Fall 2020
EE 236A
Prof. Christina Fragouli
TAs: Mine Dogan and Kaan Ozkara

EE236A Linear Programming
Quiz 3
Tuesday November 10, 2020

NAME: ____________________ **UID**: ____________________

This quiz has 3 questions, for a total of 20 points.

Open book.
The exam is for a total of 1:00 hour. **Please, write your name and UID on the top of each sheet.**

Good luck!

Problem	**Mark**	**Total**
P1		6
P2		7
P3		7
Total		20

Problem 1 (`6 points`) Prove that the optimization problem in (1) is not feasible.

$$
\begin{aligned}
\min_{x_1,x_2,x_3,x_4,x_5} \quad & \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} \\
\text{subject to} \quad & \begin{bmatrix}1 & 2 & 0 & 3 & 1\\0 & 2 & 0 & 1 & 1\\0 & 0 & 1 & 0 & 2\\0 & 1 & 0 & 1 & 0\\0 & 1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}3\\1\\1\\5\\2\end{bmatrix} \\
& x_i \geq 0 \text{ for i } = 1,\dots,5
\end{aligned}
\tag{1}
$$

Solution:

By the theorem of alternatives, either there exists an $x : Ax = b, x \geq 0$ or there exists a $y : A^T y \geq 0, b^T y < 0$. Then, to show that the above problem is infeasible we need to show there exists a $y : A^T y \geq 0, b^T y < 0$. From $A^T y \geq 0, b^T y < 0$ we have the following linear inequalities:

$$
\begin{aligned}
y_1 &\geq 0 \\
2y_1 + 2y_2 + y_4 + y_5 &\geq 0 \\
y_3 &\geq 0 \\
3y_1 + y_2 + y_4 + y_5 &\geq 0 \\
y_1 + y_2 + 2y_3 + 2y_5 &\geq 0 \\
3y_1 + y_2 + y_3 + 5y_4 + 2y_5 &< 0
\end{aligned}
\tag{2}
$$

We can satisfy these inequalities by choosing $y = (5, 0, 0, -4, 0)$. Hence, by the theorem of alternatives, the program in (1) is infeasible.

Problem 2 (`7 points`) Show that any $k+2$ points in $\mathbb{R}^k$ can be partitioned into two groups: $(v_i), i \in I$, and $(v_j), j \notin I$, whose convex hulls intersect.

(*Hint:* Argue that the vectors $\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_{k+2} \\ 1 \end{pmatrix}$ are linearly dependent, and thus there exists a set of not-all-zero coefficients $a_1, \dots, a_{k+2}$ such that: $\sum_{i=1}^{k+2} a_i \begin{pmatrix} v_i \\ 1 \end{pmatrix} = \mathbf{0}$.)

Solution:

Given the hint, we know that there exist a set of coefficients not all zeros $a_1, \dots, a_k$ such that:

$$
\sum_{i=1}^{k+2} a_i \begin{pmatrix} v_i \\ 1 \end{pmatrix} = \mathbf{0}
\tag{3}
$$

Then we have the following linear equalities:

$$\sum_{i=1}^{k+2} a_i v_i = \mathbf{0}, \quad \sum_{i=1}^{k+2} a_i = 0 \tag{4}$$

Let's denote $I = \{i|a_i > 0\}$. From the fact that $\sum_{i=1}^{k+2} a_i = 0$, we have $\sum_{i\in I} a_i = -\sum_{i\notin I} a_i$. And from the fact that $\sum_{i=1}^{k+2} a_i v_i = \mathbf{0}$ we have $\sum_{i\in I} a_i v_i = -\sum_{i\notin I} a_i v_i$. Denote $A = \sum_{i\in I} a_i = -\sum_{i\notin I} a_i$. Then, dividing both sides of $\sum_{i\in I} a_i v_i = -\sum_{i\notin I} a_i v_i$ by A, there exists some point p:

$$p = \sum_{i\in I} \frac{a_i}{A} v_i = -\sum_{i\notin I} \frac{a_i}{A} v_i \tag{5}$$

Note that, for $i \in I, a_i > 0$ and for $i \notin I, -a_i \geq 0$ also not that $\frac{\sum_{i\in I} a_i}{A} = 1$ and $\frac{\sum_{i\notin I} -a_i}{A} = 1$. Then we see that $\sum_{i\in I} \frac{a_i}{A} v_i$ is in the convex hull of $(v_i), i \in I$ and $\sum_{i\notin I} \frac{-a_i}{A} v_i$ is in the convex hull of $(v_i), i \notin I$. This implies that the point p is in the convex hull of the two sets: $(v_i), i \in I$, and $(v_j), j \notin I$.

Problem 3 (`7 points`): Assume we are given an LP in the standard form, with A an $m \times n$ matrix:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{6}$$

Recall that, as we derived in class, the associate dual program can be expressed as:

$$\begin{aligned} \max_{y} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned} \tag{7}$$

Assume now that the primal LP has a very large number of variables (i.e. n is very large). A friend of yours claims that we can use the following trick to solve this LP more efficiently: She claims that we can consider a subset of $n_I << n$ variables, say all variables x_i with $i \in I$, where I is a set of indices, and solve instead the following LP:

$$\begin{aligned} \min_{x_I\in\mathbb{R}^{n_I}} \quad & c_I^T x_I \\ \text{subject to} \quad & A_I x_I = b \\ & x_I \geq 0 \end{aligned} \tag{8}$$

where c_I only keeps the values corresponding to the indices in I, and A_I keeps the columns of A corresponding to the indices in I. Assume the optimal solution to the problem in (8) is x_I^* and the corresponding dual optimal variables are y_I^*. Your friend claims that for the dual optimal variables, if $c_i + a_i^T y_I^* \geq 0$ for all $i = 1, \ldots, n$, where a_i is the i^{th} column vector of A, then you can directly find from x_I^* the optimal solution for the problem in (6) x^*. Is your friend right or wrong? Prove your claim.

Solution:

Your friend is right. We will show that the vector $x \in R^n$ with elements

$$x_i = \begin{cases} (x_I^\star)_i & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases} \tag{9}$$

is optimal, by providing a "certificate of optimality", namely, a vector y that is feasible in the dual, and such that the dual and the primal achieve the same objective function value. (Recall that if we find a feasible solution for the primal and a feasible solution for the dual that lead to the same objective function value, these solutions are optimal).

For convenience, let P1 denote the primal in (6), D1 the associated dual in (7), P2 the primal in (8), and the associated dual in (10) as D2.

$$\begin{aligned} \max_{y_I \in \mathbb{R}^m} \quad & -y_I^T b \\ \text{subject to} \quad & A_I^T y_I + c_I \geq 0 \end{aligned} \tag{10}$$

Let $x_I^\star$ and $y_I^\star$ be the optimal solutions of P2 and D2, from strong duality we have that

$$c_I^T x_I^\star = -(y_I^\star)^T b \tag{11}$$

Note that, because the condition $c_i + a_i^T y_I^* \geq 0$ for all $i = 1, \ldots, n$ is satisfied, $y_I^\star$ is feasible in D1, and achieves the value $-(y_I^\star)^T b$. Note also that x in (9) is feasible in P1 and achieves the value $c_I^T x_I^\star$. From (11) these values are equal, and the proof is concluded.