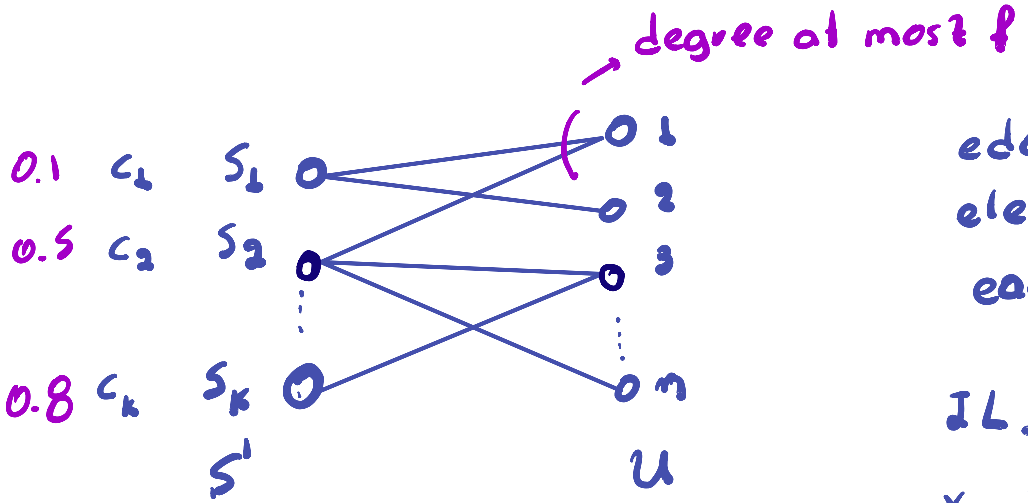
Lecture 14

Given a universe $\mathcal{U}$ of n elements, a collection of subsets of $\mathcal{U}$ $S = \{S_1, S_2, \dots, S_k\}$ and a cost function $c: S \to \mathbb{R}^+$, find a minimum cost subcollection of S' that covers all the elements in $\mathcal{U}$.



edges indicate which elements belong in each subset.

ILP

$$x_i = \begin{cases} 1 & \text{select } S_i \\ 0 & \text{otherwise} \end{cases}$$

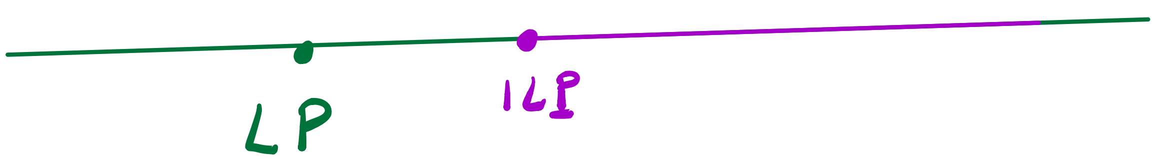
$$\min_{x_i} \sum_{i=1}^{k} c_i x_i$$

$$\text{s.t.} \quad \sum_{u \in S_i} x_i \geq 1 \leftarrow \text{all items are covered}$$

$$x_i \geq 0, \ \cancel{x_i \in \mathbb{Z}} \rightsquigarrow \ x_i \in \mathbb{R}, \text{ relaxed problem}$$

LP

fractional set cover.



Example of an approximation algorithm

- Assume that each element u_i appears in at most f sets.

Solve the LP relaxation and consider x^*, the optimal solution of the LP.
Let x_I be the solution of the ILP we will create.

