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ECE 236A
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Linear Programming

Homework 2 Solutions

Problem 1 (`3 points`, Exer. 2.6) in *Convex Optimization Book*): When does one halfspace contain another? Give conditions under which

$$\{x \mid a^T x \leq b\} \subseteq \left\{x \mid \tilde{a}^T x \leq \tilde{b}\right\}$$

(where $a \neq 0, \tilde{a} \neq 0$). Also find the conditions under which the two halfspaces are equal.

Solution:

Let $\mathcal{H} = \left\{x \mid a^T x \leq b\right\}$ and $\tilde{\mathcal{H}} = \left\{x \mid \tilde{a}^T x \leq \tilde{b}\right\}$. The conditions are:

- $\mathcal{H} \subseteq \tilde{\mathcal{H}}$ if and only if there exists a $\lambda > 0$ such that $\tilde{a} = \lambda a$ and $\tilde{b} \geq \lambda b$
- $\mathcal{H} = \tilde{\mathcal{H}}$ if and only if there exists a $\lambda > 0$ such that $\tilde{a} = \lambda a$ and $\tilde{b} = \lambda b$

Let us prove the first condition. The condition is clearly sufficient: if $\tilde{a} = \lambda a$ and $\tilde{b} \geq \lambda b$ for some $\lambda > 0$, then

$$a^T x \leq b \Longrightarrow \lambda a^T x \leq \lambda b \Longrightarrow \tilde{a}^T x \leq \tilde{b}$$

i.e., $\mathcal{H} \subseteq \tilde{\mathcal{H}}$

To prove necessity, we distinguish three cases. First suppose a and $\tilde{a}$ are not parallel. This means we can find a v with $\tilde{a}^T v \neq 0$ and $a^T v = 0$. Let $\hat{x}$ be any point in the intersection of $\mathcal{H}$ and $\tilde{\mathcal{H}}$, i.e., $a^T \hat{x} \leq b$ and $\tilde{a}^T x \leq \tilde{b}$. We have $a^T(\hat{x}+tv) = a^T \hat{x} \leq b$ for all $t \in \mathbf{R}$ However $\tilde{a}^T(\hat{x}+tv) = \tilde{a}^T \hat{x} + t\tilde{a}^T v$, and since $\tilde{a}^T v \neq 0$, we will have $\tilde{a}^T(\hat{x}+tv) > \tilde{b}$ for sufficiently large $t > 0$ or sufficiently small $t < 0$. In other words, if a and $\tilde{a}$ are not parallel, we can find a point $\hat{x} + tv \in \mathcal{H}$ that is not in $\tilde{\mathcal{H}}$, i.e., $\mathcal{H} \nsubseteq \tilde{\mathcal{H}}$ Next suppose a and $\tilde{a}$ are parallel, but point in opposite directions, *i.e.*, $\tilde{a} = \lambda a$ for some $\lambda < 0$. Let $\hat{x}$ be any point in $\mathcal{H}$. Then $\hat{x} - ta \in \mathcal{H}$ for all $t \geq 0$. However for t large enough we will have $\tilde{a}^T(\hat{x} - ta) = \tilde{a}^T \hat{x} - t\lambda\|a\|_2^2 > \tilde{b}$, so $\hat{x} - ta \notin \tilde{\mathcal{H}}$. Again, this shows $\mathcal{H} \nsubseteq \tilde{\mathcal{H}}$. Finally, we assume $\tilde{a} = \lambda a$ for some $\lambda > 0$ but $\tilde{b} < \lambda b$. Consider any point $\hat{x}$ that satisfies $a^T \hat{x} = b$. Then $\tilde{a}^T \hat{x} = \lambda a^T \hat{x} = \lambda b > \tilde{b}$, so $\hat{x} \notin \hat{\mathcal{H}}$. The proof for the second part of the problem is similar.

Problem 2 (`2 points`, Exer. 2.9 (a) in *Convex Optimization Book*):

Voronoi sets and polyhedral decomposition. Let $x_0, \ldots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, \quad i = 1, \ldots, K\}$$

V is called the Voronoi region around x_0 with respect to $x_1, \ldots, x_K$ Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$

Solution

x is closer to x_0 than to x_i if and only if

$$\begin{aligned} \|x - x_0\|_2 \leq \|x - x_i\|_2 &\iff (x - x_0)^T (x - x_0) \leq (x - x_i)^T (x - x_i) \\ &\iff x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2x_i^T x + x_i^T x_i \\ &\iff 2(x_i - x_0)^T x \leq x_i^T x_i - x_0^T x_0 \end{aligned}$$

which defines a halfspace. We can express V as $V = \{x \mid Ax \preceq b\}$ with

$$A = 2 \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix}, \quad b = \begin{bmatrix} x_1^T x_1 - x_0^T x_0 \\ x_2^T x_2 - x_0^T x_0 \\ \vdots \\ x_K^T x_K - x_0^T x_0 \end{bmatrix}$$

Problem 3 (`3 points`, Exer. 33 (b)(e) in *Linear Programming Exercises*): Which of the following sets S are polyhedra? If possible, express S in inequality form, i.e., give matrices A and b such that $S = \{x | Ax \leq b\}$.

