Fall 2020
EE 236A
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Linear Programming

Solutions of Homework 3

Problem 1 (`3 points`, Exer. 50 in *Linear Programming Exercises*): A matrix $A \in \mathbf{R}^{(mp)\times n}$ and a vector $b \in \mathbf{R}^{mp}$ are partitioned in m blocks of p rows:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

with $A_k \in \mathbf{R}^{p\times n}$, $b_k \in \mathbf{R}^p$.

(a) Express the optimization problem

$$\text{minimize} \quad \sum_{k=1}^{m} ||A_k x - b_k||_\infty \tag{1}$$

as an LP.

(b) Suppose $\mathbf{rank}(A) = n$ and $Ax_{ls} - b \neq 0$, where x_{ls} is the solution of the least-squares problem

$$\text{minimize} \quad ||Ax - b||^2.$$

Derive the dual program and show that it can be simplified as

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^m b_k^T z_k \\ \text{subject to} & \sum_{k=1}^m A_k^T z_k = 0 \\ & ||z_k||_1 \leq 1, \quad k = 1, \ldots, m \end{array}$$

(c) For the setup of (b), show that the optimal value of (1) is bounded below by

$$\frac{\sum_{k=1}^m ||r_k||^2}{\max_{k=1,\cdots,m} ||r_k||_1}$$

where $r_k = A_k x_{ls} - b_k$ for $k = 1, \cdots, m$.

Solution:

(a) The problem is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y_k \mathbf{1} \leq A_k x - b_k \leq y_k \mathbf{1}, \quad k = 1, \ldots, m, \end{array}$$

with an auxiliary variable $y \in \mathbf{R}^m$.

(b) The dual problem can be written as

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{m} b_k^T(u_k - v_k) \\ \text{subject to} & \sum_{k=1}^{m} A_k^T(u_k - v_k) = 0 \\ & \mathbf{1}^T(u_k + v_k) = 1, \quad k = 1, \ldots, m \\ & u_k \geq 0, \quad v_k \geq 0, \end{array}$$

with variables $u_k, v_k \in \mathbf{R}^p$. We can derive the simplified equivalent problem

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{m} b_k^T z_k \\ \text{subject to} & \sum_{k=1}^{m} A_k^T z_k = 0 \\ & ||z_k||_1 \leq 1, \quad k = 1, \ldots, m. \end{array}$$

This can be testified as follows (see, page 6-16). If u_k, v_k are feasible for the first problem, then $z_k = u_k - v_k$ is feasible for the second problem. If z_k is feasible for the second problem, then $(u_k)_i = \max\{(z_k)_i, 0\} + \alpha_k$ and $(v_k)_i = \max\{-(z_k)_i, 0\} + \alpha_k$ are feasible for the first problem, where $\alpha_k = (1 - ||z_k||_1)/(2p)$.

(c) For the least square solution x_{ls}, we have the following property:

$$(A^T A)x_{ls} = A^T b.$$

Plugging in the block form A and b, we have

$$\sum_{k=1}^{m} A_k^T(A_k x_{lx} - b_k) = 0.$$

i.e.,

$$\sum_{k=1}^{m} A_k^T r_k = 0.$$

We can see that r_k satisfies the first constraint of the dual problem. Then we normalize r_k to get a feasible point:

$$z_k = -\frac{r_k}{\max_{k=1,\ldots,m} ||r_k||_1}.$$

Obviously, $||z_k||_1 \leq 1$. We plug in z_k and get a feasible value of the dual problem, which is a lower bound of the primal problem.

$$\sum_{k=1}^{m} b_k^T z_k = \sum_{k=1}^{m} (b_k - A_k x_{ls})^T z_k = \frac{\sum_{k=1}^{m} ||r_k||^2}{\max_{k=1,\ldots,m} ||r_k||_1}.$$

<u>Problem 2</u> (`3 points`, Exer. 59 in *Linear Programming Exercises*): The projection of a point $x_0 \in \mathbf{R}^n$ on a polyhedron $\mathcal{P} = \{x | Ax \leq b\}$, in the l_∞-norm, is defined as the solution of the optimization problem

$$\begin{array}{ll}\text{minimize} & \|x - x_0\|_\infty \\ \text{subject to} & Ax \leq b.\end{array}$$

The variable is $x \in \mathbf{R}^n$. We assume that $\mathcal{P}$ is nonempty.

