第七章习题一

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一、1. $\bar{X} = np$

$\Rightarrow \hat{p} = \frac{\bar{X}}{n}$

2. $\bar{X} = \lambda$

$\Rightarrow \hat{\lambda} = \bar{X}$

3. $\begin{cases} \bar{X} = \frac{a+b}{2} \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \left(\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \end{cases}$

$\Rightarrow \begin{cases} a+b = 2\bar{X} \\ b-a = \sqrt{12\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right)} \end{cases}$

$\Rightarrow \begin{cases} \hat{a} = \bar{X} - \sqrt{3\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right)} \\ \hat{b} = \bar{X} + \sqrt{3\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right)} \end{cases}$

4. $\bar{X} = \theta$

$\Rightarrow \hat{\theta} = \bar{X}$

5. $\begin{cases} \bar{X} = \mu \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mu^2 + \sigma^2 \end{cases}$

$\Rightarrow \begin{cases} \hat{\mu} = \bar{X} \\ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \end{cases}$

6. $\bar{X} = \frac{1}{p}$

$\Rightarrow \hat{p} = \frac{1}{\bar{X}}$

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第七章　习题 2.

离散：$L(\theta)=\prod_{i=1}^{n} P(X_i=x_i)$　连续：$L(\theta)=\prod_{i=1}^{n} f(x_i,\theta)$

一、1. ① $L(p)=\prod_{i=1}^{m} P(X_i=x_i)=\prod_{i=1}^{m}\left[C_n^{x_i} p^{x_i}(1-p)^{n-x_i}\right]$

② $\ln L(p)=\ln\prod_{i=1}^{m} C_n^{x_i}+\sum_{i=1}^{m} x_i\cdot\ln p+\sum_{i=1}^{m}(n-x_i)\cdot\ln(1-p)$

$$\frac{d\ln L(p)}{dp}=\frac{\sum_{i=1}^{m} x_i}{p}-\frac{\sum_{i=1}^{m}(n-x_i)}{1-p}=0$$

$$\sum_{i=1}^{m} x_i\cdot(1-p)-\sum_{i=1}^{m}(n-x_i)p=0$$

$$\sum_{i=1}^{m} x_i-\sum_{i=1}^{m} x_i\cdot p-mnp+\sum_{i=1}^{m} x_i\cdot p=0$$

$$\hat{p}=\frac{\sum_{i=1}^{m} x_i}{mn}=\frac{\overline{x}}{n}$$

2. $L(\lambda)=\prod_{i=1}^{n}\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}=\frac{\lambda^{\left(\sum_{i=1}^{n} x_i\right)}}{\prod_{i=1}^{n}(x_i!)}e^{-n\lambda}$

$$\ln L(\lambda)=\sum_{i=1}^{n} x_i\cdot\ln\lambda-\ln\prod_{i=1}^{n}(x_i!)+(-n\lambda)$$

$$\frac{d\ln L(\lambda)}{d\lambda}=\frac{\sum_{i=1}^{n} x_i}{\lambda}-n=0\Rightarrow\lambda=\frac{\sum_{i=1}^{n} x_i}{n}=\overline{x}$$

3. $L(a,b)=\prod_{i=1}^{n} f(x_i,a,b)=\begin{cases}\frac{1}{(b-a)^n}, & a\le\min\{x_i\}\le\max\{x_i\}\le b\\ 0, & \text{其它}\end{cases}$

$L(a,b)$ 关于 b 是单调减，故 b 取 b 参数空间最小值，$\hat{b}=\max\{x_i\}$

$L(a,b)$ 关于 a 是单调增，故 a 取 a 参数空间最大值，$\hat{a}=\min\{x_i\}$

4. $L(\theta)=\prod_{i=1}^{n} f(x_i,\theta)=\begin{cases}\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}, & \min\{x_i\}>0 \\ 0, & \text{其它}\end{cases}$

$\ln L(\theta)=\begin{cases}-\frac{1}{\theta}\sum_{i=1}^{n} x_i - n\ln\theta, & \min\{x_i\}>0 \\ 0, & \text{其它}\end{cases}$

$\frac{d\ln L(\theta)}{d\theta}=\begin{cases}\frac{1}{\theta^2}\sum_{i=1}^{n} x_i - \frac{n}{\theta}, & \min\{x_i\}>0 \\ 0, & \text{其它}\end{cases}$

$\hat{\theta}=\overline{x}, \quad (\min\{x_i\}>0)$

5. $L(\mu,\sigma^2)=\prod_{i=1}^{n} f(x_i,\mu,\sigma^2)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

$=\left(\frac{1}{\sqrt{2\pi}}\right)^n (\sigma^2)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}}$

$\ln L(\mu,\sigma^2)=\ln\left(\frac{1}{\sqrt{2\pi}}\right)^n-\frac{n}{2}\ln\sigma^2-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$

