Stochastic-alpha-beta-rho (SABR) Model
Applied Stochastic Processes (FIN 514)

Instructor: Jaehyuk Choi

Peking University HSBC Business School, Shenzhen, China

2019-20 Module 1 (Fall 2019)

The project overview

SABR Model

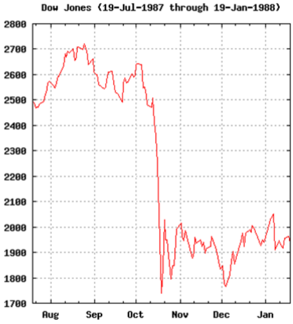
- One of the most popular stochastic volatility (SV) model
- Heavily used in trading options for interest rate and FX
- Explains volatility skew/smile with minimal and intuitive parameters

Project Goal

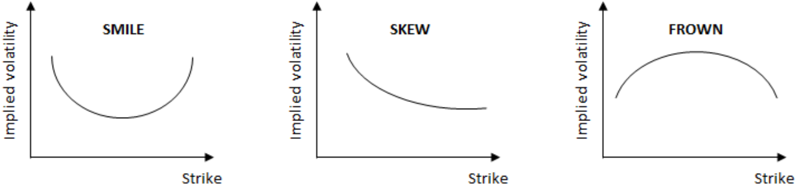
- Implement the approximation formula by Hagan (provided)
- Implement option pricing with Euler/Milstein scheme and conditional MC
- Implement a smile calibration routine based on the method of Kennedy et al (2012)

Background: volatility skew/smile

- Black Monday crash in 1987:
  DJIA -22.6% in one day!
- Overall 'short gamma' due to the portfolio insurance (put on equity index)
- Market values (down-side) tail event higher than before.
- Market sees volatility skew/smile



(From Wikipedia)



Why need model for smile? challanges in risk management

- Option trading desk (market-making/sell-side) usually accumulates option positions with different strikes.
- Under BSM model,
    - Vol σ fixed under spot change $S_0 \to S_0 + \Delta$.
    - Risk-management is easy: delta and vega clearly defined
    - One can hedge delta (with underlying stock) and vega (with ATM option)
    - However, the OTM option prices/risks are not correct!
- BSM model with different σ to each option K?
    - How do we fix the volatilities?
    - Sticky strike rule $\sigma = \sigma(K)$ vs sticky delta rule $\sigma = \sigma(S_0 - K)$.
    - Need to characterize the smile with a few minimal parameters.
- For better risk management, we need models which can capture the volatility smile.

How to model smile? Local volatility (LV)

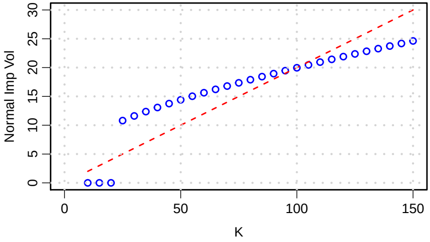
- Volatility depending on the 'current location' of S_t:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma f(S_t)\, dW_t \qquad \text{Normal: } dS_t = \sigma_{\rm N} f_{\rm N}(S_t)\, dW_t$$

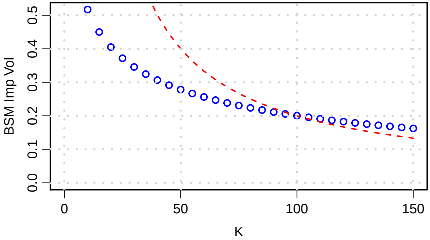
- BSM model: a trivial case with $f(x) = 1$. However, it is a local vol model under normal volatility ($f_{\rm N}(x) = x$).
- Normal model: a trivial case with $f_{\rm N}(x) = 1$. However, it is a local vol model under BSM volatility ($f(x) = 1/x$).
- What is the implied normal volatility of the Black-Scholes price on varying K? What is the relation between the implied volatility and the local vol?
- The implied volatililty is the volatility average of the in-the-money paths
- **Exercise 1**: Chart the normal implied voly of the prices under BSM model for typical parameter sets. Measure the slope, $\partial\sigma(K)/\partial K$, at the money.