(a) Write this problem as an LP in standard form.

(b) Derive the dual problem, and show it is equivalent to the following:

$$\begin{array}{ll}\text{maximize} & (Ax_0 - b)^T w \\ \text{subject to} & \|A^T w\|_1 \leq 1 \\ & w \geq 0.\end{array}$$

Solution:

(a) The problem can be written as

$$\begin{array}{ll}\text{minimize} & y \\ \text{subject to} & -y\mathbf{1} \leq x - x_0 \leq y\mathbf{1} \\ & Ax \leq b.\end{array}$$

By introducing slackness variable vectors (t_1, t_2, t_3) and writing $x = x^+ - x^-$, we can write it as

$$\begin{array}{ll}\text{minimize} & y \\ \text{subject to} & x^+ - x^- - x_0 + t_1 = y\mathbf{1} \\ & x^+ - x^- - x_0 - t_2 = -y\mathbf{1} \\ & Ax^+ - Ax^- + t_3 = b. \\ & x^+ \geq 0,\ x^- \geq 0,\ y \geq 0 \\ & t_i \geq 0, \quad i \in \{1, 2, 3\}\end{array}$$

The problem is now in the standard form

$$\begin{array}{ll}\text{minimize} & c^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = \tilde{b} \\ & \tilde{x} \geq 0\end{array}$$

where:

$$\tilde{x} = \begin{bmatrix} x^+ \\ x^- \\ t_1 \\ t_2 \\ t_3 \\ y \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} I & -I & I & 0 & 0 & -\mathbf{1} \\ I & -I & 0 & -I & 0 & \mathbf{1} \\ A & -A & 0 & 0 & I & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} x_0 \\ x_0 \\ b \end{bmatrix}.$$

(b) The dual problem of the LP in (a) is

$$\begin{array}{ll}\text{maximize} & \tilde{b}^T \tilde{z} \\ \text{subject to} & \tilde{A}^T \tilde{z} \leq c\end{array}$$

where $\tilde{A}$, $\tilde{b}$ and c are as described in part (a) and the dual variable vector is divided as $\tilde{z} = [\tilde{z}_1^T \ \tilde{z}_2^T \ \tilde{z}_3^T]^T$.

If we expand the constraints and objective function, then we get:

$$\begin{array}{ll} \text{maximize} & x_0^T \tilde{z}_1 + x_0^T \tilde{z}_2 + b^T \tilde{z}_3 \\ \text{subject to} & \begin{bmatrix} I & I & A^T \\ -I & -I & -A^T \\ I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \\ -\mathbf{1}^T & \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

This can be translated into the following:

$$\begin{array}{ll} \text{maximize} & x_0^T(\tilde{z}_1 + \tilde{z}_2) + b^T \tilde{z}_3 \\ \text{subject to} & -\mathbf{1}^T \tilde{z}_1 + \mathbf{1}^T \tilde{z}_2 = 1 \\ & \tilde{z}_1 + \tilde{z}_2 \leq -A^T \tilde{z}_3 \leq \tilde{z}_1 + \tilde{z}_2 \ \Leftrightarrow \ -A^T \tilde{z}_3 = \tilde{z}_1 + \tilde{z}_2 \\ & \tilde{z}_1 \leq 0, \quad \tilde{z}_2 \geq 0, \quad \tilde{z}_3 \leq 0. \end{array}$$

Using the transformation: $(z_1, z_2, w) = (\tilde{z}_2, -\tilde{z}_1, -\tilde{z}_3)$, we get that the dual is on the equivalent form.

$$\begin{array}{ll} \text{maximize} & x_0^T(z_1 - z_2) - b^T w \\ \text{subject to} & \mathbf{1}^T z_1 + \mathbf{1}^T z_2 = 1 \\ & A^T w = z_1 - z_2 \\ & z_1 \geq 0, \quad z_2 \geq 0, \quad w \geq 0. \end{array}$$

