Fall 2020
EE 236A
Christina Fragouli
TAs: Mine Dogan and Kaan Ozkara

Linear Programming

Homework 5 Solutions
Due: 9 AM, Dec. 4, 2020

Problem 1 (3 points): Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A *matching* on $\mathcal{G}$ is a collection of $\mathcal{M} \subseteq \mathcal{E}$ such that, no two edges in $\mathcal{M}$ share a vertex. In other words, each vertex in $\mathcal{V}$ has at most one connected edge in $\mathcal{M}$. A *maximal matching* is a matching $\mathcal{M}$ such that, if any other edge in $\mathcal{E} \setminus \mathcal{M}$ is added to $\mathcal{M}$ it no longer becomes a valid matching.

Assume that you have an algorithm that takes as input an arbitrary graph $\mathcal{G}$, and outputs a maximal matching $\mathcal{M}$. Propose a heuristic that takes as input $\mathcal{G}$ and $\mathcal{M}$, and outputs a valid vertex cover $\mathcal{C}$ (a cover is a subset of vertices $\mathcal{C} \subseteq \mathcal{V}$ such that each edge in $\mathcal{E}$ is incident to at least one vertex in $\mathcal{C}$). The size of the output vertex cover should be within an approximation factor of 2.

Solution:

Given a maximal matching $\mathcal{M}$, a valid vertex cover consists of the nodes on either side of each edge in $\mathcal{M}$; denote this vertex cover as $\hat{V}$. First, note that $\hat{V}$ is indeed a valid cover, that is, there are no edges in $\mathcal{G}$ such that neither of its associated nodes are in $\hat{V}$. If such an edge exists, let us denote it as $\hat{e}$, then this means that its associated nodes do not have any other edges in $\mathcal{M}$. Therefore, $\hat{e}$ could have been added to $\mathcal{M}$ and it would still be a valid matching. This contradicts the assumption that $\mathcal{M}$ is a maximal matching.

Note that $|\hat{V}| = 2|\mathcal{M}|$. Let an optimal (i.e. minimum) vertex cover for such a graph be denoted as $V^\star$. Since we have just found a valid vertex cover (that is not necessarily the optimal), then $|V^\star| \leq 2|\mathcal{M}|$. Note also that, for arbitrary graphs, $|\mathcal{M}| \leq |V^\star|$. Therefore we have

$$|\mathcal{M}| \leq |V^\star| \leq 2|\mathcal{M}|$$

which means that the proposed algorithm for finding a vertex cover is within an approximation factor of 2 from the optimal one.

Problem 2 (6 points):

(a) Use the simplex procedure to solve the following problem

$$\begin{array}{rl} \text{minimize} & z = x - y \\ \text{subject to} & -x + y \geq -4 \\ & -x - y \geq -6 \\ & x, y \geq 0. \end{array}$$

(b) Draw a graphical representation of the problem in X-Y space and indicate the path of the simplex steps.

(c) Repeat the problem above but using the new objective function $z = -x + y$. This problem has multiple solutions, so find all the vertex solutions and write down an expression for the full set of solutions.

(d) Solve the following problem, and graph the path followed by the simplex method:

$$\begin{array}{rl} \text{minimize} & z = -x - y \\ \text{subject to} & 2x - y \geq -2 \\ & -x + y \geq -1 \\ & x, y \geq 0. \end{array}$$

Solution: (a) We first introduce slack variables u and v, and form the following tableau:

$$\begin{array}{cc|cc|c} & & x & y & 1 \\ \hline u & = & -1 & 1 & 4 \\ v & = & -1 & -1 & 6 \\ \hline z & = & 1 & -1 & 0 \end{array}$$

This tableau is feasible. According to the pivoting rule, we choose column 2 and row 2 as the pivot column and pivot row. Using the Jordan exchange, we get the following tableau:

$$\begin{array}{cc|cc|c} & & x & v & 1 \\ \hline u & = & -2 & -1 & 10 \\ y & = & -1 & -1 & 6 \\ \hline z & = & 2 & 1 & -6 \end{array}$$

The optimality condition is satisfied, then we get the solution $x = v = 0$, $u = 10$, $y = 6$, and the optimal value $z = -6$.

(b) The graphical representation of part (a) is shown in Fig. 1. The path of the simplex steps is Path 1: from vertex 1 to vertex 2.

