(31-147)

Chapter 8

MATRICES AND APPLICATIONS

Analysis is the technically most successful and best elaborated part of mathematics — **Neumann**

INTRODUCTION

Arthur Calyey was the first person to introduce the concept of matrices. Latter Eisenberg used matrices as a tool to explain his famous Quantum Principle. The whole world knew the importance of the application of it. Here we deal with some of the operations and properties of matrices.

A matrix is a collection of numbers ordered by rows and columns. It is customary to inclose the elements of a matrix in parenthesis, brackets, braces.

For example,
$$X = \begin{pmatrix} 5 & 8 & 2 \\ 7 & 1 & 5 \end{pmatrix}$$

Hence each number within the array is called an element. The horizontal lines and the vertical lines formed by the elements are known as rows and columns respectively. If there are m rows and n columns in a matrix, then it is known as m by n matrix or a matrix of order $m \times n$. The above example has two rows and three columns. So it is known as 2×3 (read as 2 by 3) matrix.

Generally, the capital letters of English alphabets are assigned to denote matrix.

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The first subscript in a matrix refers to the row and the second subscript refers to the column. So the element a_{ij} is at the intersection of the i-th row and j-th column.

i , j

For example, construct a matrix of order 2 × 3 where $a_{ij} = i + 2j$.

Suppose the required matrix be

Now, $a_{11} = i + 2j = 1 + 2(1) = 3$ $a_{12} = i + 2j = 1 + 2(2) = 5$

$a_{21} = i + 2j = 2 + 2(1) = 4$ $a_{22} = i + 2j = 2 + 2(2) = 6$

$a_{31} = i + 2j = 2 + 2(1) = 5$ $a_{32} = i + 2j = 3 + 2(2) = 7$

Hence the required matrix is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{pmatrix}$$ Ans.

Example. *Construct 3 × 3 matrix with elements a_{ij} such that $a_{ij} = i + 2j$.*

Solution. Let us consider a 3 × 3 matrix with elements a_{ij} as follows

$$x = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Here $a_{ij} = i + 2j$

Hence $a_{11} = 1 + 2.1 = 3;\ a_{12} = 1 + 2.2 = 5;\ a_{13} = 1 + 2.3 = 7$

$a_{21} = 2 + 2.1 = 4;\ a_{22} = 2 + 2.2 = 6;\ a_{23} = 2 + 2.3 = 8$

$a_{31} = 3 + 2.1 = 5;\ a_{32} = 3 + 2.2 = 7;\ a_{33} = 3 + 2.3 = 9$

Hence the matrix is

$$x = \begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$$ Ans.

TYPES OF MATRICES

Square Matrix. A matrix is called a square matrix of the number of rows is equal to number of columns.

For example, $A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 8 & 7 \end{pmatrix}$

Row Matrix. *A* matrix of order $i \times m$ is called a row matrix.

For example, (5 6), (5 6 3) of order 1 × 2 and 1 × 3 respectively.

Column Matrix. *A* matrix of order $m \times 1$ is known as column matrix.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are column matrices of order 2 × 1 and 3 × 1 respectively.

Zero Matrix. If all the elements of a matrix are zero, then it is known as zero matrix denoted by (0). But, this matrix may be of any order.

For example, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Unit Matrix. The square matrix whose elements on its main diagonal (left top to right botom) are '1's and the rest of its elements are zeros is known as unit matrix.

For example, (1), $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Singular and Non-singular Matrices. A square matrix A is called a singular matrix *iff* its determinant is zero and is called non-singular (or regular) matrix if determinant is not equal to zero.

For example, $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \Rightarrow \det A = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

$\Rightarrow A$ is a singular matrix.

If $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = 3 - 10 = -7$

$\Rightarrow A$ is a non-singular matrix.

Symmetric Matrix. A symmetric matrix is a square matrix in which $X_{ij} = X_{ji}$ for all i and j.

For example, $A = \begin{pmatrix} 8 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 8 & 2 & 4 \\ 1 & 4 & 1 \\ 4 & 2 & 6 \end{pmatrix}$

Matrix A is symmetric; Matrix B is not symmetric

Diagonal Matrix. A diagonal matrix is a symmetric matrix where all of diagonal elements are 0.