Using the same method as in Problem 3 (b), we introduce a new variable t, and the problem can be simplified as

$$\begin{array}{ll} \text{maximize} & x_0^T t - b^T w \\ \text{subject to} & \|t\|_1 \leq 1 \\ & A^T w = t \\ & w \geq 0. \end{array}$$

This can also be written as

$$\begin{array}{ll} \text{maximize} & (Ax_0 - b)^T w \\ \text{subject to} & \|A^T w\|_1 \leq 1 \\ & w \geq 0. \end{array}$$

<u>Problem 3</u> (`3 points`, Exer. 51 in *Linear Programming Exercises*):

Let x be a real-valued random variable which takes values in $\{a_1, a_2, ..., a_n\}$ where $0 < a_1 < a_2 < ... < a_n$, and $\mathbf{prob}(x = a_i) = p_i$. Obviously p satisfies $\sum_{i=1}^n p_i = 1$ and $p_i \geq 0$ for $i = 1, ..., n$.

(a) Consider the problem of determining the probability distribution that maximizes $\mathbf{prob}(x \geq \alpha)$ subject to the constraint $\mathbf{E}x = b$, *i.e.*,

$$\begin{array}{ll} \text{maximize} & \mathbf{prob}(x \geq \alpha) \\ \text{subject to} & \mathbf{E}x = b, \end{array} \tag{2}$$

where α and b are given ($a_1 < \alpha < a_n$, and $a_1 \leq b \leq a_n$). The variable in problem (2) is the probability distribution, *i.e.*, the vector $p \in \mathbf{R}^n$. Write (2) as an LP.

(b) Take the dual of the LP in (a), and show that it can be reformulated as

$$\begin{array}{ll} \text{minimize} & \lambda b + \nu \\ \text{subject to} & \lambda a_i + \nu \geq 0 \text{ for all } a_i < \alpha, \\ & \lambda a_i + \nu \geq 1 \text{ for all } a_i \geq \alpha, \end{array}$$

The variables λ and ν. Show that the optimal value is equal to

$$\begin{cases} (b - a_1)/(\bar{a} - a_1) & b \leq \bar{a} \\ 1 & b \geq \bar{a} \end{cases}$$

where $\bar{a} = \min\{a_i | a_i \geq \alpha\}$. Also give the optimal values of λ and ν.

<u>Solution</u>:

(a) The LP formulation is

$$\begin{array}{ll} \text{maximize} & \sum_{i:a_i \geq \alpha} p_i \\ \text{subject to} & \sum_{i=1}^{n} p_i a_i = b, \\ & \sum_{i=1}^{n} p_i = 1, \\ & p \geq 0. \end{array} \tag{3}$$

(b) we can derive the dual by reducing the problem in inequality form. We can also simply note that the problem has the form of the dual of the pair of primal and dual LPs

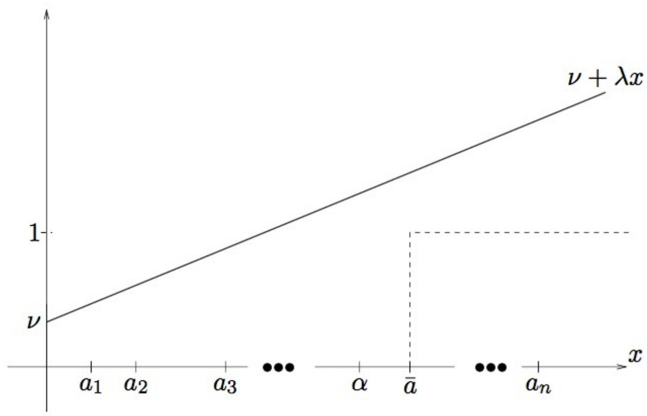
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0. \end{array}$$

The corresponding primal (or the dual of (3)) is

$$\begin{array}{ll} \text{minimize} & b\lambda + \nu \\ \text{subject to} & \lambda a_i + \nu \geq 0 \text{ if } a_i < \alpha, \\ & \lambda a_i + \nu \geq 1 \text{ if } a_i \geq \alpha. \end{array} \tag{4}$$

Now note that since a_i takes only discrete values then the constraints in (4) can be written comparing a_i to $\bar{a}$ instead of α.

