{i_2}...\partial_{i_{r-1}}$.

Consider, for example, $l=1$. The three functions $Z_i \equiv \partial_i r^{-1} = -x_i/r^3$ are given by

$$Z_1 = -\frac{\sin\theta\cos\varphi}{r^2}, \quad Z_2 = -\frac{\sin\theta\sin\varphi}{r^2}, \quad Z_3 = -\frac{\cos\theta}{r^2}, \tag{3.111}$$

when expressed in terms of spherical polar coordinates, for which $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. On the other hand, the $l=1$ spherical harmonics are given by

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}. \tag{3.112}$$

Thus we see that

$$Z_1 = \sqrt{\frac{8\pi}{3}}\frac{Y_{11}-Y_{1,-1}}{2r^2}, \tag{3.113}$$

$$Z_2 = \sqrt{\frac{8\pi}{3}}\frac{Y_{11}+Y_{1,-1}}{2ir^2}, \tag{3.114}$$

$$Z_3 = \sqrt{\frac{4\pi}{3}}\frac{Y_{10}}{r^2}, \tag{3.115}$$

Analogous relations can be seen for all higher values of $l$.

Indeed, working directly with the spherical harmonics, we may substitute the expansion for $|\mathbf{r}-\mathbf{r}'|^{-1}$ into (3.88), obtaining

$$\phi(\mathbf{r}) = 4\pi\sum_{l,m}\frac{q_{lm}}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi), \tag{3.116}$$

where the multipole moments $q_{lm}$ are given by

$$q_{lm} \equiv \int \rho(\mathbf{r}')r'^l\bar{Y}_{lm}(\theta',\varphi')d^3\mathbf{r}'. \tag{3.117}$$

Note that we have

$$\bar{q}_{lm} = (-1)^m q_{l,-m}. \tag{3.118}$$

Clearly, in view of (3.118), the total number of real quantities encoded in $q_{lm}$ for a given value of $l$ is $2l+1$, which is exactly the same as the number of independent components $Q_{i_1 i_l}$ in the $l$-th multipole moment tensor. Using the expressions for the first few spherical harmonics, we can see that the $q_{lm}$ for $l=0,1$ and 2 are related to

$Q$, $p_i$ and $Q_{ij}$.

$$q_{00} = \frac{1}{\sqrt{4\pi}}Q, \tag{3.119}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_1 - ip_2), \quad q_{10} = \sqrt{\frac{3}{4\pi}}p_3, \tag{3.120}$$

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - i2Q_{12} - Q_{22}), \quad q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}),$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{23}. \tag{3.121}$$

The expressions for $q_{lm}$ with negative $m$ follow from (3.118). Analogous relations hold for all the $q_{lm}$.

### 3.7.3 Multipole expansion of the energy in an external field

Note that the interaction energy $U$ for a localized charge distribution are

$$U = \frac{1}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})d^3\mathbf{r}. \tag{3.122}$$

We have added the subscript "int" here to emphasise that the above result gives the "self energy" or internal energy of the charge distribution itself, in its own self-generated electrostatic field. The factor 1/2 is because the interaction energy for each pair of the two charges has been doubly counted in the integral. A different question, with a different answer, concerns the energy of a charge distribution in an externally-applied electrostatic field. If the external field is expressed in terms of the potential $\Phi(\mathbf{r})$ (as opposed to the potential $\phi(\mathbf{r})$ in (3.122), which is the potential due to the charge distribution itself), then the "external" energy of the system is simply calculated by integrating up the energy of assembling all the charges that form the distribution $\rho(\mathbf{r})$. This gives

$$U_{\text{ext}} = \int \rho(\mathbf{r})\Phi(\mathbf{r})d^3\mathbf{r}. \tag{3.123}$$

We shall assume that the external electric field $\mathbf{E} = -\nabla\Phi$ is generated by distant sources (i.e. distant charges), so that we can take $\nabla^2\Phi = 0$ in the region where the localised charge distribution $\rho$ is non-zero. Let us choose the origin to lie in the vicinity of the localised charge distribution, and furthermore we assume that the external field is a slowly varying function of $\mathbf{r}$ in this region. We may then Taylor expand $\Phi(\mathbf{r})$, to give

$$\Phi(\mathbf{r}) = \Phi(0) + x_i\partial_i\Phi(0) + \frac{1}{2}x_ix_j\partial_i\partial_j\Phi(0) + .... \tag{3.124}$$

Note that in the above formula $\partial_i\Phi(0)$ means $\partial_i\Phi(\mathbf{r})$ evaluated at $\mathbf{r} = 0$, and so on.) Equation (3.124) can be written in terms of the external electric field as

$$\Phi(\mathbf{r}) = \Phi(0) - x_iE_i(0) - \frac{1}{2}x_ix_j\partial_iE_j + .... \tag{3.125}$$

Since we are assuming there are no sources for the external electric field within the localized region of interest, it follows that $\partial_i E_i = 0$, and so (3.125) may be re-expressed as

$$\Phi(\mathbf{r}) = \Phi(0) - x_i E_i(0) - \frac{1}{6}(3x_i x_j - \delta_{ij} r^2)\partial_i E_j + .... \tag{3.126}$$

The extra term added to the above formula is zero.

Now we can use the use the definitions for the multipole moments. We can see that when (3.126) is substituted into the expression (3.123) for the "external" energy, it gives

$$U_{\text{ext}} = Q\Phi(0) - p_i E_i(0) - \frac{1}{6} Q_{ij} \partial_i E_j + .... \tag{3.127}$$

The first term is the familiar result for the energy of a charge $Q$ in an electrostatic field. The second term is the energy $-\mathbf{p} \cdot \mathbf{E}$ of a dipole in an electric field. The third term, which depends on the gradient of the electric field, is the energy of a quadrupole. The higher multipole moments will be associated with higher derivatives of the electric field.