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Matrix A is diagonal.

Upper Triangular and Lower Triangular Matrix. A square matrix $A = [a_{ij}]$ is called upper triangular matrix if all the elements below the main diagonal are zero *i.e.,* if $a_{ij} = 0$ for all $i > j$

For example, $A = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Similarly, a square matrix is called lower triangular matrix if all the elements above the main diagonal are zero *i.e.*, if $a_{ij} = 0$ for all $i < j$.

For example,

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 1 & 2 \end{pmatrix}$$

MATRIX ADDITION AND SUBTRACTION

Definition : Two matrices A and B can be added or subtracted if and only if their dimensions are the same. (*i.e.*, both matrices have the identical amount of rows and columns.

Let us take

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix}$$

Addition. If A and B above are matrices of the same type, then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of adding A and B, together

$$A + B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

Subtraction. If A and B are matrices of the same type, then the subtraction is found by subtracting the corresponding elements $a_{ij} + b_{ij}$.

Here is an example of subtracting matrices ,

$$A - B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -3 \\ -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 \\ 5 & 1 & 7 \\ 8 & 0 & 0 \end{pmatrix}$$

Example 1. *If*

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

Then find out $A + B$ *and* $A - B$.

Solution. Given

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 2-1 & -3+1 \\ 3-2 & 1+0 & 4+3 \\ 5+3 & -2+2 & -1+2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & 7 \\ 8 & 0 & 1 \end{pmatrix}$$ **Ans.**

$$A - B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 5 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2 & 2+1 & -3-1 \\ 3+2 & 1-0 & 4-3 \\ 5-3 & -2-2 & -1-2 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 \\ 5 & 1 & 1 \\ 2 & -4 & -3 \end{pmatrix}$$ **Ans.**

Example 2. *If* $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$

Find (*i*) $A + B$ (*ii*) $A - 2B$ (*iii*) $A - 3B + 4C$

Solution. Given $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$

(*i*) $$A + B = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 5 \end{pmatrix}$$

(*ii*) $$A - 2B = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ -4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4 & -2+2 \\ 3+4 & 2-6 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

(*iii*) $$A - 3B + 4C = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - 3\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -6 & 9 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -12 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 1 \\ 9 & -7 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -12 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -3 & -11 \end{pmatrix}$$ **Ans.**

Example 3. *If* $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$

Then find $A + B, A + 2B, A - B$

Solution. Given $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$

Then $$A + B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 0+9 & 2+10 \\ 3+5 & 0+3 & 5+2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 9 & 12 \\ 8 & 3 & 7 \end{pmatrix}$$ **Ans.**

$$A + 2B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + 2\begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 12 & 18 & 20 \\ 10 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 & 22 \\ 13 & 6 & 9 \end{pmatrix}$$ **Ans.**

$$A - B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 9 & 10 \\ 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & 0-9 & 2-10 \\ 3-5 & 0-3 & 5-2 \end{pmatrix} = \begin{pmatrix} -5 & -9 & -8 \\ -2 & -3 & 3 \end{pmatrix}$$ **Ans.**

Properties

Commutative. The addition of matrices is commutative, that is, if A and B are two matrices of same order, then $A + B = B + A$.

From the definition of addition of matrices, it follows that $A + B$ and $B + A$ are of same order. Further if $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$

and $\quad B + A = (b_{ij} + a_{ij})$

But, $\quad (a_{ij} + b_{ij}) = (b_{ij} + a_{ij})$

i.e., each element of $(A + B)$ is equal to the corresponding element of $(B + A)$.

Hence the result.

Associative. The matrix addition is associative *i.e.*, if A, B and C are three matrices of same order, then $\quad A + (B + C) = (A + B) + C$

Since A, B and C are of same order

$A + (B + C)$ and $(A + B) + C$ are of the same order,

Further, if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$,

Then $A + (B + C) = a_{ij} + (b_{ij} + c_{ij})$ and $(A + B) + C = (a_{ij} + b_{ij}) + c_{ij}$

But, $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$

i.e., each element of $A + (B + C)$ and $(A + B) + C$ are equal *i.e.*, the addition of matrices is associative.

