Lecture 8

Primal	Dual
$\min\ c^T x$	$\max\ -b^T\lambda$
st $\ A x \leq b$, A: $m \times n$	st $\ A^T\lambda + c = 0$, $n \times m$
	$\lambda \geq 0$
$x \in \mathbb{R}^n$, x^*, p^*	$\lambda \in \mathbb{R}^m$, λ^*, q^*

$c^T = -\lambda^T A$

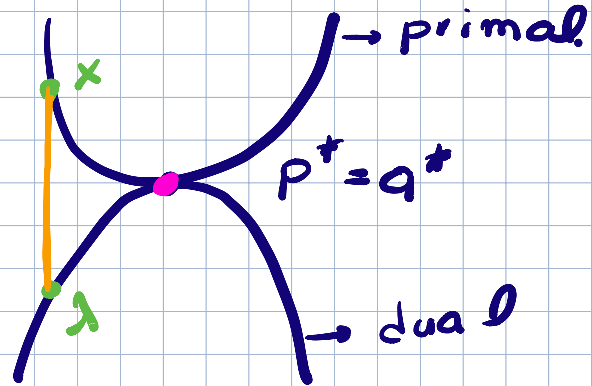
Observations

# of constraints in primal = # of variables in dual.

# of constraints in dual = # of variables in the primal.

Weak duality

Strong duality



Weak duality says:

if x is feasible in primal and λ is feasible in dual

$$\boxed{c^T x \geq -\lambda^T b}$$

Duality gap : $c^T x + \lambda^T b \geq 0$

Proof: We know that $c^T = -\lambda^T A$ since λ is feasible

$$c^T x = (-\lambda^T A)\, x \geq -\lambda^T b$$

because x feasible $Ax \leq b \Rightarrow -Ax \geq -b$ and $\lambda \geq 0$ ■

Weak duality implies

$$c^T x \geq p^* \geq q^* \geq -\lambda^T b$$

$p^* \geq q^*$

Strong duality: provided that either the primal or the dual are feasible, we have that $q^* = p^*$

→ Both feasible, $p^* = q^*$ take a finite value.
→ If primal is feasible, $p^* = -\infty$, dual infeasible.
→ If dual is feasible & $q^* = +\infty$, primal infeasible.

(Proof → we need theorem of alternatives)

Primal		Dual
$\min\ c^T x$ $\text{st}\ \underset{m \times n}{A} x \leq b$	↔	$\max\ -b^T \lambda$ $\text{st}\ \underset{n \times m}{A^T} \lambda + c = 0$ $\lambda \geq 0$
$x \in \mathbb{R}^n$, x^*, p^*		$\lambda \in \mathbb{R}^m$, λ^*, q^*

• At the optimal x^* and λ^*, the following 3 conditions hold:

$$\left.\begin{array}{ll} ① & x^* \text{ is feasible, } \quad A x^* \leq b \\ ② & \lambda^* \text{ is feasible, } \quad c + A^T \lambda^* = 0, \quad \lambda^* \geq 0 \\ ③ & c^T x^* + b^T \lambda^* = 0 \end{array}\right\} \text{certificate of optimality}$$

We can rewrite as follows:

$$\left.\begin{array}{l} c^T x^* + b^T \lambda^* = 0 \\ c^T = -\lambda^T A \end{array}\right\} \Rightarrow (-\lambda^{*T} A) x^* + \overbrace{b^T \lambda^*}^{\lambda^{*T} b} = 0$$

$$\Rightarrow \underset{\substack{\downarrow \\ (\lambda_1^*,\ \lambda_2^*, \dots, \lambda_m^*)}}{\lambda^{*T}} (A x^* - b) = 0$$

≥ 0

$$\begin{pmatrix} a_1^T x^* - b_1 \\ a_2^T x^* - b_2 \\ \vdots \\ a_m^T x^* - b_m \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\lambda_i^* (a_i^T x^* - b_i) = 0} \quad \text{for all } i = 1, \dots m$$

complementary slackness

if $\lambda_i^* > 0 \Rightarrow a_i^T x^* - b_i = 0$

if $a_i^T x^* - b_i < 0 \Rightarrow \lambda_i^* = 0$

Structure of the optimal solution.

x^* meets $J(x^*)$ constraints with equality: these exactly the indices where λ^* can take nonzero values

Second way to check optimality

Find x^* feasible and λ^* feasible such that $\lambda_i^* = 0$ when constraint i is not active for x^*.

Example.

$$\min \ (-4 \ \ -5) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}.$$

Show that $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is optimal.

x_0 is feasible ✓ it satisfies c_2 and c_4 with equality.

$\Rightarrow$ corresponding dual variables need to be of the form

$$\lambda = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ \lambda_4 \end{pmatrix}, \quad \lambda_2, \lambda_4 \geq 0$$

λ needs to be feasible $\Rightarrow$ $A^T\lambda = -c$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The vector $\lambda = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ is feasible in dual with correct sparseness

This proves the optimality of x_0.