Applied Stochastic Processes (FIN 514)
Midterm Exam

Instructor: Jaehyuk Choi

2019-20 Module 1 (2019. 10. 22)

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. You can use the following functions in your answers without further evaluation,

$$\text{Standard normal PDF: } n(x) = e^{-x^2/2}/\sqrt{2\pi}$$
$$\text{Standard normal CDF: } N(x) = \textstyle\int_{-\infty}^{x} n(s)ds.$$

1. ($2\times 2 = 2$ points) **(Random number generation)** Pareto distribution is defined by the survival function:
$$S(x) = \text{Prob}(X > x) = \begin{cases} \left(\frac{\lambda}{x}\right)^\alpha & (x \ge \lambda) \\ 1 & (x < \lambda). \end{cases}$$
   (a) (2 points) Find the mean and variance of the distribution. Clearly state the condition that the mean and variance are finite (i.e., not infinite).
   (b) (2 points) How can you generate RN following the Pareto distribution from uniform random number U ?

Solution:

(a) Based on the PDF of X,
$$f(x) = \frac{\alpha\lambda^\alpha}{x^{\alpha+1}} \quad \text{for} \quad x \ge \lambda \quad (0 \quad \text{otherwise}),$$
the mean and variance are computed as
$$E(X) = \frac{\alpha\lambda}{\alpha-1} \quad \text{for} \quad \alpha > 1 \quad (\infty \quad \text{otherwise}),$$
$$\text{Var}(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} \quad \text{for} \quad \alpha > 2 \quad (\infty \quad \text{otherwise}).$$

(b) The CDF is easily invertible. From
$$U = 1 - \left(\frac{\lambda}{X}\right)^\alpha \quad \Rightarrow \quad X = \frac{\lambda}{(1-U)^{1/\alpha}} \quad \text{or} \quad \frac{\lambda}{U^{1/\alpha}}$$

Reference: Pareto Distribution (Wikipedia)

2. ($3{\times}2 = 6$ points) **(Simulation of multidimensional normal RVs)** Suppose that $\boldsymbol{F}_t$ is a column vector of three asset prices at time t and that $\boldsymbol{F}_T$ is distributed as

$$\boldsymbol{F}_T - \boldsymbol{F}_0 = \boldsymbol{L}\,\boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a standard normal RV (column) vector of size 3 and $\boldsymbol{L}$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

(Hint: L is the lower triangular matrix in Cholesky decomposition.)

(a) Assuming that $T = 5$, what is the volatility of each asset?

(b) What is the correlation between the 2nd and 3rd asset?

(c) What is the price of the at-the-money basket call option based on the three assets with equal weight (i.e, 1/3 each)? Assume that the at-the-money option price under the normal volatility $\sigma_{\rm N}$ is $0.4\,\sigma_{\rm N}\sqrt{T}$.

Solution: The covariance of the price change is

$$\mathrm{Cov}(\boldsymbol{F}_T - \boldsymbol{F}_0) = \boldsymbol{\Sigma} = \boldsymbol{L}^T\boldsymbol{L} = \begin{pmatrix} 1 & -3 & -2 \\ -3 & 25 & 10 \\ -2 & 10 & 9 \end{pmatrix}$$

(a) The volatility of the assets are $\sqrt{1/5}$, $\sqrt{5}$, and $\sqrt{9/5} = 3/\sqrt{5}$.

(b) $10/(\sqrt{25}\,\sqrt{9}) = 2/3 \approx 66.6\%$.

(c) From

$$\sigma_{\rm N}^2 T = \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = 5 \quad \text{for} \quad \mathbf{w} = [1/3, 1/3, 1/3]^T,$$

the basket option price is $0.4\sqrt{5}$.

3. ($3{\times}2 = 6$ points) **(Euler and Milstein Schemes)** The variance process for the GARCH diffusion model is given by

$$dv_t = \kappa(\theta - v_t)dt + \xi v_t\, dZ_t$$

and you want to simulate v_t using time-discretization scheme.

(a) What is the Euler and Milstein schemes for v_t? Explicitly write down the expression for $v_{t+\Delta t} - v_t$ using standard normal RV Z_1.

(b) The SDE for v_t tells us that v_t cannot go negative. However, in the Monte-Carlo simulation with the time-discretization scheme, v_t sometimes go negative. To avoid this problem, it is better simulate $w_t = \log v_t$ instead. Derive the SDE for w_t.

