A quick note on implied volatility
Applied Stochastic Processes (FIN 514)

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Overview

- The Black-Scholes-Merton (BSM), Bachelier (Normal) models are seldom used to *predict* the option price. The option prices are determined from the supply and demand of market.
- However, the pricing models are still important because they provide a consistent measure to intuitively understand the prices of the options with different strikes, time to maturity, etc.
- Implied volatility (IV) is the value of the volatility in pricing model which returns (or solve) the price of an option given (i.e., from market):

$$C(K, S_0, \sigma, T_e) = \text{Price}$$

- How can you compare the two prices?

$$C(K = 105, S_0 = 100, T_e = 1) = 5.9 \quad P(K = 98, S_0 = 100, T_e = 1) = 8.7$$

The implied volatility is same as 20%.

General rule

- Option prices (both call and put) increase as the volatility increases: $C(K, S_0, \sigma, T_e)$ is monotonically increasing function.
- Option value = Time value + Intrinsic Value
  - Intrinsic value: the value you get by exercising option now. (> 0)
  - Time value: the extra value from the change of the underlying price until the expiry. (> 0)
- The intrinsic value can be understood as the option value wih $\sigma = 0$, $C(K, S_0, \sigma = 0, T_e)$, hence the minimum value.
- The call option value as $\sigma \to \infty$:
  - BSM model ($S_T \geq 0$): S_0. The underlying stock is always worth more than any call option with $K > 0$.
  - Bachelier (Normal) model (S_T can be negative): ∞. Call option can protect the (infinite) loss.

IV computation

- The computation of IV depends on numerical root-solving method. [Demo]
  - Newton method: using vega,

$$\sigma^{(k+1)} = \sigma^{(k)} - \frac{C(\sigma^{(k)}, \cdots) - \text{Price}}{V(\sigma^{(k)})}, \quad V(\sigma) = \frac{\partial C(\sigma, \cdots)}{\partial \sigma}$$

  - Brent's method: [Demo]
- BSM model:
  - Newton's method with good initial guess (`PyFeng` package)
  - Let's Be Rational (Jackel, 2015): Machine epsilon error within two step iterations.
- Normal model:
  - Choi et al. (2007). Numerical Approximation of the Implied Volatility Under Arithmetic Brownian Motion:
    Polynomial approximation with error ($< 10^{-9}$)

IV under the Bachelier Model (Choi et al., 2007)

$$\sigma_{\mathrm{N}} = \sqrt{\frac{\pi}{2T}}\,(2C - \theta(F_0 - K))\,h(\eta),$$

where C is the undiscounted price of either call ($\theta = 1$) or for put ($\theta = -1$) option. The function $h(\eta)$ and the argument η are defined in a sequence,

$$h(\eta) = \sqrt{\eta}\,\frac{\sum_{k=0}^{7} a_k \eta^k}{1 + \sum_{k=1}^{9} b_k \eta^k}, \quad \eta = \frac{v}{\tanh^{-1}(v)}, \quad \text{and} \quad v = \frac{|F_0 - K|}{2C - \theta(F_0 - K)},$$

with the coefficients,

$$\begin{aligned}
a_0 &= 3.99496\ 16873\ 45134\ \mathrm{e}{-1}\\
a_1 &= 2.10096\ 07950\ 68497\ \mathrm{e}{+1}\\
a_2 &= 4.98034\ 02178\ 55084\ \mathrm{e}{+1}\\
a_3 &= 5.98876\ 11026\ 90991\ \mathrm{e}{+2}\\
a_4 &= 1.84848\ 96954\ 37094\ \mathrm{e}{+3}\\
a_5 &= 6.10632\ 24078\ 67059\ \mathrm{e}{+3}\\
a_6 &= 2.49341\ 52853\ 49361\ \mathrm{e}{+4}\\
a_7 &= 1.26645\ 80513\ 48246\ \mathrm{e}{+4}
\end{aligned}$$

$$\begin{aligned}
b_1 &= 4.99053\ 41535\ 89422\ \mathrm{e}{+1}\\
b_2 &= 3.09357\ 39367\ 43112\ \mathrm{e}{+1}\\
b_3 &= 1.49510\ 50083\ 10999\ \mathrm{e}{+3}\\
b_4 &= 1.32361\ 45378\ 99738\ \mathrm{e}{+3}\\
b_5 &= 1.59891\ 96976\ 79745\ \mathrm{e}{+4}\\
b_6 &= 2.39200\ 88917\ 20782\ \mathrm{e}{+4}\\
b_7 &= 3.60881\ 71083\ 75034\ \mathrm{e}{+3}\\
b_8 &= -2.06771\ 94864\ 00926\ \mathrm{e}{+2}\\
b_9 &= 1.17424\ 05993\ 06013\ \mathrm{e}{+1}.
\end{aligned}$$