ECE 236A Shengze Ye HW1

problem 1

$$\min \quad c_1x_1 + c_2x_2 + c_3x_3$$
$$\text{s.t.} \quad x_1 + x_2 \geq 1$$
$$x_1 + 2x_2 \leq 3$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

① when $c = (-1, 0, 1)$

$$\Rightarrow \quad \min \quad -x_1 + x_3$$
$$\text{st} \quad x_1 + x_2 \geq 1$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2, x_3 \geq 0$$

$x_3 \geq 0$, and no other constraints on x_3

$\therefore x_3 = 0$

want $-x_1$ as small as possible $\Rightarrow x_1$ as large as possible

$x_1 + 2x_2 \leq 3 \quad \therefore x_1 = 1,\ x_2 = 0$

$\therefore \quad x^* = (3, 0, 0) \quad p^* = -3$

② when $c = (0, 1, 0)$

$$\Rightarrow \quad \min \quad x_2$$
$$\text{st} \quad x_1 + x_2 \geq 1$$
$$x_1 + 2x_2 \leq 3$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

want minimize x_2 and $x_2 \geq 0$, $\Rightarrow x_2 = 0$

Then $(x_1 \geq 1 \quad x_1 + 0 \leq 3) \Rightarrow 1 \leq x_1 \leq 3$

$\therefore \quad X_{optimal} = \{x \in \mathbb{R}^3 \mid x_2 = 0,\ x_3 \geq 0,\ 1 \leq x_1 \leq 3\}$

$p^* = 0$

(3) when $c = (0, 0, -1)$

$$\Rightarrow \min \ -x_3$$

$$\text{s.t.} \quad x_1 + x_2 \geq 1$$

$$x_1 + 2x_2 \leq 3$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

x_3 should be as large as possible

so there is no x^*.

since if we find $x^* = (x_1^*, x_2^*, x_3^*)$

we can always find $x_3' > x_3^*$ that has larger value

$\therefore \quad X_{optimal} = \phi$

$p^* = -\infty$ (unbounded)

problem 2

A young investor is planning to invest in the stocks of Metalco company during the week.

operation	Monday	Tuesday	wednesday	Thursday	friday
Buy	3	1.7	2.4	2.5	1.8
Sell	2.1	2.4	3	2	3.1

Let x_i $(i=1,2,3,4,5)$ be the shares that the investor will buy for ith day

Let y_i $(i=1,2,3,4,5)$ be the shares that the investors will sell for ith day

The LP can be described as

$$\boxed{\max_{\substack{x_i\\ y_i}} \quad i=1,2,3\cdots5 \quad 120+2.1y_1+2.4y_2+3y_3+2y_4+3.1y_5 -3x_1-1.7x_2-2.4x_3-2.5x_4-1.8x_5}$$

objective function

s.t. $\quad x_i \geq 0 \quad i=1,2,3,4,5$

$\quad y_i \geq 0 \quad i=1,2,3,4,5$

(satisfying constraint ③, ④)

$$3x_1 \leq 120$$

$$1.7x_2 \leq 120-3x_1+2.1y_1$$

$$2.4x_3 \leq 120-3x_1+2.1y_1-1.7x_2+2.4y_2$$

$$2.5x_4 \leq 120-3x_1+2.1y_1-1.7x_2+2.4y_2-2.4x_3+3y_3$$

$$x_5 \leq 0$$

constraints

(satisfying constraint ①)

$$y_1 \leq 0$$

$$y_2 \le x_1$$
$$y_3 \le x_1 + x_2 - y_2$$
$$y_4 \le x_1 + x_2 + x_3 - y_2 - y_3$$
$$y_5 \le x_1 + x_2 + x_3 + x_4 - y_2 - y_3 - y_4$$

Problem 3.

