

Solutions to Exercises from The Art of Computer Programming, by Donald Knuth

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1 Chapter 1

1.1 Solutions for section 1.1

1. Using a temp variable t the values can be rotated like this -
$$t \leftarrow a, a \leftarrow b, b \leftarrow c, c \leftarrow d, d \leftarrow t$$
2. At step [E3] r is assigned to n and n to m . As r is remainder of division of m by n , r should be $< n$. Hence $m < n$.
3. Below are the steps of modified algorithm [F] which takes m and n as input.
[F1] Divide m by n and let the remainder be r .
[F2] if $r = 0$ return n . Terminate
[F3] Invoke [F] with n, r as input and return result.
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5. From the procedure reading the book following properties are missing which means it's not a proper algorithm.
 - Finiteness is missing - the whole procedure goes in a loop and does not actually terminate.
 - Output is missing - the procedure does not a definite output.

- Effectiveness is missing - The steps cannot be done on pencil/paper or a real computer realistically.
Comparison with [E]: [E] terminates after finite number of steps and is effective(steps can be performed on pen and paper and has definite output.

6. The answer should be close to 3. I dint calculate the exact value though.
7. $T_m + 1 = U_m$

1.2 Mathematics Preliminaries

1.2.1 Solutions for section 1.2.1

1. Prove that $P(0)$ is true. Prove that Given $P(0) \dots P(n)$ is true than $P(n+1)$ is also true for $n \geq 0$
2. In the second step its being assumed that $\frac{a^{n-1}}{a^{(n-1)-1}}$ is a then in very next step everything is replaced by 1. This essentially means $a = 1$ all the time which is incorrect.
3. For $n = 1$ only the right hand side of the equation is being verified while on left hand side divide by zero operation occurs which is invalid.
4. for $n = 1$ $F_n = 1 \geq \phi^{-1}$ (which is 0.618) We may assume by induction that $F_n \geq \phi^{(n-2)}$
Now, $F_{n+1} = F_{n-1} + F_n \geq (\phi^{n-3} + \phi^{n-2}) = \phi^{n-2}(\frac{1}{\phi} + 1) = \phi^{n-2} \times \phi = \phi^{n-1}$
Hence $F_{n+1} \geq \phi^{n-1}$ for all $n \geq 1$
5. We may assume by induction that all the numbers till n are either primes or product of primes. Now for $n + 1$ the number can either be prime or in form $x \times y$. As $x < (n + 1), y < (n + 1)$ they themselves must be prime or product of primes. Hence $n + 1$ takes the required form.
6. The second equation gets transformed into first one once the swap of values happens in $E4.(c \leftarrow d, a' \leftarrow a, b' \leftarrow b)$. Hence it holds true.
The first equation is -

$$am + bn = d$$

Substituting values this becomes

$$(a' - qa)m + (b' - qb)n = r$$

$$a' + b' - q(am + bn)$$

$$c - qd = r \text{ (from A6).}$$

$$qd - qd + r = r$$

7. Given

$$1^2 = 1$$

$$2^2 - 1^2 = 3$$

$$3^2 - 2^2 + 1^2 = 6$$

$$4^2 - 3^2 + 2^2 - 1^2 = 10$$

$$5^2 - 4^2 + 3^2 - 2^2 + 1^2 = 15$$

The formulation for this problem will be -

$$n^2(-1)^0 + (n-1)^2(-1)^1 + (n-2)^2(-1)^2 \dots + 1(-1)^{n-1} = \frac{n(n+1)}{2}$$

$P(1)$ is valid.

We may assume by induction that this is valid for n . For $(n+1)^{th}$ term the series will be -

$$(n+1)^2(-1)^0 - 1 \times n^{th} series = (n+1)^2 - \frac{n(n+1)}{2} = \frac{2(n+1)^2 - n(n+1)}{2} = \frac{(n+1) \times (n+2)}{2}$$

Hence it takes the required form and proved.

