Lecture 10

Today

Theorem of alternatives

Farkas lemma I

① and ② cannot be true at the same time:

Farkas Lemma II

For given A and b, either

① there exist x, with $Ax=b$ and $x \geq 0$

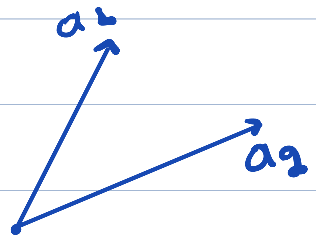
or

② there exists y with $A^T y \geq 0$, $b^T y < 0$

but not both

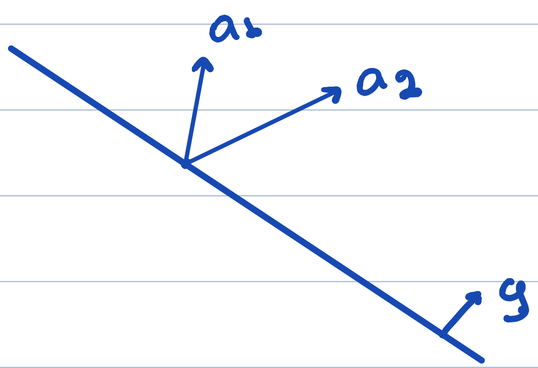
Geometric interpretation

$$Ax = b \Rightarrow \begin{matrix}|\\a_1\\|\end{matrix} x_1 + \begin{matrix}|\\a_2\\|\end{matrix} x_2 + \dots + \begin{matrix}|\\a_m\\|\end{matrix} x_m = \begin{matrix}|\\b\\|\end{matrix}$$



$x_i \geq 0$

(2) there exists a hyperplane with normal vector y that strictly separates the a's & b



Proof: Apply Farkas lemma I to

$$\left.\begin{matrix} Ax = b \\ x \geq 0 \end{matrix}\right\} \Rightarrow \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

Mixed inequalities & equalities

Either

1. there exists x

2. there exist y and z

Example 1 Let P be a matrix with elements p_{ij}, such that $p_{ij} \geq 0$ and the columns of P sum to 1, namely: $\sum_{i=1}^{n} p_{ij} = 1$, for $j=1,\dots,n$

Show that there exists $y \in \mathbb{R}^n$ such that

$$Py = y, \quad y \geq 0, \quad \sum_{i=1}^{n} y_i = 1$$

Strong duality proof

$$\begin{aligned} \min \;& c^T x \\ \text{st} \;& Ax \leq b \end{aligned} \qquad\qquad \begin{aligned} \max \;& -\lambda^T b \\ \text{st} \;& c + A^T \lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} &A^T u + t c = 0 \\ &A w \le t b \\ &b^T u + c^T w < 0 \\ &u \geqslant 0, \quad t \geqslant 0 \end{aligned}}$$

① $t > 0 \Rightarrow$

② $t = 0 \Rightarrow$