ECE 236A HW3 Shengze Ye 205418959

problem 1

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \begin{matrix} A_k \in \mathbb{R}^{p \times n} \\ b_k \in \mathbb{R}^p \end{matrix}$$

(a) Express the optimization problem

$$\min \sum_{k=1}^{m} \|A_k x - b_k\|_\infty \quad \text{as an LP}$$

$$\|A_k x - b_k\|_\infty = \max\{a_{ki}^T x - b_{ki}, -a_{ki}^T x - b_{ki}\}$$
$$i = 1, \dots p$$

Let $y_k = \|A_k x - b_k\|_\infty$

$$\therefore \Rightarrow \min \quad y_1 + y_2 + \cdots y_m$$

$$\text{s.t.} \quad -y_k \cdot \mathbf{1} \le A_k x - b_k \qquad y_k \in \mathbb{R}^m$$
$$A_k x - b_k \le y_k \cdot \mathbf{1} \qquad k = 1, 2, \dots m$$

(b) Suppose $\text{rank}(A) = n$

$$\min \quad \|Ax - b\|^2$$

Derive the dual problem and show it can be simplified as

$$\max \sum_{k=1}^{m} b_k^T z_k$$
$$\text{s.t.} \sum_{k=1}^{m} A_k^T z_k = 0$$
$$\|z_k\|_1 \le 1 \qquad k = 1, \dots, m$$

primal : $\min \ y_1+y_2+\cdots+y_m$

st $-y_k\cdot \mathbf{1} \le A_k x - b_k$

$A_k x - b_k \le y_k \cdot \mathbf{1}$

in matrix form :

$$\min \ (\underbrace{0,0,\cdots,0}_{n},\ \underbrace{1,1,1,\cdots,1}_{m}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{pmatrix}$$

s.t

$$\begin{bmatrix} -a_{11}^T- & -1 & 0 & 0 & \cdots & 0 \\ -a_{12}^T- & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \\ -a_{1p}^T & -1 & 0 & 0 & \cdots & 0 \\ -a_{21}^T- & 0 & -1 & 0 & \cdots & 0 \\ -a_{22}^T- & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \\ -a_{2p}^T- & 0 & -1 & 0 & \cdots & 0 \\ & \vdots & & & & \\ -a_{m1}^T- & 0 & \cdots & \cdots & 0 & -1 \\ -a_{m2}^T- & 0 & \cdots & \cdots & 0 & -1 \\ \vdots & & & & & \\ -a_{mp}^T- & 0 & \cdots & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{bmatrix} \le \begin{array}{c} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ -b_1 \\ -b_2 \\ \vdots \\ -b_m \end{bmatrix} \end{array} \begin{array}{l} \to u_1 \\ \\ \to u_k \\ \to u_m \\ \to v_1 \\ \\ \to v_k \\ \to v_m \end{array}$$

$\therefore$ The dual problem can be written as

$$\max \quad \sum_{k=1}^{m} b_k^T (u_k - v_k)$$

$$\text{s.t} \quad \sum_{k=1}^{m} A_k^T (u_k - v_k) = 0 \qquad ①$$

$$1^T (u_k + v_k) = 1 \qquad k = 1, \dots m$$

$$u_i \geq 0 \quad v_i \geq 0 \qquad i = 1, \dots, m$$

$$\max \quad \sum_{k=1}^{m} b_k^T z_k$$

$$\text{s.t} \quad \sum_{k=1}^{m} A_k^T z_k = 0 \qquad ②$$

$$\|z_k\|_1 \leq 1 \qquad k = 1, 2, \dots m$$

① $\Rightarrow$ ②

if u_k, v_k are feasible for ①,

let $z_k = u_k - v_k$

$$\sum_{k=1}^{m} A_k^T z_k = \sum_{k=1}^{m} A_k^T (u_k - v_k) = 0$$

$$\|z_k\|_1 = \|u_k - v_k\|_1 \leq \|u_k\|_1 + \|v_k\|_1 \leq 1 \qquad (u_k, v_k) \geq 0$$

$\therefore$ z_k is feasible in ②

② $\Rightarrow$ ①

if z_k is feasible for ②, then

let $(u_k)_i = \max\{(z_k)_i, 0\} + d_k$

$(v_k)_i = \max\{-(z_k)_i, 0\} + d_k$

$$\alpha_k = (1 - \|z_k\|_1) / 2p$$

$\therefore 1^T(u_k + v_k) = 1 \quad \sum_{k=1}^{m} (A_k^T (u_k - v_k)) = 0$

$\therefore u_k, v_k$ are feasible in (1)

