# Spread and Basket Option Pricing
# Applied Stochastic Processes (FIN 514)

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# Overview

- Options written on multiple underlying assets
- Extension of Black-Scholes model from uni-variate to multi-variate
- Write a MC pricing routine with control variate
- Analytic approximations (to be used as control variates)
    - Normal model price
    - Kirk's approximation (Margrabe's formula)
    - Geometric basket option

# Background: popular derivatives in non-vanilla class

## Spread options: $(S_1 - S_2 - K)^+$

- Crack spread option: (P of oil products − P of oil − $K)^+$
- Spark spread option: (P of electricity − P of gas − $K)^+$
- Non-inversion note: digital call on rates term-structure, $(30y - 2y)^+$

## Basket options: $(\sum w_k\, S_k - K)^+$ with $w_k > 0$

- Popular as OTC derivatives in FX and commodities market
- Index options

## Asian options (path-dependent): $(\sum_1^N S(t_k)/N - K)^+$

- Efficient hedge over average cost, safe from market manipulation.
- Fed fund swaps: daily compounded trade-averaged FF rate
- China interest swaps: 3-month average of 7-day repo rate as a standard floating rate (cf. 3-month LIBOR)

# Problem setup

- $N$ asset prices, $S_k(t)$, following the correlated geometric Brownian motions (GBM)

$$\frac{dS_k(t)}{S_k(t)} = (r - q_k)\,dt + \sigma_k\,dW_k(t) \quad \text{for} \quad 1 \le k \le N$$

  for volatilities $\sigma_k$, dividend rates $q_k$, risk-free rate $r$ and BMs $W_k(t)$, the correlation $\mathbb{E}\{dW_k(t)dW_j(t)\} = \rho_{kj}dt$ ($\rho_{kk} = 1$).
- $N$ observation times: $t_k$ ($1 \le k \le N$), with expiry at $T$
- The payout of the option at the maturity $T$

$$C(T) = \Big(\sum_{k=1}^{N} w_k\,S_k(t_k) - K\Big)^+$$

# Option types

$$C(T) = \Big(\sum_{k=1}^{N} w_k\, S_k(t_k) - K\Big)^+$$

- **European basket option**:
  $w_k > 0$ and $t_k = T$ for all $k$; $N < 10$
- **European spread option**:
  $w_k < 0$ for some $k$, $t_k = T$ for all $k$; $N = 2$
- **Discretely monitored Asian option** (covered later):
  $w_k = 1/N$, $0 \le t_1 < \cdots < t_N = T$ and $S_k(t)$'s are identical; $N \gg 10$

# Challenges

## Mathematical problem

Multi-dimensional integration over the domain of positive payoff

## Difficulties

- The lognormal RV sum is neither lognormal nor has analytic distribution.
- Numerical valuation is cursed by dimensionality: $O(M^N)$. E.g., $h=0.1$ grid between $\pm 7$ std. dev. for $4$ assets: $140^4 \approx 400 \times 10^6$
- Monte-Carlo simulation is used for pricing in industry and academics.

## Quote from Broadie and Detemple (2004, MS Survey)

*"Many problems are effectively exponential $\cdots$. Efficient and convergent methods for pricing high-dimensional and path-dependent American securities depend on the development of new algorithms, not faster computers."*

# Normal model approximation

- Spread Option: $\sigma_{\mathrm{N}1} = \sigma_1 S_1(0),\ \sigma_{\mathrm{N}2} = \sigma_2 S_2(0)$

$$\mathsf{Var}(S_1(T) - S_2(T)) = (\sigma_{\mathrm{N}1}^2 + \sigma_{\mathrm{N}2}^2 - 2\rho\sigma_{\mathrm{N}1}\sigma_{\mathrm{N}2}) \cdot T$$

$$\sigma_{\mathrm{N}} = \sqrt{\sigma_{\mathrm{N}1}^2 + \sigma_{\mathrm{N}2}^2 - 2\rho\sigma_{\mathrm{N}1}\sigma_{\mathrm{N}2}}$$

  and use the normal model formula.