(c) We perform the simplex procedure by first forming a tableau:

$$\begin{array}{cc|cc|c} & & x & y & 1 \\ \hline u & = & -1 & 1 & 4 \\ v & = & -1 & -1 & 6 \\ \hline z & = & -1 & 1 & 0 \end{array}$$

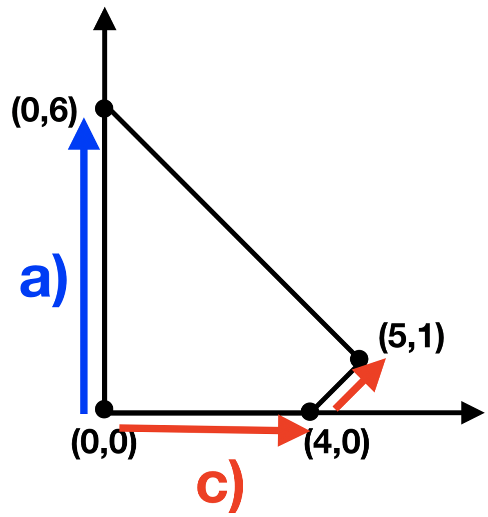

Figure 1: The simplex path.

We choose column 1 and and row 1 as the pivot column and the pivot row according to the pivoting rule, and get the following tableau by Jordan exchange:

$$\begin{array}{cc|cc|c} & & u & y & 1 \\ \hline x & = & -1 & 1 & 4 \\ v & = & 1 & -2 & 2 \\ \hline z & = & 1 & 0 & -4 \end{array}$$

This tableau satisfies the optimality condition, so we get a vertex solution $x = 4$, $y = u = 0$, $v = 2$, and the optimal value $z = -4$. We can apply the pivoting rule to exchange the variables y and v, and get the tableau:

$$\begin{array}{cc|cc|c} & & u & v & 1 \\ \hline x & = & -1/2 & -1/2 & 5 \\ y & = & 1/2 & -1/2 & 1 \\ \hline z & = & 1 & 0 & -4 \end{array}$$

This also satisfies the optimality condition, so we get another vertex solution $x = 5$, $y = 1$, $u = v = 0$, and the optimal value $z = -4$. If we apply the pivoting rule again, we will exchange the variables v and y. This is already done. So we get all the vertex solutions.

The graphical representation is shown in Fig. 1. The path of simplex steps is shown as Path 2: from vertex 1 to vertex 4 to vertex 3. The full set of solution can be written as $\{(x, y)|y = x-4, 4 \leq x \leq 5\}$.

(d) We first form the tableau:

$$\begin{array}{cc|cc|c} & & x & y & 1 \\ \hline u & = & 2 & -1 & 2 \\ v & = & -1 & 1 & 1 \\ \hline z & = & -1 & -1 & 0 \end{array}$$

This is feasible. Next, we exchange the variables x and v according to the pivoting rule. We have the following tableau by Jordan exchange:

$$\begin{array}{cc|cc|c} & & v & y & 1 \\ \hline u & = & -2 & 1 & 4 \\ x & = & -1 & 1 & 1 \\ \hline z & = & 1 & -2 & -1 \end{array}$$

We then choose column 2 as the pivot column, but the pivot row does not exist. Therefore, the problem is unbounded. The path is shown in Fig. 2 as Path 1: from vertex 1 to vertex 2 to infinity.

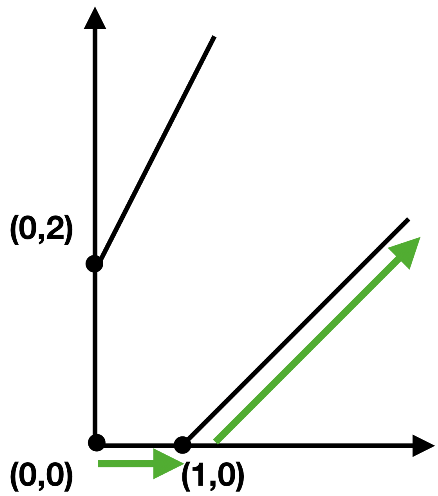

Figure 2: The simplex path.

