Fall 2020
EE 236A
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EE236A Linear Programming
Quiz 5 Solutions
Thursday December 10, 2020

NAME: ____________________ **UID**: ____________________

This quiz has 3 questions, for a total of 20 points.

Open book.
The exam is for a total of 1:00 hour. **Please, write your name and UID on the top of each sheet.**

Good luck!

Problem	**Mark**	**Total**
P1		6
P2		7
P3		7
Total		20

Problem 1 (`6 points`) Consider the following tableau

		x_1	x_2	1
y_1	$=$	-2	1	0
y_2	$=$	5	5	5
y_3	$=$	2	-2	2
z	$=$	2	1	-5

1) Find an LP this tableau corresponds to and then, find the associated dual program.
2) Find the optimal solutions for your primal; are there multiple or a unique solution for the primal?
3) Find the optimal solutions for your dual; are there multiple or a unique solution for the dual?

Solution:

1) The LP is as follows:

$$\begin{aligned} \min_{x} \quad & 2x_1 + x_2 - 5 \\ \text{subject to} \quad & -2x_1 + x_2 \geq 0 \\ & 5x_1 + 5x_2 + 5 \geq 0 \\ & 2x_1 - 2x_2 + 2 \geq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned} \tag{1}$$

We can equivalently represent the LP in the inequality form as:

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned} \tag{2}$$

where $A = \begin{bmatrix} 2 & -1 \\ -5 & -5 \\ -2 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 5 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. Then the dual is:

$$\begin{aligned} \max_{x} \quad & -b^T z \\ \text{subject to} \quad & A^T z + c = 0 \\ & z \geq 0 \end{aligned} \tag{3}$$

where $z \in \mathcal{R}^5$. In expanded form we have:

$$
\begin{aligned}
\max_{x} \quad & -5z_2 - 2z_3 \\
\text{subject to} \quad & 2z_1 - 5z_2 - 2z_3 - z_4 = -2 \\
& -z_1 - 5z_2 + 2z_3 - z_5 = -1 \\
& z_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0, z_5 \geq 0
\end{aligned}
\tag{4}
$$

2) The tableau corresponds to an optimal solution since there are no negative coefficients in the last row, in columns corresponding to x_1, x_2. Also note that there are no 0's in the last row, hence, there is only one optimal solution and it is the current vertex: $(x_1, x_2) = (0, 0)$.

3) From tableau we know that $x_1^\star = 0, x_2^\star = 0$ and $-2x_1^\star + x_2^\star = 0$ then due to complementary slackness conditions we have $z_2^\star = 0, z_3^\star = 0$. Furthermore, complementary slackness and equality constraint of the dual program gives us the following 4 linear inequalities in 5 variables:

$$-z_1 - 5z_2 + 2z_3 - z_5 = -1 \tag{5}$$

$$2z_1 - 5z_2 - 2z_3 - z_4 = -2 \tag{6}$$

$$z_2 = 0 \tag{7}$$

$$z_3 = 0 \tag{8}$$

For some $\alpha \geq 0$ let's set $z_1 = \alpha$, this gives us $z_4 = 2 + 2\alpha$ and $z_5 = 1 - \alpha$. We also have non-negativity constraints: $z_1 \geq 0, z_4 \geq 0, z_5 \geq 0$, as a result we have the the following set of optimal solutions: $z^\star = \left(\begin{bmatrix} \alpha \\ 0 \\ 0 \\ 2+2\alpha \\ 1-\alpha \end{bmatrix} : 1 \geq \alpha \geq 0 \right)$.

<u>Problem 2</u> (`7 points`)

Assume that $x \in R^n$; A_1, A_2, A_3 are constant matrices of dimension $m_1 \times n$, $m_2 \times n$ and $m_3 \times n$; and b_1, b_2, b_3, and c are constant vectors of appropriate dimension. Prove that the problem (9):

$$
\begin{aligned}
\min_{x} \quad & c^T x \\
\text{subject to} \quad & A_1 x = b_1 \\
& A_2 x \leq b_2 \\
& A_3 x \geq b_3 \\
& x \geq 0
\end{aligned}
\tag{9}
$$

is infeasible if and only if there exists a vector $\tilde{y} = (y_1, y_2, y_3)$, where each subvector y_i has m_i elements, $i = 1, \ldots 3$, such that:

$$
\begin{aligned}
& b_1^T y_1 + b_2^T y_2 + b_3^T y_3 > 0 \\
& A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \leq 0 \\
& y_2 \leq 0 \\
& y_3 \geq 0
\end{aligned}
\tag{10}
$$

Solution:

By the theorem of alternatives, either there exists an $x : Ax = b, x \geq 0$ or there exists a $y : A^T y \geq 0, b^T y < 0$. We will first show that the constraints of the program in (9) can be written as the first statement of the Farkas lemma and when we write the second statement of the Farkas lemma, we will obtain the inequalities in (10).