$\therefore$ The dual problem is equivalent to

$$\text{(2)} \quad \begin{aligned} &\max \ \sum_{k=1}^{m} b_k^T z_k \\ &\text{s.t} \ \ A_k^T z_k = 0 \\ &\|z_k\|_1 \le 1 \qquad k = 1, \cdots, m \end{aligned}$$

3. For the setup of (b), show that the optimal value of (1) is bounded by

$$\sum_{k=1}^{m} \|r_k\|^2 / \max_{k=1,\dots,m} \|r_k\|_1$$

$r_k = A_k x_{ls} - b_k \quad$ for $k = 1, \dots, m$

The least square solution x_{ls} satisfy :

$$(A^T A) x_{ls} = A^T b$$

$$A^T (A x_{ls} - b) = 0$$

$$\therefore \sum_{k=1}^{m} A_k^T (A_k x_{ls} - b_k) = 0$$

$$\therefore \sum_{k=1}^{m} A_k^T r_k = 0$$

let $z_k = \dfrac{r_k}{\max \|r_k\|_1}$

$\|z_k\| \le 1$ and $\sum_{k=1}^{m} A_k^T z_k = 0$

$\therefore$ z_k is feasible in dual problem

$\therefore$ we plug in z_k in dual problem, with get a lower bound for the primal

$$\sum_{k=1}^{m} b_k^T z_k = \sum_{k=1}^{m} (b_k - A_k x_{ls})^T z_k$$

$$= \sum_{k=1}^{m} \|r_k\|^2 \Big/ \max \|r_k\|_1$$

Problem 2

$$\begin{aligned} \text{minimize} \quad & \|x - x_0\|_\infty \\ \text{s.t} \quad & Ax \le b \end{aligned}$$

(a) write this problem as an LP in standard form

$\Rightarrow$
$$\begin{aligned} \min \quad & y \\ \text{s.t} \quad & -y \cdot \mathbb{1} \le x - x_0 \qquad y \in \mathbb{R} \\ & x - x_0 \le y \cdot \mathbb{1} \\ & Ax \le b \end{aligned}$$

(b) Derive the dual problem, and show it is equivalent to the following

①

$$\max \quad (Ax_0 - b)^T w$$
$$\text{s.t.} \quad \|A^T w\|_1 \le 1$$
$$w \ge 0$$

$$L(x, y, z_1, z_2, w)$$
$$= y + w^T(Ax - b) + z_1^T(-y \cdot \mathbb{1} - (x - x_0)) + z_2^T(x - x_0 - y \cdot \mathbb{1})$$

$$= y + (z_1^T \cdot \mathbb{1} + z_2^T \cdot \mathbb{1}) \cdot (-y) + (w^T A + z_2^T - z_1^T) x - w^T b + x_0 (z_1^T - z_2^T)$$

$$g(\lambda) = \inf_{x, y} L(x, y, z_1, z_2, w)$$

$$\Rightarrow g(\lambda) = \begin{cases} x_0^T(z_1 - z_2) - b^T w & \text{if } \mathbb{1}^T z_1 + \mathbb{1}^T z_2 = 1 \\ & A^T w = z_1 - z_2 \\ & z_1, z_2, w \ge 0 \\ -\infty & \text{otherwise} \end{cases}$$

$\therefore$ The dual problem is

$$\max_{z_1, z_2, w} \quad x_0^T(z_1 - z_2) - b^T w$$

$$\text{s.t} \quad \mathbf{1}^T z_1 + \mathbf{1}^T z_2 = 1$$

$$A^T w = z_1 - z_2$$

$$z_1, z_2, w \geq 0$$

Similar to the previous problem, we can use
$t = z_1 - z_2$

$$\Rightarrow \quad \max \quad x_0^T t - b^T w$$

$$\|t\|_1 \leq 1$$

$$A^T w = t \qquad w \geq 0$$

$$x_0^T \cdot (A^T w) - b^T w = (x_0^T A^T - b^T) w$$

$$= (A x_0 - b)^T w$$

$\therefore$ This could be written as

$$\max_{w} \quad (A x_0 - b)^T w$$

$$\text{s.t} \quad \|A^T w\|_1 \leq 1$$

$$w \geq 0$$

Problem 3

X be an r.v in $\{a_1, a_2, \cdots, a_n\}$, $0 < a_1 < a_2 < \cdots < a_n$
and $\text{prob}(x = a_i) = p_i$