As an application of the result in equation (3.127), we may calculate the interaction energy between a pair of electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$. Suppose they are located, respectively, at points $\mathbf{r} = \mathbf{r}_1$ and $\mathbf{r} = \mathbf{r}_2$. From the expression (3.108) for the electric field due to a dipole, we see that the electric field at $\mathbf{r}_1$ due to a dipole moment $\mathbf{p}_2$ located at $\mathbf{r} = \mathbf{r}_2$ is given by

$$\mathbf{E}(\mathbf{r}_1) = \frac{3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}_2) - \mathbf{p}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{3.128}$$

where $\mathbf{n}$ is the unit vector in the direction from $\mathbf{r}_2$ to $\mathbf{r}_1$. From (3.127), we then see that the energy of the dipole $\mathbf{p}_1$ in this electric field is given by

$$U_{12} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{n} \cdot \mathbf{p}_1)(\mathbf{n} \cdot \mathbf{p}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{3.129}$$

As one would expect, this expression is completely symmetrical between $\mathbf{p}_1$ and $\mathbf{p}_2$. Two special examples are particularly interesting. First, if the two dipoles $\mathbf{p}_{1,2}$ are parallel in direction, but they are perpendicular to the the vector $\mathbf{n}$ which characterizes the relative direction between $\mathbf{r}_2$ and $\mathbf{r}_1$. In this case we have

$$U_{12} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \frac{p_1 p_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} > 0. \tag{3.130}$$

This implies that the interaction is repulsive. On the other hand, if $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{n}$ are parallel, the interaction reads

$$U_{12} = -2\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -2\frac{p_1 p_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} < 0. \tag{3.131}$$

Thus the interaction is attractive. The above results show that the dipole-dipole interaction is highly anisotropic. For a dipolar system, by controlling the polarization of the system with an external field, one can tune the interactions in the system. This

property can have interesting applications.

**An example**

**Dielectric sphere.**—Consider a dielectric sphere with radius $R_0$ put in an external uniform electric field $\mathbf{E}_{\rm ex} = E_0\hat{e}_z$, along $z$ direction. The relative permittivity of the dielectric medium in the sphere is $\epsilon$, while outside the sphere is $\epsilon_0(=1)$. The general solution of the potential can be given by Legendre polynomials that

$$\phi_1 = \sum_l (a_n r^l + \frac{b_l}{r^{l+1}}) P_l(\cos\theta), \quad \text{for} \quad r > R_0, \tag{3.132}$$

and

$$\phi_2 = \sum_l (c_l r^l + \frac{d_l}{r^{l+1}}) P_l(\cos\theta), \quad \text{for} \quad r < R_0. \tag{3.133}$$

The coefficients $a_l, b_l, c_l$ and $d_l$ can be determined by boundary conditions at $r \to \infty, r = 0$, and $r = R_0$. First of all, for $r \to \infty$, the electric field determined by the potential $\phi_1$ must approach the external field $\mathbf{E}_{\rm ex}$. This requires that

$$\phi_1(\infty) \to -E_0 r\cos\theta = -E_0 r P_1(\cos\theta). \tag{3.134}$$

From the above formula we can have that

$$a_1 = -E_0, \quad a_l = 0, \quad \text{for} \quad l \neq 1. \tag{3.135}$$

Furthermore, at $r = 0$, we require that $\phi_2$ be finite. This implies that

$$d_l = 0. \tag{3.136}$$

Finally, for the connection conditions at $r = R_0$, we require that

$$\phi_1(R_0) = \phi_2(R_0), \tag{3.137}$$

$$\epsilon_0 \frac{\partial\phi_1}{\partial r}|_{r=R_0} = \epsilon \frac{\partial\phi_2}{\partial r}|_{r=R_0}. \tag{3.138}$$

It is noteworthy that under the condition (3.136) the electric field along the tangent direction automatically satisfies the boundary condition. With the above two equations we have

$$-E_0 R_0 + \frac{b_1}{R_0^2} = c_1 R_0, \tag{3.139}$$

$$-E_0 R_0 - \frac{2b_1}{R_0^3} = \frac{\epsilon}{\epsilon_0} c_1, \tag{3.140}$$

from which we get

$$b_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 R^3, \quad c_1 = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0. \tag{3.141}$$

The remaining coefficients are zero: $b_l = c_l = 0$, for $l \neq 0$. In this way we obtain the solution for the whole space that

$$\phi_1 = -E_0 r\cos\theta + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\frac{E_0 R_0^3\cos\theta}{r^2}, \tag{3.142}$$

$$\phi_2 = -\frac{3\epsilon_0}{\epsilon+2\epsilon_0}E_0 r\cos\theta. \tag{3.143}$$

Note that the permittivity can be set as $\epsilon_0 = 1$ in the Gaussian units.

The second term in solution $\phi_1$ represents the contribution by the dielectric sphere to the potential in the region $r > R_0$. Compare this term with the Eq. (3.103) we can find that this contribution is actually given by the electric dipole induced in the dielectric sphere. The total electric dipole induced in the dielectric sphere is that

$$\mathbf{p} = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}R_0^3\mathbf{E}_{\rm ex}. \tag{3.144}$$

If the sphere is a conductor, we have $\epsilon = \infty$. Then $\phi_2 = 0$, and outside the sphere we have

$$\phi_1(\mathbf{r}) = -E_0 r\cos\theta + \frac{E_0 R_0^3}{r^2}\cos\theta. \tag{3.145}$$

Actually this results can be easily obtained by the boundary condition $\phi_1(R_0) = 0$ for the conductor. The surface charge density is given by

$$\sigma = -\epsilon_0\frac{\partial\phi}{\partial r}|_{r=R_0} = 3\epsilon_0 E_0\cos\theta. \tag{3.146}$$

It is interesting that for the present dielectric sphere, the contribution by the electric polarization to the electric potential at $r > R_0$ can also be studied by the methods of imagine, similar as the results given in Sec.3.4.3. In this case a pair of image charges $q_1$ and $q_2$ with the charge magnitudes

$$q_1 = Q\frac{R_0}{b}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}, \quad q_2 = -Q\frac{R_0}{b}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}. \tag{3.147}$$

Similar as the Sec.3.4.3, we have $E_0 = 2Q/b^2$, with $b \to \pm\infty$ being the positions of two infinite charges $\pm Q$ which create the electric field $\mathbf{E}_{\rm ex}$. Thus the only difference between the conductor and a dielectric sphere is the factor $\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}$. For the conductor, this factor is unit since $\epsilon \to \infty$.

# Chapter 4

# Magnetostatics

We now turn to the magnetic analogue of electrostatics, namely magnetostatics. This describes the situation when everything is independent of time, and only magnetic fields and electric currents are present.