Additive Identity. The identity matrix for addition is the zero matrix or null matrix denoted by '0'. Thus, if A is a matrix, then

$A + 0 = A$, provided the order of the zero matrix is same as that of A.

Thus,
$$\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}$$

and
$$(x \quad y)+(0 \quad 0) = (x \quad y)$$

Additive Inverse. The matrix in which each element is the negative of the corresponding element of a given matrix, A is called the inverse of A and is denoted by $(-A)$.

Thus if
$$A = \begin{pmatrix}2 & 1 & 0\\-3 & 2 & 1\\1 & 2 & 0\end{pmatrix}$$

Then
$$(-A) = \begin{pmatrix}-2 & -1 & 0\\3 & -2 & -1\\-1 & -2 & 0\end{pmatrix}$$

Further,
$$A + (-A) = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

and
$$A + (-A) = (-A) + A$$

i.e.,
$$A + (-A) = 0 = (-A) + A$$

MATRIX MULTIPLICATION

Feasibility. For matrices A and B, AB is possible only when number of columns of A = number of rows of B and then the product AB has as many rows as A has and as many columns as B has.

Procedure. The $(i, j)^{\text{th}}$ element of the product AB is obtained by dot multiplication of i^{th} row of the first matrix with j^{th} column of the 2nd.

Here is an example of matrix multiplication for two 2 × 2 matrices.

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}e & f\\g & h\end{pmatrix} = \begin{pmatrix}ae+bg & af+bh\\ce+dg & cf+dh\end{pmatrix}$$

Here is an example of matrix multiplication for a 3 × 3 matrices.

$$\begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}\begin{pmatrix}j & k & l\\m & n & o\\p & q & r\end{pmatrix} = \begin{pmatrix}aj+bm+cp & ak+bn+cq & ai+bo+cr\\dj+em+fp & dk+en+fq & di+eo+fr\\gj+hm+ip & gk+hn+iq & gi+ho+ir\end{pmatrix}$$

Now, lets look at the $n \times n$ matrix case, where A has dimensions $m \times n$, B has dimensions $n \times p$. Then the product of A and B is the matrix C, which has dimensions $m \times p$. The ij^{th} element of matrix C is found by multiplying the entries of the i-th row of A with the corresponding entries in the j-th column of B and summing the n terms. The elements of C are,

$$C_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots\dots\dots\dots + a_{1n}b_{n1} = a_{ij}b_{ji}$$

$$C_{12} = a_{11}b_{12} + a_{12}b_{22} + \dots\dots\dots\dots + a_{1n}b_{n2}$$

$$C_{mp} = a_{m1}\, b_{1p} + a_{m2}\, b_{2p} + \dots\dots\dots\dots + a_{mn}b_{np}$$

NOTE That $A \times B$ is not the same as $B \times A$.

Properties

Commutative. The multiplication of matrices is not commutative *i.e.*, if A and B are tw matrices, then A may not be equal to BA.

From the defination of product of two matrices, it follows that if A and B are of differen orders and the product AB is defined, then BA may not be defined. Similarly, whenever B is defined, AB matrix be defined. Hence both the products AB and BA are defined when A and B are square matrices of same order.

For example, let
$$A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix},\ B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

Then
$$AB = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+2\times1 & 1\times(-1)+2\times3 \\ -1\times2+(-2)\times1 & (-1)\times(-1)+(-2)\times3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ -4 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+(-1)\times(-1) & 2\times2+(-1)\times(-2) \\ 1\times1+3\times(-1) & 1\times2+3\times(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$$

From the above, we get $AB \neq BA$.

But when
$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},\ B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Then
$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$AB = BA.$$

Associative. The multiplication of matrices is associative *i.e.*, if A, B, C are three matrices, then $AB(C) = A(BC)$, provided the products are defined.