(a) $\max \ \text{prob}(x \geq \alpha)$ (2)

s.t $Ex = b$

write (2) as LP, α and b are given

The problem can be written as

$$\max \sum_{i: a_i \geq \alpha} p_i$$

$$\text{s.t} \quad \sum_{i=1}^{n} p_i a_i = b$$

$$\sum_{i=1}^{r} p_i = 1$$

$$p \geq 0$$

(b) Take the dual of the LP in (a), show that it can be reformulated as

$$\min \ \lambda b + v$$

$$\text{s.t} \quad \lambda a_i + v \geq 0 \quad \text{for all } a_i < \alpha$$

$$\lambda a_i + v \geq 1 \quad \text{for all } a_i \geq a$$

The problem has the form of dual problem pair

$\min \ c^T x$ $\qquad\qquad$ $\max \ -b^T z$

s.t $Ax \leq b$ $\qquad\qquad$ s.t $A^T z + c = 0$

$\qquad\qquad\qquad\qquad\qquad z \geq 0$

we first rewrite the LP problem in (a) in matrix format

$$\begin{array}{c} -\beta^T \\ \max \; -[0,\cdots 0,\underset{\substack{\uparrow \\ i:\,a_i\geq\alpha}}{-1},\cdots -1] \end{array} \begin{array}{c} z \\ \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \end{array}$$

$$\text{s.t} \quad \begin{array}{c} A^T \\ \begin{bmatrix} a_1, & a_2 & \cdots & a_n \\ 1, & 1 & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{c} z \\ \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \end{array} = \begin{array}{c} -C \\ \begin{bmatrix} b \\ 1 \end{bmatrix} \end{array} \qquad p_i \geq 0$$

let $\lambda = \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}$

$\therefore$ the primal problem is

$$\min \; (-b \;\; -1) \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}$$

$$\text{s.t} \; \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix} \begin{bmatrix} -\lambda_1 \\ -\lambda_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{array}{ll} \min & b\cdot\lambda_1 + \lambda_2 \\ \text{s.t} & -a_i\lambda_1 - \lambda_2 \leq 0 \\ & -a_i\lambda_1 - \lambda_2 \leq -1 \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & b\cdot\lambda_1 + \lambda_2 & \\ & \lambda_1 a_i + \lambda_2 \geq 0 & a_i < \alpha \\ & \lambda_1 a_i + \lambda_2 \geq 1 & a_i \geq \alpha \end{array}$$

$\lambda = \lambda_1$, $v = \lambda_2$

$\therefore \Rightarrow$ $\min \quad b\lambda + v$

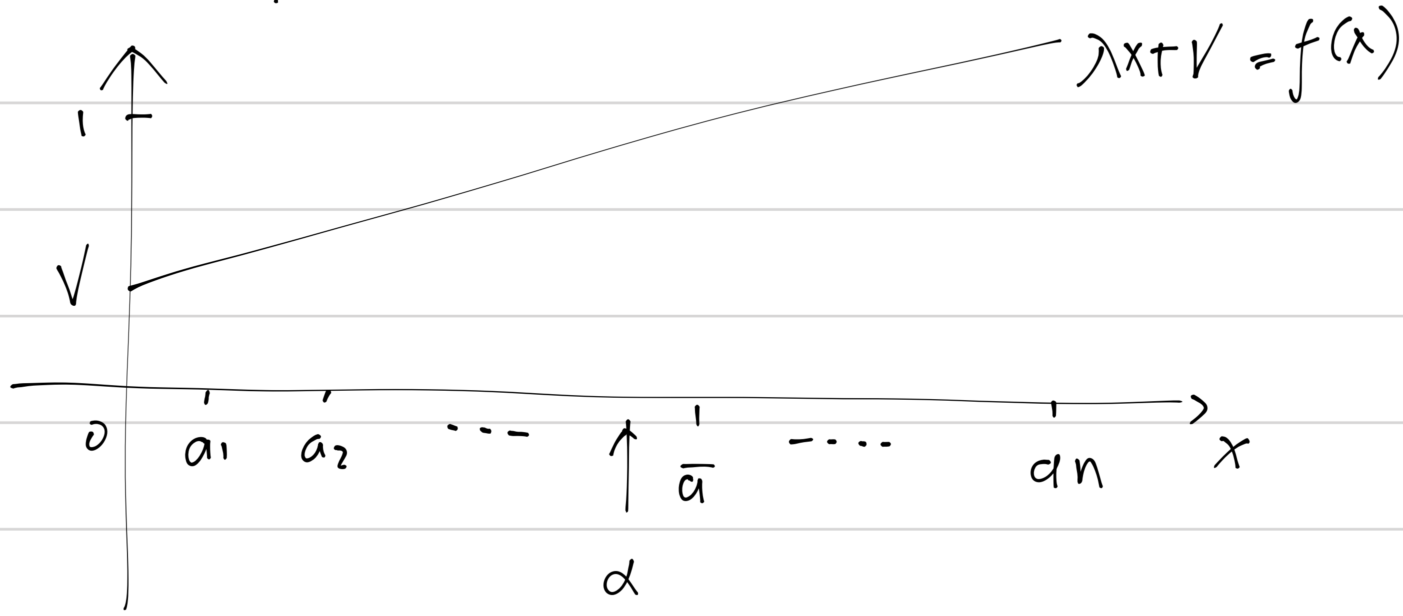
s.t $\quad \lambda a_i + v \geq 0 \quad$ if $a_i < \alpha$

