ECE 236A Shengze Ye HW4 205418959

problem 1

let us consider the primal and dual

$$\min \; c^T x \quad \text{s.t } Ax \le b \quad \text{(primal)}$$

$$\max \; -\lambda^T b \quad \text{s.t } A^T\lambda + c = 0, \; \lambda \geq 0 \quad \text{(dual)}$$

From weak duality, we have

$$-\lambda^{*T} b = d^* \le p^* = c^T x^*$$

The dual can be written as

$$\min \; \lambda^T b$$

$$\text{s.t } A^T\lambda + c = 0 \quad \text{and } \lambda \geq 0$$

Then we derive the dual of dual

$$L(\lambda, x, v) = \lambda^T b + x^T(A^T\lambda + c) - v\lambda$$

$$g(x, v) = \inf_{\lambda} L(\lambda, x, v)$$

$$= \begin{cases} c^T x & \text{if } (Ax+b = v) \\ -\infty & \text{otherwise} \end{cases}$$

$\therefore$ The dual of dual is

$$\max \; c^T x \quad \text{s.t } Ax+b = v, \; v \geq 0 \quad \Rightarrow \quad \max \; c^T x \quad \text{s.t } Ax+b \geq 0$$

if we take $x = -x$

The dual of dual is the same of primal

$$\max \quad -c^T x$$

$$\text{s.t} \quad Ax \le b$$

From dual and dual of dual, we have

$$x^{*T} b = -d^* \geq -p^* = -c^T x^*$$

$$\therefore \quad -d^* \geq -p^*$$

we still have $p^* \geq d^*$, we cannot conclude $p^* \le d^*$

$\therefore$ This proof of strong duality is flawed.

problem 2

$$\text{maximize} \quad f_{ts}$$

$$\text{s.t} \quad f_{ij} \le c_{ij} \quad \forall ij \in \mathcal{E}$$

$$\sum_{j,i \in \mathcal{E}} f_{ji} - \sum_{i,k \in \mathcal{E}} f_{ik} \le 0 \quad \forall i \in V$$

$$f_{ij} \geq 0 \quad (i,j) \in \mathcal{E}$$

(max-flow)

$$\min \quad c^T d$$

$$\text{s.t} \quad p_s - p_t \geq 1$$

$$d_{ij} \geq p_i - p_j$$

$$d_{ij} \in \{0,1\}$$

$$p_i \in \{0,1\}$$

(min-cut)

(a) Prove that the constraint matrix of the max-flow is TUM

As we did in the class, we should first write the max-flow LP in matrix format

$$\min \quad -e_n^T f$$
$$\text{s.t} \quad (I \quad 0) f \le c \qquad f \in \mathbb{R}^n$$
$$M^T f = 0$$
$$-f \le 0$$

M is the edge-vertex adjacency matrix $\qquad M: |\mathcal{E}|+1 \times |V|$

(1 for f_{ts})

$$\Rightarrow \quad \min \quad -e_n^T f$$
$$\text{s.t} \quad \underbrace{\begin{bmatrix} I_{|\mathcal{E}|} & 0 \\ -I_{|\mathcal{E}|+1} & \\ M^T & \end{bmatrix}}_{A} f \le \begin{bmatrix} c \\ \\ 0_{|\mathcal{E}|+1+|V|} \end{bmatrix}$$

we have known that M is TUM

Now consider the sub-matrix of A

if all rows belong to M, then det $=0$ (M is TUM)

if all rows belong to the above part $\begin{pmatrix} I_{|\mathcal{E}|} & 0 \\ -I_{|\mathcal{E}|+1} & \end{pmatrix}$

we know that det $=0$, and only contains $\{1, -1, 0\}$

if some rows belong to above and some come from M, we can expand along the column that contains the $1/-1$, and check the det of the small sub-matrix

$\therefore$ matrix A is TUM, the constraint matrix is TUM

(b) Prove that the dual of the max-flow LP gives the same solution

The dual LP:

$$\min\ c^T d$$
$$\text{s.t}\quad p_s - p_t \geq 1$$
$$d_{ij} \geq p_i - p_j$$
$$d_{ij} \geq 0 \quad p_i \geq 0$$
①

