Lecture 6

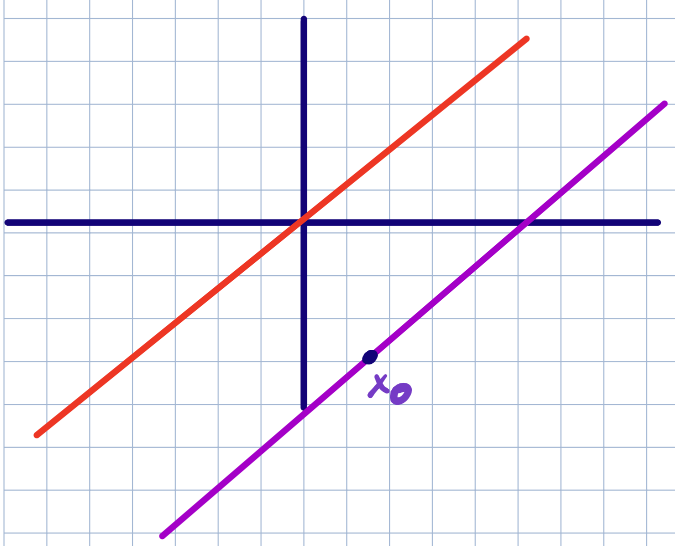
In the last lecture:

Convex :

Affine :

Subspace :

For every affine set we have a parallel subspace



"Dimension" of affine set

$v_1, v_2, \dots, v_k$ "affinely independent"

Convex Cone

Problem 1

Polyhedron

$$P = \{ x \mid a_i^T x \le b_i \underset{i=1,\dots,m}{} , \; c_i^T x = d_i \underset{i=1,\dots,k}{} \}$$

$$P = \{ x \mid Ax \le b, \; Cx = d \}$$

Linearity Space $\quad \mathcal{L} = N\begin{pmatrix} A \\ C \end{pmatrix}$

Pointed Polyhedron $\quad \mathcal{L} = \{0\}$

Face of a polyhedron.

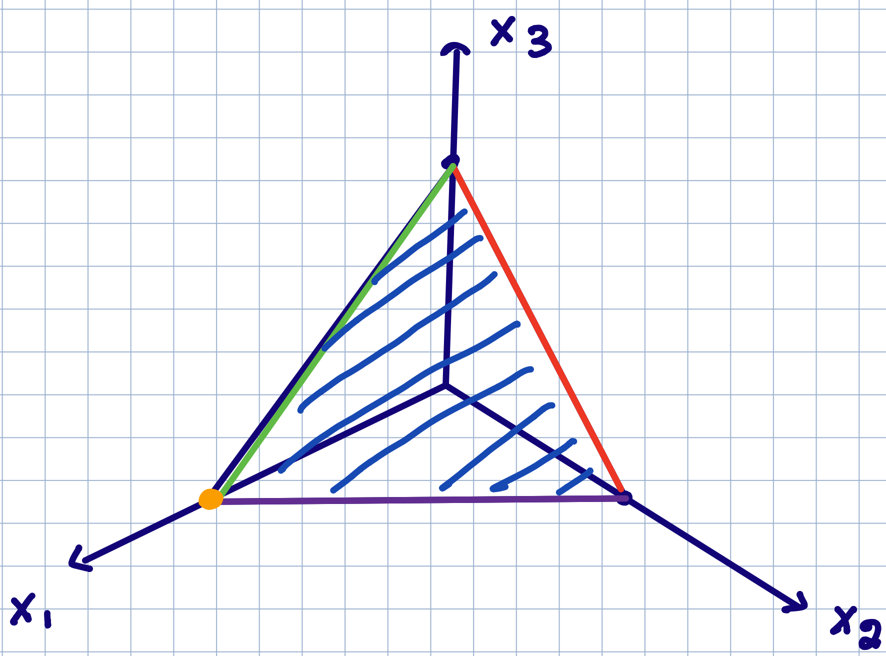
$$P = \{ x \mid a_i^T x \le b_i \underset{i=1,\dots,m}{} , \; c_i^T x = d_i \underset{i=1,\dots,k}{} \} =$$

$$\mathbb{F}_J = \{ \qquad \}$$

if this set is <u>not empty</u> it is called a face of P.