For example, let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Then
$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_3b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$$

$$AB(C) = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_3b_3 & a_3b_2 + a_4b_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_2 + a_2b_4c_2 \\ a_3b_1c_1 + a_3b_3c_1 + a_3b_2c_2 + a_4b_4c_2 \end{pmatrix}$$

$$BC = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1c_1 + b_2c_2 \\ b_3c_1 + b_3c_2 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1c_1 + b_2c_2 \\ b_3c_1 + b_4c_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 + a_1b_2c_2 + a_2b_3c_1 + a_2b_4c_2 \\ a_3b_1c_1 + a_3b_2c_2 + a_4b_3c_1 + a_4b_4c_2 \end{pmatrix}$$

From the above, we get

$$(AB)C = A(BC)$$

Example. *If*
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & -2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix}$$

Then show that, $(AB)C = A(BC)$.

Solution.

$$AB = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & -2 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -4 \\ -6 & 4 & 3 \\ -14 & 4 & 8 \end{pmatrix}$$

$$AB(C) = \begin{pmatrix} 5 & -1 & -4 \\ -6 & 4 & 3 \\ -14 & 4 & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & -1 \\ 6 & 4 \end{pmatrix}$$

IDENTITY MATRIX

The identity matrix for multiplication for the set of all square matrices of a given order is the unit matrix of the same order.

For example, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a square matrix of order 2.

Taking $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it can be easily shown that $AI = A = IA$.

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\times 1 + b\times 0 & a\times 0 + b\times 1 \\ c\times 1 + d\times 0 & c\times 0 + d\times 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1\times a + 0\times c & 1\times b + 0\times d \\ 0\times a + 1\times c & 0\times b + 1\times d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Similarly, if A is a square matrix of order 3,

Taking $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, it can be proved that

$\therefore$ $$AI = A = IA.$$

Cancellation Law. The cancellation law does not hold for matrix multiplication *i.e.*, $CA = CB$ does not imply $A = B$, even if the products are defined.

For example, let $A = \begin{pmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$

Then, $$CA = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$\therefore$ $$CA = CB, \text{ when } A \neq B.$$

Distributive Law. The distributive laws hold for matrices *i.e.*, if A, B and C are three matrices, then

$$A(B + C) = AB + AC, \quad (A + B)\,C = AC + BC$$

Provided, the addition and multiplication in above equations are defined.

For example, let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Then $$B + C = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \end{pmatrix}$$

and $$A(B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 \\ a_3b_1 + a_3c_1 + a_4b_2 + a_4c_2 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_2 \\ a_3b_1 + a_4b_2 \end{pmatrix}$$

$$AC = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_2 \\ a_3c_1 + a_4c_2 \end{pmatrix}$$

Then, $$AB + AC = \begin{pmatrix} a_1b_1 + a_2b_2 \\ a_3b_1 + a_4b_2 \end{pmatrix} + \begin{pmatrix} a_1c_1 + a_2c_2 \\ a_3c_1 + a_4c_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2 \\ a_3b_1 + a_4b_2 + a_3c_1 + a_4c_2 \end{pmatrix}$$

$\therefore$ $$A(B + C) = AB + AC$$

Taking $$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \text{and } C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Similar way, we can prove that,

$$(A + B)C = AC + BC$$

Product of Matrices

THEOREM 1. *If A and B are two square matrices of the same order, then*

$$\det (AB) = (\det A)(\det B)$$

Proof. Let us consider two matrices A and B of order 2×2 as

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \text{and } B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

Now, $$AB = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} \end{pmatrix}$$

$\Rightarrow$ $$\det (AB) = (\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21})(\alpha_{21}\beta_{12} + \alpha_{22}\beta_{22}) - (\alpha_{11}\beta_{12} + \alpha_{12}\beta_{22})(\alpha_{21}\beta_{11} + \alpha_{22}\beta_{21})$$
$$= \alpha_{11}\beta_{11}\alpha_{22}\beta_{22} + \alpha_{12}\beta_{21}\alpha_{21}\beta_{21} - \alpha_{12}\beta_{22}\alpha_{21}\beta_{11} - \alpha_{11}\beta_{12}\alpha_{22}\beta_{21}$$

R.H.S. $$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

Then $\det A = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})$ and $\det B = (\beta_{11}\beta_{22} + \beta_{12}\beta_{21})$

$\Rightarrow$ $$(\det A)(\det B) = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})(\beta_{11}\beta_{22} + \beta_{12}\beta_{21})$$
$$= \alpha_{11}\alpha_{22}\alpha_{11}\alpha_{22} - \alpha_{11}\alpha_{22}\beta_{12}\beta_{21} - \alpha_{12}\alpha_{21}\beta_{11}\beta_{22} + \alpha_{12}\alpha_{21}\beta_{12}\beta_{21}$$
$$= \det (AB)$$

Proved.