<u>Problem 3</u> (`4 points`):

Consider the following LP:

$$\begin{array}{ll} \text{minimize} & z = x_1 - x_2 \\ \text{subject to} & 0 \le x_i \le \frac{1}{2}, \quad i = 1, 2, 3 \\ & \sum_{i=1}^{3} x_i = 1 \end{array}$$

Given an initial feasible point $(1/2, 1/2, 0)$, use the simplex method to find an optimal solution to this LP.

<u>Solution</u>:

We can reformulate the problem as follows:

$$
\begin{array}{ll}
\text{minimize} & z = x_1 - x_2 \\
\text{subject to} & x_4 = \frac{1}{2} - x_1 \geq 0 \\
& x_5 = \frac{1}{2} - x_2 \geq 0 \\
& x_6 = \frac{1}{2} - x_3 \geq 0 \\
& x_7 = -1 + \sum_{i=1}^{3} x_i \geq 0 \\
& x_8 = 1 + \sum_{i=1}^{3} -x_i \geq 0
\end{array}
$$

Our initial tableau is as follows:

$$
\begin{array}{cc|ccc|c}
 & & x_1 & x_2 & x_3 & 1 \\
\hline
x_4 & = & -1 & 0 & 0 & 1/2 \\
x_5 & = & 0 & -1 & 0 & 1/2 \\
x_6 & = & 0 & 0 & -1 & 1/2 \\
x_7 & = & 1 & 1 & 1 & -1 \\
x_8 & = & -1 & -1 & -1 & 1 \\
\hline
z & = & 1 & -1 & 0 & 0
\end{array}
$$

This tableau corresponds to the point $(0, 0, 0)$, which is infeasible. In order to pivot to our initial feasible point $(1/2, 1/2, 0)$, we need to exchange rows and columns. For example, we can exchange x_4 and x_1, and then x_5 and x_2, so that our active constraints are $x_4 = 1/2 - x_1 \geq 0$, $x_5 = 1/2 - x_2 \geq 0$, and $x_3 \geq 0$:

$$
\begin{array}{cc|ccc|c}
 & & x_4 & x_2 & x_3 & 1 \\
\hline
x_1 & = & -1 & 0 & 0 & 1/2 \\
x_5 & = & 0 & -1 & 0 & 1/2 \\
x_6 & = & 0 & 0 & -1 & 1/2 \\
x_7 & = & -1 & 1 & 1 & -1/2 \\
x_8 & = & 1 & -1 & -1 & 1/2 \\
\hline
z & = & -1 & -1 & 0 & 1/2
\end{array}
$$

$$
\begin{array}{cc|ccc|c}
 & & x_4 & x_5 & x_3 & 1 \\
\hline
x_1 & = & -1 & 0 & 0 & 1/2 \\
x_2 & = & 0 & -1 & 0 & 1/2 \\
x_6 & = & 0 & 0 & -1 & 1/2 \\
x_7 & = & -1 & -1 & 1 & 0 \\
x_8 & = & 1 & 1 & -1 & 0 \\
\hline
z & = & -1 & 1 & 0 & 0
\end{array}
$$

From here, we select columns and rows according to the simplex method. We begin by pivoting around the row containing x_7 and the column containing x_4:

		x_7	x_5	x_3	1
x_1	$=$	1	1	-1	$1/2$
x_2	$=$	0	-1	0	$1/2$
x_6	$=$	0	0	-1	$1/2$
x_4	$=$	-1	-1	1	0
x_8	$=$	-1	0	0	0
z	$=$	1	0	-1	0

Next, we exchange x_1 and x_3:

		x_7	x_5	x_1	1
x_3	$=$	1	1	-1	$1/2$
x_2	$=$	0	-1	0	$1/2$
x_6	$=$	-1	-1	1	0
x_4	$=$	0	0	-1	$1/2$
x_8	$=$	-1	0	0	0
z	$=$	0	-1	1	$-1/2$

Finally, we exchange x_6 and x_5:

		x_7	x_6	x_1	1
x_3	$=$	0	-1	0	$1/2$
x_2	$=$	1	1	-1	$1/2$
x_5	$=$	-1	-1	1	0
x_4	$=$	0	0	-1	$1/2$
x_8	$=$	-1	0	0	0
z	$=$	1	1	2	$-1/2$

This satisfies our optimality condition. We read off the optimal extreme point as $(x_1^*, x_2^*, x_3^*) = (0, 1/2, 1/2)$, with an optimal value $z^* = -1/2$.