Integer LP:

$$\min\ c^T d$$
$$\text{s.t}\quad p_s - p_t \geq 1$$
$$d_{ij} \geq p_i - p_j$$
$$d_{ij} \in \{0,1\}, \quad p_i \in \{0,1\}$$
②

The dual LP could be written as:

$$\min\ c^T d$$

$$\text{s.t}\quad \underbrace{\begin{bmatrix} 0_{1\times|E|} & \\ I_{|E|} & M^T \\ & \\ -I_{|E|+|V|} & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d \\ p \end{bmatrix}}_{x} \geq \underbrace{\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}}_{b}$$

consider an extreme point in ①. let J be the set of active constraints at the extreme point.

A_J is non-singular submatrix of A, A_J is TUM

from the class we know that

if A_J is non-singular submatrix of A

A_J^{-1} is an integral matrix

$$A_J \cdot x_J = b' \qquad x_J = A_J^{-1} \cdot b'$$

b' has only single non-zero element (1)

$\therefore$ the extreme point x_J is integral

The optimal value obtained from (1) and (2) should be same,

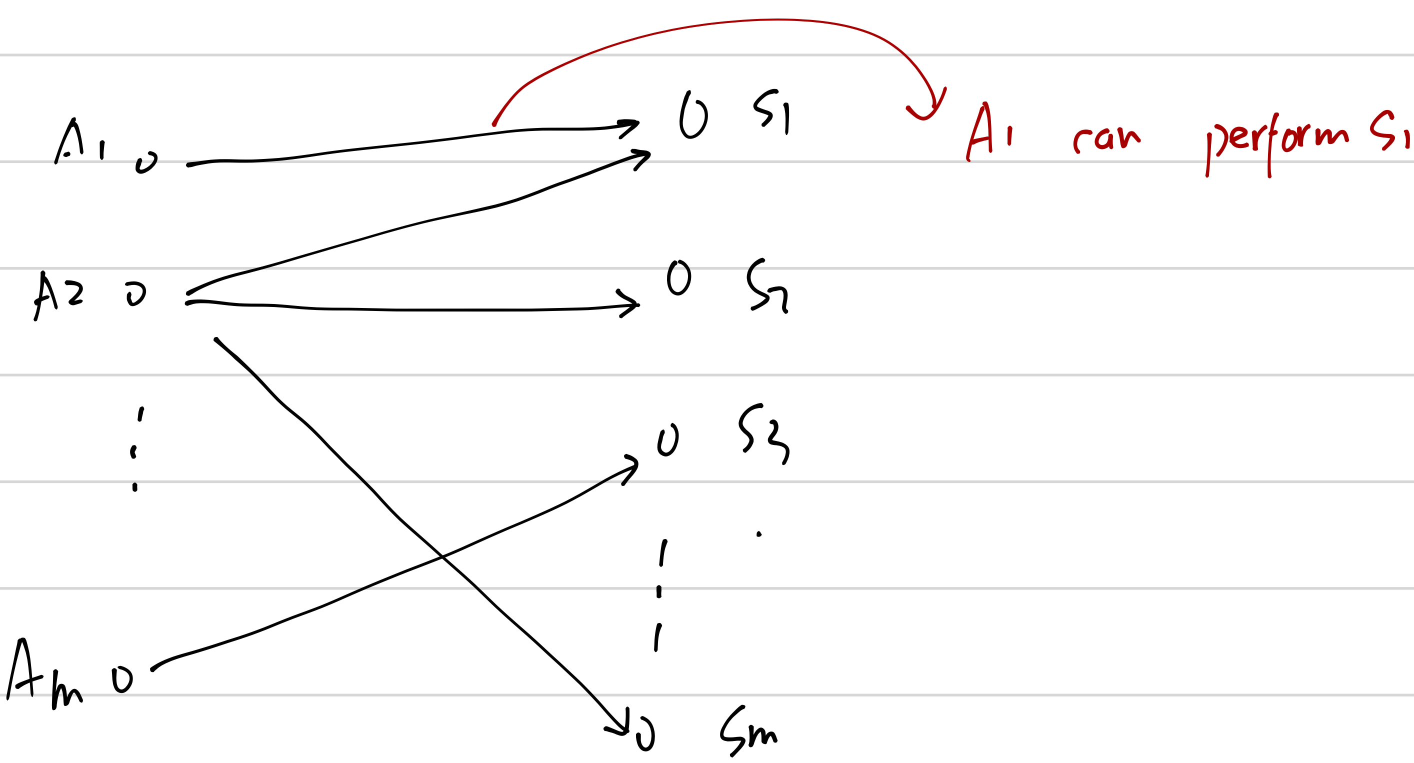
$\therefore$ The dual of max-flow LP gives the same solution as its ILP.