8. Given -

$$1^3 = 1$$

$$2^3 = 3 + 5$$

$$3^3 = 7 + 9 + 11$$

$$4^3 = 13 + 15 + 17 + 19$$

The formulation will be -

$$n^3 = n(n-1) + 1 + n(n-1) + 3 + \dots n(n-1) + (2n-1) \text{ till } n^{th} \text{ term}$$

Above is true for 1 hence P(1) is true.

We may assume by induction that this is valid for n

For $n+1$ $(n+1)^3 = (n+1)(n) + 1 + (n+1)(n) + 3 \dots (n+1)n + (2n+1)$
till $(n+1)^{th}$ term

$$= (n+1)(n+1)(n)1 + 3 + \dots + 2n + 1$$

$$= n(n+1)^2 + (n+1)^2 \text{ By Eq (2)}$$

$$= (n+1)^3$$

Hence proved.

For b -

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$$

P(1) is true.

We may assume by induction that this is valid for n .

For $(n+1)$ -

$$1^3 + 2^3 + 3^3 + \dots + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{(n+1)^2 n^2}{4} + (n+1)^3$$

$$= \frac{(n+1)^2 (n^2 + 4n + 4)}{4}$$

$$= \frac{(n+1)^2 (n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

9. Inequality to prove is — if $0 < a < 1$, then $(1-a)^n \geq 1-na$

The base case for $n = 1$ is true as $(1-a)^1 \geq (1-1 \times a)$

We may assume by induction that this is valid for n

Now for $n+1$ the inequality we need to prove is -

$$(1-a)^{(n+1)} \geq (1-(n+1)a)$$

$$\text{which is } (1-a)^n \times (1-a) \geq 1-a-na$$

dividing both sides by $(1-a)$

$$(1-a)^n \geq 1 - \frac{na}{1-a}$$

The above inequality is true because $(1-a)^n \geq (1-na)$ and $1-na \geq 1 - \frac{na}{1-a}$ as $0 < a < 1$ Hence the proof.

10. To prove that if $n \geq 10$, then $2^n > n^3$

The base case is true, as $2^{10} = 1024 > 10^3 = 1000$ We may assume by induction that this is valid for n

For $n+1$ the inequality will be $2^{n+1} > (n+1)^3$

INCOMPLETE

1.2.2 Solutions for section 1.2.2

1. There is no smallest rational number.
2. $1 + 0.239999999 \dots$ is not a decimal expansion as it ends with infinite 9s sequence.
3. Applying eq 4 $(-3)^{-3} = \frac{(-3)^{-2}}{-3} = \frac{(-3)^{-1}}{-3 \times -3} = \frac{(-3)^0}{-3 \times -3 \times -3} = -\frac{1}{27}$
4. By eq 4 and 6 $(0.125)^{-\frac{2}{3}} = \sqrt[3]{0.125^{-2}} = \sqrt[3]{\frac{1}{0.125} \times \frac{1}{0.125}} = \sqrt[3]{10^6/5^6} = \frac{100}{25} = 4$
5. Decimal expansion -
 $n + 0.d_1d_2 \dots$
 Binary expansion - convert each digit to binary form of 0 and 1 once you have decimal expansion for a real number.
 $p_1p_2 \dots + 0.q_1q_2 \dots$ where $p_i, q_i \in \{0, 1\}$
6. $x = m + 0.d_1d_2d_3 \dots$ $y = n + 0.e_1e_2e_3 \dots$
 $x = y$ if $m = n$ and $d_i = e_i$ for all i
 $x < y$ if $m < n$ or if $m = n$ and $d_i < e_i$ for some i such that $d_1d_2d_3 \dots d_{i-1} = e_1e_2e_3 \dots e_{i-1}$
 $x > y$ if $m > n$ or if $m = n$ and $d_i > e_i$ for some i such that $d_1d_2d_3 \dots d_{i-1} = e_1e_2e_3 \dots e_{i-1}$