## 4.1 Biot-Savat law

From the general form of Maxwells equations, we see that the only non-trivial equations to be considered are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}. \tag{4.1}$$

Note that we are using Gaussian units, and that c is the speed of light. In view of $\nabla \cdot \mathbf{B} = 0$, we can write $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$, as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.2}$$

As discussed in the chapter 2, the choice of vector potential is not unique, and we can perform a gauge transformation of the form $\mathbf{A} \to \mathbf{A} + \nabla\lambda$, with $\lambda$ being any smooth function of $\mathbf{r}$, which does not change the formulas in Eq. (4.1).

We may employ the gauge invariance of the system under the gauge transformations in order to impose a convenient gauge condition called the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0. \tag{4.3}$$

This can always be achieved. Indeed, supposing that we start with a gauge potential $\tilde{\mathbf{A}}$ that was not in the Coulomb gauge, we can transform it according to

$$\mathbf{A} = \tilde{\mathbf{A}} + \nabla\lambda, \tag{4.4}$$

so that $\mathbf{A}$ satisfies $\nabla \cdot \mathbf{A} = 0$. Taking the divergence of the above equation we therefore see that

$$\nabla^2\lambda = -\nabla \cdot \tilde{\mathbf{A}}. \tag{4.5}$$

This Poisson equation can always be solved; in fact, we are familiar with its solution from solving the same equation in electrostatics (i.e. $\nabla^2\phi = -4\pi\rho$). Thus we can immediately write down the solution for the required gauge transformation function :

$$\lambda(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\nabla'\cdot\tilde{\mathbf{A}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'. \tag{4.6}$$

We can now proceed under the assumption that $\mathbf{A}$ is in Coulomb gauge. Substituting into the second equation in (4.1), we find

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{4.7}$$

This equation also can be solved immediately; it is just a vector-valued version of the equation $\nabla^2\phi = -4\pi\rho$ from electrostatics. Thus we have

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'. \tag{4.8}$$

We may easily calculate the magnetic field $\mathbf{B}$ from (4.8), by taking the curl. It is useful to note that for any vector $\vec{V}$ and any scalar $f$, we have the identity

$$\nabla\times(f\vec{V}) = (\nabla f)\times\vec{V} + f\nabla\times\vec{V}. \tag{4.9}$$

Using this identity we can obtain that (note that $\nabla$ only acts on $\mathbf{r}$)

$$B = -\frac{1}{c}\int \mathbf{J}(\mathbf{r}')\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'. \tag{4.10}$$

This yields that

$$B = \frac{1}{c}\int \mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'. \tag{4.11}$$

Suppose we consider an infinitesimal current element $Id\vec{l}$, where $I$ is the current and $d\vec{l}$ is the infinitesimal element of length pointing along the direction of the current flow. Without loss of generality, we set that the coordinate of the wire which carries this current is given by $\mathbf{r} = \mathbf{r}_l$. The current density $\mathbf{J}(\mathbf{r})$ describing the current element can therefore be written as

$$\mathbf{J}(\mathbf{r}) = I\delta^{(2)}_{\perp}(\mathbf{r}-\mathbf{r}_l)\hat{e}_{\vec{l}}, \tag{4.12}$$

where $\delta^{(2)}_{\perp}(\mathbf{r})$ denotes the two-dimensional delta function in the plane orthogonal to $d\vec{l}$ and $\hat{e}_{\vec{l}}$ is the unit vector along the direction of $d\vec{l}$. This delta function describes the way in which the current density is zero everywhere except along the infinitesimal line element. For example, if the element $d\vec{l}$ lies along the $z$ direction, then

$$J_x = 0, \quad J_y = 0, \quad J_z = I\delta(x)\delta(y)\hat{e}_z. \tag{4.13}$$

Substituting (4.12) into (4.11), we can perform the two integrations in the plane orthogonal to $d\vec{l}$, and thus obtain the infinitesimal contribution

$$d\mathbf{B}(\mathbf{r}) = \frac{I\big[d\vec{l}\times(\mathbf{r}-\mathbf{r}_{\vec{l}})\big]}{c|\mathbf{r}-\mathbf{r}_{\vec{l}}|^3}. \tag{4.14}$$

For convenience, we reexpress the relative position $\mathbf{r} - \mathbf{r}_{\vec{l}}$, i.e. the position that we consider relative to the position of the differential current, as $\mathbf{r}$. Then the above formula can be simplified as

$$d\mathbf{B}(\mathbf{r}) = \frac{I(d\vec{l} \times \mathbf{r})}{cr^3}. \tag{4.15}$$

This result is known as the Biot-Savat law, and it can be used, by integrating along a loop, to calculate the magnetic field resulting from current flow around the loop. (Historically, of course, the Biot-Savat law came first, and (4.11) was deduced from it.)

One way in which a current loop can arise is if a particle of electric charge $q$ moves with velocity $\mathbf{v}$. Suppose that at a given instant the moving particle is located at the origin, with the current density being described as $\mathbf{J}(\mathbf{r}) = q\mathbf{v}\delta^{(3)}(\mathbf{r})$. It then follows that the moving charge give rise to a magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{q\mathbf{v} \times \mathbf{r}}{cr^3}. \tag{4.16}$$

One should keep in mind that here $\mathbf{r}$ is the position that we consider relative to the position of the moving charge. We note that this result assumes that the velocity $\mathbf{v}$ is small compared with the speed of light, so that relativistic effects can be neglected.