$\quad \lambda a_i + v \geq 1 \quad$ if $a_i \geq \alpha$

Show that the optimal value is

$$\begin{cases} (b-a_1)/(\bar{a}-a_1) & b \leq \bar{a} \\ 1 & b \geq \bar{a} \end{cases}$$

$\bar{a} = \min\{a_i \mid a_i \geq \alpha\}$



$\therefore$ $f(x) \geq 0$ when $x < \bar{a}$

$f(x) \geq 1$ when $x \geq \bar{a}$

and we want $f(b)$ as small as possible

if $b \leq \bar{a}$

the optimal choice is $f(a_1) = 0$ and $f(\bar{a}) = 1$

$\therefore$ $v = -a_1/(\bar{a}-a_1)$ $\quad \lambda = 1/(\bar{a}-a_1)$

the optimal value is

$f(b) = (b-a_1)/(\bar{a}-a_1)$

if $b > \bar{a}$, the optimal choic is

$v = 1 \quad \lambda = 0$

The optimal value is $f(b) = 1$

Problem 4

Consider the following problem in x:

$$\min \quad c^T x$$
$$s.t \quad \|Ax + b\|_1 \le 1$$

$$\|Ax + b\|_1 = \sum_{i=1}^{m} |a_i^T x + b_i|$$
$$= \sum_{i=1}^{m} \max\{a_i^T x + b_i, -(a_i^T x + b_i)\}$$

let $y_i = \max\{a_i^T x + b_i, -(a_i^T x + b_i)\}$

$\therefore$ $a_i^T x + b_i \le y_i$

$-y_i \le a_i^T x + b_i$

$y_1 + y_2 + \cdots + y_m \le 1 \qquad \mathbb{1} \cdot y \le 1$

$\therefore \quad \Rightarrow \quad \min \quad c^T x$

$$\begin{aligned} s.t \quad & Ax + b \le y \\ & -(Ax + b) \le y \\ & \mathbb{1} \cdot y \le 1 \end{aligned}$$

in matrix format

$\Rightarrow$ $\min \ c^T x$

s.t $\begin{bmatrix} A & -I \\ -A & -I \\ 0 & 1^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -b \\ b \\ 1 \end{bmatrix}$

(b) Derive the dual LP and show that it is equivalent to the problem

$$\max \quad b^T z - \|z\|_\infty \qquad (2)$$

$$\text{s.t} \quad A^T z + c = 0$$

The dual problem can be written as

$$\max \quad b^T u - b^T v - w$$

$$\text{s.t} \quad A^T u - A^T v + c = 0 \qquad (1)$$

$$-u - v + w \cdot 1 = 0$$

$$u \geq 0 \quad , v \geq 0 \quad w \geq 0$$

(1) $\rightarrow$ (2)

assume u, v, w are feasible in (1)

let $z = u - v$

$A^T z + c = A^T(u - v) + c = 0$

for all j $|z_j| = |u_j - v_j| \leq |u_j + v_j| = w$

$\therefore \|z\|_\infty \leq w$

z is feasible in (2)

(2) → (1)

Assume z is feasible in (2)

let $w = \|z\|_\infty$

$u_i = \max\{z_i, 0\} + 0.5(w - |z_i|)$

$v_i = \max\{-z_i, 0\} + 0.5(w - |z_i|)$

$\therefore u_i \geq 0 \quad v_i \geq 0$

$A^T u - A^T v + c = A^T z + c = 0$

$-u - v + w\mathbb{1} = -w\mathbb{1} + w \cdot \mathbb{1} = 0$

$\therefore u, v, w$ are feasible in (1)

$\therefore$ The LP problem (1) and (2) are equivalent and

$$b^T u^* - b^T v^* - w^* = b^T z^* - \|z^*\|_\infty$$

Problem 5

consider the robust LP

$$\min \ c^T x$$

s.t $\quad \max_{a \in P_i} a^T x \leq b_i$

$P_i = \{a \mid C_i a \leq d_i\} \qquad c \in \mathbb{R}^n \qquad C_i \in \mathbb{R}^{m_i \times n}$

