Fall 2020
ECE 236A
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Linear Programming

Homework 3
Due Friday Nov. 6, 2020

Problem 1 (`3 points`, Exer. 50 in *Linear Programming Exercises*): A matrix $A \in \mathbf{R}^{(mp)\times n}$ and a vector $b \in \mathbf{R}^{mp}$ are partitioned in m blocks of p rows:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

with $A_k \in \mathbf{R}^{p\times n}$, $b_k \in \mathbf{R}^p$.

(a) Express the optimization problem

$$\text{minimize} \quad \sum_{k=1}^{m} ||A_k x - b_k||_\infty \tag{1}$$

as an LP.

(b) Suppose $\mathbf{rank}(A) = n$ and $Ax_{ls} - b \neq 0$, where x_{ls} is the solution of the least-squares problem

$$\text{minimize} \quad ||Ax - b||^2.$$

Derive the dual program and show that it can be simplified as

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^m b_k^T z_k \\ \text{subject to} & \sum_{k=1}^m A_k^T z_k = 0 \\ & ||z_k||_1 \leq 1, \quad k = 1, \ldots, m \end{array}$$

(c) For the setup of (b), show that the optimal value of (1) is bounded below by

$$\frac{\sum_{k=1}^m ||r_k||^2}{\max_{k=1,\cdots,m} ||r_k||_1}$$

where $r_k = A_k x_{ls} - b_k$ for $k = 1, \cdots, m$.

Problem 2 (`3 points`, Exer. 59 in *Linear Programming Exercises*): The projection of a point $x_0 \in \mathbf{R}^n$ on a polyhedron $\mathcal{P} = \{x | Ax \leq b\}$, in the l_∞-norm, is defined as the solution of the optimization problem

$$\begin{array}{ll} \text{minimize} & ||x - x_0||_\infty \\ \text{subject to} & Ax \leq b. \end{array}$$

The variable is $x \in \mathbf{R}^n$. We assume that $\mathcal{P}$ is nonempty.

(a) Write this problem as an LP in standard form.

(b) Derive the dual problem, and show it is equivalent to the following:

$$\begin{array}{ll} \text{maximize} & (Ax_0 - b)^T w \\ \text{subject to} & ||A^T w||_1 \leq 1 \\ & w \geq 0. \end{array}$$

Problem 3 (`3 points`, Exer. 51 in *Linear Programming Exercises*):

Let x be a real-valued random variable which takes values in $\{a_1, a_2, ..., a_n\}$ where $0 < a_1 < a_2 < ... < a_n$, and $\mathbf{prob}(x = a_i) = p_i$. Obviously p satisfies $\sum_{i=1}^n p_i = 1$ and $p_i \geq 0$ for $i = 1, ..., n$.

(a) Consider the problem of determining the probability distribution that maximizes $\mathbf{prob}(x \geq \alpha)$ subject to the constraint $\mathbf{E}x = b$, *i.e.*,

$$\begin{array}{ll} \text{maximize} & \mathbf{prob}(x \geq \alpha) \\ \text{subject to} & \mathbf{E}x = b, \end{array} \tag{2}$$

where α and b are given ($a_1 < \alpha < a_n$, and $a_1 \leq b \leq a_n$). The variable in problem (2) is the probability distribution, *i.e.*, the vector $p \in \mathbf{R}^n$. Write (2) as an LP.

(b) Take the dual of the LP in (a), and show that it can be reformulated as

$$\begin{array}{ll} \text{minimize} & \lambda b + \nu \\ \text{subject to} & \lambda a_i + \nu \geq 0 \text{ for all } a_i < \alpha, \\ & \lambda a_i + \nu \geq 1 \text{ for all } a_i \geq \alpha, \end{array}$$

The variables λ and ν. Show that the optimal value is equal to

$$\begin{cases} (b - a_1)/(\bar{a} - a_1) & b \leq \bar{a} \\ 1 & b \geq \bar{a} \end{cases}$$

where $\bar{a} = \min\{a_i | a_i \geq \alpha\}$. Also give the optimal values of λ and ν.

Problem 4 (`3 points`, Exer. 48 [(a), (b)], in *Linear Programming Exercises*): Consider the following optimization problem in x:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \|Ax + b\|_1 \leq 1
\end{array} \tag{3}
$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$.

(a) Formulate this problem as an LP in inequality form and explain why your LP formulation is equivalent to problem (3).

(b) Derive the dual LP, and show that it is equivalent to the problem

$$
\begin{array}{ll}
\text{maximize} & b^T z - \|z\|_\infty \\
\text{subject to} & A^T z + c = 0.
\end{array} \tag{4}
$$

What is the relation between the optimal z and the optimal variables in the dual LP?

Problem 5 (`3 points`, Exer. 63 in *Linear Programming Exercises*): Consider the robust LP

$$
\begin{array}{ll}
\min & c^T x \\
\text{subject to} & \max_{a \in \mathcal{P}_i} a^T x \leq b_i, \quad i = 1, \cdots, m
\end{array}
$$

with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_i a \leq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i \times n}$, $d_i \in \mathbf{R}^{m_i}$ and $b \in \mathbf{R}^m$. We assume the polyhedra $\mathcal{P}_i$ are nonempty.

Show that this problem is equivalent to the LP

$$
\begin{array}{ll}
\min & c^T x \\
\text{subject to} & d_i^T z_i \leq b_i, \quad i = 1, \cdots, m \\
& C_i^T z_i = x, \quad i = 1, \cdots, m \\
& z_i \geq 0, \quad i = 1 \cdots, m
\end{array}
$$

with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1, \cdots, m$. *Hint: Find the dual of the problem of maximizing a_i^x over $a_i \in \mathcal{P}_i$ (with variable a_i).*

Problem 6 (`2 points`, Exer. 40 in *Linear Programming Exercises*): Prove the following result. If a set of m linear inequalities in n variables is infeasible, then there exists an infeasible subset of no more than $n + 1$ of the m inequalities.

Problem 7 (`3 points`, Exer. 46 in *Linear Programming Exercises*): For the following two LPs, check (and prove whether) the proposed solution is optimal, by using duality:

1. For the LP

$$\begin{array}{ll} \text{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{subject to} & \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} \end{array}$$

Is $x = (1, 1, 1, 1)$ optimal?

2. For the LP

$$\begin{array}{ll} \text{maximize} & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\ \text{subject to} & \begin{bmatrix} 1 & 3 & 5 & -2 & 3 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 5 \\ 1 \end{bmatrix}, x_i \geq 0, i = 1 \ldots 5 \end{array}$$

Is $x = (0, 4/3, 2/3, 5/3, 0)$ optimal?