## 4.2 Ampère's law

Suppose that a current element $I_1 d\vec{l}_1$ is placed in a magnetic field $\mathbf{B}$. Experiments by Ampère in the 19′th century established that it would experience a force $dF$ given by

$$d\mathbf{F} = \frac{I_1}{c} d\vec{l}_1 \times \mathbf{B}. \tag{4.17}$$

If the magnetic field is itself due to an infinitesimal current element $I_2 d\vec{l}_2$, then from (4.14), the (doubly infinitesimal) force $d\mathbf{F}_{12}$ experienced by the first current element will be given by

$$d\mathbf{F}_{12} = \frac{I_1 I_2}{c^2} \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{r}_{12})}{r_{12}^3}, \tag{4.18}$$

where $r_{12}$ is the vector from $d\vec{l}_2$ to $d\vec{l}_1$. In other words, $\vec{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. Note that $d\vec{l}_2$ is the same thing as $d\mathbf{r}_2$. The expression (4.18) can be integrated up around the two current loops, to give

$$\mathbf{F}_{12} = \frac{I_1 I_2}{c^2} \oint \oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{r}_{12})}{r_{12}^3}. \tag{4.19}$$

As it stands, this expression is not manifestly (anti)symmetric under the exchange of the roles of the two current loops. However, this can be made manifest as follows: Using the standard identity for the vector triple product, we can write

$$d\vec{l}_1 \times (d\vec{l}_2 \times \vec{r}_{12}) = d\vec{l}_2 (d\vec{l}_1 \cdot \vec{r}_{12}) - (d\vec{l}_1 \cdot d\vec{l}_2)\vec{r}_{12}, \tag{4.20}$$

and then we obtain that

$$d\vec{l_1} \times (d\vec{l_2} \times \vec{r}_{12}) = \frac{I_1 I_2}{c^2} d\vec{l_2} \frac{d\vec{l_1} \cdot \vec{r}_{12}}{r_{12}^3} - \frac{I_1 I_2}{c^2} \frac{d\vec{l_1} \cdot d\vec{l_2}}{r_{12}^3} \vec{r}_{12}, \tag{4.21}$$

If we consider just the Loop 1 integration for now, the first term in (4.21) can be written as the exact differential

$$-\frac{I_1 I_2}{c^2} d\vec{l_2} d\big(\frac{1}{r_{12}}\big) \tag{4.22}$$

This follows from the fact that $\vec{r}_{12}^{\,2} = (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)$, and so (with $\mathbf{r}_2$ held fixed),

$$2 r_{12} dr_{12} = 2 d\mathbf{r}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1) = 2 d\vec{l_1} \cdot \vec{r}_{12}. \tag{4.23}$$

Thus if we first integrate (4.21) around Loop 1, the first term gives zero (since $\oint df = 0$, whenever any exact differential $df$ is integrated around a closed loop). Thus we arrive at the alternative expression for the total force,

$$\mathbf{F}_{12} = -\frac{I_1 I_2}{c^2} \oint \oint \frac{(d\vec{l_1} \cdot d\vec{l_2}) \vec{r}_{12}}{r_{12}^3}. \tag{4.24}$$

This makes manifest the total symmetry between the roles of the two loops.

The expression (4.17) for the force on a current element $Id\vec{l}$ in a magnetic field $\mathbf{B}$ can be generalised immediately to the situation where there is a current density $\mathbf{J}$ in an external $\mathbf{B}$ field. The infinitesimal force on the current density in the volume element $d^3\mathbf{r}$ will be given by

$$d\mathbf{F} = \frac{1}{c} \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3\mathbf{r}, \tag{4.25}$$

and so the total force on the current distribution will be given by

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3\mathbf{r}. \tag{4.26}$$

It also follows from (4.26) that the infinitesimal torque on the element $d^3\mathbf{r}$ will be

$$d\vec{N} = \mathbf{r} \times \mathbf{F} = \frac{1}{c} \mathbf{r} \times \big[\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\big] d^3\mathbf{r}, \tag{4.27}$$

and so the total torque (measured relative to the origin) is

$$\vec{N} = \int d^3\mathbf{r} (\mathbf{r} \times \mathbf{F}) = \frac{1}{c} \int \mathbf{r} \times \big[\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\big] d^3\mathbf{r}. \tag{4.28}$$

A further application of the expression (4.17) for the force on a current element in an external magnetic field is to the situation where a particle of charge $q$ is moving with velocity $\mathbf{v}$ in the field $\mathbf{B}$. Let the position of the charge be $\mathbf{r} = \mathbf{r}_q$, and the current density reads $\mathbf{J} = q\mathbf{v}\delta^{(3)}(\mathbf{r} - \mathbf{r}_q)$. The differential force is $d\mathbf{F} = (1/c) q\mathbf{v} \times \mathbf{B} \delta^{(3)}(\mathbf{r} - \mathbf{r}_q) d^3\mathbf{r}$. It then follows that the moving charge experiences a force given by

$$\mathbf{F} = \frac{q}{c} \mathbf{v} \times \mathbf{B}. \tag{4.29}$$

This is known as the Lorentz force. Again, we are assuming here that the velocity $\mathbf{v}$ is small in comparison to the speed of light.

## 4.3 Magnetically permeable media and boundary conditions

In the electrostatics, we discussed the phenomenological description of dielectric media, in which one introduces a macroscopic **D** field in addition to the fundamental electric field **E**. The essential idea is that for many purposes, one can give a macroscopic description of the effect of a piece of dielectric medium, such as a salt crystal or a block of glass, in which the microscopic contributions of each atom or molecule within the medium are averaged over, so that on the large scale a relatively simple description of the electrical properties of the material as a whole can be given.

In a very similar vein, in magnetostatics one may give a macroscopic description of magnetically permeable materials. As introduced in the second chapter, we can introduce a phenomenological field **H** in addition to the fundamental field **B** of magnetostatics. The two physical quantities satisfy the equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_f. \tag{4.30}$$

Here $\mathbf{J}_f$ is the free electric current. Note that **H** is called the magnetic field, while the fundamental field **B** is called the magnetic induction.

The effect of the ensemble of atoms or molecules in the medium is to give rise to a magnetic moment density $\mathbf{M}(\mathbf{r})$, which comes from an averaging over all the atoms or molecules. The relation between **B** and **H** are defined as

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}. \tag{4.31}$$

From (4.30) and (4.31) we can find that

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}J_f + 4\pi\nabla \times \mathbf{M}. \tag{4.32}$$

We note that the magnetic induction **B** still satisfies the original Maxwell equation $\nabla \cdot \mathbf{B} = 0$, and so we may still write $\mathbf{B} = \nabla \times \mathbf{A}$. Under the Coulomb gauge condition, we further derive that

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}_f - 4\pi\nabla \times \mathbf{M}, \tag{4.33}$$

which is followed by the solution as

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{J}_f(\mathbf{r}') + c\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}d^3\mathbf{r}'. \tag{4.34}$$

From the above solution one can see that the magnetic moment density **M** can thus be thought of as giving rise to an effective current density

$$\mathbf{J}_M = c\nabla \times \mathbf{M}(\mathbf{r}). \tag{4.35}$$

In order to apply the phenomenological description of magnetostatics, it is necessary to know the so-called *constitutive* relation between **B** and **H**. This can be quite a complicated business in general, especially in the case of ferromagnetic materials where

there is not even a single-valued functional relation between $\mathbf{B}$ and $\mathbf{H}$. The simplest materials to consider are paramagnetic and diamagnetic media. For these, there is a linear relation between $\mathbf{B}$ and $\mathbf{H}$, with

$$\mathbf{B} = \mu\mathbf{H}. \tag{4.36}$$

where $\mu$ is a constant called the magnetic permeability. For paramagnetic materials, $\mu$ is slightly greater than 1, whilst for diamagnetic materials, $\mu$ is slightly less than 1. (The deviations from 1 are in the order of $10^{-5}$.)