An advertising agency has M products to advertise at N locations

advertising time $x_{i,j}$ product i being advertised at location j

cost : c_j

multiplier effect : $u_{i,j}$

advertising budget for i product : b_i

total advertising time for j location : t_j

formulate the LP problem

$$\max \sum_{i=1}^{M} \sum_{j=1}^{N} x_{i,j} \cdot u_{i,j} - \sum_{i=1}^{M} \sum_{j=1}^{N} x_{i,j} \cdot c_j$$

$$\text{s.t.} \quad \sum_{j=1}^{N} c_j x_{i,j} \le b_i \qquad i = 1, 2, \cdots, M$$

$$\sum_{i=1}^{M} x_{i,j} \le t_j \qquad j = 1, 2, \cdots, N$$

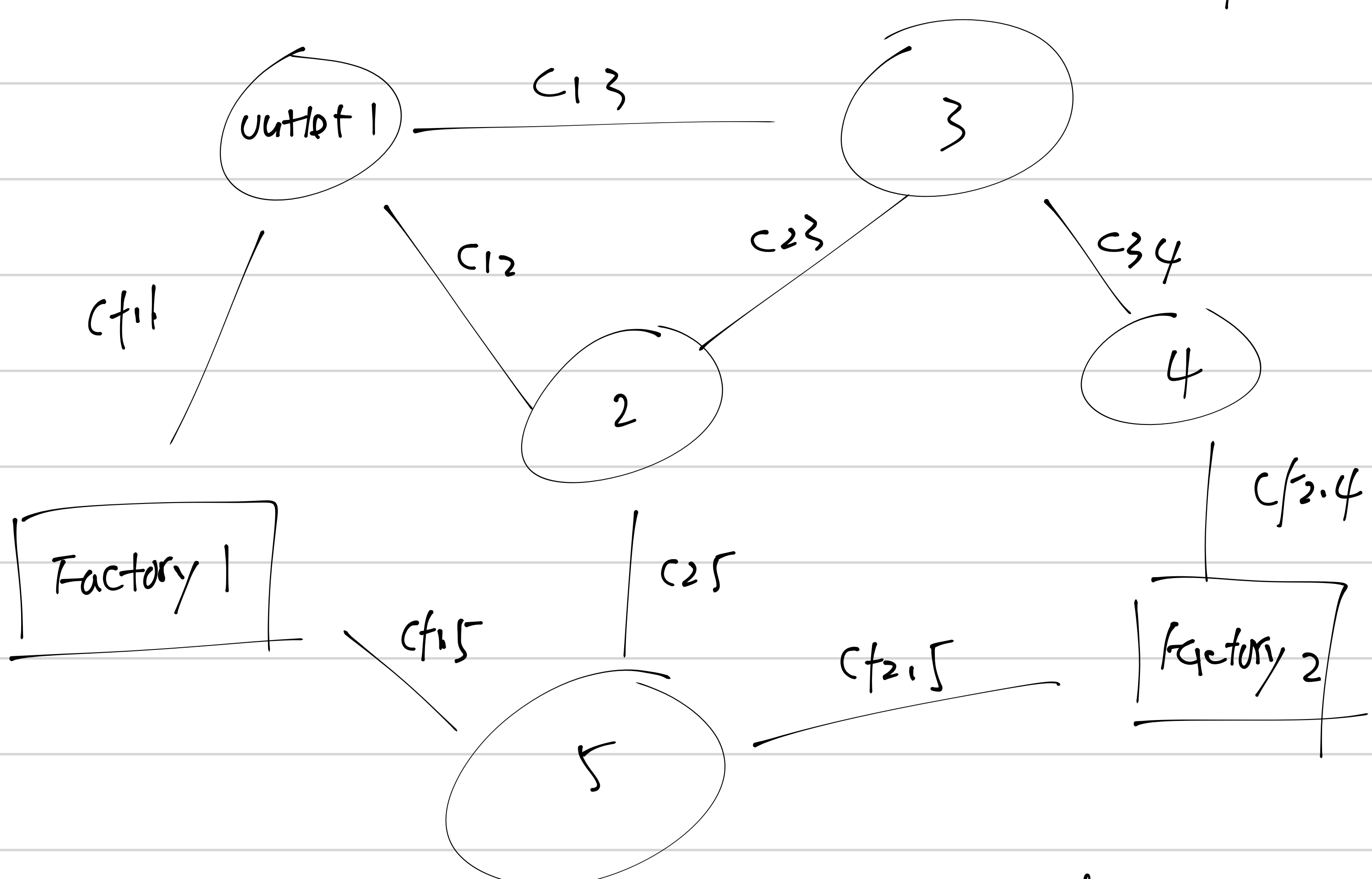
$$x_{i,j} \ge 0 \qquad i = 1, \cdots, M, \quad j = 1, 2, \cdots, N$$