→ if $x_i^* \geq \frac{1}{f}$, set $x_{Ii} = 1$

otherwise, set $x_{Ii} = 0$

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1) We need to argue that x_I is feasible in the ILP

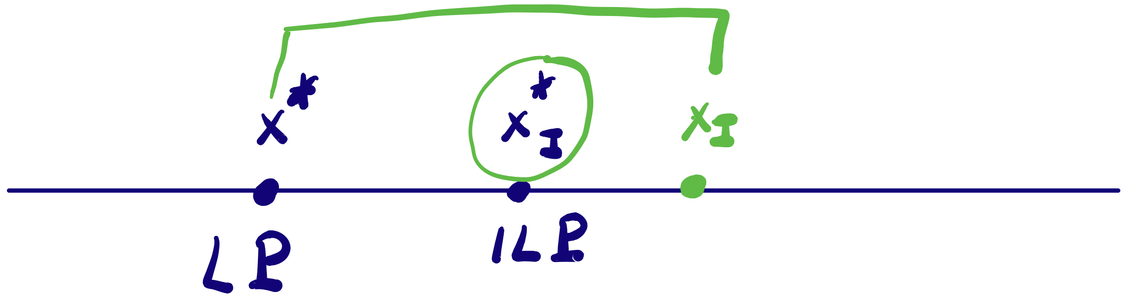
$$\min_{x_i} \sum_{i=1}^{K} c_i x_i$$

$$\text{s.t.} \quad \sum_{u \in S_i} x_i \geq 1$$ → for every item

$$x_i \geq 0, \quad x_i \in \mathbb{Z}$$

in the relaxed LP, we sum $\leq f$ values to something ≥ 1 ⇒ at least one of these values has to be $\geq \frac{1}{f}$ ⇒ this value will become 1 in the ILP and satisfy the constrain-

We need to check how much we increase the obj function value



in the worst case we will increase the obj function value by a factor of f.

$$\sum c_i \underbrace{x_i^*}_{\frac{1}{f}} \rightsquigarrow 1$$

$$R(I) = \frac{A(I)}{OPT(I)}$$

our alg achieves

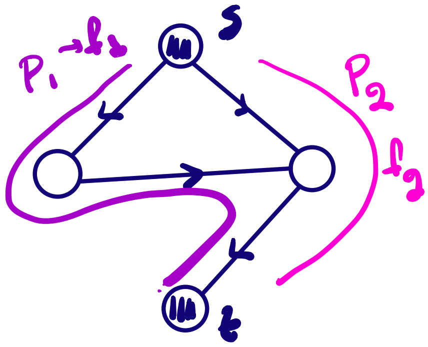
optimal ILP

approx gap depends instance I

specific bipartite graph costs

$$\max_I R(I) \le f$$

Equivalent formulation of the min-cut max-flow problem with paths.



Let $P = \{p_i\}$ be the set of all paths that connect s to t

Let f_i be the flow that we send from s to t using path p_i.

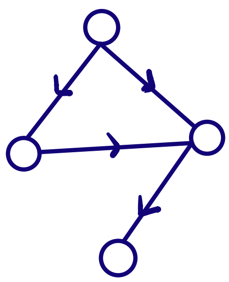
max flow

$$\max \sum_i f_i$$
st
$$\sum_{e \in p_i} f_i \leq c_e \sim d_e$$
$$f_i \geq 0 \quad \text{for each path}$$

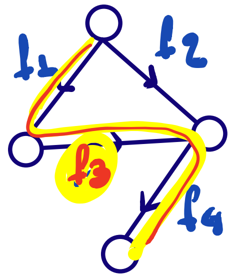
dual

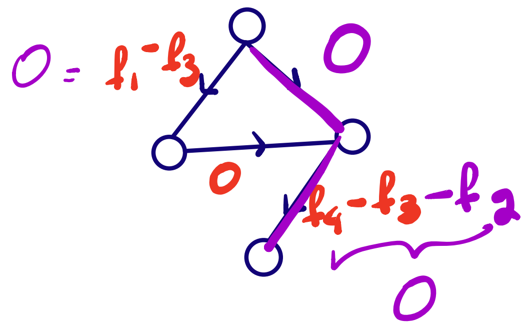
$$\min \sum c_e d_e$$
st
$$\sum_{e \in p} d_e \geq 1 \quad \text{for each path}$$
$$d_e \geq 0$$

Example



How do we prove that the flow and path formulations are equivalent?





path decomposition.

$\boxed{P_3 \rightsquigarrow f_3}$

$\boxed{P_2 \rightsquigarrow f_2}$