$\begin{cases}\frac{\partial\ln L(\mu,\sigma^2)}{\partial\mu}=\sum_{i=1}^{n}\frac{1}{2\sigma^2}2(x_i-\mu)=0 \\ \frac{\partial\ln L(\mu,\sigma^2)}{\partial\sigma^2}=-\frac{n}{2}\frac{1}{\sigma^2}+\sum_{i=1}^{n}(x_i-\mu)^2\cdot\frac{1}{2}\frac{1}{\sigma^4}=0\end{cases}$

$\Rightarrow\begin{cases}\hat{\mu}=\overline{x} \\ \hat{\sigma^2}=S_n^2\end{cases}$

6. $L(p)=\prod_{i=1}^{n}P(X_i=x_i)=\prod_{i=1}^{n}(1-p)^{x_i-1}p=(1-p)^{\sum_{i=1}^{n}x_i-n}p^n$

$\ln L(p)=(\sum_{i=1}^{n}x_i-n)\ln(1-p)+n\ln p$

$\frac{d\ln L(p)}{dp}=-\frac{\sum_{i=1}^{n}x_i-n}{1-p}+\frac{n}{p}=0$

$n(1-p)=p\cdot\sum_{i=1}^{n}x_i-np$

$\hat{p}=\frac{\sum_{i=1}^{n}x_i}{n}=\overline{x}$

二、$\frac{1}{n}\sum_{i=1}^{n}x_i=\frac{\theta}{2}\Rightarrow\hat{\theta}=2\overline{x}$，$E(\hat{\theta})\neq\theta$，不是无偏估计

矩估计：

最大似然估计：$f(x)=\begin{cases}\frac{1}{\theta}, & 0\leq x\leq\theta\\ 0, & 其它\end{cases}$

$L(\theta)=\prod_{i=1}^{n}f(x_i,\theta)=\begin{cases}\frac{1}{\theta^n}, & 0\leq\min\{x_i\}\leq\max\{x_i\}\leq\theta\\ 0, & 其它\end{cases}$

由上式知，$L(\theta)$关于θ单调递减，故$\hat{\theta}=\max\{x_i\}$，令$Y=\max\{x_i\}$

$E(\hat{\theta})=E(\max\{x_i\})=EY$，$F_Y(y)=P(x_1\leq y,x_2\leq y,\cdots,x_n\leq y)$

$=P(x_1\leq y)\cdot P(x_2\leq y)\cdots P(x_n\leq y)=\prod_{i=1}^{n}F_{x_i}(y)=\begin{cases}0, & y<0\\ \frac{y^n}{\theta^n}, & 0\leq y\leq\theta\\ 1, & y>\theta\end{cases}$

$f_Y(y)=\begin{cases}n\theta^{-n}y^{n-1}, & 0\leq y\leq\theta\\ 0, & 其它\end{cases}$

$E(\hat{\theta})=E(Y)=\int_{-\infty}^{+\infty}yf_Y(y)dy=\int_0^\theta y\cdot n\cdot\theta^{-n}\cdot y^{n-1}dy=n\cdot\theta^{-n}\int_0^\theta y^n dy$

$=\frac{n}{n+1}\theta^{-n}\,y^{n+1}\Big|_0^\theta=\frac{n}{n+1}\theta^{-n}\cdot\theta^{n+1}=\frac{n}{n+1}\theta\neq\theta$

故不是无偏估计

三.(1) $\frac{1}{n}\sum_{i=1}^{n}X_i = EX \Rightarrow \bar{x} = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 (\theta+1)x^{\theta+1}\,dx$

$= \frac{\theta+1}{\theta+2} x^{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2} \Rightarrow \hat{\theta} = \frac{2\bar{x}-1}{1-\bar{x}}$

(2) $L(\theta) = \prod_{i=1}^{n} f(x_i,\theta) = \begin{cases} (\theta+1)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}, & 0 < \min\{x_i\} \le \max\{x_i\} < 1 \\ 0, & \text{其它} \end{cases}$

当 $0 < \min\{x_i\} \le \max\{x_i\} < 1$ 时:

$\ln L(\theta) = n\ln(\theta+1) + \theta \ln \prod_{i=1}^{n} x_i$

$\frac{d\ln L(\theta)}{d\theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n}(\ln x_i) = 0$

$\hat{\theta} = -1 - \frac{n}{\sum_{i=1}^{n}\ln x_i}$

五、$EX = 2\theta(1-\theta) + 2\theta^2 + 3(1-2\theta) = 2\theta - 2\theta^2 + 2\theta^2 + 3 - 6\theta = 3 - 4\theta$

$\bar{x} = 2$

矩估计: $\bar{x} = EX \Rightarrow 3 - 4\theta = 2 \Rightarrow \hat{\theta} = \frac{1}{4}$

最大似然估计: $L(\theta) = \prod_{i=1}^{n} P(X_i = x_i)$

$= \theta^2 \cdot [2\theta(1-\theta)]^2 \cdot \theta^2 \cdot (1-2\theta)^4 = 4\theta^6 (1-\theta)^2 (1-2\theta)^4$

$\ln L(\theta) = 6\ln 4\theta + 2\ln(1-\theta) + 4\ln(1-2\theta)$

$\frac{d\ln L(\theta)}{d\theta} = 6\frac{1}{4\theta}\cdot 4 - 2\frac{1}{1-\theta} + 4\frac{1}{1-2\theta}(-2)$

$= \frac{24\theta^2 - 28\theta + 6}{\theta(\theta-1)(2\theta-1)} = 0 \Rightarrow \hat{\theta} = \frac{7\pm\sqrt{13}}{12}.$

又 $\because \frac{7+\sqrt{13}}{12} > \frac{1}{2}$ $\therefore \hat{\theta} = \frac{7-\sqrt{13}}{12}$

第七章　习题7.