Problem 4 (`4 points`):

(a) Demonstrate that

$$\begin{array}{rl} \text{minimize} & z = -3x_1 + 4x_2 \\ \text{subject to} & -x_1 - x_2 \geq -1 \\ & -2x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0. \end{array}$$

is infeasible using the Phase I procedure.

(b) Demonstrate that

$$\begin{aligned} \text{minimize} \quad & z = -2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \geq 1 \\ & x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

is unbounded using the Phase I procedure.

Solution: (a) We first formulate the Phase I problem as follows:

$$\begin{aligned} \text{minimize} \quad & z_0 = x_0 \\ \text{subject to} \quad & x_3 = -x_1 - x_2 + 1 \\ & x_4 = -2x_1 + x_2 - 2 + x_0 \\ & x_0, x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

We use the Phase I procedure to form the following tableau:

		x_1	x_2	x_0	1
x_3	$=$	-1	-1	0	1
x_4	$=$	-2	1	1	-2
z	$=$	-3	4	0	0
z_0	$=$	0	0	1	0

We perform the special pivot to exchange the variables x_0 and x_4 and get the following tableau:

		x_1	x_2	x_4	1
x_3	$=$	-1	-1	0	1
x_0	$=$	2	-1	1	2
z	$=$	-3	4	0	0
z_0	$=$	2	-1	1	2

According to the pivoting rule, we choose column 2 and row 1 as the pivot column and the pivot row. Then we get the following tableau:

		x_1	x_3	x_4	1
x_2	$=$	-1	-1	0	1
x_0	$=$	3	1	1	1
z	$=$	-7	-4	0	4
z_0	$=$	3	1	1	1

This tableau satisfies the optimality condition. The optimal value is $z_0 = x_0 = 1$. Hence, the original problem is infeasible.

(b) We first formulate the following Phase I problem:

$$\begin{aligned} \text{minimize} \quad & z_0 = x_0 \\ \text{subject to} \quad & x_3 = 2x_1 - x_2 - 1 + x_0 \\ & x_4 = x_1 + 2x_2 - 2 + x_0 \\ & x_0, x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

We can form the following tableau:

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_2 & x_0 & 1 \\ \hline
x_3 & = & 2 & -1 & 1 & -1 \\
x_4 & = & 1 & 2 & 1 & -2 \\ \hline
z & = & -2 & 1 & 0 & 0 \\
z_0 & = & 0 & 0 & 1 & 0
\end{array}$$

Next, we perform the special pivot to exchange the variables x_0 and x_4. Then we have the following tableau:

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_2 & x_4 & 1 \\ \hline
x_3 & = & 1 & -3 & 1 & 1 \\
x_0 & = & -1 & -2 & 1 & 2 \\ \hline
z & = & -2 & 1 & 0 & 0 \\
z_0 & = & -1 & -2 & 1 & 2
\end{array}$$

We choose column 2 and row 1 as the pivot column and the pivot row. Performing the Jordan exchange, we get the tableau:

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_3 & x_4 & 1 \\ \hline
x_2 & = & 1/3 & -1/3 & 1/3 & 1/3 \\
x_0 & = & -5/3 & 2/3 & 1/3 & 4/3 \\ \hline
z & = & -5/3 & -1/3 & 1/3 & 1/3 \\
z_0 & = & -5/3 & 2/3 & 1/3 & 4/3
\end{array}$$

Then choose column 1 and row 2 as the pivot column and the pivot row. We perform the Jordan exchange to get:

$$\begin{array}{cc|ccc|c}
 & & x_0 & x_3 & x_4 & 1 \\ \hline
x_2 & = & -1/5 & -1/5 & 2/5 & 3/5 \\
x_1 & = & -3/5 & 2/5 & 1/5 & 4/5 \\ \hline
z & = & 1 & -1 & 0 & -1 \\
z_0 & = & 1 & 0 & 0 & 0
\end{array}$$

This satisfies the optimality condition. We get the optimal value $z_0 = x_0 = 0$, which shows that the original problem is feasible.