THEOREM 2. *If A is a non-singular matrix,*

then $$\det (A)^{-1} = (\det A)^{-1} = \frac{1}{\det A}$$

Proof. Suppose A is a non-singular matrix,

$$AA^{-1} = I$$

$\Rightarrow$ $$\det (AA^{-1}) = \det I = 1$$

$\Rightarrow$ $(\det A)(\det A^{-1}) = 1$

Since, we know that if A and B are two square matrices of the same order, then

$$\det(AB) = (\det A)(\det B)$$

$\Rightarrow$ $(\det A^{-1}) = \dfrac{1}{\det A} = (\det A)^{-1}$ as $\det A \neq 0$ **Proved.**

Example. *If* $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$, *then find* AB.

Solution. Given $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 & -3 \\ 0 & -2 & 1 \\ 3 & 11 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times(-1)+1\times0+(-2)\times3 & 2\times(-2)+1\times(-2)+(-2)\times11 & 2\times(-3)+1\times1+(-2)\times0 \\ 1\times(-1)+2\times0+3\times3 & 1\times(-2)+2\times(-2)+3\times11 & 1\times(-3)+2\times1+3\times0 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -28 & -5 \\ 8 & 27 & -1 \end{pmatrix}$$

MATRIX TRANSPOSE

Transpose of an $m \times n$ matrix A is the matrix of order $n \times m$ obtained by interchanging the rows and columns of $A = (a_{ij})$. The transpose of a matrix is denoted by a prime (A′) or a superscript t or T ($A^t = A^T$).

For example. The transpose of a matrix would be,

If $A = \begin{pmatrix} 2 & 1 & 2 & 0 \\ -2 & 1 & 3 & 1 \\ 1 & -3 & 1 & 2 \end{pmatrix}, A^T = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -3 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, A^T = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 5 & 6 & 3 \\ 7 & 4 & 8 \end{pmatrix}, A^T = \begin{pmatrix} 4 & 5 & 7 \\ 1 & 6 & 4 \\ 2 & 3 & 8 \end{pmatrix}$$

Example. *If* $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$, *show that*

(*i*) $(A')' = A$ (*ii*) $(A + B)' = A' + B'$ (*iii*) $(AB)' = B'A'$

Solution. (*i*) If
$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$
$$(A')' = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow (A')' = A$$

(*ii*) Here
$$A = \begin{pmatrix} -1 & 3 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$$

We have to show that,
$$(A + B)' = A' + B'$$

L.H.S.
$$(A + B) = \begin{pmatrix} -1 & 3 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 3 & 6 \end{pmatrix}$$
$$(A + B)' = \begin{pmatrix} -4 & 3 \\ 1 & 6 \end{pmatrix}$$

R.H.S.
$$A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}, B' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}$$
$$A' + B' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 1 & 6 \end{pmatrix}$$

From the above, we get $(A + B)' = A' + B'$

(*iii*) Here given that
$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix}$$

We have to show that, $(AB)' = B'A'$

L.H.S.
$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & 4 \end{pmatrix}$$
$$(AB)' = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

L.H.S.
$$B' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}, A' = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$
$$B'A' = \begin{pmatrix} -3 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

Hence, $(AB)' = B'A'$

Now, it follows immediately from the definition of transpose that,

THEOREM. *If A and B are two N × n matrices, then*

(i) $(A + B)^T = A^T + B^T$

(ii) $(\alpha\ A)^T = \alpha\ A^T$ *where* α *is any scalar*

(iii) $(A^T) = A$

The proof is left to the reader.