(a) $S = \{x \in \mathbf{R}^n |\ x \geq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_i \in \mathbf{R}$, $i = 1, \cdots, n$, $b_1 \in \mathbf{R}$, and $b_2 \in \mathbf{R}$ are given.

(b) $S = \{x \in \mathbf{R}^n |\ ||x - x_0|| \leq ||x - x_1||\}$, where $x_0, x_1 \in \mathbf{R}^n$ are given. S is the set of points that are closer to x_0 than to x_1.

Solution:

(a) S is a polyhedron. In fact the definition involves linear inequalities $x > 0$ and three equality constraints, so we only have to write the equality constraints as two inequalities. This yields a set of $n + 6$ inequalities:

$$\begin{array}{rcl} -x_i & \leq & 0, \quad i = 1, \cdots, n \\ \mathbf{1}^T x & \leq & 1 \\ -\mathbf{1}^T x & \leq & -1 \\ \sum_i a_i x_i & \leq & b_1 \\ -\sum_i a_i x_i & \leq & -b_1 \\ \sum_i a_i^2 x_i & \leq & b_2 \\ -\sum_i a_i^2 x_i & \leq & -b_2. \end{array}$$

(b) This set is a polyhedron (in fact, a halfspace). By squaring both sides of the inequality $||x - x_0|| \leq ||x - x_1||$, we obtain the equivalent condition

$$\begin{aligned} ||x - x_0||^2 \leq ||x - x_1||^2 \quad &\Leftrightarrow \quad -2x_0^T x + x_0^T x_0 \leq -2x_1^T x + x_1^T x_1 \\ &\Leftrightarrow \quad -2(x_0 - x_1)^T x \leq ||x_1||^2 - ||x_0||^2, \end{aligned}$$

which is a linear inequality.

Problem 4 (`3 points`, Exer. 35 (a) in *Linear Programming Exercises*):

Is $\tilde{x} = (1,1,1,1)$ an extreme point of the polyhedron $\mathcal{P}$ defined by the linear inequalities

$$\begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} ?$$

If it is, find a vector c such that $\tilde{x}$ is the unique minimizer of $c^T x$ over $\mathcal{P}$.

Hint: If the objective function is parallel to one of the hyperplanes defining the feasibility region, Do you get an unique minimizer ? Try to think of the solution of this problem graphically.

Solution:

To show that $\tilde{x}$ is an extreme point, we apply the rank criterion of page 3-23. The set of active constraints at $\tilde{x}$ is $J = \{1,2,3,4\}$, so we have to check the rank of the matrix

$$A_J = \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix}.$$

The rank of A_J is 4; therefore $\tilde{x}$ is an extreme point.

As vector c we can choose any (strictly) negative combination of the rows of A_J, for example,

$$c = -A_J^T \mathbf{1} = \begin{bmatrix} 8 \\ 16 \\ 4 \\ -15 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & -6 \\ -6 & -2 & 3 & -11 \\ 1 & 7 & -10 & -2 \\ 3 & 1 & -1 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

With this choice of c, we have

$$c^T \tilde{x} = -\mathbf{1}^T A_J \tilde{x} = -\sum_{i \in J} b_i = 13.$$

To show that no other feasible x can have the same or a lower value of $c^T x$, note that if $Ax \leq b$, then

$$c^T x = -\mathbf{1}^T A_J x \geq -\sum_{i \in J} b_i = 13,$$

with equality only if $a_i^T x = b_i$ for $i = 1,2,3,4$. However, this is a set of four equations in four variables with a nonsingular coefficient matrix A_J. Therefore $x = \tilde{x}$.

Problem 5 (4 points, Exer. 47 in *Linear Programming Exercises*)

Consider the polyhedron

$$\mathcal{P} = \{x \in \mathbf{R}^4 | Ax \leq b, Cx = d\}$$

where

$$A = \begin{bmatrix} -1 & -1 & -3 & -4 \\ -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 13 & 11 & 12 & 12 \end{bmatrix}, \quad d = 48$$

Prove that $\hat{x} = (1, 1, 1, 1)$ is an extreme point of $\mathcal{P}$.

Solution:

We have

$$b - A\hat{x} = (1, 0, 0, 0),$$

i.e., all inequalities except the first one are active. We therefore have to examine the rank of the matrix

$$\begin{bmatrix} -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \\ 13 & 11 & 12 & 12 \end{bmatrix}$$

The rank is four. Hence $\hat{x}$ is an extreme point.

Problem 6 (3 points, Exer. 36 in *Linear Programming Exercises*):

We define the polydedron

$$\mathcal{P} = \{x \in \mathbb{R}^5 \mid \quad Ax \leq b, \ -1 \leq x \leq 1\},$$

with

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The following three vectors x are in $\mathcal{P}$:

(a) $\hat{x} = (1, -1/2, 0, -1/2, -1)$

(b) $\hat{x} = (0, 0, 1, 0, 0)$

(c) $\hat{x} = (0, 1, 1, -1, 0)$.

