1. $\min \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$

s.t $AX = b \quad x_i \geq 0$

$$\begin{array}{c} \\ \\ \\ ① \rightarrow \\ ② \rightarrow \end{array} \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

① : $x_2 + x_4 = 5$

$x_2 + x_4 + 2x_5 = 2$

$\therefore 2x_5 = -3$

$x_5 = -\frac{3}{2} < 0$

contradict with $x_i \geq 0$

$\therefore$ it is not feasible

2. The convex hull for (v_i), $i \in I$ is

$S_1 = \{ y = \sum_{i \in I} \theta_i v_i \quad \sum \theta_i = 1 \}$

for v_j, $j \notin I$

$S_2 = \{ y = \sum_{j \notin I} \theta_j v_j \quad \sum \theta_j = 1 \}$

we need to find a vector y'
s.t y' could be expressed as $\sum_{i \in I} \theta_i v_i$ $\sum \theta_i = 1$
and also be ~~d~~ expressed as $\sum_{j \in J} \theta_j v_j$ $\sum \theta_j = 1$

$\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_{k+2} \\ 1 \end{pmatrix}$

$v_i = \begin{pmatrix} v_i \\ 1 \end{pmatrix} \in \mathbb{R}^{k+1}$ and there are $k+2$ points

$\therefore \begin{pmatrix} v_1 \\ 1 \end{pmatrix} \dots, \begin{pmatrix} v_{k+2} \\ 1 \end{pmatrix}$ must be linearly dependent

$\therefore$ exists not-all-zero coefficients $a_1, a_2, \dots a_{k+2}$

s.t $a_1 \begin{pmatrix} v_1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} v_2 \\ 1 \end{pmatrix} + \dots + a_{k+2} \begin{pmatrix} v_{k+2} \\ 1 \end{pmatrix} = 0$

$\therefore a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_{k+2} \cdot v_{k+2} = 0$

$a_1 + a_2 + \dots + a_{k+2} = 0$

we just assume that $a_1, a_2, \dots a_t$ are not zero
$a_1^+ \; a_2^+ \dots a_m^+$ are positive
$a_1^- \; a_2^- \dots a_n^-$ are negtive

$m+n = t$

let $S^+ = a_1^+ + \cdots + a_m^+$

$S^- = a_1^- + \cdots + a_n^-$

$\therefore \vec{0} = \frac{a_1^+}{S^+} v_1^+ + \frac{a_2^+}{S^+} v_2^+ + \cdots \frac{a_m^+}{S^+} \cdot v_m^+$ $\Leftarrow$ corresponding vector of v

$$\vec{0} = \frac{a_1^-}{S^-} v_1^- + \frac{a_2^-}{S^-} v_2^- + \cdots \frac{a_n^-}{S^-} v_n^-$$

$\therefore S_1 \cap S_2 \neq \phi$

$\therefore$ convex hull intersect

3. friend is right

At the optimal x^* and y^*

$$c^T x^* + y^{*T} \cdot b = 0$$

$b = Ax$

$$\therefore \quad c^T x^* + y^{*T} A x^* = 0$$

$$(c^T + y^{*T} A) x^* = 0$$

$$\therefore \quad ((c_i^T + y_i^{*T} a_i) \cdot x_i^* = 0$$

if $x_i^* > 0 \Rightarrow (c_i^T + y_i^* a_i) = 0$

if $c_i^T + y_i^* a_i > 0 \Rightarrow x_i^* = 0$

it is claimed that $c_i + a_i^T y_I^* \geq 0$

for all $i = 1, \cdots, n$

for $i \in I$ $\quad x_i^* = x_{I_i}^*$

for $j \notin I$ $\quad x_j^* = 0$

$\therefore$ Once we known the x_I^*, we know the $i \in I$ indicies that could be non-zero in x^*, we set all the indices that $j \notin I$ $x_j^* = 0$

Then we can get x^* from x_I