We can interpret the objective function graphically as a discrete linear function $f(x) = \lambda x + v$ with parameters λ and v.



The constraints state that the value of the function $f(x)$ must be at least zero for $x < \bar{a}$ and at least one for $x \geq \bar{a}$. We want to minimize the function value at $x = b$.

If $b \leq \bar{a}$, the optimal choice is $f(a_1) = 0$ and $f(\bar{a}) = 1$, Since we now have two points on the line, we can estimate the parameters λ and v. *i.e.*, $\nu = -a_1/(\bar{a} - a_1)$ and $\lambda = 1/(\bar{a} - a_1)$. The function value at b is $f(b) = (b - a_1)/(\bar{a} - a_1)$.

If $b \geq \bar{a}$, the optimal choice is $\nu = 1$, $\lambda = 0$, which makes $f(b) = 1$.

To determine the primal optimal solution, we can use complementary slackness. p_i is the multiplier associated with the ith inequality in (4). Therefore the optimal p_i can be positive only if $a_i\lambda + \nu = 0$ (if $a_i < \alpha$) or $a_i\lambda + \nu = 1$ (if $a_i \geq \alpha$).

Let's first apply this to the case where $b \leq \bar{a}$. Suppose $\bar{a} = a_j$. Then complementary slackness tells us that only p_1 and p_j are nonzero. The values of p_1 and p_j are uniquely determined by the constraints

$$p_1 + p_j = 1, \qquad\qquad a_1 p_1 + \bar{a} p_j = b,$$

i.e., $p_1 = (\bar{a} - b)/(\bar{a} - a_1)$ and $p_j = (b - \bar{a})/(\bar{a} - a_1)$.

In the second ($b \geq \bar{a}$) complementary slackness tells us that $p_i = 0$ for all $a_i < \alpha$. This leaves us many choices for the remaining values of p_i, as long as $\sum_i p_i = 1$ and $\sum_i a_i p_i = b$. All those solutions are optimal for the primal problem. For example, assuming that a_j and a_k satisfy $\bar{a} \leq a_j \leq b$ and $b \leq a_k$, we can take $p_j = (a_k - b)/(a_k - a_j)$ and $p_k = (b - a_j)/(a_k - a_j)$.

<u>Problem 4</u> (`3 points`, Exer. 48 [(a), (b)], in *Linear Programming Exercises*): Consider the following optimization problem in x:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|Ax + b\|_1 \leq 1 \end{array} \tag{5}$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$.

(a) Formulate this problem as an LP in inequality form and explain why your LP formulation is equivalent to problem (5).

(b) Derive the dual LP, and show that it is equivalent to the problem

$$\begin{array}{ll} \text{maximize} & b^T z - \|z\|_\infty \\ \text{subject to} & A^T z + c = 0. \end{array} \tag{6}$$

What is the relation between the optimal z and the optimal variables in the dual LP?

<u>**Solution**</u>:

(a) The problem can be formulated as the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -b \\ b \\ 1 \end{bmatrix}. \end{array}$$

with variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$.

(b) The dual LP is

$$\begin{array}{ll} \text{maximize} & b^T u - b^T v - w \\ \text{subject to} & A^T u - A^T v + c = 0 \\ & -u - v + w\mathbf{1} = 0 \\ & u \geq 0, \quad v \geq 0, \quad w \geq 0 \end{array} \tag{7}$$

with variables $u \in \mathbf{R}^m$, $v \in \mathbf{R}^m$, $w \in \mathbf{R}$.