For many ferromagnetic materials, a relation of the form (4.36) is approximately valid, provided the fields are sufficiently weak. In these materials, $\mu$ is typically in the range between 10 and $10^4$.

In the remainder of our consideration of magnetically-permeable media, we shall assume that the simple linear constitutive relation (4.36) holds.

## 4.4 Solving boundary problems with magnetic vector and scalar potentials

### 4.4.1 Boundary conditions

The boundary conditions for the magnetostatics can be found in the 2nd chapter. These were obtained by considering a "pill-box" volume integration of the equation $\nabla\cdot\mathbf{B} = 0$, and integrations of $\nabla\times\mathbf{H} = (4\pi/c)\mathbf{J}_f$ over slender rectangular loops straddling the boundary. These lead, respectively, to the conditions

$$\vec{n}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0, \tag{4.37}$$

$$\vec{n}\times(\mathbf{H}_1 - \mathbf{H}_2) = \frac{4\pi}{c}\vec{K}_f, \tag{4.38}$$

where $\vec{n}$ is the unit vector normal to the interface, pointing from medium 1 into medium 2, and $\vec{K}_f$ is the surface free current density. The first equation in (4.37) says that the normal component of $\mathbf{B}$ must be continuous across the boundary. In many common situations there will be no surface currents at the interface, and then the second equation in (4.37) just says that the tangential components of $\mathbf{H}$ must be continuous across the boundary.

There are various techniques that can be applied in order to solve for the magnetic fields in a boundary-value problem in magnetostatics. Depending upon the the circumstances, one or another may be more convenient. For concreteness and simplicity, we shall typically focus on situations where the simple relation $\mathbf{B} = \mu\mathbf{H}$ holds, with the assumption that in any given region $\mu$ is a constant. (We shall allow $\mu$ to take different constant values in different regions, as would be the case in a typical boundary-value problem with an interface between media.)

### 4.4.2 Solving boundary-value problems with vector potential

As we have obtained in the former section, the solution to the magnetostatics can be expressed in terms of the vector potential based on the equation

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}_f - 4\pi\nabla\times\mathbf{M}, \tag{4.39}$$

and the solution is

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{J}_f(\mathbf{r}') + c\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'. \tag{4.40}$$

In the free space, the magnetization $\mathbf{M}$ is zero, and the magnetic field is induced solely by the free current $\mathbf{J}_f$.

**Example.**–Consider a current along $\hat{e}_z$ direction with current strength $I$. We can solve the magnetic field according to the formula (4.40). The current density is given by

$$\mathbf{J} = I\delta(x)\delta(y)\hat{e}_z. \tag{4.41}$$

The vector potential is given by

$$\begin{aligned} \mathbf{A}(\rho) &= \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \\ &= \frac{I}{c}\int_{-\infty}^{+\infty} \frac{dz}{(\rho^2+z^2)^{1/2}}\hat{e}_z. \end{aligned} \tag{4.42}$$

The above integral gives $\ln(z+\sqrt{z^2+\rho^2})$ which diverges at infinity. To remove the unphysical divergence, we can choose a zero point for $\mathbf{A}(\rho)$, say at $\rho = \rho_0$. Then we can obtain that

$$\begin{aligned} \mathbf{A}(\rho) &= \hat{e}_z\frac{I}{c}\ln(z+\sqrt{z^2+\rho^2})|_{z=-\infty}^{z=+\infty} - \hat{e}_z\frac{I}{c}\ln(z+\sqrt{z^2+\rho_0^2})|_{z=-\infty}^{z=+\infty} \\ &= \hat{e}_z\frac{I}{c}\ln\frac{z+\sqrt{z^2+\rho_0^2}}{z+\sqrt{z^2+\rho^2}}|_{z=-\infty} \\ &= \frac{I}{c}\ln\frac{\rho_0^2}{\rho^2}\hat{e}_z \\ &= \frac{2I}{c}\ln\frac{\rho_0}{\rho}\hat{e}_z. \end{aligned} \tag{4.43}$$

The magnetic field can then be calculated by

$$\begin{aligned} B_x &= \partial_y A_z \\ &= -\frac{2I}{c\rho^2}y, \end{aligned} \tag{4.44}$$

$$\begin{aligned} B_y &= -\partial_x A_z \\ &= \frac{2I}{c\rho^2}x, \end{aligned} \tag{4.45}$$

$$\tag{4.46}$$

and $B_z = 0$. Together with the above results we have that

$$\mathbf{B} = \frac{2I}{c\rho}\hat{e}_\varphi, \tag{4.47}$$

where $\hat{e}_\varphi = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y$, with $\varphi$ being the azimuthal angle. It is easy to see that

$$\oint \mathbf{B}\cdot dl = \frac{4\pi I}{c}, \tag{4.48}$$

when the integral encloses the current.

### 4.4.3 Solving boundary-value problems with scalar potential

If there are no free currents in the problem, as is often the case, the Maxwell field equation becomes

$$\nabla \times \mathbf{H} = 0, \tag{4.49}$$

and this can be solved by writing $\mathbf{H}$ as the gradient of a scalar, just as one does for the electric field in electrostatics. Thus we may write

$$\mathbf{H} = -\nabla \Phi_M, \tag{4.50}$$

where $\Phi_M$ is the magnetic scalar potential. If we assume the simple constitutive relation $\mathbf{B} = -\mu\mathbf{H}$ between $\mathbf{B}$ and $\mathbf{H}$, where $\mu$ is piecewise constant, then in each region the remaining Maxwell equation $\nabla \cdot \mathbf{B} = 0$ becomes

$$\nabla^2 \Phi_M = 0. \tag{4.51}$$

Solving boundary-value magnetostatics problems where $\mathbf{J}_f = 0$ is thus closely analogous to solving boundary-value problems in electrostatics; one just has to solve Laplaces equation in the various regions, and then impose the boundary conditions at the interfaces.

