HW2 ECE 236A Shengze Ye 205418959

problem 1

when does one half space contain another

$$\underbrace{\{x \mid a^T x \le b\}}_{H_1} \subseteq \underbrace{\{x \mid \tilde{a}^T x \le \tilde{b}\}}_{H_2}$$

(1)

if H_1 is contained in H_2, a and $\bar{a}$ should be parallel, otherwise we could find some vector x with $a^T x = 0$ and $\bar{a}^T x > 0$

Now, for any point w in H_1, we have

$$a^T(w + \lambda x) = a^T w + \underbrace{\lambda a^T x}_{=0} = a^T w \le b$$

but $\bar{a}^T(w + \lambda x) = \underbrace{\bar{a}^T w}_{\le b} + \underbrace{\lambda \bar{a}^T x}_{>0}$ could $\ge b$ when λ is very large, so a and $\bar{a}$ should be $\boxed{\text{parallel}}$

$\Rightarrow$ we can find som λ $\boxed{a = \lambda \bar{a}}$

if $H_1 \subseteq H_2$

$\forall h \in H_1$, $h \in H_2$

$\Rightarrow \lambda \bar{a}^T h \le b \qquad \bar{a}^T h \le \bar{b}$

$\Rightarrow b \le \lambda \bar{b}$

$\therefore H_1 \subseteq H_2$ if and only if we can find some $\lambda > 0$.

such that $a = \lambda \bar{a}$, $b \le \lambda \bar{b}$

(2) when are the half space equal?

$H_1 = H_2$

$\Rightarrow$ $H_1 \subseteq H_2$ and $H_2 \subseteq H_1$ by observation the above

$\therefore$ $H_1 = H_2$ if and only if we can find some $\lambda > 0$

such that $a = \lambda \bar{a}$, $b = \lambda \bar{b}$

problem 2

let $x_0, \cdots x_k \in \mathbb{R}^n$

$$V = \{ x \in \mathbb{R}^n \mid \|x - x_0\|_2 \le \|x - x_i\|_2 \quad i = 1, \cdots k \}$$

show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \le b\}$

$$\|x - x_0\|_2 \le \|x - x_i\|_2$$

$$\therefore \quad (x - x_0)^T (x - x_0) \le (x - x_i)^T (x - x_i)$$

$$\Rightarrow \quad x^T x - 2x_0^T x + x_0^T x_0 \le x^T x - 2x_i^T x + x_i^T x_i$$

$$\Rightarrow \quad 2x_i^T x - 2x_0^T x \le x_i^T x_i - x_0^T x_0$$

$$2(x_i - x_0)^T x \le x_i^T x_i - x_0^T x_0$$

Let

$$A = 2\begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_k - x_0 \end{bmatrix}^T \qquad b = \begin{bmatrix} x_1^T x_1 - x_0^T x_0 \\ x_2^T x_2 - x_0^T x_0 \\ \vdots \\ x_k^T x_k - x_0^T x_0 \end{bmatrix}$$

$\therefore$ V is a polyhedron with

$V = \{x \mid Ax \leq b\}$ A and b are the matrix above

problem 3

which of the following sets S are polyhedra?

(a) $S = \{ x \in \mathbb{R}^n \mid x \succeq 0,\ 1^T x = 1,\ \sum_{i=1}^{n} a_i x_i = b_1,\ \sum_{i=1}^{n} x_i a_i^2 = b_2 \}$

$a_i \in \mathbb{R},\ i = 1, \cdots n,\quad b_1, b_2 \in \mathbb{R}$ are given

S is a polyhedron,

the conditions can be expressed as inequalities

$-x_i \leq 0 \quad i = 1, \cdots, n$

$1^T x \leq 1$

$-1^T x \leq -1$

$\sum_i a_i x_i \leq b_1 \qquad -\sum_i a_i x_i \leq -b_1$

$\sum_i a_i^2 x_i \leq b_2 \qquad -\sum_i a_i^2 x_i \leq -b_2$

let $A = \begin{bmatrix} \left.\begin{matrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & 1 \end{matrix}\right\} \text{n row} \\ \begin{matrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ a_1 & a_2 & \cdots & a_n \\ -a_1 & -a_2 & \cdots & -a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ -a_1^2 & -a_2^2 & \cdots & -a_n^2 \end{matrix} \end{bmatrix}$ $b = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \text{n row} \\ 1 \\ -1 \\ b_1 \\ -b_1 \\ b_2 \\ -b_2 \end{bmatrix}$

$\therefore$ S is a polyhedron

$S = \{x \mid Ax \le b$

(b) $S = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le \|x - x_1\|\}$, x_0, x_1 are given,

Same with the problem 2

$\Leftrightarrow \|x - x_0\|^2 \le \|x - x_1\|^2$

$\Leftrightarrow \quad 2(x_1 - x_0)^T x \le x_1^T x_1 - x_0^T x_0$

$\therefore$ S is a polyhedron

Problem 4

Is $\bar{x} = (1,1,1,1)$ an extreme point of the polyhedron P defined by the linear inequalities

$$\begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The set of active constraints at $\tilde{x}$ is $J = \{1,2,3,4\}$