first constraint describe the cost for each product cannot exceed bi

second constraint describe that time for each location

Problem 4

Write an LP that will enable Solodrex to minimize the cost needed to meet the new market requirement



	current (Lb)	Needed (Lb)
Outlet 1	1200	2450
2	1800	1100
3	1200	1600
4	1500	3300
5	1100	2100

1. variables

let $\boxed{a_1, a_2}$ be the amount of cheese produced at factory 1 and 2

Let $\boxed{X_{i,j}}$ be the amount of cheese moved from node i to j. i and j cound be

$i, j \in \{1, 2, 3, 4, 5, f_1, f_2\}$

p_1, p_2 are constants ($ per lb)

2. objective function

we want the cost (production + transportant) to be minimized

$$\boxed{\therefore \min_{\substack{a_1, a_2 \\ X_{i,j} \\ i,j \in \{1,2,\cdots,5,f_1,f_2\}}} a_1 p_1 + a_2 p_2 + \sum_{i,j \in \{1,\cdots,5,f_1,f_2\}} c_{i,j} (X_{ij} + X_{ji})}$$

3. constraints

$$X_{i,j} \geq 0 \quad i, j \in \{1, 2, 3, 4, 5, f_1, f_2\}$$

$$a_1 \geq 0 \quad , a_2 \geq 0$$

① The total amount of cheese should meet the demand

$$a_1 + a_2 \geq (2450 + 1100 + 1600 + 3300 + 2100) - (1200 + 1800 + 1200 + 1500 + 1100)$$

② At each outlet, the difference amount between incoming and outcoming should be greater than demand and cost

$$X_{f_1,1} + X_{31} + X_{21} - (X_{f_1,f_1} + X_{13} + X_{12}) \geq 2450 - 1200$$

$$X_{12} + X_{32} + X_{52} - (X_{21} + X_{23} + X_{25}) \geq 1100 - 1800$$

$$X_{13} + X_{23} + X_{43} - (X_{31} + X_{32} + X_{34}) \geq 1600 - 1200$$

$$X_{34} + X_{f_2 4} - (X_{43} + X_{4f_2}) \geq 3300 - 1500$$

$$X_{f_1 5} + X_{f_2 5} + X_{25} - (X_{5f_1} + X_{5f_2} + X_{52}) \geq 2100 - 1100$$

(3) For factories the outcoming and incoming amount' difference should be smaller than the production at each factory

$$X_{f_1 1} + X_{f_1 5} - (X_{1f_1} + X_{5f_1}) \leq a_1$$

$$X_{f_2 4} + X_{f_2 5} - (X_{5f_2} + X_{4f_2}) \leq a_2$$

Problem 5

we are given p matrices $A_i \in \mathbb{R}^{m \times n}$, to find $X \in \mathbb{R}^{n \times m}$ that $A_i X \simeq I \quad i = 1, \dots, p$

$$\text{minimize} \quad \max_{i=1,\dots,p} \|I - A_i X\|_\infty$$

$$\|H\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |H_{ij}|$$

express the problem as LP

Let $t = \max_{i=1,\cdots p} \| I - A_i X \|_\infty$

$$\Rightarrow \quad \min \; t$$
$$\text{s.t.} \quad \| I - A_i X \|_\infty \le t$$
$$\text{for } i=1, \cdots p$$

Then we can introduce $B \in \mathbb{R}^{n \times n}$

$$|(I - A_i X)_{ij}| \le (B_i)_{ij}$$

and $\sum_{j=1}^{n} (B_i)_{ij} \le t$

$$\therefore \Rightarrow \quad \min \; t$$
$$\text{s.t} \quad -B_k \le I - A_k X \le B_k \quad k=1,2, \cdots p$$
$$\sum_{j=1}^{n} (B_k)_{ij} \le t$$