Problem 3.

Surgeries $S = \{S_1, S_2, \cdots S_n\}$

Surgeon $A = \{A_1, A_2 \cdots, A_m\}$

each candidate can perform several surgeries, this relation can be illustrated by a graph



A connection means that surgeon A_i could perform S_j

we can use a vector to denote that

$$p_i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \begin{array}{l} \\ \to \text{2th surgery} \\ \\ \vdots \\ \to \text{n-1 th surgery} \\ \to \text{n th} \end{array}$$

-> means that ith surgeon can perform 2th and n-1 th surgery

once we have the set S and A,

$\Rightarrow$ we know $p_1, p_2 \cdots p_m$ (constant)

Then we introduce the salary $\Rightarrow S_i$ be the salary required by surgeon number i

and X_i to denote whether we select surgeon number i

$\therefore x_i \in \{0,1\}$

$\therefore$ We can write the ILP

$$\min \sum_{i=1}^{m} X_i \cdot S_i$$

s.t

$$X_1 \cdot \begin{vmatrix} \\ p_1 \\ \\ \end{vmatrix} + X_2 \cdot \begin{vmatrix} \\ p_2 \\ \\ \end{vmatrix} + \cdots + X_m \begin{vmatrix} \\ p_m \\ \\ \end{vmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{(n \times 1)}$$

each $p_i \in \mathbb{R}^n$

$\to$ (handle all the surgeries)

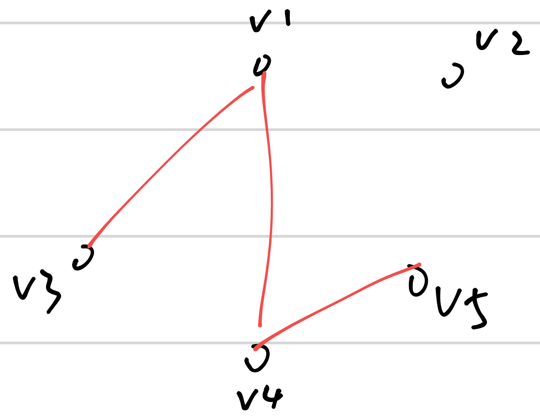
$$s_i \geq 0 \qquad i = 1, 2, \cdots m$$

$$b_i \in \{0,1\}$$

Problem 4

a) Write an ILP associated with the question

The problem can be described as a graph. we use the node v_i to denote the player i. we use the connection e_{ij} to note that match-ups between i and j.



Use x_e to denote whether we use the match-up

let $N(v) \subseteq \mathcal{E}$ be the set of edges that are adjacent to v

$\therefore$ we can formulate the ILP

$$\max \sum_{e \in \mathcal{E}} x_e$$

$$\text{s.t.} \quad \sum_{e \in N(\mathcal{E})} x_e \le 1 \quad \forall v$$

$$x_e \in \{0, 1\}$$

We can express it using a matrix (TUM)

let $M = \begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_n \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & \cdots & & v_m \end{array} \\ \left[\begin{array}{ccccc} 1 & & & & 1 \\ & 1 & & 1 & \\ & & & & \\ & & & & \\ & & & 1 & 1 \end{array} \right] \end{array}$

$|\mathcal{E}| \cdot |V|$

e_1: v_1 —— v_m

each match-up will set two "1" to the corresponding player

$\therefore$ The ILP can be writen as

$$\max \quad \mathbb{1} \cdot x$$

$$\text{s.t.} \quad \underbrace{M^T}_{|V| \times |E|} \cdot \underbrace{x}_{|E| \times 1} \le 1$$

$$x_i \in \{0, 1\}$$

(b) Coach P want to arrange matches on three days,

$\therefore \Rightarrow$ each player can play in up to 3 different matches

Write the ILP.