As an illustration of the method, consider a spherical shell of material with magnetic permeability $\mu$, with inner radius $a$ and outer radius $b$. The regions inside and outside the shell are assumed to be empty space. Outside the shell, we take the magnetic field to be asymptotically uniform, $\mathbf{B} \to \mathbf{B}_0$, with $\mathbf{B}_0$ directed along the $z$ axis. The problem is to solve for the magnetic fields everywhere. A particular point of interest will be to calculate the magnetic field inside the shell, in order to study the phenomenon of *magnetic shielding*.

In the regions $0 \leq r \leq a$ and $r > b$ we have $\mathbf{B} = \mathbf{H}$, whilst in the region $a < r < b$ we have $\mathbf{B} = \mu\mathbf{H}$. There are no external currents, and so the conditions discussed above apply. There is azimuthal symmetry around the z axis, and so in spherical polar coordinates the magnetic scalar potential will depend only on $r$ and $\theta$. Since it must satisfy Laplaces equation in the three regions, it must take the form for $0 < r < a$

$$\Phi_M(r, \theta) = \sum_{l \geq 0} \alpha_l r^l P_l(\cos\theta), \tag{4.52}$$

for $a < r < b$ it is

$$\Phi_M(r, \theta) = \sum_{l \geq 0} (\beta_l r^l + \frac{\gamma_l}{r^{l+1}}) P_l(\cos\theta), \tag{4.53}$$

and for $r > b$ it is

$$\Phi_M(r, \theta) = -B_0 r \cos\theta + \sum_{l \geq 0} \frac{\delta_l}{r^{l+1}} P_l(\cos\theta). \tag{4.54}$$

Note that in Eq. (4.54) we have already considered the boundary condition $\mathbf{B} \to \mathbf{B}_0$ at infinity.

The boundary conditions at $r = a$ and $r = b$ require that the radial components of $\mathbf{B}$ be continuous, and the tangential components of $\mathbf{H}$ be continuous since the surface current $\mathbf{K}_f = 0$. We therefore have the conditions

$$\mu \frac{\partial \Phi_M}{\partial r}|_{r=a^+} = \frac{\partial \Phi_M}{\partial r}|_{r=a_-}, \tag{4.55}$$

$$\mu \frac{\partial \Phi_M}{\partial r}|_{r=b^-} = \frac{\partial \Phi_M}{\partial r}|_{r=b_+}, \tag{4.56}$$

$$\frac{\partial \Phi_M}{\partial \theta}|_{r=a^+} = \frac{\partial \Phi_M}{\partial \theta}|_{r=a_-}, \tag{4.57}$$

$$\frac{\partial \Phi_M}{\partial \theta}|_{r=b^-} = \frac{\partial \Phi_M}{\partial \theta}|_{r=b_+}, \tag{4.58}$$

where $a^\pm$ and $b^\pm$ mean that $a$ and $b$ are approached from above (plus sign) or below (minus sign), respectively. The later two equations, i.e. the continuous conditions for tangential components of $\mathbf{H}$, are indeed equivalent to the continuous conditions of the magnetic scalar potential $\Phi_M$ itself at the interfaces. Plugging in the expansions, and using the linear independence of the Legendre polynomials (and their derivatives with respect to $\theta$), we obtain the equations

$$l\mu\beta_l a^{l-1} - (l+1)\mu\gamma_l a^{-l-2} - l\alpha_l a^{l-1} = 0, \tag{4.59}$$

$$\beta_l a^l + \gamma_l a^{-l-1} - \alpha_l a^l = 0, \tag{4.60}$$

$$l\mu\beta_l b^{l-1} - (l+1)\mu\gamma_l b^{-l-2} + (l+1)\delta_l b^{-l-2} = -B_0\delta_{l,1} \tag{4.61}$$

$$\beta_l b_l + \gamma_l b^{-l-1} - \delta_l b^{-l-1} = -bB_0\delta_{l,1}, \tag{4.62}$$

where $\delta_{l,1}$ is a Kronecker delta which vanishes except when $l = 1$, in which case it equals 1. It is straightforward to solve the above four equations for the four unknowns $\alpha_l, \beta_l, \gamma_l$ and and $\delta_l$, for each value of $l$. The result is

$$\alpha_l = \beta_l = \gamma_l = \delta_l = 0, \quad \text{for } \ l \neq 1, \tag{4.63}$$

while for $l = 1$ we obtain

$$\alpha_1 = -\frac{9\mu}{(\mu+2)(2\mu+1) - 2a^3b^{-3}(\mu-1)^2} B_0, \tag{4.64}$$

$$\beta_1 = -\frac{3(2\mu+1)}{(\mu+2)(2\mu+1) - 2a^3b^{-3}(\mu-1)^2} B_0, \tag{4.65}$$

$$\gamma_1 = -\frac{3(\mu-1)a^3}{(\mu+2)(2\mu+1) - 2a^3b^{-3}(\mu-1)^2} B_0, \tag{4.66}$$

$$\delta_1 = \frac{(2\mu+1)(\mu-1)(b^3-a^3)}{(\mu+2)(2\mu+1) - 2a^3b^{-3}(\mu-1)^2} B_0. \tag{4.67}$$

$$\tag{4.68}$$

The magnetic field inside the shell is uniform, parallel to $\mathbf{B}_0$, and has magnitude of $-\alpha_1$. If $\mu \gg 1$, as is easily possible for a high permeability material, which might have $\mu \sim 10^6$, we see that

$$\alpha_1 = -\frac{9}{2\mu(1 - a^3b^{-3})} B_0. \tag{4.69}$$

This shows that even with a fairly thin shell of the permeable medium, a high degree of magnetic shielding can be achieved. Outside the shell, the magnetic field is the sum of the uniform field $\mathbf{B}_0$ plus a magnetic dipole term of magnitude $\alpha_1$.