Problem 6. For each LPs, express the optimal value an optimal solution in terms of problem parameter (c, k)

(i) $$\min \; c^T X$$
$$\text{s.t} \quad -1 \le X \le 1$$

$$\Rightarrow \quad X_i^* = \begin{cases} -1 & c_i > 0 \\ 1 & c_i \le 0 \end{cases}$$

$$p^* = -\sum_{i=1}^{n} |c_i| = -\|c\|_1$$

(ii) maximize $c^T x$

s.t $1^T x = k$ $\quad x \in \mathbb{R}^n$

$0 \le x \le 1$ $\quad 1 \le k \le n$ $\quad k$ is integer

$\Rightarrow$ we need to allocat "1's" to largest c_i

assume that vector $\vec{c}$ is sorted

which means $c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_n$

we want larger c_i as many as possible

$\therefore p^* = c_1 + c_2 + \cdots + c_k$

$x_1^* = \cdots = x_k^* = 1$

$x_{k+1}^* = \cdots = x_n^* = 0$

The optimal value is the largest k components of c

(iii) maximize $c^T x$

s.t. $1^T x \le k$ $\quad x \in \mathbb{R}^n$

$0 \le x \le 1$ $\quad 1 \le k \le n$ $\quad k$ is integer

The optimal value should be the largest k positive components of c

Similar to (ii), assume that the components of c are sorted

$c_1 \geq c_2 \geq \cdots \geq c_n$

if $c_1 \leq 0$ then $x^* = 0$ $p^* = 0$

if $c_n \geq 0$ the $x^* = (1, 1, \cdots \underset{\uparrow \atop k\text{th}}{1}, 0, 0 \cdots 0)$ $p^* = c_1 + c_2 + \cdots + c_k$

else let $c_i \geq 0 \geq c_{i+1}$

if $i \geq k$

$x = (1, 1, \cdots \overset{k\text{th}}{1}, 0, 0, \cdots 0)$ $p^* = c_1 + \cdots + c_k$

if $i < k$

$x = (1, 1, \cdots \overset{i\text{th}}{1}, 0, \cdots 0)$ $p^* = c_1 + \cdots + c_i$

(iv) maximize $c^T x$

s.t $d^T x = \alpha$

$0 \leq x \leq 1$

$x \in \mathbb{R}$ α and the components of d are positive

let $y_i = d_i x_i$

$\Rightarrow$ min $\sum_{i=1}^{n} c_i \cdot x_i = \sum_{i=1}^{n} \frac{c_i}{d_i} \cdot y_i$

s.t $1^T y = \alpha$

$0 \leq y_i \leq d_i$ $i = 1, 2, \cdots n$

Then we assume that c_i / d_i is sorted

$$\frac{c_1}{d_1} \geq \frac{c_2}{d_2} \geq \cdots \geq \frac{c_n}{d_n}$$

let k_{max} denote the largest value such that $\sum_{i=1}^{k} d_i \le \alpha$

$$\Leftrightarrow \quad d_1 + d_2 + \cdots + d_{k_{max}} \le \alpha$$

$$d_1 + d_2 + \cdots + d_{k_{max}+1} \ge \alpha$$

$$\therefore \; y_i^* = \begin{cases} d_i & \text{when } i \le k_{max} \\ \alpha - \sum_{j=1}^{k_{max}} d_j & \text{when } i = k_{max} + 1 \\ 0 & \text{else} \end{cases}$$

$$\because \; y_i = d_i \cdot x_i \qquad \therefore \; x_i^* = \frac{y_i^*}{d_i}$$

$$x_i^* = \begin{cases} 1 & i \le k_{max} \\ \left(\alpha - \sum_{j=1}^{k_{max}} d_j\right) / d_i & i = k_{max} + 1 \\ 0 & \text{else} \end{cases}$$

$$\therefore \; \underset{\uparrow}{p^*} = c_1 + c_2 + \cdots + c_{k_{max}} + \left[c_{k_{max}+1} \cdot \frac{\left(\alpha - \sum_{j=1}^{k_{max}} d_j\right)}{d_i} \right]$$

The optimal value

Problem 7.