Example

$$\{ x \in \mathbb{R}^3 \mid x_i \geq 0, \; \sum x_i = 1 \} \quad i = 1, 2, 3$$



Minimal face :

Extreme point (or vertex) :

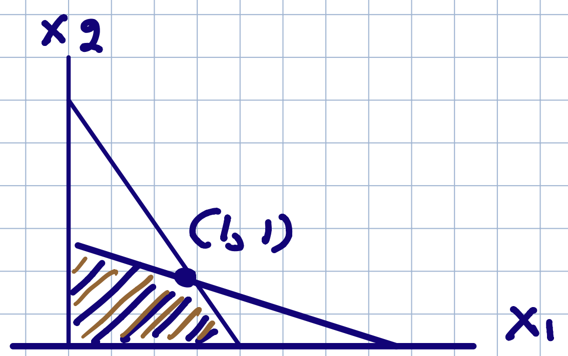
Properties

Break

Problem 2

1) Given $P = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0,\ 2x_1 + x_2 \leq 3,\ x_2 \geq 0,\ x_1 + 2x_2 \leq 3 \}$

Is the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a vertex?



2) $\mathbb{P} = \{ x \mid x \geq 0,\ \underset{m \times n}{C}\, x = d \}$ $\quad x \in \mathbb{R}^n$

vertices: have at least $n-m$ zero entries

3) How many vertices can the polyhedron

$$P = \{ x \mid \underset{m \times n}{A} x \leq b \} \quad \text{have?}$$

4) How many vertices does the polyhedron

$$S = \{ x \mid 0 \leq x \leq 1 \} \quad \text{have?}$$

Caratheodory's theorem

Consider the set $\mathcal{G}$ of matrices $\mathbb{P} \in \mathbb{R}^{K \times K}$ with elements $p_{ij} \geq 0$, and $\sum_{j=1}^{K} p_{ij} = 1$ (sum of elements in each row equals one)

That is

$$\mathcal{G} = \left\{ \mathbb{P} \in \mathbb{R}^{K \times K} \;\middle|\; p_{ij} \geq 0, \sum_{j=1}^{K} p_{ij} = 1 \right\}$$

(i) is $\mathcal{G}$ a convex set? (why?)

(ii) can every $\mathbb{P}$ in $\mathcal{G}$ be expressed as a convex combination of matrices with exactly one 1 per row?

Problems from past exams

Prove that, for $x \in R^n$, if the function $f(x)$ is a convex function, then the set $C = \{x | f(x) \le b\}$ is a convex set, with $b \in R$ a given constant.

Can you find the solution to the following problem (call this P1), by solving an LP?

$$\begin{array}{ll} \text{minimize}_x & ||x||_1^2 + 2||x||_1 \\ \text{subject to} & Ax = b, \end{array} \tag{1}$$

where $x \in R^n$, A is an $m \times n$ matrix and $b \in R^m$. If yes, explain which LP you can solve, if not, explain why.

Consider the optimization problem

$$\min \quad \frac{3x_1^2 + 5x_2^2 + 1.2x_3 + x_4 + 6}{4x_1^2 + x_2^2 + x_4 + 1.3} \qquad (1)$$

$x_i \in \mathbb{R}$

$$\text{st} \quad x_3 - x_4 \leq 8$$
$$0 \leq x_i \leq 10, \quad i = \{1, 2, 3, 4\}$$

Show that it is equivalent to the LP

$$\begin{aligned}
\min \quad & 3y_1 + 5y_2 + 1.2y_3 + y_4 + 6y_5 \\
\text{st} \quad & \\
& y_3 - y_4 \leq 8y_5 \\
& 4y_1 + y_2 + y_4 + 13y_5 = 1 \\
& y_i \leq 100\, y_5, \quad i \in \{1, 2\} \\
& y_i \leq 10\, y_5, \quad i \in \{3, 4\} \\
& y_i \geq 0 \quad i \in \{1, 2, 3, 4, 5\}
\end{aligned} \qquad (2)$$