THEOREM. *If A is a non-singular square matrix, then* A^T *is also non-singular, and* $(A^T)^{-1} = (A^{-1})^T$.

Proof. Suppose A is a non-singular square matrix. Let there exists a matrix B such that

$$AB = I = BA$$

$\Rightarrow$ $$(AB)^T = I^T = (BA)^T$$

$\Rightarrow$ $$A^TB^T = I = A^TB^T$$

$\Rightarrow$ A has the inverse *i.e.*, B^T

and $$(A^T)^{-1} = B^T = (A^{-1})^T \text{ as } B = A^{-1}$$ **Proved.**

INVERSE OF A MATRIX

Definition : Assuming, we have a square matrix A, which is non-singular (*i.e.*, det (A) does not equal zero), then there exists an $n \times n$ matrix A^{-1} which is called the inverse of A, such that this property holds :

$$AA^{-1} = A^{-1}A = I \text{ where } I \text{ is the identity matrix.}$$

Inverse of a 2×2 matrix

Take for example, a arbitrary 2×2 matrix A whose determinant $(ad - bc)$ is not equal to zero.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are numbers. The inverse is :

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a $n \times n$ matrix

The inverse of a general $n \times n$ matrix A can be found by using the following equation

$$A^{-1} = \frac{Adj\,(A)}{Det\,(A)}$$

where $Adj\ (A)$ denotes the adjoint (or adjugate) of a matrix. It can be calculated by the following method.

1. Given the $n \times n$ matrix A, define

$$B = (b_{ij})$$

to be the matrix whose co-efficients are found by taking the determinant of the $(n^{-1}) \times (n^{-1})$ matrix obtained by deleting the i^{th} row and j^{th} column of A. The terms of B (*i.e.*, $B = b_{ij}$) are known as the co-factors of A.

2. And define the matrix C, where

$$C_{ij} = (-1)^{i+j} b_{ij}.$$

3. The transpose of C (*i.e.*, C^T) is called the adjoint of matrix A.

Lastly, to find the inverse of A, divide the matrix C^T by the determinant of A to give its inverse.

Example 1. *Find the inverse of the following 2×2 matrix*

(*a*) $\begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}$ (*b*) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (*c*) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution. (*a*) Let
$$A = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}$$

$$\therefore \quad |A| = \text{det. } A = \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} = 0 + 4 = 4 \neq 0.$$

Hence A^{-1} exists.

(*b*) Let
$$A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

$$\therefore \quad |A| = \text{det. } A = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 6 - 5 = 1 \neq 0$$

Hence A^{-1} exists.

(*c*) Let
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|A| = \text{det } A = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 0 \neq 0$$

Hence A^{-1} exists.

Example 2. *Find the inverse of the following 3×3 matrix.*

(*a*) $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (*b*) $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

Solution. (*a*) Let
$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Then we find $$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$$

$$= 1(3-1) - 2(2+1) + 5(2+3) = 2 - 6 + 25 = 21$$

Since $|A| \neq 0$, therefore A^{-1} exist.

We know that $$A^{-1} = \frac{Adj\ A}{|A|}$$

For this, we find co-factors of A.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3$$

$$A_{31} = (-1)^{1+3} M_{31} = (-1)^4 \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13$$

$$A_{12} = (-1)^{2+1} M_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -3$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6$$

$$A_{32} = (-1)^{2+3} M_{22} = (-1)^5 \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -9$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$Adj\ A = \begin{vmatrix} 2 & 3 & -13 \\ -3 & 6 & -9 \\ 5 & 3 & -1 \end{vmatrix}$$

Hence $$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{21}\begin{vmatrix} 2 & 3 & -13 \\ -3 & 6 & -9 \\ 5 & -3 & -1 \end{vmatrix}$$

(*b*) Let $$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 3(-3 + 4) + 3(2 - 0) + 4(-2 - 0)$$

$$= 3 + 6 - 8 = 1 \neq 0$$

Hence, A^{-1} exist.

We know that $$A^{-1} = \frac{Adj\ A}{|A|}$$

Now find *Adj A*. For this, we find co-factors of *A*.

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$$

Now, $$Adj\ A = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{1}\begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{vmatrix}$$