Lecture 2

Today:

1) Equivalence
2) Approximation

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① General form

② Standard form

③ Inequality form

Basic equivalence transformations

- transformation of objective function

- Constraints

1) write $a_i^T x \leq b_i$ as equality:

2) Constraints of the form $a_i^T x \geq b$ as equality:

3) Constraints of the form $a_i^T x = b_i$ as inequality:

4) Making all variables non-negative

Example 1: from ③ inequality form to ② standard form

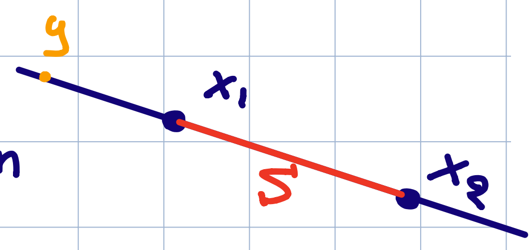
$$\begin{aligned} \min \ & c^T x \\ \text{st} \ & \underset{m\times n}{A} x \leq b \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \min \ & c^T x \\ \text{st} \ & A x = b \\ & x \geq 0 \end{aligned}$$

BREAK

We will now introduce some classes of functions that lead to equivalent or approximate LPs.

Notation/background

1) Line and line segment



Consider two points, $x_1, x_2 \in \mathbb{R}^n$

Line segment between x_1 and x_2:

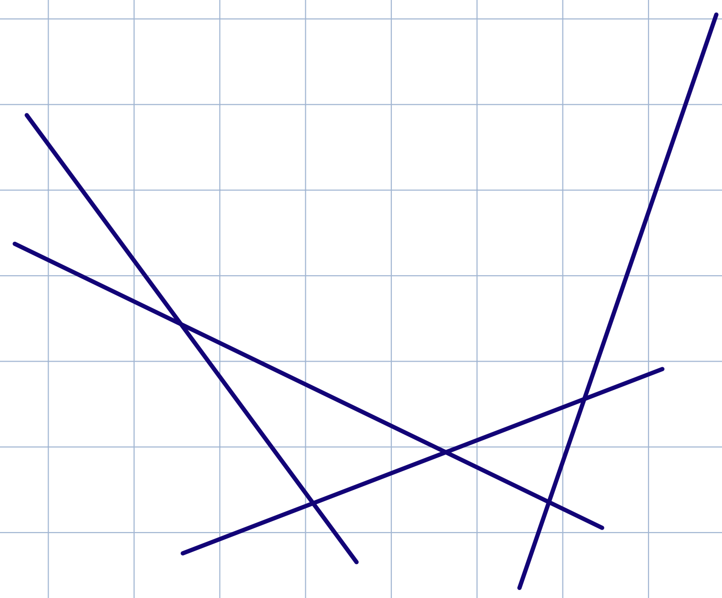
2) Convex function $f: \mathbb{R}^n \to \mathbb{R}$

A function is convex if

3) Linear functions

4) Affine functions

5) Piecewise linear functions



Piecewise-linear functions are convex.

Proof

6) Norms

"express distance"
"length"

A function $f: \mathbb{R}^m \to \mathbb{R}$ is called a norm iff

i) $f(x) \geq 0$ for all $x \in \mathbb{R}^m$

ii) $f(x) = 0 \iff x = 0$

iii) $f(ax) = |a| f(x)$, $a \in \mathbb{R}$,

iv) $f(x+y) \leq f(x) + f(y)$

ℓ_2 - norm

ℓ_1 - norm

ℓ_p - norm

ℓ_∞ - norm

For finite-dimensional vectors x, all the norms are "equivalent", in the following sense:
There exist constants A and B such that

$$A \|x\|_q \leq \|x\|_p \leq B \|x\|_q$$

break

Prove that $x^T y \leq \|x\|_1$ for all y with $\|y\|_\infty \leq 1$.

Piecewise linear minimization can be expressed as (is equivalent to) an LP.

(1) $\boxed{\min_x f(x) = \min_x \left\{ \max_i (a_i^T x + b_i) \right\}}$ p_1^*

(2) $\boxed{}$ p_2^*

To prove that (1) and (2) are equival.
we will prove that

$$\left.\begin{array}{l} p_2^* \leq p_1^* \\ \text{and} \\ p_1^* \leq p_2^* \end{array}\right\} \Rightarrow p_1^* = p_2^*$$

- Assume we solve (1) and find some optimal value x^*, $p_1^* = f(x^*)$

We conclude: $p_1^* = p_2^*$

We can write (2) in matrix format as follows:

$$\min \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} a_1^T & -1 \\ a_2^T & -1 \\ \vdots & \\ a_m^T & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq \begin{pmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_m \end{pmatrix}$$

Why useful

→ some norms can be expressed as piece-wise linear functions

→ can be used to approximate convex optimization problems

Sum of piecewise functions

$$\min_x \left[\max_{i=1\ldots m} (a_i^T x + b_i) + \max_{i=1\ldots K} (c_i^T x + d_i) \right]$$

equivalent problem $\qquad \min$

$\qquad \qquad \qquad \quad$ st

Can be expressed in matrix form

$$\min \tilde{c}^T \tilde{x}$$

$$\text{st } A\tilde{x} \leq \tilde{b}$$

$$\tilde{x} = \begin{pmatrix} \\ \\ \end{pmatrix}, \quad \tilde{c}^T = (\qquad)$$

$$A = \begin{bmatrix} a_1^T & -1 & 0 \\ \vdots & & \\ a_m^T & -1 & 0 \\ c_1^T & 0 & -1 \\ \vdots & & \\ c_K^T & 0 & -1 \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \\ -d_1 \\ \vdots \\ -d_K \end{bmatrix}$$

Norm minimization