<u>Proving equivalence:</u> We wish to prove the equivalence of problems (7) and (6), which we re-write here for convenience,

$$\text{P1:} \quad \begin{array}{ll} \text{maximize} & f_1(u, v, w) = b^T u - b^T v - w \\ \text{subject to} & A^T u - A^T v + c = 0 \\ & -u - v + w\mathbf{1} = 0 \\ & u \geq 0, \quad v \geq 0, \quad w \geq 0 \end{array}$$

$$\text{P2:} \quad \begin{array}{ll} \text{maximize} & f_2(z) = b^T z - \|z\|_\infty \\ \text{subject to} & A^T z + c = 0 \end{array}$$

The notion of equivalence required in this problem is the following:
instead of solving P1 to find the optimal value and optimal dual variables, we need to prove

that we can solve P2 and get the optimal value, and we can also use the optimal variable z to construct optimal dual variables u, v and w. Put more formally, we need to prove that $f_1(u^\star, v^\star, w^\star) = f_2(z^\star)$, where $(u^\star, v^\star, w^\star)$ and $z^\star$ are the optimal variables for problems P1 and P2 respectively. We need to also show how from $z^\star$, we can get construct $(u^\star, v^\star, w^\star)$ (but not necessarily the other way around).

Note that the main difference between this notion of equivalence and the notion of equivalence discussed in the class is the lack of one step: how to construct $z^\star$ from $(u^\star, v^\star, w^\star)$. Therefore, it seems that proving this (weaker) notion of equivalence could be easier. In addition, it is a convenient notion of equivalence in the context of our problem: because P1 is a dual program, then it is instructive to learn the values the variables (u, v, w) (for example to know which constraints are the active ones at the optimal point and so on), and therefore we would need to have a way of constructing $(u^\star, u^\star, w^\star)$ from $z^\star$. However, it is not as important to do the opposite.

Now, to prove this equivalence, we need to show that

(a) $f_1(u^\star, v^\star, w^\star) = f_2(z^\star)$,

(b) From $z^\star$ we can construct $(u^\star, v^\star, w^\star)$.

We prove (a) by showing that $f_1(u^\star, v^\star, w^\star) \leq f_2(z^\star)$ and $f_1(u^\star, v^\star, w^\star) \geq f_2(z^\star)$, and therefore the only possibility for these two conditions to hold simultaneously is to have $f_1(u^\star, v^\star, w^\star) = f_2(z^\star)$. Then we prove (b) by showing an actual construction from $z^\star$ to $(u^\star, v^\star, w^\star)$.

(1) Proving $f_1(u^\star, v^\star, w^\star) \leq f_2(z^\star)$:

Given $(u^\star, v^\star, w^\star)$, we can construct $\hat{z} = u^\star - v^\star$ (note that this is not necessarily the optimal z). Since $(u^\star, v^\star, w^\star)$ are feasible in P1, then we have $A^T u^\star - A^T v^\star + c = A^T \hat{z} - c = 0$. Therefore $\hat{z}$ is feasible in P2. We can note also that

$$|\hat{z}_j| = |u_j^\star - v_j^\star| \leq |u_j^\star + v_j^\star| \overset{(i)}{=} u_j^\star + v_j^\star = w^\star, \qquad \text{for all } j \tag{8}$$

where (i) follows because $u^\star$ and $v^\star$ are non-negative. Since (8) holds for all j, then $\|\hat{z}\|_\infty \leq w^\star$. Therefore, we can write

$$f_2(z^\star) \overset{(ii)}{\geq} f_2(\hat{z}) = b^T \hat{z} - \|\hat{z}\|_\infty \geq b^T (u^\star - v^\star) - w^\star = f_1(u^\star, v^\star, w^\star),$$

where (ii) holds by definition of $z^\star$ being the optimal point.

(2) Proving $f_2(z^\star) \leq f_1(u^\star, v^\star, w^\star)$:

Given $z^\star$, we can construct $(\hat{u}, \hat{v}, \hat{w})$ as follows:

$$\begin{aligned}
\hat{w} &= \|z^\star\|_\infty, \\
\hat{u}_i &= \max\{z_i^\star, 0\} + 0.5(\hat{w} - |z_i^\star|), \\
\hat{v}_i &= \max\{-z_i^\star, 0\} + 0.5(\hat{w} - |z_i^\star|).
\end{aligned}$$