$\therefore$ we have to check the rank of matrix

$$A_J = \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix}$$

The rank is 4 $\therefore$ $\tilde{x} = (1,1,1,1)$ is indeed an extreme point

Let $c = -A_J^T \mathbf{1}$

$$= \begin{bmatrix} -1 & -1 & 0 & 6 \\ -6 & -2 & 3 & -11 \\ 1 & 7 & -10 & -2 \\ 3 & 1 & -1 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 4 \\ -15 \end{bmatrix}$$

with $c = [8, 16, 4, -15]^T$

$c^T \tilde{x} = -1^T A_J \tilde{x} = -\sum_i b_i$

$\because Ax \le b \quad \therefore -1^T A_J x \ge -\sum_i b_i$

$\therefore$ with this c, $\tilde{x}$ is the unique minimizer of $c^T x$ over P

Problem 5

Consider the polyhedron

$$P = \{ x \in \mathbb{R}^4 \mid Ax \le b,\ cx = d \}$$

$$A = \begin{bmatrix} -1 & -1 & -3 & -4 \\ -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix}$$

$c = [13 \quad 11 \quad 12 \quad 12] \qquad d = 48$

Prove $\hat{x} = (1,1,1,1)$ is an extreme point of P

$c^T\hat{x} = 13+11+12+12 = 48 \qquad c^T\hat{x} = d$

$$b - A\hat{x} = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix} - \begin{bmatrix} -1 & -1 & -3 & -4 \\ -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first one is not active, others are active

we then have to examine the rank of the matrix

$$\begin{bmatrix} -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \\ 13 & 11 & 12 & 12 \end{bmatrix}$$

rank is 4

$\therefore \hat{x}$ is an extreme point

Problem 6

$$P = \{x \in \mathbb{R}^5 \mid Ax \le b, -1 \le x \le 1\}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) $\hat{x} = (1, -\frac{1}{2}, 0, -\frac{1}{2}, -1)$

$-1 \le x \le 1$

$\therefore$ check $\begin{bmatrix} A \\ I \\ -I \end{bmatrix} \hat{x} \le \begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix}$ $A_{J(\hat{x})} = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

rank $[A_{J(\hat{x})}] \neq 5$

$\therefore$ not a extreme point

(b) $\hat{x} = (0, 0, 1, 0, 0)$

check $\begin{bmatrix} A \\ I \\ -I \end{bmatrix} \hat{x} \le \begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix}$ $A_{J(\hat{x})} = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

rank $[A_{J(\hat{x})}] \neq 5$

$\therefore \Rightarrow$ not a extreme point

(c) $\hat{x} = (0, 1, 1, -1, 0)$

Also check $\begin{bmatrix} A \\ I \\ -I \end{bmatrix} \hat{x} \le \begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix}$

$$A_J(\hat{x}) = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad \text{rank}(A_J(\hat{x})) = 5$$

$\therefore$ $\hat{x}$ is an extreme point

we can take

$$c = A^T 1 - A^T 1 + e_2 + e_3 - e_4 = (0, 1, 1, -1, 0)$$

The $c^T x$ is maximized with

$$x = \hat{x}$$

Problem 7

Formulate the following problem as an LP.

maximize R

s.t. $B(x_c, R) \subseteq P$

$B(x_c, R) = \{x \mid \|x - x_c\| \le R\}$

$P = \{x \mid a_i x^T \le b_i, \ i = 1, \dots m\}$

P is a polyhedron with m inequalities.

$a_i^T x \le b_i$

x in B could be express as $x_c + y \cdot R \quad \|y\| \le 1$

$\therefore \quad B(x_c, R) \subseteq P$

$\Rightarrow \quad a_i^T (x_c + yR) \le b_i \quad$ for $i = 1, 2, \cdots, m$

$\Rightarrow \quad a_i^T x_c + R \cdot \max_{\|y\| \le 1} a_i^T y \le b_i$

$\because \|y\| \le 1 \quad \therefore \max a_i^T y = \|a_i\|$

The value of y that achieves the maximum

is $y = a_i / \|a_i\|$

$\|y\| = 1$ and $a_i^T y = a_i^T \cdot a_i / \|a_i\| = \|a_i\|$

$\therefore \quad a_i^T x_c + R \cdot \|a_i\| \le b_i$

$\therefore$ if $B(x_c, R) \le P$

$\Downarrow$ $a_i^T x_c + R \cdot \|a_i\| \le b_i \quad$ for $i = 1, 2, \cdots, m$

The formulated LP is:

maximize R

s.t. $\quad a_i^T x_c + R\|a_i\| \le b_i$

for $i = 1, 2, \cdots, m$