(c) What is the Euler and Milstein schemes for w_t?

Solution:

(a) The Euler and Milstein schemes for v_t is given by

$$v_{t+\Delta t} - v_t = \kappa(\theta - v_t)\Delta t + \xi v_t Z_1 \sqrt{\Delta t} + \boxed{\frac{\xi^2}{2} v_t (Z_1^2 - 1)\Delta t}\,,$$

where the boxed term is only for the Milstein scheme.

(b) Applying Itô's lemma, we obtain

$$\begin{aligned} dw_t &= \frac{dv_t}{v_t} - \frac{1}{2}\frac{(dv_t)^2}{v_t^2} = \kappa\left(\frac{\theta}{v_t} - 1\right)dt + \xi dZ_t - \frac{\xi^2}{2}dt \\ &= \left(\kappa\theta e^{-w_t} - \kappa - \xi^2/2\right)dt + \xi dZ_t. \end{aligned}$$

(c) The Euler and Milstein scheme is same for w_t and they are given by

$$w_{t+\Delta t} - w_t = \left(\kappa\theta e^{-w_t} - \kappa - \xi^2/2\right)\Delta t + \xi Z_1 \sqrt{\Delta t}.$$

So it is better to simulate w_t first and obtain $v_t = e^{w_t}$, which is always positive. Also note that the Milstein scheme for v_t in (a) can be recovered by the Taylor expansion of e^x:

$$\begin{aligned} v_{t+\Delta t} &= v_t \ \exp\left(w_{t+\Delta t} - w_t\right) \\ &= v_t\left(1 + \left(\kappa\theta e^{-w_t} - \kappa - \frac{\xi^2}{2}\right)\Delta t + \xi Z_1\sqrt{\Delta t} + \frac{\xi^2}{2}Z_1^2\Delta t + o(\Delta t)\right) \\ v_{t+\Delta t} - v_t &= \kappa(\theta - v_t)\Delta t + \xi v_t Z_1\sqrt{\Delta t} + \frac{\xi^2}{2}v_t(Z_1^2 - 1)\Delta t\,, \end{aligned}$$

4. $(2 + 4 = 6$ points) **(Conditional Monte Carlo Simulation)** We are going to formulate the conditional Monte Carlo simulation for the Ornstein-Uhlenbeck stochastic volatility (OUSV) model. The dynamics for the OUSV model is given by

$$\begin{gathered} \frac{dF_t}{F_t} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}, \\ d\sigma_t = \kappa(\theta - \sigma_t)dt + \xi\, dZ_t \end{gathered}$$

(a) Derive the SDE for σ_t^2.

(b) Based on the answer of (a), express F_T in terms of (σ_T, U_T, V_T) where U_T and V_T are give by

$$U_T = \int_0^T \sigma_t\, dt \quad \text{and} \qquad V_T = \int_0^T \sigma_t^2\, dt.$$

What are $E(F_T)$ and the volatility of F_T conditional on the triplet (σ_T, U_T, V_T)?

Solution:

(a) Using Itô's lemma,

$$d\sigma_t^2 = (\xi^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\xi\sigma_t dZ_t.$$

(b) Integrating the result of (a),

$$\sigma_t^2 - \sigma_0^2 = \xi^2 T + 2\kappa(\theta U_T - V_T) + 2\xi \int_0^T \sigma_t dZ_t.$$

Therefore,

$$\log\left(\frac{F_T}{F_0}\right) = \frac{\rho}{2\xi}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\xi}{2}T - \frac{\rho\kappa\theta}{\xi}U_T + \left(\frac{\rho\kappa}{\xi} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T}\,X_1$$

and we obtain

$$E(F_T) = F_0 \exp\left(\frac{\rho}{2\xi}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\xi}{2}T - \frac{\rho\kappa\theta}{\xi}U_T + \left(\frac{\rho\kappa}{\xi} - \frac{\rho^2}{2}\right)V_T\right)$$
$$\mathrm{Vol}(F_T) = \rho_*\sqrt{V_T/T}.$$

Reference: Li, C., Wu, L., 2019. **Exact simulation of the Ornstein–Uhlenbeck driven stochastic volatility model**. European Journal of Operational Research 275, 768–779. https://doi.org/10.1016/j.ejor.2018.11.057