Solutions to Exercises from The Art of Computer Programming, by Donald Knuth

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1 Chapter 1

1.1 Solutions for section1.1

1. Using a temp variable t the values can be rotated like this -

   $t \leftarrow a, a \leftarrow b, b \leftarrow c, c \leftarrow d, d \leftarrow t$

2. At step [**E3**] r is assigned to n and n to m. As r is reminder of division of m by n, r should be $< n$. Hence $m < n$.

3. Below are the steps of modified algorithm [**F**] which takes m and n as input.

   [**F1**] Divide m by n and let the reminder by r.

   [**F2**] if $r = 0$ return n. Terminate

   [**F3**] Invoke [**F**] with n, r as input and return result.

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5. From the procedure reading the book following properties are missing which means it's not a proper algorithm.

   - Finiteness is missing - the whole procedure goes in a loop and does not actually terminate.
   - Output is missing - the procedure does not a definite output.

- Effectiveness is missing - The steps cannot be done on pencil/paper or a real computer realistically.

  Comparison with **[E]**: **[E]** terminates after finite number of steps and is effective(steps can be performed on pen and paper and has definite output.

6. The answer should be close to 3. I dint calculate the exact value though.

7. $T_m + 1 = U_m$

1.2 Mathematics Preliminaries

1.2.1 Solutions for section 1.2.1

1. Prove that $P(0)$ is true. Prove that Given $P(0) \dots P(n)$ is true than $P(n+1)$ is also true for $n \geq 0$

2. In the second step its being assumed that $\frac{a^{n-1}}{a^{(n-1)-1}}$ is a then in very next step everything is replaced by 1. This essentially means $a = 1$ all the time which is incorrect.

3. For $n = 1$ only the right hand side of the equation is being verified while on left hand side divide by zero operation occurs which is invalid.

4. for $n = 1$ $F_n = 1 \geq \phi^{-1}$ (which is 0.618) We may assume by induction that $F_n \geq \phi^{(n-2)}$

   Now, $F_{n+1} = F_{n-1} + F_n \geq (\phi^{n-3} + \phi^{n-2}) = \phi^{n-2}(\frac{1}{\phi} + 1) = \phi^{n-2} \times \phi = \phi^{n-1}$

   Hence $F_{n+1} \geq \phi^{n-1}$ for all $n \geq 1$

5. We may assume by induction that all the numbers till n are either primes or product of primes. Now for n + 1 the number can either be prime or in form $x \times y$. As $x < (n+1), y < (n+1)$ they themselves must be prime or product of primes. Hence $n + 1$ takes the required form.

6. The second equation gets transformed into first one once the swap of values happens in *E4*.($c \leftarrow d, a' \leftarrow a, b' \leftarrow b$). Hence it holds true.

   The first equation is -

$am + bn = d$

Substituting values this becomes

$(a' - qa)m + (b' - qb)n = r$

$a' + b' - q(am + bn)$

$c - qd = r$ (from A6).

$qd - qd + r = r$

7. Given

$$1^2 = 1$$
$$2^2 - 1^2 = 3$$
$$3^2 - 2^2 + 1^2 = 6$$
$$4^2 - 3^2 + 2^2 - 1^2 = 10$$
$$5^2 - 4^2 + 3^2 - 2^2 + 1^2 = 15$$

The formulation for this problem will be -

$n^2(-1)^0 + (n-1)^2(-1)^1 + (n-2)^2(-1)^2 \ldots + 1(-1)^{n-1} = \frac{n(n+1)}{2}$

$P(1)$ is valid.

We may assume by induction that this is valid for n. For $(n+1)^{th}$ term the series will be -

$(n+1)^2(-1)^0 - 1 \times n^{th} series = (n+1)^2 - \frac{n(n+1)}{2} = \frac{2(n+1)^2 - n(n+1)}{2} = \frac{(n+1)\times(n+2)}{2}$

Hence it takes the required form and proved.

8. Given -

$$1^3 = 1$$
$$2^3 = 3 + 5$$
$$3^3 = 7 + 9 + 11$$
$$4^3 = 13 + 15 + 17 + 19$$

The formulation will be -

$n^3 = n(n-1)+1+n(n-1)+3+\ldots n(n-1)+(2n-1)$ till n^{th} term

Above is true for 1 hence P(1) is true.