It is noteworthy that for a superconductor put in an external magnetic field, we may also treat the superconductor as magnetically permeable medium with permeability $\mu = 0$. It is then straightforward to find that

$$\alpha_1 = 0, \tag{4.70}$$

$$\beta_1 = -\frac{3}{2}\frac{b^3}{b^3 - a^3}B_0, \tag{4.71}$$

$$\gamma_1 = -\frac{3}{2}\frac{a^3 b^3}{b^3 - a^3}B_0, \tag{4.72}$$

$$\delta_1 = -\frac{1}{2}b^3 B_0. \tag{4.73}$$

The above results show that the magnetic field inside the superconducting shell is zero (since $\alpha_1 = 0$). On the other hand, the magnetic field in the superconductor itself is also zero since $\mu = 0$. As a result, the magnetic field (flux) can be fully excluded by a superconductor, known as the Meissner effect.

Note that for the superconducting case, it is more convenient to treat the superconductor as a normal system with $\mu = 1$, and then the surface current (polarized by the external magnetic field) is effectively regarded as the "free current". In this case we shall have the boundary condition that $B_r(r = b^+) = \partial_r \Phi_M(b^+) = 0$, which quickly gives the correct solution of $\Phi_M(r)$ and then the magnetic field $\mathbf{B}(\mathbf{r})$ outside the superconductor.

## 4.5 Magnetic field and local currents

Similar as the situation in electrostatics, we can apply the magnetic multipole expansion investigate the magnetic field when the currents $\mathbf{J}(\mathbf{r})$ are flowing in a localized region. Outside this region, it is assumed that $\mathbf{J}(\mathbf{r}) = 0$. We can proceed in a manner that is precisely analogous to earlier discussion we gave of localised charge distributions, using the expression

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}d^3\mathbf{r}'. \tag{4.74}$$

for the magnetic vector potential, and then Taylor expanding $|\mathbf{r} - \mathbf{r}'|^{-1}$ in inverse powers of $r$. Thus, keeping just the first couple of orders in the expansion, we shall have

$$A_i(\mathbf{r}) = \frac{1}{cr}\int J(\mathbf{r}')d\mathbf{r}' + \frac{x_j}{cr^3}\int x'_j J_i(\mathbf{r}')d^3\mathbf{r}' + .... \tag{4.75}$$

The first term in (4.75) vanishes, as can be seen from the following argument. We know that for the static regime, the current density is conserved, $\nabla \cdot \mathbf{J} = 0$. Now consider the quantity $\partial_i(x_j J_i)$, which is therefore given by

$$\partial_i(x_j J_i) = J_i \partial_i x_j + x_j \partial_i J_i = \delta_{ij} J_i = J_j. \tag{4.76}$$

If we integrate $\partial_i(x_j J_i)$ over all space it will give zero by the divergence theorem, since $\mathbf{J}$ vanishes outside some bounded domain:

$$\int \partial_i(x_j J_i) d^3\mathbf{r} = \int_{\text{spherical boundary at } \infty} x_j J_i d\Sigma_i = 0. \tag{4.77}$$

Hence we conclude that $\int J_i d^3\mathbf{r} = 0$.

To discuss the second term in (4.75), we also apply the condition $\partial_i J_i = 0$, which gives that

$$\partial_i(x_j x_k J_i) = x_j J_k + x_k J_j. \tag{4.78}$$

Using the divergence theorem, the integral of the left-hand side over all space is zero, and so we conclude that

$$\int x_j J_k d^3\mathbf{r} = -\int x_k J_j d^3\mathbf{r}. \tag{4.79}$$

This means that the integral in the second term in (4.75) can be written as

$$\begin{aligned} \int x'_j J_i(\mathbf{r}') d^3\mathbf{r}' &= -\frac{1}{2}\int (x'_i J_j - x'_j J_i) d^3\mathbf{r}' \\ &= -\frac{1}{2}\int \epsilon_{ijk}(\mathbf{r}' \times \mathbf{J})_k d^3\mathbf{r}', \end{aligned} \tag{4.80}$$

and so we conclude that the second term in (4.75) can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \tag{4.81}$$

where we have defined the magnetic moment $\mathbf{m}$ as

$$\mathbf{m} = \frac{1}{2c}\int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' \tag{4.82}$$

of the current distribution $\mathbf{J}$. Prior to performing the volume integration, we may define the magnetisation $\vec{\mathcal{M}}$ by

$$\vec{\mathcal{M}} = \frac{1}{2c}\mathbf{r} \times \mathbf{J}. \tag{4.83}$$

From the above result we can see that the leading-order term corresponds to the magnetic dipole term in the multipole expansion describing the magnetic vector potential $\mathbf{A}$ of a localised current distribution. Let us now calculate the magnetic field $\mathbf{B} = \nabla \times A$. In index notation we have

$$\begin{aligned} B_i &= \epsilon_{ijk}\epsilon_{klm}\partial_j(\frac{m_l x_l}{r^3}) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(\frac{m_l x_m}{r^3}) \\ &= m_i\partial_j(\frac{x_j}{r^3}) - m_j\partial_j(\frac{x_i}{r^3}). \end{aligned} \tag{4.84}$$

Now, away from the origin (i.e. for $r > 0$) we have

$$\partial_j(\frac{x_j}{r^3}) = 0, \quad \partial_j(\frac{x_i}{r^3}) = \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}. \tag{4.85}$$

Thus we obtain for $r > 0$ that

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}, \tag{4.86}$$

where as usual we define the unit vector $\mathbf{n} = \mathbf{r}/r$. We can see that $\mathbf{B}$ has exactly the same form as the electric field of an electric dipole $\mathbf{p}$.