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一、1. 最有效

2. $an+bm=1$　$\frac{4}{4n+m}$　$\frac{1}{4n+m}$

3. $\hat{\mu}_3$

二、$E\left(C\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2\right)=\sigma^2$

$X_{i+1}-X_i\sim N(0,2\sigma^2)$

$\Rightarrow E(X_{i+1}-X_i)^2=(E(X_{i+1}-X_i))^2+D(X_{i+1}-X_i)=2\sigma^2$

$E\left(C\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2\right)=C\sum_{i=1}^{n-1}E(X_{i+1}-X_i)^2=C(n-1)2\sigma^2=\sigma^2$

$\Rightarrow C=\frac{1}{2(n-1)}$

三、$L(\theta)=\prod_{i=1}^{n}f(x_i,\theta)=\begin{cases}2^n e^{-2\sum_{i=1}^{n}(x_i-\theta)}, & \min\{x_i\}>\theta\\ 0, & 其它\end{cases}$

当$\min\{x_i\}>\theta$时：$\ln L(\theta)=n\ln 2-2\sum_{i=1}^{n}(x_i-\theta)$

$=n\ln 2-2\sum_{i=1}^{n}x_i+2n\theta$

$\frac{d\ln L(\theta)}{d\theta}=2n>0$，$L(\theta)$随$\theta$增大而增大，$\hat{\theta}=\min\{X_i\}$

令$Y=\min\{X_i\}$，$F_Y(y)=1-(1-F_X(y))^n\Rightarrow f_Y(y)=-n(1-F_X(y))^{n-1}[-f_X(y)]$

$F_X(x)=\int_{-\infty}^{x}f(t)d(t)=\begin{cases}0, & x\le\theta\\ \int_\theta^x 2e^{-2(t-\theta)}dt=1-e^{-2(x-\theta)}, & x>\theta\end{cases}$

$f_Y(y)=\begin{cases}0, & y\le\theta\\ 2ne^{-2n(x-\theta)}, & y>\theta\end{cases}$

$EY=\int_{-\infty}^{+\infty}yf_Y(y)dy=\int_\theta^{+\infty}2nye^{-2n(y-\theta)}dy$

$=\int_0^{+\infty}2nye^{-2ny+2n\theta}dy$

$= e^{2n\theta}\int_{0}^{+\infty} 2ny e^{-2ny} dy = -e^{2n\theta}\int_{0}^{+\infty} y de^{-2ny} = e^{2n\theta}[ye^{-2ny}|_{0}^{+\infty} - \int_{0}^{+\infty} e^{-2ny} dy]$

$= -e^{2n\theta}[0 - \theta e^{-2n\theta} + \frac{1}{2n}\int_{\theta}^{+\infty} e^{-2ny} d(-2ny)]$

$= -e^{2n\theta}[-\theta e^{-2n\theta} + 0 - \frac{1}{2n} e^{-2n\theta}] = \theta + \frac{1}{2n} \neq \theta$

不是无偏估计

四. 令 $Y = \max\{X_i\}$, $F_Y(y) = (F_X(y))^3$

$F_X(x) = \begin{cases} 0, & x<0 \\ \frac{x}{\theta}, & 0<x\le\theta \\ 1, & x>\theta \end{cases}$ $F_Y(y) = \begin{cases} 0, & y<0 \\ (\frac{y}{\theta})^3, & 0\le y\le\theta \\ 1, & y>\theta \end{cases}$ $f_Y(y) = \begin{cases} 3\theta^{-3}y^2, & 0\le y\le\theta \\ 0, & 其它 \end{cases}$

$EY = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_{0}^{\theta} 3\theta^{-3} y^3 dy = \frac{3}{4}\theta$

$E(\hat{\theta}) = \frac{4}{3}EY = \theta$ 故 $\hat{\theta}$ 是 θ 无偏估计

五、$E\hat{\mu} = (a_1 + \cdots + a_n)\mu = \mu \Rightarrow a_1 + \cdots + a_n = 1$

$D\hat{\mu} = (a_1^2 + \cdots + a_n^2)\sigma^2 \geq \frac{(a_1 + \cdots + a_n)^2}{n}\sigma^2 = \frac{1}{n}\sigma^2$

上式取等条件为 $a_1 = a_2 = \cdots = a_n$

又 $\because a_1 + \cdots + a_n = 1$

$\therefore a_i = \frac{1}{n}$ 时, $D\hat{\mu}$ 最小, $\hat{\mu}$ 最有效