- Basket Option: $\sigma_{\mathrm{N}j} = \sigma_j S_j(0)$ or $= \sigma_j F_j(0)$

$$\mathsf{Var}(\sum_k w_k S_k(T)) = \Big(\sum_j w_j^2 \sigma_{\mathrm{N}j}^2 + 2\sum_{j\neq k} \rho_{jk} w_j w_k \sigma_{\mathrm{N}j}\sigma_{\mathrm{N}k}\Big) \cdot T$$

$$\Sigma_{jk} = \rho_{jk}\sigma_{\mathrm{N}j}\sigma_{\mathrm{N}k}, \quad \sigma_{\mathrm{N}} = \sqrt{\mathbf{w}^T\, \mathbf{\Sigma}\, \mathbf{w}}$$

  and use the normal model formula.

# Normal model approximation for Control Variate

- Use the result as a control variate of Spread and Asiain option:

$$C_{\rm BS}^{\rm CV}(T,K) = C_{\rm BS}^{\rm MC}(T,K) + \Big(C_{\rm N}^{\rm EXACT}(T,K) - C_{\rm N}^{\rm MC}(T,K)\Big)$$

  Use the same sequence of RNs for $C^{MC}$ and $C_N^{MC}$. [Py Demo]

- **Homework Set 2**:
  Implement Monte-Carlo pricer for Spread and Basket options with control variate with normal model price.
- Final project (past years): implement other analytic approximation methods or CV methods (e.g., Kirk's approximation for spread options)

# Exchange option: Margrabe's formula

- Option to exchange one asset $S_1$ for another $S_2$: $\big(S_1(T) - S_2(T)\big)^+$
- Spread option with zero strike $K = 0$: $\big(S_1(T) - S_2(T) - 0\big)^+$
- Max (best-of) option in terms of exchange option:

$$\max(S_1(T), S_2(T)) = S_2(T) + \big(S_1(T) - S_2(T)\big)^+,$$

where $(x)^+ = \max(x, 0)$.

## Margrabe's exchange option formula

$$C_{\text{EX}} = S_1(0)N(d_+) - S_2(0)N(d_-),$$

where $d_\pm = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

# SDE on $S_1/S_2$

$$\frac{dS_k(t)}{S_k(t)} = r\,dt + \sigma_k dW_k(t)\ (k=1,2), \quad dW_1 dW_2 = \rho dt$$

Applying Itô's lemma to $S_1/S_2$,

$$d\left(\frac{S_1}{S_2}\right) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2}dS_2 + \frac{S_1}{S_2^3}(dS_2)^2 - \boxed{\frac{dS_1 dS_2}{S_2^2}}$$

For $R = S_1/S_2$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + (\sigma_1 dW_1 - \sigma_2 dW_2)$$

Alternatively,

$$d\log R = d\log S_2 - d\log S_1 = -\frac{1}{2}(\sigma_1^2 - \sigma_2^2)\,dt + \sigma_1 dW_1 - \sigma_2 dW_2$$

$$\frac{dR}{R} = d\log R + \frac{1}{2}(\sigma_1 dW_1 - \sigma_2 dW_2)^2 = \text{same result}$$

# Equivalent Martingale Measure with Numeraire $S_2$

Decorrelating the SDE on $R$, for $dW_1' dW_2 = 0$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + \sigma_1(\rho\,dW_2 + \sqrt{1-\rho^2}\,dW_1') - \sigma_2\,dW_2.$$

Now we change the measure from $P$ (risk-less saving numeraire) to $Q$ ($S_2$);

$$\begin{aligned} C_{\text{EX}} &= 1 \cdot E^P\left(\frac{(S_1(T) - S_2(T))^+}{e^{rT}}\right) \\ &= S_2(0) E^Q\left(\frac{(S_1(T) - S_2(T))^+}{S_2(T)}\right) = S_2(0) E^Q\left((R(T) - 1)^+\right) \end{aligned}$$

