Lecture 7

Today:

Problems from past exams

Prove that, for $x \in R^n$, if the function $f(x)$ is a convex function, then the set $C = \{x | f(x) \le b\}$ is a convex set, with $b \in R$ a given constant.

Can you find the solution to the following problem (call this P1), by solving an LP?

$$\begin{aligned} &\text{minimize}_x \quad ||x||_1^2 + 2||x||_1 \\ &\text{subject to} \quad Ax = b, \end{aligned} \tag{1}$$

where $x \in R^n$, A is an $m \times n$ matrix and $b \in R^m$. If yes, explain which LP you can solve, if not, explain why.

Consider the optimization problem

(1)

$$
\begin{aligned}
\min \quad & \frac{3x_1^2 + 5x_2^2 + x_3 + x_4 + 6}{4x_1^2 + x_2^2 + x_4 + 1} \\
\text{st} \quad & x_3 - x_4 \leq 8 \\
& 0 \leq x_i \leq 10, \quad i \in \{1, 2, 3, 4\}
\end{aligned}
$$

Show that it is equivalent to the LP

(2)

$$
\begin{aligned}
\min \quad & 3y_1 + 5y_2 + y_3 + y_4 + 6y_5 \\
\text{st} \quad & y_3 - y_4 \leq 8y_5 \\
& y_i \leq 100y_5 \quad i \in \{1, 2\} \\
& y_i \leq 10y_5 \quad i \in \{3, 4\} \\
& y_i \geq 0 \quad i \in \{1, 2, 3, 4\}
\end{aligned}
$$

Duality

Assume we are given the LP:

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{st} \quad & x_1 + 2x_2 \geq 4 \\ & x_1 \geq 2 \\ & x_1 + 3x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Let x_1^*, x_2^* achieve optimal $p^* = 2x_1^* + 3x_2^*$

Upper bound on p^*:

How do we find systematically the dual program.

Start primal

inequality form

$$\min\ c^T x \quad \text{s.t.}\ \underset{m \times n}{A} x \leq b, \quad x \in \mathbb{R}^n$$

primal

$$\max\ -b^T \lambda \quad \text{s.t.}\ A^T \lambda + c = 0, \quad \lambda \geq 0$$

dual

To find the dual:

1) form Lagrangian:

2) form Lagrange dual function

Claim $\quad g(\lambda) \leq p^*$, for any $\lambda \geq 0$

proof

To find the best lower bound,

dual program

Apply formula to previous example:

$$\min \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\text{st} \quad \begin{pmatrix}-1 & -2\\ -1 & 0\\ -1 & -3\\ -1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \leq \begin{pmatrix}-4\\ -2\\ -3\\ 0\\ 0\end{pmatrix}$$

Standard form

primal
p^* optimal value, x^* optimal

$$\boxed{\begin{aligned} &\min\ c^T x \\ &\text{s.t.}\ \underset{m\times n}{A} x = b \\ &\qquad -x \le 0 \end{aligned}}$$

Dual

$$\boxed{\begin{aligned} \max\ & -b^T\lambda_1 \\ \text{st}\ & A^T\lambda_1 + c = \lambda_2 \\ & \lambda_2 \geq 0 \end{aligned}}$$