Sums and Products

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Sums

Sum of $a_1, a_2 \ldots a_n$ is written as

$$\sum_{j=1}^{n} a_j$$

or

$$\sum_{1 \le j \le n} a_j$$

j is *dummy* or *index* variable introduced just for the purpose of notation.
Context is important - *e.g.*

$$\sum_{j \le k} \binom{j+k}{2j-k}$$

In the above case only if either i or j (not both) have exterior significance.
Finite/Infinte summation -
Example of finite summation —

$$\sum_{1 \le j < 10} a_j$$

Example of infinite summation —

$$\sum_{j \ge 1} a_j = \sum_{j=1}^{\infty} a_j$$

Divergent and convergent summation -

$$\sum_{R(j)} a_j = \left(\lim_{n\to\infty} \sum_{\substack{R(j)\\ 0\le j<n}} a_j \right) + \left(\lim_{n\to\infty} \sum_{\substack{R(j)\\ -n\le j<0}} a_j \right)$$

If the above limits exits the summation is called *convergent* otherwise its a *divergent* summation.

Four algebraic operations on summations —

1. *The distributive law*, for product of sums

$$\left(\sum_{R(i)} a_i\right)\left(\sum_{S(j)} b_j\right) = \sum_{R(i)}\left(\sum_{S(j)} a_i b_j\right) = \sum_{R(i)}\sum_{S(j)} a_{ij}$$

2. *The Change of variable*

$$\sum_{R(i)} a_i = \sum_{R(j)} a_j = \sum_{R(p(j))} a_{p(j)}$$

$$\sum_{1\le j\le n} a_j = \sum_{1\le j-1\le n} a_{j-1} = \sum_{2\le j\le n+1} a_{j-1}$$

3. *Interchanging order of summation*

$$\sum_{R(i)}\sum_{S(j)} a_{ij} = \sum_{S(j)}\sum_{R(i)} a_{ij}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} = \sum_{j=1}^{n}\sum_{i=j}^{n} a_{ij}$$

4. *Manipulating the domain*

$$\sum_{R(j)} a_j + \sum_{S(j)} a_j = \sum_{R(j) or S(j)} a_j + \sum_{R(j) and S(j)} a_j$$

Bracket Notation and *Kronecker delta*

$$[\text{statement}] = \begin{cases} 1 & \text{if the statement is true} \\ 0 & \text{if the statement is false.} \end{cases}$$

$$\delta_{ij} = [i=j] = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\ne j \end{cases}$$

Products

Default value of Product is 1 not 0

$$\prod_{R(j)} a_j \tag{1}$$

Notes from Concrete Math

Delimited form of summation

$$\sum_{k=1}^{n} a_k \tag{2}$$

(2) is delemited form of summation. In Above each a_k is a term. Incidentally the quantity after $\sum$ (here a_k) is called *summand*. The index variable k is said to be *bound* to $\sum$ sign because k in a_k is unrelated to appearances of k outside Sigma-notation.

Generalized Form of summation It turns of that a generazlized form of summation is even more useful than the delimited form: We simply write one or more conditions under the $\sum$ to specify the set of indices over which summation should take place.Example

$$\sum_{\substack{1 \leq k \leq n \\ k odd}} a_k \tag{3}$$

Efficiency of computation is not same as efficiency of understanding.