Formulate the following problems as LPs

(a) minimize $\|Ax-b\|_1$ st $\|x\|_\infty \le 1$

$\|Ax-b\|_1 = \sum_{j=1}^{m} | a_j^T x - b_j |$ $A \in \mathbb{R}^{m\times n}$ $b \in \mathbb{R}^m$

for $j = 1, 2, \cdots, m$ $x \in \mathbb{R}^n$

$= \sum_{i=1}^{n} \max \{ a_j^T x - b_j, -(a_j^T x - b_j) \}$

let $t_i = \max\{ a_i^T x - b_i, -(a_i^T x - b_i) \}$

$\therefore$ formulated as LP : $t_i \in \mathbb{R}^m$

$\Rightarrow$ min $t_1 + t_2 + \cdots + t_m$

s.t $-t_i \le a_i x^T - b_i \le t_i$ for $i = 1, 2, \cdots m$

and $-1 \le x_i \le 1$ for $i = 1, 2, \cdots n$

(b) minimize $\|x\|_1$ s.t. $\|Ax-b\|_\infty \le 1$

similar to (a)

it can be formulated as

min $t_1 + t_2 + \cdots + t_m$ $t_i \in \mathbb{R}^n$

s.t. $-t_i \le x_i \le t_i$ $i = 1, 2, \cdots n$

and $-1 \le a_i^T x - b_i \le 1$ for $i = 1, 2, \cdots m$

if we express it as matrices

$\Rightarrow$ $\min \ 1^T t$

s.t. $-t \le x \le t$

$-1 \le Ax - b \le 1$

(c) minimize $\|Ax-b\|_1 + \|x\|_\infty$

we can introduce $y \in \mathbb{R}^m$ and $t \in \mathbb{R}$

$\Rightarrow$ $\min \ y_1 + y_2 + \cdots + y_m + t$

s.t. $-y_i \le a_i^T x - b_i \le y_i \quad i=1,2,\cdots,m$

$-t \le x_i \le t \quad i=1,2,\cdots n$

if we express it in matrix form

$\Rightarrow$ $\min \ 1^T y + t$

s.t $-y \le Ax - b \le y$

$-t\vec{1} \le x \le t\vec{1}$

problem 8

Formulate the following problems as LPs

(a) Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{m} \max\{0, a_i^T x + b_i\}$$

let $t_i = \max\{0, a_i^T x + b_i\}$

then $\Rightarrow$ $\min\ t_1 + t_2 + \cdots t_m$

s.t. $a_i^T x + b_i \le t_i \qquad i = 1, 2, \cdots m$

$t_i \geq 0$

(b) Given $p+1$ matrices $A_0, A_1, \ldots A_p \in \mathbb{R}^{m\times n}$, find $x \in \mathbb{R}^p$ that

$$\min \left(\max_{\|y\|_1 = 1} \|(A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_p A_p) y\|_1 \right)$$

we use $\max_{\|y\|_1 = 1} \|Ay\|_1 = \max_{j=1,\cdots n} \sum_{i=1,\cdots m} |A_{ij}|$

Then we can formulate the problem as

$$\min \max_{j=1,\cdots n} \sum_{i=1}^{m} |(A_0 + A_1 x_1 + \cdots A_p x_p)_{ij}|$$

let $t = \max_{j=1\cdots n} \sum_{i=1}^{m} \left| (A_0 + A_1 x_1 + \cdots + A_p x_p)_{ij} \right|$

LP ↘

$\therefore \Rightarrow$ minimize t

s.t. $-s_{ij} \le (A_0 + A_1 x_1 + \cdots + A_p x_p)_{ij} \le s_{ij}$

$\sum_{i=1}^{m} s_{ij} \le t \quad j = 1, 2, \cdots n$

$x, t \in \mathbb{R} \quad S \in \mathbb{R}^{m \times n}$