

# Lecture 7

Today:

## Problems from past exams

Prove that, for  $x \in R^n$ , if the function  $f(x)$  is a convex function, then the set  $C = \{x | f(x) \leq b\}$  is a convex set, with  $b \in R$  a given constant.

Can you find the solution to the following problem (call this P1), by solving an LP?

$$\begin{aligned} & \text{minimize}_x \quad ||x||_1^2 + 2||x||_1 \\ & \text{subject to} \quad Ax = b, \end{aligned} \tag{1}$$

where  $x \in R^n$ ,  $A$  is an  $m \times n$  matrix and  $b \in R^m$ . If yes, explain which LP you can solve, if not, explain why.

Consider the optimization problem

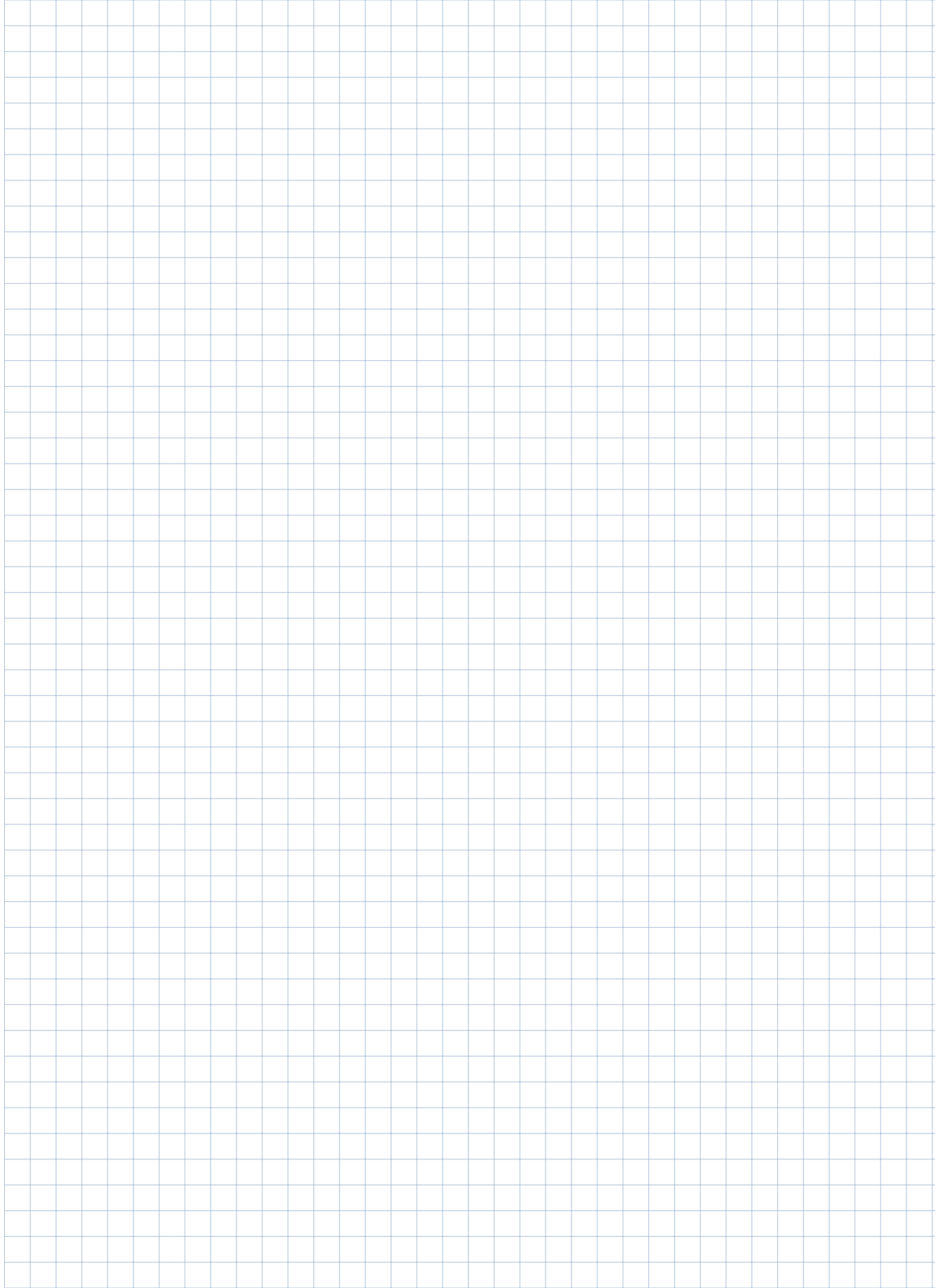
(1)