We may assume by induction that this is valid for n

For $n+1$ $(n+1)^3 = (n+1)(n)+1+(n+1)(n)+3\ldots(n+1)n+(2n+1)$ till $(n+1)^{th}$ term

$= (n+1)(n+1)(n)1+3+\ldots+2n+1$

$= n(n+1)^2+(n+1)^2$ By *Eq (2)*

$= (n+1)^3$

Hence proved.

For b -

$1^3+2^3+3^3+\ldots+n^3 = (1+2+\ldots+n)^2$

P(1) is true.

We may assume by induction that this is valid for n.

For (n+1) -

$1^3+2^3+3^3+...+(n+1)^3 = (\frac{n(n+1)}{2})^2+(n+1)^3$

$= \frac{(n+1)^2n^2}{4}+(n+1)^3$

$= \frac{(n+1)^2(n^2+4n+4)}{4}$

$= \frac{(n+1)^2(n+2)^2}{4}$

$= (\frac{(n+1)(n+2)}{2})^2$

9. Inequality to prove is — if $0 < a < 1$, then $(1-a)^n \geq 1-na$

   The base case for $n=1$ is true as $(1-a)^1 \geq (1-1\times a)$

   We may assume by induction that this is valid for n

   Now for $n+1$ the inequality we need to prove is -

$$(1-a)^{(n+1)} \geq (1-(n+1)a)$$

$$\text{which is } (1-a)^n \times (1-a) \geq 1-a-na$$

$$\text{dividing both sides by } (1-a)$$

$$(1-a)^n \geq 1 - \frac{na}{1-a}$$

The above inequality is true because $(1-a)^n \geq (1-na)$ and $1 - na \geq 1 - \frac{na}{1-a}$ as $0 < a < 1$ Hence the proof.

10. To prove that if $n \geq 10$, then $2^n > n^3$

    The base case is true, as $2^{10} = 1024 > 10^3 = 1000$ We may assume by induction that this is valid for n

    For $n+1$ the inequality will be $2^{n+1} > (n+1)^3$

    INCOMPLETE

1.2.2 Solutions for section 1.2.2

1. There is no smallest rational number.

2. $1 + 0.239999999\ldots$ is not a decimal expansion as it ends with infinite 9s sequence.

3. Applying *eq 4* $(-3)^{-3} = \frac{(-3)^{-2}}{-3} = \frac{(-3)^{-1}}{-3\times-3} = \frac{(-3)^0}{-3\times-3\times-3} = -\frac{1}{27}$

4. By *eq 4 and 6* $(0.125)^{-\frac{2}{3}} = \sqrt[3]{0.125^{-2}} = \sqrt[3]{\frac{1}{0.125} \times \frac{1}{0.125}} = \sqrt[3]{10^6/5^6} = \frac{100}{25} = 4$

5. Decimal expansion -

   $n + 0.d_1d_2\ldots$

   Binary expansion - convert each digit to binary form of 0 and 1 once you have decimal expansion for a real number.

   $p_1p_2\ldots + 0.q_1q_2\ldots$ where $p_i, q_i \in \{0, 1\}$

6. $x = m + 0.d_1d_2d_3\ldots$ $y = n + 0.e_1e_2e_3\ldots$

   $x = y$ if $m = n$ and $d_i = e_i$ for all i

   $x < y$ if $m < n$ or if $m = n$ and $d_i < e_i$ for some i such that $d_1d_2d_3\ldots d_{i-1} = e_1e_2e_3\ldots e_{i-1}$

   $x > y$ if $m > n$ or if $m = n$ and $d_i > e_i$ for some i such that $d_1d_2d_3\ldots d_{i-1} = e_1e_2e_3\ldots e_{i-1}$

7. To Prove that $b^{x+y} = b^x b^y$

   For $x = 1$ its trivial that $b^{1+y} = bb^y = b^1 b^y$ (from *Eq 4*)

   We can assume by induction that $b^{x+y} = b^x b^y$

   Now $b^{x+y+1} = (b^x b^y)b = (b^x b^y)b^1$ Hence it takes the same form and proved.

   To Prove $(b^x)^y = b^{xy}$

   Its straightforward to show that for 1 $(b^1)^y = b^{1y} = b^y$

   By induction we may assume that $(b^x)^y = b^{xy}$

   Now for $(b^{x+1})^y = (bb^x)^y = b^y b^{xy}$ INCOMPLETE