Scientific computation/Monte Carlo method
Applied Stochastic Processes (FIN 514)

Instructor: Jaehyuk Choi

Peking University HSBC Business School, Shenzhen, China

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Scientific Computation

- Most of science (including finance) problems can not be solved analytically. Even analytic/algebraic solutions still requires to be computed!
- We need problem solving methods tailored for computation.
- Scientific computation is a branch of (applied)mathematics/computer science to research efficient computation method.
- Scientific computation is also an important part of financial engineering/quantitative finance.

Example: numerical root finding

- Abel (1802-1829) proved that there is no general *algebraic* solution for the roots of a quintic equation ($a_5x^5 + \cdots + a_1x + a_0 = 0$)
- However, we can find the root numerically:
  - Bisection

$$x_{n+1} = \frac{a + x_n}{2} \quad \text{or} \quad \frac{x_n + b}{2}$$

  - Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  - Newton-Raphson method

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

  - Square root: $x = \sqrt{y}$ [Demo]

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{y}{x_n}\right)$$

Tips and cautions

- In general, carefully translate math formulas to computer implementation
- Avoid repeated evaluation. Special functions, $\exp, \sin, \cos, \log$, are several times expensive than that of multiplication.
- Even polynomials (Horner's rule):

$$a_0 + a_1 x + \cdots + a_n x^n \quad \text{v.s.} \quad a_0 + x(a_1 + x(\cdots(a_{n-1} + x\ a_n)))$$

- In computer, small difference (machine epsilon) is rounded to 0, which is unexpected and dangerous. For example,

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

we get $f(10^{16}) = 0$. So better use the equivalent same formula, [Python Demo]

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

Monte-Carlo (MC) Method

- Even with computer, many problems can not be solved with deterministic (or convergent) method. So we need Monte-Carlo simulation.
- In finance, the focus is on the price of derivative, which is the expected value of stochastic payout

$$E(f(x)) \quad \text{or} \quad E(f(x_1, \cdots, x_D))$$

  where $x, x_1, \cdots, x_D$ are (possibly correlated) D-dimensional random numbers.
- MC is favored in a large dimensions (D) because the cost of MC glows linearly ($\sim D$) whereas the cost of the deterministic method grows exponentially ($\sim e^D$), which is called the curse of dimensionality.
- Example: MC computation of π [Python Demo]

Generating Uniform Random Number (RN)

The uniform random number between 0 and 1 is the mother of all RNs. By nature, computer cannot generate true RNs. In fact, true RNs are not even desirable as we want to control and reproduce the result. So we generate pseudo-RNs in the following way:

$$X_{n+1} = AX_n + B \mod C, \quad U_n = X_n/C$$

X_0 is the **seed** of the RN generation. Using a fixed seed, you get the same result. Two popular, but outdated, choices of (A, B, C) are

- $A = 65,539 = 2^{16} + 3, \quad C = 2^{31}, \quad (B = 0)$
- $A = 16,807 = 7^5, \quad C = 2^{31} - 1, \quad (B = 0)$

The Mersenne Twister is the standard pseudo-RN generation method ($C = 2^{19937} - 1$). It is implemented in modern programming languages including python.

Generating Discrete RN

Bivariate RNs, e.g., random walk takes $+1$ and -1 with probabilities p and $1-p$ respectively can be generated as

$$X_k = \begin{cases} 1 & \text{if} \quad U_k < p \\ -1 & \text{if} \quad U_k > p \end{cases},$$

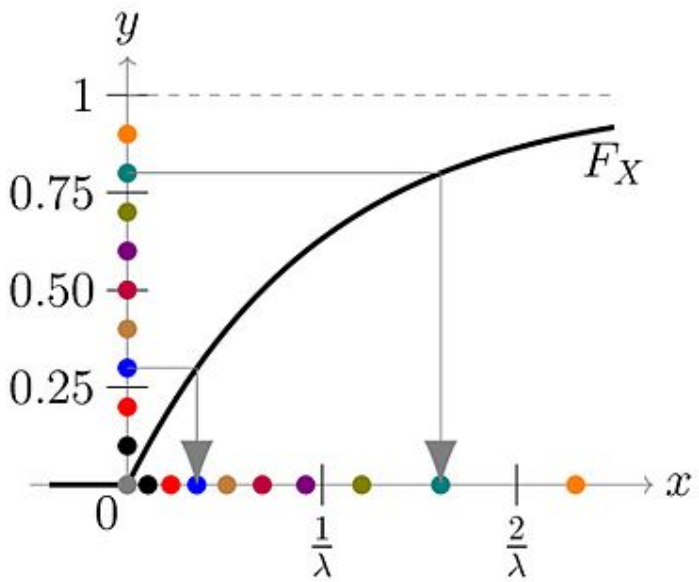
where U_k is uniform RNs. Multiple values can be similarly generated.

Generating RN from a General Distribution

If we know the cumulative distribution function (CDF) of a distribution $P(x)$, the inverse CDF gives random numbers following the distribution,

$$X = P^{-1}(U)$$

However, probability distribution with analytically invertible CDF is rare. When inverse CDF is not available, the last resort is numerical root-finding, $P(X) = U$ but this is computationally expensive. Fining efficient sampling method is an important area of research.