Under the new measure, the standard BM is defined as

$$dW_2^P = dW_2^Q + \sigma_2 dt \quad \text{and} \quad dW_1'^P = dW_1'^Q.$$

Then the SDE on $R$ becomes drift-less and can be written with a single BM

$$\begin{aligned} \frac{dR}{R} &= (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + \sqrt{1-\rho^2}\sigma_1 dW_1'^P - (\sigma_2 - \rho\sigma_1)(dW_2^Q + \sigma_2\,dt) \\ &= \sqrt{1-\rho^2}\sigma_1 dW_1'^Q + (\sigma_2 - \rho\sigma_1) dW_2^Q = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,dZ^Q \end{aligned}$$

# Margrabe's exchange option formula

The exchange option price is obtained from just another Black-Scholes formula on the ratio $R(t) = S_1(t)/S_2(t)$ with

- $K = 1$
- $\sigma_R = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$
- Prefactor (unit of options): $S_2(0)$.

Finally we obtain

$$C_{\text{EX}} = S_2(0)\, E^Q\Big((R(T) - 1)^+\Big) = S_2(0)\, \left(\frac{S_1(0)}{S_2(0)} N(d_+) - 1 \cdot N(d_-)\right)$$

where $d_\pm = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

# Spread Option: Kirk's Approximation

If we assume $S_2$ follows displace GBM with $L = K$, $S_2^D = S_2 + K$, then we can apply Margrabe's formula!

$$(S_1 - S_2 - K)^+ = (S_1 - \boxed{(S_2 + K)})^+ = (S_1 - S_2^D)^+$$

The volatility of $S^D$ should be 'calibrated'. We match the local vol at ATM

$$\sigma_2^D (S_2(0) + K) = \sigma_2 S_2(0)$$

Plug in $S_2(0) + K \to S_2(0)$ and $\sigma_2 S_2(0)/(S_2(0) + K) \to \sigma_2$ to Margrabe:

## Kirk's Approximation Formula

$$C_{\text{KIRK}} = S_1(0)N(d_+) - (S_2(0) + K)N(d_-),$$

$$d_\pm = \log(\frac{S_1(0)}{S_2(0) + K})/\sigma_R\sqrt{T} \pm \frac{1}{2}\sigma_R\sqrt{T}$$

$$\sigma_R = \sqrt{\sigma_1^2 + \sigma_2'^2 - 2\rho\sigma_1\sigma_2'}, \quad \sigma_2' = \sigma_2 S_2(0)/(S_2(0) + K)$$

# Basket Option: Levy's lognormal approximation

- The first two moments of a lognormal distribution with ($\lambda = \sigma\sqrt{T}$), $Y = \mu_1 \exp(\lambda Z - \lambda^2/2)$ for standard normal $Z$ are

$$E(Y) = \mu_1, \quad E(Y^2) = \mu_2 = \exp(\lambda^2) \quad \Rightarrow \quad \lambda = \sqrt{\log(\mu_2/\mu_1^2)}$$

- Approximate the final basket price $B(T)$ by a lognormal distribution.

$$B(T) = \textstyle\sum_{k=1}^N w_k\, S_k(T) \sim \mu_1 \exp(\lambda Z - \lambda^2/2)$$

- Obtain the first two moments from the original variables:

$$E(B(T)) = \textstyle\sum_{k=1}^N w_k\, F_k(T), \quad E(B^2(T)) = \textstyle\sum_{i,j} w_i w_j F_i F_j e^{\sigma_i \sigma_j \rho_{ij} T}.$$

- Use Black-Scholes formula with $\sigma = \lambda/\sqrt{T}$. (Implemented in `PyFENG`.)

# Asian Option

- Discretely monitored:
$$A(T) = \tfrac{1}{N}\textstyle\sum_{k=1}^{N} S(t_k)$$
- Continuously monitored:
$$\begin{aligned} A(T) &= \frac{1}{T-T_0}\int_{t=T_0}^{T} S(t)dt \\ &= \frac{\Delta t}{T-T_0}\left(\frac{S(T_0)}{2} + S(T_0+\Delta t) + \cdots S(T-\Delta t) + \frac{S(T)}{2}\right) \end{aligned}$$
- A special case of basket option:
$S(t_i)$ and $S(t_j)$ with $t_i < t_j$ are correlated by $t_i/t_j$.
- Suggested topics for final project.