It is of interest also to consider the expression for $\mathbf{B}$ including the origin $r = 0$. Note that it only really makes sense to do this in the case of a point dipole of zero size. Recalling that $\nabla^2(1/r) = -4\pi\delta^{(3)}(\mathbf{r})$, we see that

$$\partial_j(\frac{x_j}{r^3}) = 4\pi\delta^{(3)}(\mathbf{r}). \tag{4.87}$$

This implies that the first term in Eq. (4.84) gives a delta function if $r = 0$ is included. What about the other term $\partial_j(x_i/r^3)$ at $r = 0$? Note that if $i \neq j$, the result of $\partial_j(x_i/r^3)$ calculated in Eq. (4.85) is exact. However, in the case with $i = j$, this term becomes $\partial_i(x_i/r^3) = 0$ for $r \neq 0$. This implies that the second formula in Eq. (4.85), which vanishes if summing over $i = j$ terms, actually missed the case for $r = 0$ and for $i = j$. Thus the origin point $r = 0$ has to be considered additionally when $i = j$. It is clear that $\partial_j(x_i/r^3)$ must also have a delta-function term at the origin, since we already know that taking its trace gives $\partial_j(x_j/r^3) = 4\pi\delta^{(3)}(\mathbf{r})$. Indeed, if we define the matrix $M_{ij} = \partial_j(x_i/r^3)$, we have that

$$Tr(M) = 4\pi\delta^{(3)}(\mathbf{r}). \tag{4.88}$$

With the above analysis, the total result of $\partial_j(x_i/r^3)$ must also take into account of $Tr(M)$ to ensure the consistence. Furthermore, it is known that no direction in 3-dimensional space can be preferred over any other, and so the term $\partial_j(x_i/r^3)$ with $i = j$ must be isotropic for $i = x, y, z$, i.e. proportional to $\delta_{ij}$ and being $1/3$ of $Tr(M)$. As a result, the total result of $\partial_j(x_i/r^3)$ gives

$$\begin{aligned} \partial_j(\frac{x_i}{r^3}) &= \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} + \frac{1}{3}\delta_{ij}Tr(M) \\ &= \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} + \frac{4\pi}{3}\delta_{ij}\delta^{(3)}(\mathbf{r}). \end{aligned} \tag{4.89}$$

Putting all the above results together, it follows that

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta^{(3)}(\mathbf{r}). \tag{4.90}$$

If the current distribution takes the form of a planar closed loop of current (i.e. a current flowing round a planar wire loop), then the general expression for the magnetic moment reduces to

$$\mathbf{m} = \frac{I}{2c}\oint \mathbf{r}\times d\mathbf{r}. \tag{4.91}$$

where $I$ is the current. Note that the magnetic moment is perpendicular to the plane of the loop. Since $(1/2)\mathbf{r}\times d\mathbf{r}$ is the area element of the triangular wedge whose vertices lie at the origin, and the points $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ on the loop, it follows that $(1/2)\mathbf{r}\times d\mathbf{r}$ gives the area of the loop, and so the magnitude $m$ of the magnetic moment for a planar current loop of area $A$ is just given by

$$m = \frac{IA}{c}. \tag{4.92}$$

Note that this result is valid when the current loop in-plane.

### 4.5.1 Force on a current distribution in an external B field

Suppose a localised current distribution is located in a region where there is an externallygenerated magnetic field $\mathbf{B}(\mathbf{r})$ (which may be position dependent). If we assume that the magnetic field varies slowly with position, then we can make a Taylor expansion of $\mathbf{B}(\mathbf{r})$ around some point (which can conveniently be taken to be the origin), and keep just the leading-order terms. Thus we shall have

$$B_i(\mathbf{r}) = B_i(0) + \mathbf{r}\cdot\nabla B_i(0) + ..., \tag{4.93}$$

where, of course, in the second term the argument is set to zero after taking the gradient. Substituting into the expression

$$\mathbf{F} = \frac{1}{c}\int \mathbf{J}(\mathbf{r})\times\mathbf{B}(\mathbf{r})d^3\mathbf{r}, \tag{4.94}$$

for the force on a current distribution, we therefore find

$$\mathbf{F} = -\frac{1}{c}\mathbf{B}(0)\times\int J(\mathbf{r}')d^3\mathbf{r}' + \frac{1}{c}\int \mathbf{J}(\mathbf{r}')\times[\mathbf{r}'\cdot\nabla B_i(0)]d^3\mathbf{r}' + .... \tag{4.95}$$

As we already saw earlier, the integral in the first term vanishes, and so the leading-order contribution to the force comes from the second term. In index notation, the second term is

$$F_i = \frac{1}{c}\epsilon_{ijk}(\partial_l B_k)(0)\int x'_l J_j(\mathbf{r}')]d^3\mathbf{r}'. \tag{4.96}$$

This gives that

$$\begin{aligned} F_i &= -\epsilon_{ijk}\epsilon_{jlq}m_q(\partial_l B_k)(0), \\ &= -m_i\partial_k B_k(0) + m_k\partial_i B_k(0), \\ &= m_k\partial_i B_k(0), \end{aligned} \tag{4.97}$$

where, in getting to the final line, we have used $\nabla\cdot\mathbf{B}=0$. Thus, since $\mathbf{m}$ is a constant, we can write

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}). \tag{4.98}$$

The expression (4.98) for the force on a magnetic dipole $\mathbf{m}$ in a magnetic field $\mathbf{B}$ shows that we can define a potential energy

$$U = -\mathbf{m}\cdot\mathbf{B}. \tag{4.99}$$

which gives the force by $\mathbf{F} = -\nabla U$. Then, we consider two magnetic dipole moments $\mathbf{m}_1$ and $\mathbf{m}_2$, separated by a relative position $\mathbf{r}_{12} = r_{12}\mathbf{n}$. The interaction energy between the two magnetic dipoles reads

$$U_{12} = \frac{\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \mathbf{n})(\mathbf{m}_2 \cdot \mathbf{n})}{r_{12}^3}. \tag{4.100}$$

This is exactly similar as the interaction between two electric dipoles. Similarly, if the two dipoles are parallel in direction, and perpendicular to $\mathbf{n}$, we have

$$U_{12} = \frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{r_{12}^3} = \frac{m_1 m_2}{r_{12}^3} > 0, \tag{4.101}$$

which is repulsive. On the other hand, if the two dipoles are parallel in direction, and also parallel to $\mathbf{n}$, we have

$$U_{12} = -2\frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{r_{12}^3} = -2\frac{m_1 m_2}{r_{12}^3} < 0, \tag{4.102}$$

and thus the interaction is attractive. If the configuration of $\mathbf{n}$ is not changed, but the two dipoles are antiparallel, the above results are changed with an additional minus sign.

# Chapter 5

# Special Relativity and Electrodynamics