Generating Exponential Distribution RNs

Exponential/Poisson distribution

- Distribution for the survival time or arrival time T with intensity λ.
- PDF: $f(t) = \lambda e^{-\lambda t}$, CDF: $F(t) = 1 - e^{-\lambda t}$
- $E(T) = 1/\lambda$, $\mathrm{Var}(T) = 1/\lambda^2$.

The CDF function is easily invertible. The RN can be generated with

$$T_k = -\frac{1}{\lambda} \log U_k = \left(-\frac{1}{\lambda} \log(1 - U_k) \right)$$

where U_k is uniform RNs.

Gaussian Distribution

- Given the wide usage, efficient evaluation of Gaussian distribution is critically important.
- How to evaluate CDF? Integration is very expensive!

$$N(z) = \int_{-\infty}^{z} n(s)\, ds$$

- An accurate approximation is used instead ($x > 0$):

$$1 - N(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2/2}, \quad t = \frac{1}{1 + px/\sqrt{2}}$$

where $a_1 = 0.3480242$, $a_2 = -0.0958798$, $a_3 = 0.7478556$, and $p = 0.47047$.

Gaussian RN (Box-Muller Method)

- Reference: Box, George EP, and Mervin E Muller. A Note on the Generation of Random Normal Deviates. The Annals of Mathematical Statistics 29, no. 2 (1958): 610–11.
- Evaluation of the normal inverse CDF, N^{-1}, is expensive.
- There is a brilliant trick of generating Gaussian RN:

For two-dimensional Gaussian random numbers (Z_1, Z_2)

$$P\{z_1^2 + z_2^2 < R^2\} = \int_{z_1^2+z_2^2<R^2} \frac{1}{\sqrt{2\pi}^2} e^{-\frac{1}{2}(z_1^2+z_2^2)} dz_1 dz_2$$
$$= \frac{2\pi}{2\pi} \int_{r=0}^{R} r e^{-r^2/2} dR = 1 - e^{-R^2/2}$$

The squared radius, $Y = R^2 = Z_1^2 + Z_2^2$, follows exponential distribution and we know the inverse CDF is available:

$$P(y) = 1 - e^{-y/2}, \quad P^{-1}(u) = -2\log(1-u).$$

Gaussian RN (Box-Muller Method) cont.

- This means we can accurately sample $Y = R^2$ and $R = \sqrt{Y}$:

$$Y = -2\log(1 - U_1) \quad \text{for a uniform RV } U_1$$

$$R = \sqrt{Y} = \sqrt{-2\log U_1} \quad (\text{symmetry: } U_1 \leftrightarrow 1 - U_1)$$

- Then, we extract two normal RNs, Z_1 and Z_2, as $x-$ and $y-$axis projections with a random angle $2\pi U_2$:

$$Z_1 = \sqrt{-2\log U_1}\,\cos(2\pi U_2), \quad Z_2 = \sqrt{-2\log U_1}\,\sin(2\pi U_2).$$

We get two Gaussian RNs from two uniform RNs.

Gaussian RN (Marsaglia-Polar Method)

- Reference: Marsaglia, George, and Thomas A Bray. A Convenient Method for Generating Normal Variables. SIAM Review 6, no. 3 (1964): 260–64.
- The Box-Muller method can be further improved!
- $V_{1,2} = 2U_{1,2} - 1$ so that $V_{1,2}$ is uniform RVs between -1 and 1.
- Take if $0 < W = V_1^2 + V_2^2 < 1$ and reject otherwise so that (V_1, V_2) is uniform random point on the unit circle.
- W has a uniform distribution on $[0, 1]$,

$$P\{W < x\} = \pi(\sqrt{x})^2/\pi = x,$$

so W can replace U_1.

Gaussian RN (Marsaglia-Polar Method) cont.

- Using the trigonometric properties,

$$\left(\frac{V_1}{\sqrt{W}},\ \frac{V_2}{\sqrt{W}}\right) = \Big(\cos(2\pi U_2), \sin(2\pi U_2)\Big)$$

- Finally we obtain

$$Z_1 = \sqrt{-2\log W}\, V_1/\sqrt{W} = V_1\sqrt{-2(\log W)/W}$$
$$Z_2 = \sqrt{-2\log W}\, V_2/\sqrt{W} = V_2\sqrt{-2(\log W)/W}$$

- Even after *wasting* 21% of (V_1, V_2) pairs, it is still more efficient by avoiding expensive $\sin$ and $\cos$ evaluations!