$$\begin{array}{ll}\min & \frac{3x_1^2 + 5x_2^2 + x_3 + x_4 + 6}{4x_1^2 + x_2^2 + x_4 + 1} \\ \text{s.t.} & x_3 - x_4 \leq 8 \\ & 0 \leq x_i \leq 10, \quad i \in \{1, 2, 3, 4\}\end{array}$$

Show that it is equivalent to the LP

(2)

$$\begin{array}{ll}\min & 3y_1 + 5y_2 + y_3 + y_4 + 6y_5 \\ \text{s.t.} & y_3 - y_4 \leq 8y_5 \\ & y_i \leq 100y_5 \quad i \in \{1, 2\} \\ & y_i \leq 10y_5 \quad i \in \{3, 4\} \\ & y_i \geq 0 \quad i \in \{1, 2, 3, 4\}\end{array}$$



## Duality

Assume we are given the LP:

$$\begin{array}{ll}\min & 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 4 \\ & x_1 \geq 2 \\ & x_1 + 3x_2 \geq 3 \\ & x_1, x_2 \geq 0\end{array}$$

Let  $x_1^*, x_2^*$  achieve optimal  $p^* = 2x_1^* + 3x_2^*$

Upper bound on  $p^*$ :

How do we find systematically the dual program.

Start primal

inequality  
form

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & \downarrow \\ & m \times n \\ & x \in \mathbb{R}^n \\ & \text{primal} \end{array}$$

$$\begin{array}{ll} \max & -b^T \lambda \\ \text{s.t.} & A^T \lambda + c = 0 \\ & \lambda \geq 0 \\ & \text{dual} \end{array}$$

To find the dual:

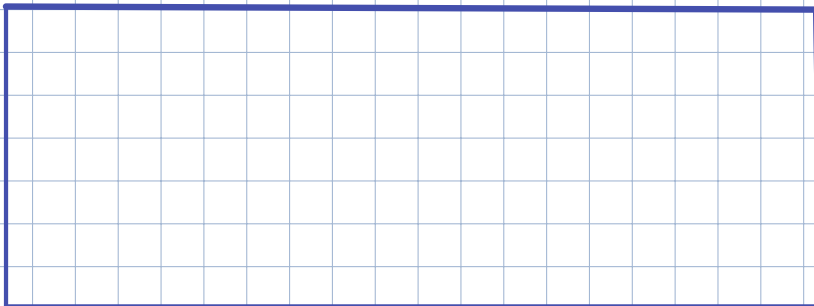
1) form Lagrangian:

2) form Lagrange dual function

Claim  
proof

$g(\lambda) \leq p^*$ , for any  $\lambda \geq 0$

To find the best lower bound,

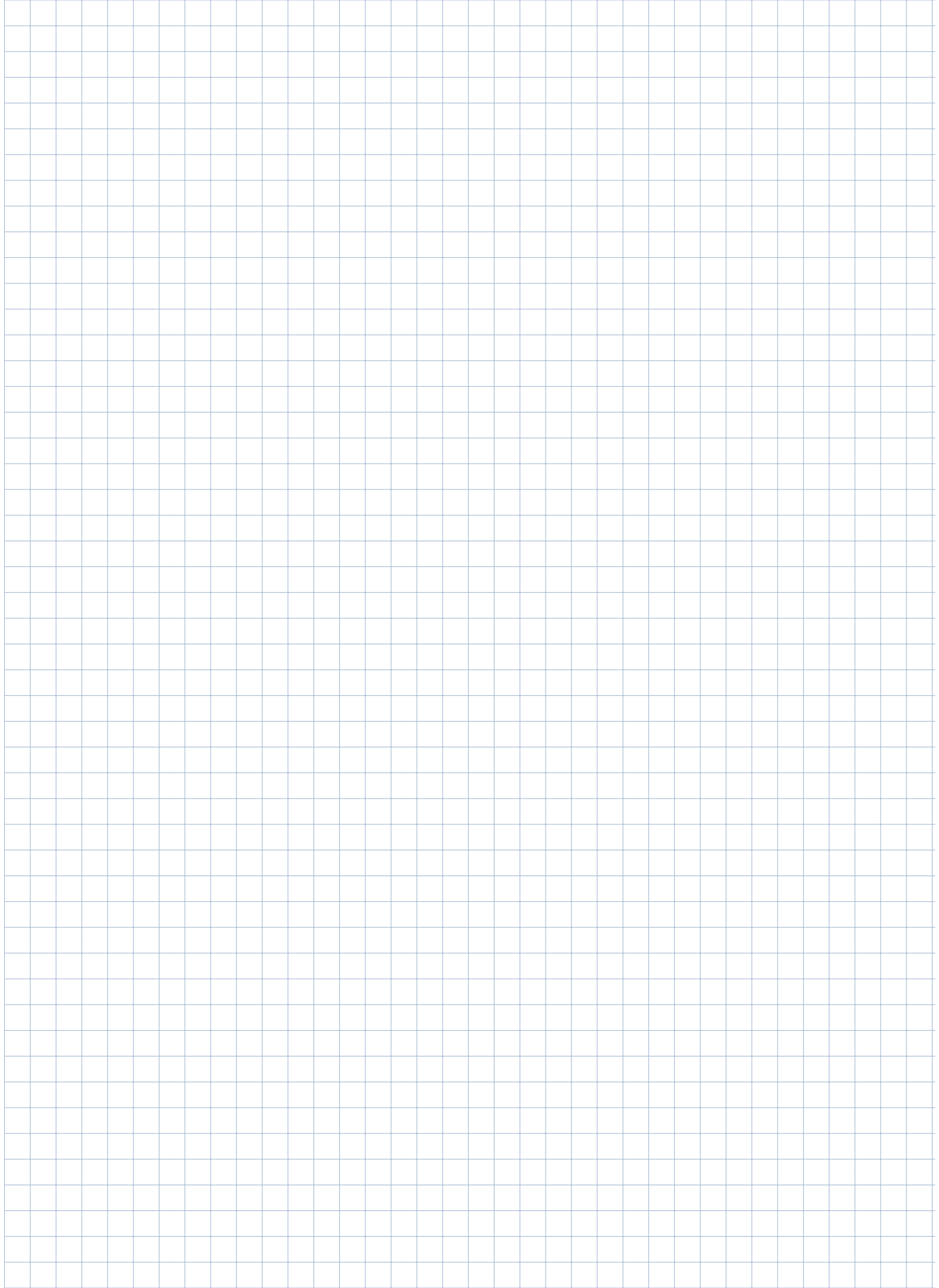


dual  
program

Apply formula to previous example:

$$\min (2 \ 3) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} -1 & -2 \\ -1 & 0 \\ -1 & -3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -4 \\ -2 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$





# Standard form

$p^*$  primal  
optimal value,  $x^*$  optimal

$$\min c^T x$$

$$s.t. \quad \underset{m \times n}{A} x = b$$

$$-x \leq 0$$

Dual

$$\begin{array}{ll}\max & -b^T \lambda_1 \\ \text{s.t.} & A^T \lambda_1 + c = \lambda_2 \\ & \lambda_2 \geq 0\end{array}$$