It can be verified that $\hat{u}_i - \hat{v}_i = z_i^\star$ and $\hat{u}_i + \hat{v}_i = \hat{w}_i$. If can also be verified that $\hat{u}$, $\hat{v}$ and $\hat{w}$ satisfy the constraints in P1, namely

$$\begin{aligned}
&A^T\hat{u} - A^T\hat{v} + c = A^T z^\star + c = 0, \\
&-\hat{u} - \hat{v} + \hat{w}\mathbf{1} = -\hat{w}\mathbf{1} + \hat{w}\mathbf{1} = 0, \\
&\hat{u} \geq 0, \quad \hat{v} \geq 0, \quad \hat{w} \geq 0.
\end{aligned}$$

Therefore, $(\hat{u}, \hat{v}, \hat{w})$ is a feasible point. Therefore, we can write

$$f_1(u^\star, v^\star, w^\star) \overset{(iii)}{\geq} f_1(\hat{u}, \hat{v}, \hat{w}) = b^T(\hat{u} - \hat{v}) - \hat{w} = b^T z^\star - \|z^\star\|_\infty = f_2(z^\star),$$

where (iii) holds by definition of $u^\star, v^\star, w^\star$ being the optimal point.

Combining (1) and (2) proves that $f_2(z^\star) = f_1(u^\star, v^\star, w^\star)$, and thus proves requirement (a). To prove (b), i.e. to show how we can construct $(u^\star, v^\star, w^\star)$ from $z^\star$, note that the point $(\hat{u}, \hat{v}, \hat{w})$ constructed in (2) has an objective value $f_1(\hat{u}, \hat{v}, \hat{w}) = f_2(z^\star) = f_1(u^\star, v^\star, w^\star)$. Therefore, $(\hat{u}, \hat{v}, \hat{w})$ is indeed an optimal point, and the construction provided in (2) is the required construction.

Problem 5 (`3 points`, Exer. 63 in *Linear Programming Exercises*): Consider the robust LP

$$\begin{array}{ll}
\min & c^T x \\
\text{subject to} & \max_{a \in \mathcal{P}_i} a^T x \leq b_i, \quad i = 1, \cdots, m
\end{array}$$

with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_i a \leq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i \times n}$, $d_i \in \mathbf{R}^{m_i}$ and $b \in \mathbf{R}^m$. We assume the polyhedra $\mathcal{P}_i$ are nonempty.

Show that this problem is equivalent to the LP

$$\begin{array}{ll}
\min & c^T x \\
\text{subject to} & d_i^T z_i \leq b_i, \quad i = 1, \cdots, m \\
& C_i^T z_i = x, \quad i = 1, \cdots, m \\
& z_i \geq 0, \quad i = 1 \cdots, m
\end{array}$$

with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1, \cdots, m$. *Hint: Find the dual of the problem of maximizing a_i^x over $a_i \in \mathcal{P}_i$ (with variable a_i).*

Solution:

The problem can be expressed as

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \cdots, m \end{array}$$

if we define $f_i(x)$ as the optimal value of the LP

$$\begin{array}{ll} \max & x^T a \\ \text{subject to} & C_i a \le d_i, \end{array}$$

where a is the variable, and x is treated as a problem parameter. The dual of this LP is given by

$$\begin{array}{ll} \min & d_i^T z \\ \text{subject to} & C_i^T z = x, \\ & z \ge 0. \end{array}$$

The optimal value of this dual LP is also equal to $f_i(x)$, so we have $f_i(x) \le b_i$ if and only if there exists a z_i with

$$d_i^T z_i \le b_i, \quad C_i^T z_i = x, \quad z_i \ge 0.$$

Problem 6 (`2 points`, Exer. 40 in *Linear Programming Exercises*): Prove the following result. If a set of m linear inequalities in n variables is infeasible, then there exists an infeasible subset of no more than $n+1$ of the m inequalities.