We can write the following equivalent program of the program in (9) by using slack variables s_1 and s_2.

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{subject to} \quad & A_1 x = b_1 \\ & A_2 x + s_1 = b_2 \\ & A_3 x - s_2 = b_3 \\ & x, s_1, s_2 \geq 0 \end{aligned} \tag{11}$$

If we define $\tilde{x} = \begin{bmatrix} x \\ s_1 \\ s_2 \end{bmatrix}$, $\tilde{c} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$, A $= \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & I & 0 \\ A_3 & 0 & -I \end{bmatrix}$ and b $= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then we can write the program in (11) as follows.

$$\begin{aligned} \min_{\tilde{x}} \quad & \tilde{c}^T \tilde{x} \\ \text{subject to} \quad & A\tilde{x} = b \\ & \tilde{x} \geq 0 \end{aligned} \tag{12}$$

The constraints of the program in (12) are the same as the first statement of the Farkas Lemma. Now, let's choose $y = \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \end{bmatrix}$ and write the second statement of the Farkas Lemma by using matrix A and vector b.

$$A^T y = \begin{bmatrix} A_1^T & A_2^T & A_3^T \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \end{bmatrix} = \begin{bmatrix} -A_1^T y_1 - A_2^T y_2 - A_3^T y_3 \\ -y_2 \\ y_3 \end{bmatrix} \geq 0 \tag{13}$$

$$b^T y = -b_1^T y_1 - b_2^T y_2 - b_3^T y_3 < 0 \tag{14}$$

We can see that the second statement is the same as the inequalities in (10). This proves that the program in (9) is infeasible if and only if there is a vector $\tilde{y} = (y_1, y_2, y_3)$ such that

$$\begin{aligned} & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 > 0 \\ & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \leq 0 \\ & y_2 \leq 0 \\ & y_3 \geq 0 \end{aligned} \tag{15}$$

Problem 3 (`7 points`): Facebook is looking to expand their TerraGraph network by installing mmWave nodes in 2354 cells. 513 of these cells already host 5 such nodes each; and no cell can host more than 10 nodes. Can you write an ILP that identifies how many nodes should Facebook install in each cell, so that the average number of nodes per cell is between 6 and 7, the minimum number of nodes per cell is 3, and we minimize the number of cells that have an odd number of nodes?

Solution:

We can solve this problem in two different ways. In the first solution, we can express the number of nodes at each cell by using two variables x_i and y_i such that $x_i \in \{0, 2, 4, 6, 8, 10\}$, $y_i \in \{0, 1\}$ where $i = 1, .., 2354$. Therefore, we can write the program that will minimize the number of cells that have an odd number of nodes as follows.

$$\begin{array}{ll} \min\limits_{x,y,\lambda} & 1^T y \\ \text{subject to} & 3 \leq x + y \leq 10 \\ & 6 \leq \frac{1}{2354} \sum_{i=1}^{2354} (x_i + y_i) \leq 7 \\ & x_i + y_i \geq 5 \qquad \forall i \in \delta \\ & x_i = 2\lambda_i, \quad i = 1, ..., 2354 \\ & \lambda_i \geq 0,\ \lambda_i \in \mathbb{Z}, \quad i = 1, ..., 2354 \\ & y_i \in \{0, 1\}, \qquad i = 1, ..., 2354 \end{array} \tag{16}$$

where set δ keeps the indices of 513 cells that had already 5 nodes each.

Since the number of nodes at each cell can be expressed as $x_i + y_i$ and x_i can take only even values, there will be an odd number of nodes in i^{th} cell if y_i becomes 1. Therefore, we can select the objective function as the minimization of the summation of y_i values so that we can make most of the y_i values 0 and minimize the number of cells that have an odd number of nodes. Moreover, the first constraint is put to make sure that the minimum number of nodes per cell is 3 and no cell can host more than 10 nodes. The second constraint is put to have the average number of nodes per cell between 6 and 7. The third constraint is put since 513 cells had already 5 nodes each.

In the second solution, our variables x_i corresponds to number of cells that have i number of nodes where $i = 3, ..., 10$. For example, x_3 corresponds to the number of cells that have 3 nodes each. Therefore, we can write the LP that can solve this problem as follows.

$$\begin{array}{ll} \min\limits_{x_3,\dots,x_{10}} & x_3 + x_5 + x_7 + x_9 \\ \text{subject to} & 6 \leq \frac{1}{2354} \sum_{i=3}^{10} x_i i \leq 7 \\ & \sum_{i=5}^{10} x_i \geq 513 \\ & \sum_{i=3}^{10} x_i = 2354 \\ & x_i \geq 0,\ x_i \in \mathbb{Z}, \qquad i = 3, ..., 10 \end{array} \tag{17}$$

The first constraint is to make sure that the average number of nodes per cell is between 6 and 7. Since we know that 513 cells had already 5 nodes each, we put the second constraint. The third constraint is to make sure that total number of cells is equal to 2354.