Case: $S_0 = 100, \sigma = 20\% (\sigma_N = 20), r = q = 0$:

- Implied normal vol for constant BSM vol ($\sigma = 20\%$):



- Implied normal vol for constant normal vol ($\sigma_N = 20$):



Displaced GBM (shifted BSM) model

- A quick local vol model
- 'Displaced asset price' $S_t + L$ follows GBM:

$$dS_t = \sigma_L(S_t + L)\, dW_t$$

- Somewhere between normal ($L \to \infty$) and log-normal model ($L = 0$).
- Can reuse BS formula with $S_0 + L \to S_0$ and $K + L \to K$.
- Calibration of σ_L (ATM option price on target):

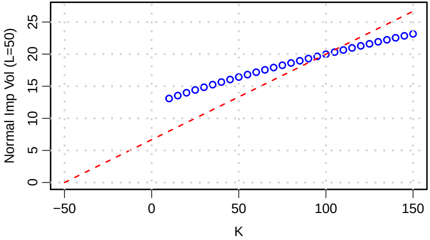
$$\sigma_{\rm N} \approx \sigma_L(S_0 + L) \approx \sigma\, S_0$$

  But, needs an exact calibration of σ_L for a given σ_{BS}.

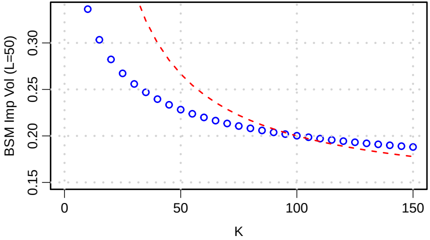
- **Exercise 2**: Chart the BSM implied vol of the prices under displaced GBM model. Using the implemented implied vol function, exactly calibrate σ_L to the ATM price.

Case: $S_0 = 100, L = 50, \sigma = 20\%, r = q = 0$:

- $\sigma_L = \sigma S_0/(S_0 + L) = 13.33\%$
- Implied normal vol: (red line: $\sigma_L(K + L)$)



- Implied BSM vol: (red line: $\sigma_L(K + L)/K$)



How to model smile? Stochastic volatility (SV)

- Volatility changing over time:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma_t\, dW_t \qquad \text{Normal: } dS_t = \sigma_t\, dW_t$$

- Many models proposed (mostly for BSM). For $dW_t dZ_t = \rho\, dt$,
  - Hull-While (SABR):

$$\frac{d\sigma_t}{\sigma_t} = \alpha\, dZ_t$$

  - Heston: $V_t = \sigma_t^2$ follows Cox-Ingersoll-Ross (CIR) process,

$$dV_t = \kappa(V_\infty - V_t)dt + \alpha\sqrt{V_t}dZ_t$$

- SV model correctly captures the smile, α for curvature and ρ for skewness.

SABR model: LV + SV

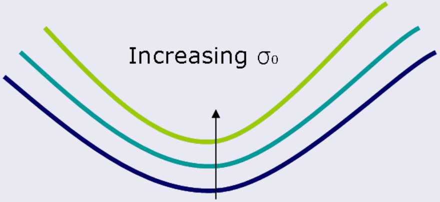
Stochastic–α, β, ρ model SDE:

$$dS_t = \sigma_t S_t^\beta dW_t$$
$$d\sigma_t = \alpha \sigma_t dZ_t$$
$$dW_t \, dZ_t = \rho \, dt$$

- Parameters: σ_0, α, β, ρ.
- σ_0: ovarall volatility, calibrated to ATM implied vol
- β: elasticity or 'backbone'. (Normal: $\beta = 0$, BSM: $\beta = 1$)
- α: volatility of volatility, σ following a GBM
- ρ: correlation between asset price and volatility

The impact of parameters

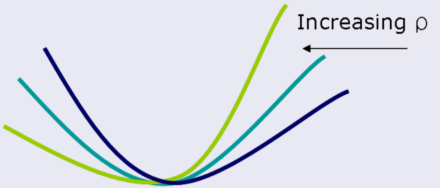
Starting vol σ_0



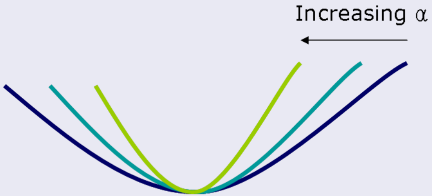
Backbone β

- Fixed or infrequently changed
- BSM moel: $\beta = 1$ (Equity, FX)
- Normal model: $\beta = 0$ (Interest Rate)

Correlation ρ



'Vol of vol' α



Implied vol formula (Hagan et al, 2002)

The first few terms of the 'Taylor expansion' near $\alpha\sqrt{T} \approx 0$.

$$\sigma_B(K, f)$$
$$= \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K + \cdots\right\}} \cdot \left(\frac{z}{x(z)}\right)$$
$$\cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{ex} + \cdots .\right. \tag{2.17a}$$