Solution: By the theorem of alternatives the system

$$A^T z = 0, \quad b^T z < 0, \quad z \ge 0$$

is true, that is, there is a vector z which satisfies this. This can be equivalently written as

$$A^T z = 0, \quad b^T z \le -\epsilon, \quad z \ge 0$$

for some $\epsilon > 0$. We can even claim that the following is true for this vector z

$$A^T z = 0, \quad b^T z = -\epsilon, \quad z \ge 0.$$

Define $P = \{z \mid A^T z = 0, \, b^T z = -\epsilon, \, z \ge 0\}$. A vector z in the polyhedron P is an extreme point if

$$\textbf{rank}\left(\begin{bmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_k} \\ b_{i_1} & b_{i_2} & \cdots & b_{i_k} \end{bmatrix}\right) = k$$

where $\{i_1, \cdots, i_k\} = I = \{i \mid z_i > 0\}$ (lecture notes, page 3-27) and a_i^T is row i of A. Since the vectors a_i have length n, the rank condition can only be satisfied if $k \le n+1$. Therefore, the extreme point z satisfies

$$\sum_{i \in I} z_i a_i = 0, \quad \sum_{i \in I} b_i z_i = -1, \quad z_i > 0 \text{ for } i \in I.$$

Such a vector z certifies that the inequalities $a_i^T x \le b_i$, $i \in I$ are infeasible. The set I has $k \le n+1$ elements.

<u>Problem 7</u> (`3 points`, Exer. 46 in *Linear Programming Exercises*): For the following two LPs, check (and prove whether) the proposed solution is optimal, by using duality:

1. For the LP

$$\begin{array}{ll} \text{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{subject to} & \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \le \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} \end{array}$$

   Is $x = (1, 1, 1, 1)$ optimal?

2. For the LP

$$\begin{array}{ll} \text{maximize} & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\ \text{subject to} & \begin{bmatrix} 1 & 3 & 5 & -2 & 3 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \le \begin{bmatrix} 4 \\ 3 \\ 5 \\ 1 \end{bmatrix}, x_i \ge 0, i = 1 \ldots 5 \end{array}$$

   Is $x = (0, 4/3, 2/3, 5/3, 0)$ optimal?

<u>Solution</u>:

1. Clearly, $x^* = (1, 1, 1, 1)$ is feasible: it satisfies the first four constraints with equality and the fifth constraint with strict inequality. To prove that x^* is optimal, we construct a dual optimal z^* as a certificate of the optimality. z^* must satisfy:

$$A^T z^* + c = 0, \quad z^* \ge 0, \quad z_k^*(b_k - a_k^T x^*) = 0, \quad k = 1, \ldots, 5.$$

From the complementarity conditions we see that $z_5^* = 0$, and the dual equality constraints reduce to a set of four equations in four variables

$$\begin{bmatrix} -1 & -1 & 0 & -6 \\ -6 & -2 & 3 & -11 \\ 1 & 7 & -10 & -2 \\ 3 & 1 & -1 & 12 \end{bmatrix} \begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \\ z_4^* \end{bmatrix} + \begin{bmatrix} 47 \\ 93 \\ 17 \\ -93 \end{bmatrix} = 0.$$

These equations have a unique solution $(3, 2, 2, 7, 0)$. Therefore,

$$z^* = (3, 2, 2, 7, 0).$$

This implies that the optimality condition holds, and $x^* = (1, 1, 1, 1)$ is optimal.

2. We consider the given problem as the dual problem and we can form the primal problem as

$$\begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & Ay \leq b \\ & y \leq 0, \end{array}$$

where

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 3 & 1 & 5 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -7 \\ -6 \\ -5 \\ 2 \\ -3 \end{bmatrix}, \quad c = \begin{bmatrix} -4 \\ -3 \\ -5 \\ -1 \end{bmatrix}.$$

Let us suppose $\tilde{x} = (0, 4/3, 2/3, 5/3, 0)^T$ is dual optimal, and $\tilde{y}$ is primal optimal. Then according to the optimality condition, we have

$$\begin{array}{l} \tilde{x}_i(b - A\tilde{y})_i = 0, \quad i = 1, \dots, 5, \\ c^T \tilde{y} = -b^T \tilde{x}, \end{array}$$

where the first constraints are according to the complementary slackness and the second constraint is due to the strong duality. We can solve the above equations, which contain four equations and four variables. Then we can get the solution: $\tilde{y} = (-1, -1, 0, -1)^T$.

Obviously, $\tilde{y}$ is not primal feasible, since the last constraint of $Ay \leq b$ is not satisfied. Therefore, $\tilde{x} = (0, 4/3, 2/3, 5/3, 0)^T$ is not optimal.