Are these vectors extreme points in $\mathcal{P}$? For each $\hat{x}$, if it is an extreme point, give a vector c for which $\hat{x}$ is the unique solution of the optimization problem

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & -1 \leq x \leq 1. \end{array}$$

Solution:

We first work out a simplified version of the general rank test applied to a polyhedron in $\mathbf{R}^n$ defined by a set of inequalities and equalities

$$-\mathbf{1} \leq x \leq \mathbf{1}, \quad Ax = b$$

Let x be a feasible point. We partition its components in three groups:

$$x_k = -1 \text{ for } k \in J_-, \quad -1 < x_k < 1 \text{ for } k \in J_0, \quad x_k = 1 \text{ for } k \in J_+$$

and denote the sizes of the three sets by $n_- = |J_-|$, $n_0 = |J_0|$, $n_+ = |J_0|$. To check whether x is an extreme point we have to examine the submatrix of

$$\begin{bmatrix} -I \\ I \\ A \end{bmatrix},$$

obtained by keeping only the rows of the first block indexed by J_-, the rows of the second block indexed by J_+, and all rows of the third block. Up to a reordering of the columns, this is the matrix

$$\begin{bmatrix} -I_{n_-} & 0 & 0 \\ 0 & I_{n_+} & 0 \\ A_- & A_+ & A_0 \end{bmatrix}, \tag{1}$$

where the matrix A_- is the submatrix of A with the columns indexed by J_-, A_+ is the submatrix of A with the columns indexed by J_+, and A_0 is the submatrix of A with the columns indexed by J_0. The rank of the matrix (1) is $n_- + n_+ + \mathbf{rank}(A_0)$. The point x is an extreme point if this rank is n, i.e. if $\mathbf{rank}(A_0) = n_0$.

To summarize, x is an extreme point if

$$\mathbf{rank}\left(\begin{bmatrix} a_{k_1} & a_{k_2} & \cdots & a_{k_{n_0}} \end{bmatrix}\right) = n_0,$$

where $k_1, \cdots, k_{n_0}$ are the indices for which $-1 < x_k < 1$, and a_k denotes column k of A.

(a) Not an extreme point. The submatrix formed by columns 2,3 and 4 of A has rank 2.

(b) Not an extreme point. The submatrix formed by columns 1,2,4,5 has rank 3.

(c) Extreme point. The submatrix formed by columns 1 and 5 has rank 2. For c we can take the positive sum of the active constraints:

$$c = A^T\mathbf{1} - A^T\mathbf{1} + e_1 + e_3 - e_4 = (0, 1, 1, -1, 0).$$

Problem 7 (`2 points`, Exer. 28 in *Linear Programming Exercises*): Formulate the following problem as an LP. Find the largest ball

$$\mathcal{B}(x_c, R) = \{x \mid \quad \|x - x_c\| \leq R\},$$

enclosed in a given polyhedron

$$\mathcal{P} = \{x | a_i^T x \leq b_i,\ i = 1, \cdots, m\}.$$

In other words, express the problem

$$\begin{array}{ll} \text{maximize} & R \\ \text{subject to} & \mathcal{B}(x_c, R) \subseteq \mathcal{P} \end{array}$$

as an LP. The problem variables are the center $x_c \in \mathbb{R}^n$ and the radius R of the ball.

Solution:

The difficult part of this problem is expressing the constraints

$$\mathcal{B}(x_c, R) \subseteq \mathcal{P} \tag{2}$$

as a set of linear inequalities in R and x_c. We first consider the simpler constraints

$$\mathcal{B}(x_c, R) \subseteq \mathcal{H}_i \tag{3}$$

where $\mathcal{H}_i$ is the halfspace define by the inequality $a_i^T x \leq b_i$.
$\mathcal{B}(x_c, R) \subseteq \mathcal{H}_i$ if and only if

$$a_i^T (x_c + Ru) \leq b_i,\ \text{for all } u \text{ with } \|u\| \leq 1,$$

i.e.,

$$a_i^T x_c + R \max_{\|u\| \leq 1} a_i^T u \leq b_i.$$

The first term is linear in x_c. To simplify the second term we use the fact that

$$\max_{\|u\| \leq 1} a_i^T u = \|a\|_i.$$

(The value of u that achieves the maximum is $i = a_i / \|a_i\|$.)
This means we can express (3) as a linear inequality in x_c and R:

$$a_i^T x_c + R\|a\|_i \leq b_i.$$

It is now easy to express the constraint (2), by writing at as

$$\mathcal{B}(x_c, R) \subseteq \mathcal{H}_i, \quad i = 1, \cdots, m,$$

and using the equivalent expression we just derived: $\mathcal{B}(x_c, R) \subseteq \mathcal{P}$ if and only if

$$a_i^T x_c + R\|a_i\| \leq b_i, \quad i = 1, \cdots, m.$$

In conclusion, we can find the largest ball in the polyhedron $\mathcal{P}$ by solving

$$\begin{array}{ll} \text{maximize} & R \\ \text{subject to} & a_i^T x_c + R\|a_i\| \leq b_i, \quad i = 1, \cdots, m. \end{array}$$

which is an LP in x_c and R.