Here

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \tag{2.17b}$$

and $x(z)$ is defined by

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}. \tag{2.17c}$$

Success of SABR model

- Volatility smile information encoded into three parameters σ_0, α, ρ !!
- Vega (volatility) risk managed by the three parameters rather than each individual vol.
- Three implied vols (or option prices) on the smile can calibrate the parameters. $\rightarrow$ An effective interpolation method for implied volatility (or option price)

Limitation of Hagan's formula

- Arbitrage is equivalent to some event happening with negative probability. The price of a derivative paying $\$1$ on that event is negative (should be free at most)!
- Digital (Call/Put) option from call spread

$$\text{Prob}(S_T > K) = D(K) = \frac{C_{\text{BS}}(K) - C_{\text{BS}}(K+\Delta K)}{\Delta K} = -\frac{\partial C_{\text{BS}}(K)}{\partial K}$$

- When $\alpha\sqrt{T} \gg 1$, Hagan's formula sometimes implies $\text{P}(S_T > K) < 0$ from

$$C_{\text{BS}}(K, \sigma(K)) < C_{\text{BS}}(K+\Delta K, \sigma(K+\Delta K)).$$

The volatility effect $\sigma(K+\Delta K)$ overcomes (should NOT!) the moneyness effect $K+\Delta K$.

MC Simulation (Euler method)

- Unlike normal or BSM model (as in spread/basket option project), we can not jump directly from $t=0$ to T.
- Divide the interval $[0,T]$ into N small steps, $t_k=(k/N)T$ and $\Delta t_k = T/N$ and simulate each time step,

$$S_t:\begin{cases}\beta=0: & S_{t_{k+1}} = S_{t_k} + \sigma_{t_k} Z_1 \sqrt{\Delta t_k} \\ \beta=1: & \log S_{t_{k+1}} = \log S_{t_k} + \sigma_{t_k}\sqrt{\Delta t_k}\, Z_1 - \tfrac{1}{2}\sigma_{t_k}^2 \Delta t_k,\end{cases}$$

$$\sigma_t : \sigma_{t_{k+1}} = \sigma_{t_k} \exp\left(\alpha\sqrt{\Delta t_k}\, Z_2 - \frac{1}{2}\alpha^2 \Delta t_k\right),$$

  where $Z_1,\ Z_2 \sim N(0,1)$ with correlation ρ.
- Typically, $\Delta t_k \approx 0.25$. For $T=30$, $N=120$, quite time-consuming.
- Any good control variate?

$$C(K) = \frac{1}{N}\sum_{i=1}^{N}(S_T^{(i)} - K)^+$$

Stochastic integral of σ_t

From Itô's lemma,

$$d\sigma_t = \alpha\sigma_t \, dZ_t \quad \rightarrow \quad d\log\sigma_t = -\frac{1}{2}\alpha^2 dt + \alpha dZ_t$$

and we know the final distribution,

$$\sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right).$$

We also know

$$\alpha \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right) - \sigma_0,$$

which will be useful for the integration of S_t.

Stochastic integral of S_t (normal: $\beta = 0$)

Writing the SDE in a de-correlated form,

$$dS_t = \sigma_t \left(\rho dZ_t + \sqrt{1-\rho^2} dW_t\right) \quad \text{for} \quad dW_t\, dZ_t = 0.$$

Integrating S_t, we get so far as

$$\begin{aligned} S_T - S_0 &= \rho \int_0^T \sigma_t dZ_t + \sqrt{1-\rho^2} \int_0^T \sigma_t dW_t \\ &= \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{1-\rho^2} \int_0^T \sigma_t dW_t \end{aligned}$$

From Itô's Isometry, the integrated variance is $I_T := \int_0^T \sigma_t^2 dt$. With some more work, the box can be expressed as

$$\int_0^T \sigma_t dW_t = W\sqrt{I_T}, \text{ where } W \sim N(0,1) \text{ intependent from } I_T \text{ and } \sigma_T$$

Conditional MC method (normal $\beta = 0$)

Given (σ_T, I_T), the distribution of S_T is

$$S_T = S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{(1-\rho^2) I_T}\, W$$

and the option price is from the normal model:

$$C_{\mathrm{N}}\left(K, S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0),\ \sigma_{\mathrm{N}} := \sqrt{(1-\rho^2) I_T / T}\right)$$