Show that this problem is equivalent to the LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t} \quad & d_i^T z_i \le b_i & j=1,2,\cdots,m \\ & c_i^T z_i = x & i=1,2,\cdots,m \\ & z_i \geq 0 & j=1,\dots,m \end{aligned}$$

let $t = \max_{a \in P_i} a^T x$

The problem is $\Rightarrow$ $\min \; c^T x$

$\text{s.t} \quad t \le b_i$

t is the optimal value of another LP

$$\begin{aligned} \max \quad & a^T x \\ \text{s.t} \quad & c_i a \le d_i \end{aligned}$$

where a is the variable

x is treated as constant

The dual of this LP is

$$\Rightarrow \begin{aligned} \min \quad & d_i^T z \\ \text{s.t} \quad & c_i^T z = x \\ & z \geq 0 \end{aligned}$$

The optimal value is equal to the optimal value of primal problem

$\therefore$ The whole problem is:

$$\Rightarrow \begin{aligned} \min \quad & c^T x \\ \text{s.t} \quad & d_i^T z_i \le b_i \\ & c_i^T z_i = x & i=1,2,\cdots,m \\ & z_i \geq 0 \end{aligned}$$

Problem 6

Prove the following results. If a set of m linear inequalities in n variables is infeasible, then there exists an infeasible subset of no more than n+1 of the m inequalities

Using the theorem of alternatives :

$$A^T z = 0 \quad , b^T z < 0 \quad , z \geq 0$$

the following is true for vector z

$$A^T z = 0 \, , \quad b^T z = -\varepsilon \quad z \geq 0$$

Define $P = \{ z \mid A^T z = 0, \ b^T z = -\varepsilon, \ z \geq 0 \}$ z is vertex

iff $\text{rank}\left(\begin{bmatrix} a_{i1}, & a_{i2} & \dots & a_{ik} \\ b_{i1}, & b_{i2} & \dots, & b_{ik} \end{bmatrix} \right) = k$

a_i^T is the ith row of A

Since vector a_i have length n, the rank condition can only be satisfied when

$$k \leq n+1$$

∴ There exists an infeasible subset of no more than n+1 of m inequalities.

Problem 7

Check the proposed solution is optimal

1. For the LP

$$\min \quad 47x_1 + 93x_2 + 17x_3 - 93x_4$$

$$\text{s.t} \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \le \begin{bmatrix} -3 \\ 5 \\ 8 \\ -7 \\ 4 \end{bmatrix}$$

Is $x = (1,1,1,1)$ optimal ?

$$Ax = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 3 \end{bmatrix} \le \begin{bmatrix} -3 \\ 5 \\ 8 \\ -7 \\ 4 \end{bmatrix}$$

$\rightarrow$ active

$\rightarrow$ not active

we construct a dual optimal $\lambda^* = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0)$

and $A^T\lambda^* = -C$

$$\begin{bmatrix} -1 & -1 & 0 & -6 \\ -6 & -2 & 3 & -11 \\ 1 & 7 & -10 & -2 \\ 3 & 1 & -1 & 12 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} -47 \\ -93 \\ -17 \\ 93 \end{bmatrix}$$

there are 4 equations and 4 variables

$\therefore$ we can find a unique solution

$x^* = (3, 2, 2, 7, 0)$

$\therefore$ The optimality condition holds, $x^* = (1, 1, 1, 1)$ is optimal

2. For the LP

$$\max \quad 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$$

primal

$$\text{s.t} \quad \underbrace{\begin{bmatrix} 1 & 3 & 5 & -2 & 3 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} \leq \underbrace{\begin{bmatrix} 4 \\ 3 \\ 5 \\ 1 \end{bmatrix}}_{b}$$

Is $x = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ optimal?

for the dual problem

we need that $A^T y + c = 0$

$$\begin{bmatrix} 1 & 3 & 5 & -2 & 3 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{4}{3} \\ \frac{2}{3} \\ \frac{5}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ \frac{14}{3} \\ 1 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 5 \\ 1 \end{bmatrix}$$

$\leftarrow$ not active

$\therefore$ The optimal solution for dual problem should have the form

$$y^* = (y_1^*, y_2^*, 0, y_4^*)$$

and

$$\begin{bmatrix} 1 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & -2 & 2 \\ -2 & 1 & -1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \\ y_4^* \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ -5 \\ 2 \\ -3 \end{bmatrix}$$

There is no solution for this equation

$\therefore$ $x = (0, \frac{6}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ is not optimal