Next, we perform the Phase II procedure. We remove the first column and the last row to get a tableau for the original problem.

$$\begin{array}{cc|cc|c}
 & & x_3 & x_4 & 1 \\ \hline
x_2 & = & -1/5 & 2/5 & 3/5 \\
x_1 & = & 2/5 & 1/5 & 4/5 \\ \hline
z & = & -1 & 0 & -1
\end{array}$$

According to the pivoting rule, we exchange variables x_3 and x_2, and get the following tableau:

$$\begin{array}{rc|rr|r} & & x_2 & x_4 & 1 \\ \hline x_3 & = & -5 & 2 & 3 \\ x_1 & = & -2 & 1 & 2 \\ \hline z & = & 5 & -2 & -4 \end{array}$$

We can find column 2 as the pivot column, but no pivot row exists. Then the original problem is unbounded.

<u>Problem 5</u> (`3 points`): Solve the following linear program using the simplex algorithm with Bland's pivoting rule. Start the algorithm at the extreme point $x = (2, 2, 0)$, with active set $I = \{3, 4, 5\}$.

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}. \end{array}$$

<u>Solution</u>: We can start from the tableau for $x = (0, 0, 0)$ and perform Jordan exchange to get a feasible tableau for $x = (2, 2, 0)$. We can introduce slack variables x_4, x_5, x_6 and x_7 to write the following tableau for point $x = (0, 0, 0)$.

$$\begin{array}{rc|rrr|r} & & x_1 & x_2 & x_3 & 1 \\ \hline x_4 & = & -1 & 0 & 0 & 2 \\ x_5 & = & 0 & -1 & 0 & 2 \\ x_6 & = & 0 & 0 & -1 & 2 \\ x_7 & = & -1 & -1 & -1 & 4 \\ \hline z & = & 1 & 1 & -1 & 0 \end{array}$$

We can first perform Jordan exchange between x_1 and x_4 and then, perform Jordan exchange between x_2 and x_5 to get the following feasible tableau for $x = (2, 2, 0)$.

$$\begin{array}{rc|rrr|r} & & x_4 & x_5 & x_3 & 1 \\ \hline x_1 & = & -1 & 0 & 0 & 2 \\ x_2 & = & 0 & -1 & 0 & 2 \\ x_6 & = & 0 & 0 & -1 & 2 \\ x_7 & = & 1 & 1 & -1 & 0 \\ \hline z & = & -1 & -1 & -1 & 4 \end{array}$$

Now, we can perform Jordan exchange by following Bland's rule. We first need to exchange x_3

with x_7.

$$\begin{array}{cc|ccc|c}
 & & x_4 & x_5 & x_7 & 1 \\
\hline
x_1 & = & -1 & 0 & 0 & 2 \\
x_2 & = & 0 & -1 & 0 & 2 \\
x_6 & = & -1 & -1 & 1 & 2 \\
x_3 & = & 1 & 1 & -1 & 0 \\
\hline
z & = & -2 & -2 & 1 & 4
\end{array}$$

Now, we need to exchange x_4 with x_1 and get the following tableau.

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_5 & x_7 & 1 \\
\hline
x_4 & = & -1 & 0 & 0 & 2 \\
x_2 & = & 0 & -1 & 0 & 2 \\
x_6 & = & 1 & -1 & 1 & 0 \\
x_3 & = & -1 & 1 & -1 & 2 \\
\hline
z & = & 2 & -2 & 1 & 0
\end{array}$$

We can exchange x_5 with x_6.

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_6 & x_7 & 1 \\
\hline
x_4 & = & -1 & 0 & 0 & 2 \\
x_2 & = & -1 & 1 & -1 & 2 \\
x_5 & = & 1 & -1 & 1 & 0 \\
x_3 & = & 0 & -1 & 0 & 2 \\
\hline
z & = & 0 & 2 & -1 & 0
\end{array}$$

Finally, we can exchange x_7 with x_2 to get the optimal tableau.

$$\begin{array}{cc|ccc|c}
 & & x_1 & x_6 & x_2 & 1 \\
\hline
x_4 & = & -1 & 0 & 0 & 2 \\
x_7 & = & -1 & 1 & -1 & 2 \\
x_5 & = & 0 & 0 & -1 & 2 \\
x_3 & = & 0 & -1 & 0 & 2 \\
\hline
z & = & 1 & 1 & 1 & -2
\end{array}$$

The optimal solution is $x = (0, 0, 2)$ and the optimal value is -2.