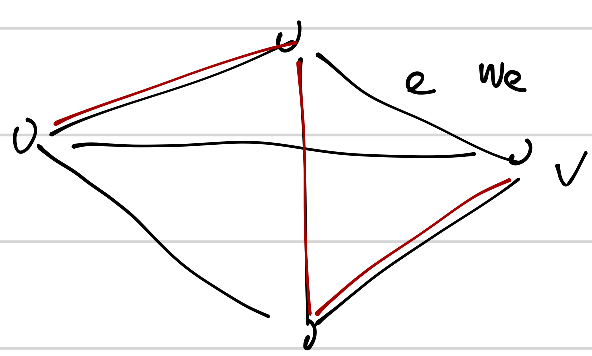
$$\therefore \Rightarrow \quad \max \quad \mathbb{1} \cdot x$$

$$\text{s.t.} \quad M^T \cdot x \le 3$$

$$x_i \in \{0, 1\} \quad \text{for all } i$$

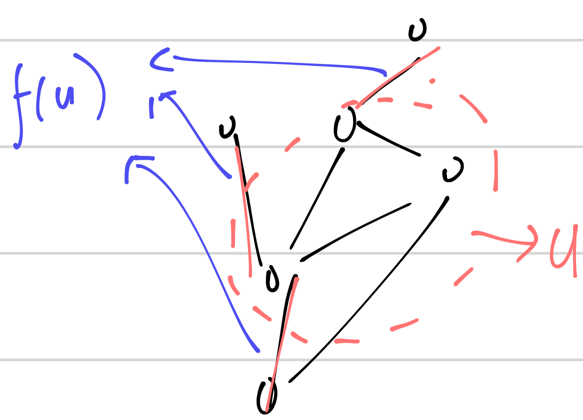
Problem 5

We can model the problem using an undirected graph



The nodes denote the houses, and we have edges between nodes, for each edge, we could set a penalty term, represents the distance between two nodes we want to find a subset of $\mathcal{E}$ to across all the nodes with minimum sum of penalty, and do not go back to the nodes that visited.

define $f(U) = \{(i,j) \in \mathcal{E} \mid i \in U, j \notin U\}$



∴ The problem can be formulated as

$$\min \sum_{e \in \mathcal{E}} w_e x_e \qquad (x_e : \text{whether we select this edge or not})$$

s.t.

① $\sum_{e \in f(\{v\})} x_e = 2 \quad \forall v \in V$

② $\sum_{e \in f(U)} x_e \geq 2 \quad \forall U \subset V \quad 2 \leq |U| \leq |V|-1$

$x_e \in \{0,1\}$

① because each node in the solution has degree 2

② because we need to exclude the situation of multiple disjoint cycles

Problem 6

$$\max \sum f_i$$
$$\text{st} \quad \sum_{i: e \in P_i} f_i \le c_e \qquad \forall \text{ edge } e$$
$$f_i \ge 0$$

$P = \{P_i\}$ is the set of all paths from s to t.

and f_i is the flow associated with the ith path.

① prove that the LP is equivalent to the max-flow

The max-flow LP is

$$\max \quad f_{ts}$$
$$\text{s.t.} \quad f_{ij} \le c_{ij} \qquad \forall (i,j) \in E$$
$$\sum_{j \in E} f_{ji} - \sum_{k \in E} f_{ik} \le 0 \qquad \forall i \in V$$
$$f_{ij} \ge 0 \quad (i,j) \in E \qquad ①$$