Quiz 2 ECE 236A Shangze Ye 205418959

Problem 1

1) $C = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq r\}$

$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$

$= \max\{x_1, -x_1, x_2, -x_2, \cdots, x_n, -x_n\}$

$\|x\|_\infty \leq r$

$\Rightarrow$ $x_i \leq r$ for $i = 1, 2, \cdots n$

$-x_i \leq r$

$\therefore$ set $C = \{x \in \mathbb{R}^n \mid x_i \leq r, -x_i \leq r$

$i = 1, 2, \cdots r\}$

could be expressed as a set of linear inequalities

It's linearity space only contain $\{0\}$

$\Rightarrow$ pointed polyhedron in $\mathbb{R}^n$

The vertices should satisfy n equalities

$$\hat{x} = \begin{bmatrix} r & r & \cdots & r \\ \text{or} & \text{or} & & \text{or} \\ -r & -r & & -r \end{bmatrix}$$

each position has 2 choices

$\Rightarrow$ 2^n vertices

2) $P = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_i \geq 0,\ x_1 + x_2 \geq 2,\ x_1 + x_3 \geq 2,\ 3x_1 + 2x_2 + x_3 \geq 6 \right\}$

we can find matrix A. and column vector b

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

s.t $\Rightarrow P = \{ x \in \mathbb{R}^3 \mid Ax \geq b \}$

we check $x_0 = (1, 1, 1)^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

row 1, 2, 3 are active constraints

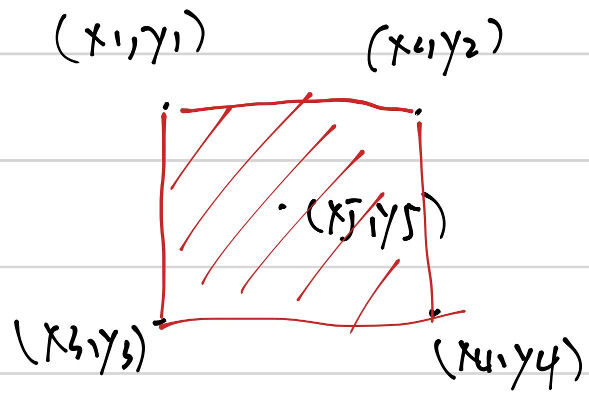
$$A_J(\hat{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} \qquad \operatorname{rank}(A_J(\hat{x})) = 2 \neq 3$$

$\therefore$ x_0 is not a vertex

Problem 2

(i) m points in $\mathbb{R}^n$ $m > n$, The convex hull of these points in a polyhedron with m vertices

False



consider a case in $\mathbb{R}^2$
we can have 5 points
$(m=5)$
but only 4 vertex

(ii) The minimal face of a polyhedron is always a single point

False. The minimal face is defined for a polyhedron, if the linearity space of that polyhedron contains a line, its minimal face also contains a line

(iii) $P \subseteq \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes. The the following $S \subseteq \mathbb{R}^{d+e+1}$ is also a poly tope

$$S = \left\{ z \in \mathbb{R}^{d+e+1} \middle| z = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix} \quad x \in P \text{ and } y \in Q \right\}$$

True.

polytopes are bounded polyhedron, so they are affine set.

we need to prove that $\forall z_1, z_2 \in S$

$\theta_1 z_1 + \theta_2 z_2 \in S \quad \theta_1 + \theta_2 = 1$

($\because$ need to prove S is an affine set)

suppose $z_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$

$z_2 = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$

$\theta_1 z_1 + \theta_2 z_2 = \begin{pmatrix} \theta_1 x_1 + \theta_2 x_2 \\ \theta_1 y_1 + \theta_2 y_2 \\ \theta_1 + \theta_2 \end{pmatrix}$

$\because x \in P$ $\quad P$ is a polytope (affine set)

$\therefore \theta_1 x_1 + \theta_2 x_2 = x_3 \in P$

$y \in Q$ $\quad Q$ is an affine set

$\therefore \theta_1 y_1 + \theta_2 y_2 = y_3 \in Q$

$\theta_1 + \theta_2 = 1$

$\therefore \theta_1 z_1 + \theta_2 z_2 = \begin{pmatrix} x_3 \\ y_3 \\ 1 \end{pmatrix} \in S$

$\therefore$ set S is an affine set, and $z = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.

because P, Q are polytope (bounded) $x \in P$, $y \in Q$ cannot be infinity number $z \in S$. S is also a polytope

Problem 3

prove the two program (4) and (5) are equivalent

$$\min_{x_1, x_2, y_1, y_2} \quad -c^T(x_1 - x_2)$$

$$\text{s.t.} \quad A(x_1 - x_2) + y_1 - y_2 = 0 \qquad (4)$$

$$y_1 + y_2 = \lambda I$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad y_1 \geq 0, \quad y_2 \geq 0$$

$$\max_z \quad c^T z \qquad (5)$$

$$\text{s.t.} \quad \|Az\|_\infty \leq \lambda$$

minimize a value is equivalent to maximize its negative value

$$\max_{x_1, x_2, y_1, y_2} \quad c^T(x_1 - x_2)$$

$$\text{s.t.} \quad A(x_1 - x_2) + y_1 - y_2 = 0 \qquad (1)$$

$$y_1 + y_2 = \lambda I$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad y_1 \geq 0, \quad y_2 \geq 0$$

(4) is equivalent to (1)

consider a mapping $z = x_1 - x_2$

$$\Rightarrow \max_{z, y_1, y_2} \quad c^T z$$

$$\text{s.t.} \quad Az + y_1 - y_2 = 0 \qquad (2)$$

$$y_1 + y_2 = \lambda \mathbf{1}$$

$$y_1 \geq 0, \; y_2 \geq 0$$

(1) $\Rightarrow$ (2)

Assume x_1, x_2 is a feasible point in (1)

$$z = x_1 - x_2$$

feasible x_1, x_2 $\rightarrow$ gives us feasible z in (2)

(2) $\Rightarrow$ (1)

Assume z is a feasible point in (2)

we can always find 2 vector $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$

$x_1 \geq 0$ and $x_2 \geq 0$

s.t $z = x_1 - x_2$

feasible z in (2) $\rightarrow$ gives us feasible x_1, x_2 in (1)

(1) and (2) are equivalent

Then we notice that the constraints

$$Az + y_1 - y_2 = 0$$

$$\therefore Az = y_2 - y_1$$

and $y_1 + y_2 = \lambda \mathbb{1}$

$$y_1 + y_2 = \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix} \in \mathbb{R}^n \qquad \therefore y_2 = \lambda \mathbb{1} - y_1$$

$$\therefore Az = y_2 - y_1 = \lambda \mathbb{1} - 2y_1$$

$$Az = \lambda \mathbb{1} - 2y_1 \qquad \because y_1 \geq 0$$

$$\therefore Az \leq \lambda \mathbb{1}$$

we notice that $2y_1$ here is the slack variable

$$\begin{aligned} Az &= \lambda \mathbb{1} - 2y_1 \\ y_1 &\geq 0 \end{aligned} \quad \rightarrow \quad Az \leq \lambda \mathbb{1}$$

standard form $\qquad$ inequality form

$\therefore$ (2) is equivalent to

$$\max_z \ c^T z$$
$$\text{s.t} \ Az \leq \lambda \mathbb{1}$$

$$Az \leq \lambda \mathbb{1} \qquad Az \leq \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix} \qquad \therefore \|Az\|_\infty \leq \lambda$$

$\therefore$ it is also equivalent to

$$\max_z \ c^T z \qquad (5)$$
$$\text{s.t} \ \|Az\|_\infty \leq \lambda$$