Reducing MC variance

- Anti-thetic method; (π by MC)
  - By symmetry, U is a uniform random number, so is $1 - U$
  - By symmetry, Z is a Gaussian random number, so is $-Z$
- Importance sampling method: ($\mathrm{P}(Z > 30)$ by MC)
  - Original probability = Modified probability $\times$ Correction
- Low-disccrepancy sequence
  - Qusai-random numbers evenly distributed over $(0, 1)$.
- Control Variate: for $g(x)$ similar to $f(x)$ and known $E(g(X))$,

$$E_{CV}(f(X)) = E_{MC}(f(X)) + \Big(E(g(X)) - E_{MC}(g(X))\Big)$$

[Python Demo]

Generation of correlated Gaussian RNs ($N = 2$)

- We want to generate two standard Gaussian RNs, W_1 and W_2, correlated by ρ,

$$\frac{E(W_1 W_2)}{E(W_1)E(W_2)} = E(W_1 W_2) = \rho.$$

- We can achieve that by separating correlate and de-correlated parts:

$$W_1 = Z_1,\ W_2 = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2,$$

  where Z_1 and Z_2 are independent and standard RNs.
- You can easily prove that (see StoFin mid-term exam):

$$E(W_2^2) = 1 \quad \text{and} \quad E(W_1 W_2) = \rho$$

- You'll see this trick in spread option and stochastic differential equations for stochastic volatility models (Heston and SABR model)

Covariance/correlation matrix

For N-dimensional samples, $X = (X_1, \cdots, X_N)^T$, with mean 0 and stdev σ_k, the covariance matrix $\boldsymbol{\Sigma}$ is given as

$$\boldsymbol{\Sigma}_{ij} = \mathrm{Cov}(X_i, X_j) = E\left((X_i - \bar{X}_i)(X_i - \bar{X}_j)\right) \quad (\boldsymbol{\Sigma}_{ij} = \sigma_i^2).$$

The correlation matrix R is similarly defined as

$$R_{ij} = \mathrm{Corr}(X_i, X_j) = \frac{\boldsymbol{\Sigma}_{ij}}{\sigma_i \sigma_j} \quad (R_{ii} = 1)$$

- If $\{X_i\}$ are independent standard normals, $\Sigma = I$
- Symmetric: $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T$ $(\Sigma_{ij} = \Sigma_{ij})$
- Positive-definite: $\boldsymbol{a}^T \Sigma \boldsymbol{a} \geq 0$ for any (row) vector $\boldsymbol{a}$:

$$\mathrm{Var}(\boldsymbol{a}X) = \mathrm{Var}(a_1 X_1 + \cdots + a_N X_N) = \boldsymbol{a}\boldsymbol{\Sigma}\boldsymbol{a}^T \geq 0$$

- For a linear combination, $\boldsymbol{L}X$, $\mathrm{cov}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{\Sigma}\boldsymbol{L}^T$.

Generation of correlated Gaussian RNs (N in general)

- $X = (X_i)^T$ with covariance matrix $\boldsymbol{\Sigma}$: we want to find such that $X = \boldsymbol{L}Z$ where $Z = (Z_i)^T$ is independent standard normals.
- Then, $\boldsymbol{L}$ should satisfy the covariance matrix condition:

$$\boldsymbol{\Sigma} = \text{Var}(\boldsymbol{L}Z) = \boldsymbol{L}\boldsymbol{I}\boldsymbol{L}^T = \boldsymbol{L}\boldsymbol{L}^T$$

  Therefore, $\boldsymbol{L}$ should be a square-root matrix of $\boldsymbol{\Sigma}$: $\boldsymbol{L} = \sqrt{\boldsymbol{\Sigma}}$.
- The square-matrix $\boldsymbol{L}$ is not unique. Cholesky decomposition (lower triangular matrix) is a numerically stable and fast solution. [Python Demo]
- The 2 asset case ($\rho = \rho_{12}$):

$$\boldsymbol{L} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}, \quad \boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

MC of Brownian Motion(s)

- Standard BM: $W_T \sim \sqrt{T} Z_1$ where Z_1 is a standard normal RV.
- Arithmetic Brownian motion (Bachelier model)
  - $$dF_t = \sigma_{\text{N}}\, dW_t \quad \Rightarrow \quad F_T \sim F_0 + \sigma_{\text{N}} \sqrt{T} Z_1$$
  - From $t = S$ to $t = T$: $F_T \sim F_S + \sigma_{\text{N}} \sqrt{T - S} Z_1$
  - For N assets: $\boldsymbol{F}_T = \boldsymbol{F}_0 + \boldsymbol{L}\sqrt{T}\, Z$, where $\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{\Sigma}$
- Geometric Brownian motion (BSM model)
  - $$\frac{dF_t}{F_t} = \sigma\, dW_t \quad \Rightarrow \quad F_T \sim F_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} Z_1\right)$$
  - From $t = S$ to $t = T$: $F_T \sim F_S \exp\left(-\frac{1}{2}\sigma^2 (T - S) + \sigma\sqrt{(T - S)} Z_1\right)$
  - For N assets: $F_k(T) = F_k(0) \exp\left(-\frac{1}{2} T \Sigma_{kk} + \boldsymbol{L}_{k*}\sqrt{T}\, Z\right)$, where $\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{\Sigma}$ and $\boldsymbol{L}_{k*}$ is the k-th row vector of $\boldsymbol{L}$.
- [Python Demo]