Lecture 12

Integer Programming

$$\begin{aligned} \max\ & c^T x \\ \text{st}\ & Ax \le b \\ & x \in \mathbb{Z}^n \end{aligned} \qquad\qquad \begin{aligned} \max\ & c^T x \\ \text{st}\ & Ax \le b \\ & x \in \mathbb{R}^n \end{aligned}$$

When do these problems have the <u>same solution</u>?

<u>Sufficient condition</u> :

Theorem: If the matrix A is totally unimodular (TUM) then for all integer vectors b the polyhedron $P = \{x \mid Ax \leq b\}$ is an integral polyhedron.

A matrix is TUM if the determinant of every square submatrix in A is in $\{0, 1, -1\}$. In particular, all elements in A are in $\{0, 1, -1\}$

Examples:

Proof of theorem.

Proposition If A is TUM then for all integral vector a, b, c, d the polyhedron $\{x \mid a \le x \le b,\ c \le Ax \le d\}$ is integral.

Proof

Example Consider the graph adjacency matrix, we will prove it is TUM.

$$M^T = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ \\ v_m \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{pmatrix} +1 & 0 & & \\ 0 & +1 & & \\ \vdots & -1 & & \\ -1 & 0 & & \\ 0 & \vdots & & \end{pmatrix} \end{array}$$

$$m_{e,v} = \begin{cases} 1 & \text{if } e \text{ enters } v \\ -1 & \text{if } e \text{ exits } v \\ 0 & \text{otherwise} \end{cases}$$

Min-cut Max-flow problem

Max flow problem

$$\min \ -e_m^T f$$
$$\text{s t} \quad (I \ \ 0) f \leq d$$
$$M^T f \leq 0$$
$$-I f \leq 0$$

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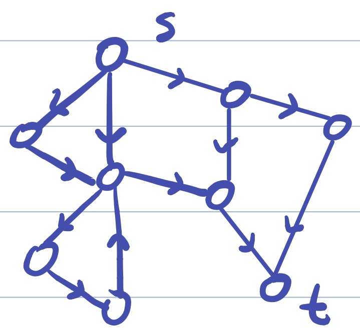
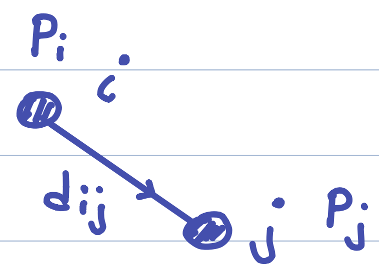
• If the capacities are all 1 (unit rate edges)



Claim: in a graph with unit capacity edges we can find h edge-disjoint paths that connect s to t, where h = mincut between s & t.

Dual LP

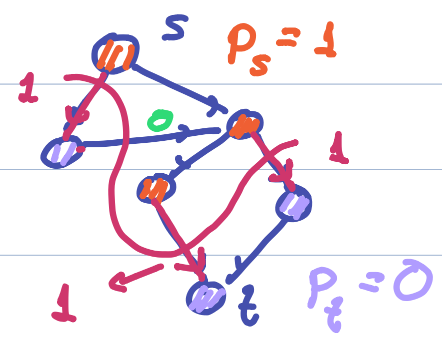
interpretation:



$$\min\ c^T d$$
$$\text{s.t.}\quad P_s - P_t \geq 1$$
$$d_{ij} \geq P_i - P_j$$
$$d_{ij} \geq 0,\quad P_i \geq 0$$

Integer LP

$$\min\ c^T d$$
$$\text{s.t.}\quad P_s - P_t \geq 1$$
$$d_{ij} \geq P_i - P_j$$
$$d_{ij} \in \{0, 1\}$$
$$P_i \in \{0, 1\}$$



max flow LP	dual-relaxed min-cut LP
$\max f_{ts}$	$\min \sum c_{ij} d_{ij}$
st $\sum f_{ki} - \sum f_{ij} \leq 0$	st $p_s - p_t \geq 1$
$f_{ij} \leq c_{ij}$	$d_{ij} \geq p_i - p_j$
$f_{ij} \geq 0$	$d_{ij}, p_i \geq 0$

KKT conditions: $d_{ij}(f_{ij}-c_{ij})=0$

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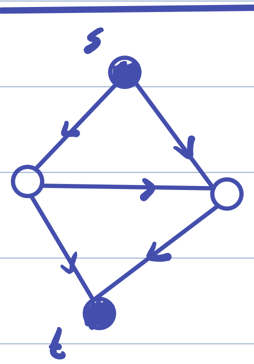
Equivalent formulation of the min-cut max-flow problem with paths.



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