Then, the price is obtained as an expectation over MC simulation of I_T:

$$C_{\beta=0} = E\left(C_{\mathrm{N}}(\sigma_T, I_T)\right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t. Given (σ_T, I_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

Conditional MC method (BSM $\beta = 1$)

Given (σ_T, I_T), the distribution of S_T is

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1-\rho^2}\, W\sqrt{I_T}$$

and the option price is from the BSM model:

$$C_{\text{BS}}\left(K, S_0 e^{\frac{\rho}{\alpha}\left(\sigma_T - \sigma_0\right) - \frac{\rho^2}{2}I_T}, \sqrt{(1-\rho^2)I_T/T}\right)$$

Then, the price is obtained as an expectation over MC simulation of I_T:

$$C_{\beta=1} = E\left(C_{\text{BS}}(\sigma_T, I_T)\right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t. Given (σ_T, I_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

The conditional distribution of I_T on σ_T (Kennedy et al)

The conditional mean of I_T on σ_T is known as

$$E(I_T|\sigma_T) = \frac{\sigma_0^2\sqrt{T}}{2\alpha}\frac{N(d_\alpha+\alpha\sqrt{T}) - N(d_\alpha-\alpha\sqrt{T})}{n(d_\alpha+\alpha\sqrt{T})}$$
$$\text{for}\quad d_\alpha = \log(\sigma_T/\sigma_0)/(\alpha\sqrt{T}).$$

The distribution of S_T is approximated

$$S_T = S_0 + \frac{\rho}{\alpha}\big(\sigma_T - \sigma_0\big) + \sqrt{1-\rho^2}\,\eta(\sigma_T)W\sqrt{T}$$

for $\eta(\sigma_T) = E(I_T|\sigma_T)/\sqrt{T}$. For a given σ_T, S_T follows a normal distribution, so we now the option

$$C_\mathrm{N}(\sigma_T) = C_\mathrm{N}\left(S_0 := S_0 + \frac{\rho}{\alpha}\big(\sigma_T - \sigma_0\big),\ \sigma_\mathrm{N} := \sqrt{1-\rho^2}\,\eta(\sigma_T)\right)$$

Option price as an integration (Kennedy et al)

$$C_{\beta=0} = E\Big((S_T - K)^+\Big) = E\Big((S_T - K)^+ | \sigma_T\Big) = E\Big(C_{\text{N}}(\sigma_T)\Big)$$
$$= \int_{-\infty}^{\infty} C_{\text{N}}\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z) - \sigma_0), \sqrt{1-\rho^2}\,\eta(\sigma_T(z))\right)\, n(z)\, dz$$
$$\text{where} \quad \sigma_T(z) = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha\sqrt{T}\, z\right)$$

Using Gauss-Hermite quadrature (GHQ), [Py Demo]

$$C_{\beta=0} = \sum_m C_{\text{N}}\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z_m) - \sigma_0), \sqrt{1-\rho^2}\,\eta(z_m)\right)\, w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$, and $\eta(z_m) := \eta(\sigma_T(z_m))$.

BSM: $\beta = 1$

The results are similar:

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1-\rho^2}\, W\sqrt{I_T}$$

$$C_{\beta=1} = \sum_m C_{BS}\left(S_0 e^{\frac{\rho}{\alpha}\left(\sigma_T(z_m)-\sigma_0\right)-\frac{\rho^2}{2}\eta(z_m)}, \sqrt{1-\rho^2}\,\eta(z_m)\right)\, w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$.

- Implement the method of Kennedy et al and compare it against the Monte Carlo result for both normal ($\beta = 0$) and BSM backbone ($\beta = 1$).

Smile Calibration

- When β is given (0 or 1), three parameters, σ_0, ρ and α, can be calibrated to three option prices (or implied volatilities), typically at $K = S_0$ (ATM), $S_0 - \Delta$ and $S_0 + \Delta$.

$$\text{SABR}(\sigma_0, \rho, \alpha) \to \sigma(S_0), \sigma(S_0 - \Delta), \sigma(S_0 + \Delta)$$

- Write a calibration routine in R to solve σ_0, ρ and α.

Projects

Validation

- $C_{\beta=0}$ should converge to the normal model price when α is very small.
- Test against the result from Korn & Tang (Wilmott)

Homework

- First focus on the normal backbone $\beta = 0$.
- Make sure to use antithetic method (create Z, then add $-Z$).
- Short (1 page) write-up briefly explaining the code.
- Reproduce the graphs in 'The impact of parameters'. Fix your normal implied vol at ATM.
- Your script should be self-complete and should run without error.