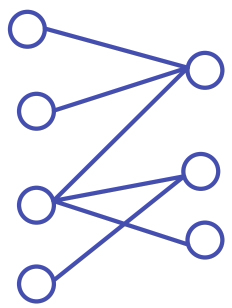
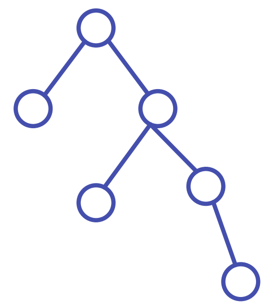
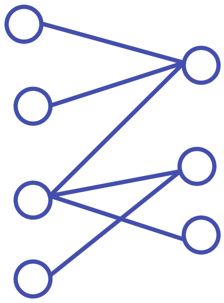
# Lecture 13 Bipartite Graphs and matching





A **vertex cover** $S$ is a subset of the vertices such that every edge has at least one endpoint incident to $S$

A **matching** $M$ is a subset of the edges so that no vertex in $G$ is incident to more than one edges in $M$



Königs theorem for bipartite graphs

Applications

① Personnel assignment: in a company n workers are available for m jobs and each worker is qualified for one or more jobs. Can all workers be assigned, one person per job, to jobs they are qualified for?

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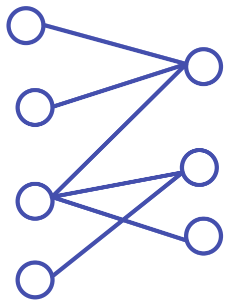
ILP for max matching

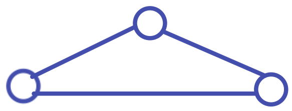
$$
\begin{array}{ll}
 & \begin{matrix} e_1 & e_2 & \cdots & e_n \end{matrix} \\
\begin{matrix} V_1 \left(\begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{matrix}\right. \\ V_2 \left(\begin{matrix} v_{k+1} \\ \vdots \\ v_m \end{matrix}\right. \end{matrix}
& \underbrace{\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \end{bmatrix}}_{m}
\end{array}
\begin{pmatrix} x(e_1) \\ x(e_2) \\ \vdots \\ x(e_n) \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
$$

Derive the dual of the matching LP



$$M = \begin{array}{r l} & \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{array}{r} v_1 \left\{ \vphantom{\begin{array}{c}1\\0\\0\\0\end{array}} \right. \\ v_2 \left\{ \vphantom{\begin{array}{c}0\\1\\0\end{array}} \right. \end{array} & \begin{bmatrix} 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ 1 & 0 & & \\ 0 & 0 & & \end{bmatrix} \end{array}$$

Why bipartite graphs



$$\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline v_1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 \end{array}$$

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KKT conditions $\quad x(e)\left[\lambda(u)+\lambda(v)-1\right]=0$

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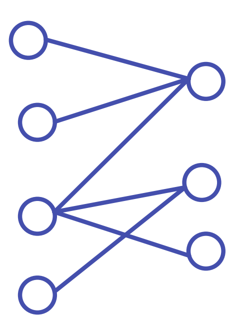
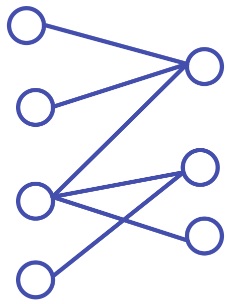
We can solve the max matching problem by solving a max-flow problem.





Is matching optimal?

## Set Cover Problem

Given a universe $\mathcal{U}$ of $n$ elements, a collection of subsets of $\mathcal{U}$ $S = \{S_1, S_2, \dots, S_k\}$ and a cost function $c: S \to \mathbb{R}^+$, find a minimum subcollection of $S$ that covers all the elements in $\mathcal{U}$.

# Spanning tree ILP formulation