Fall 2020
ECE 236A
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Linear Programming

Homework 2

Due: 9 am, Friday Oct. 23rd

Problem 1 (`3 points`, Exer. 2.6) in *Convex Optimization Book*): When does one halfspace contain another? Give conditions under which

$$\left\{x \mid a^T x \leq b\right\} \subseteq \left\{x \mid \tilde{a}^T x \leq \tilde{b}\right\}$$

(where $a \neq 0, \tilde{a} \neq 0$). Also find the conditions under which the two halfspaces are equal.

Problem 2 (`2 points`, Exer. 2.9 (a) in *Convex Optimization Book*):
Voronoi sets and polyhedral decomposition. Let $x_0, \ldots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_2\|_2, \quad i = 1, \ldots, K\}$$

V is called the Voronoi region around x_0 with respect to $x_1, \ldots, x_K$ Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$

Problem 3 (`3 points`, Exer. 33 (b)(e) in *Linear Programming Exercises*): Which of the following sets S are polyhedra? If possible, express S in inequality form, i.e., give matrices A and b such that $S = \{x | Ax \leq b\}$.
(a) $S = \{x \in \mathbf{R}^n |\ x \geq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_i \in \mathbf{R}$, $i = 1, \cdots, n$, $b_1 \in \mathbf{R}$, and $b_2 \in \mathbf{R}$ are given.
(b) $S = \{x \in \mathbf{R}^n |\ ||x - x_0|| \leq ||x - x_1||\}$, where $x_0, x_1 \in \mathbf{R}^n$ are given. S is the set of points that are closer to x_0 than to x_1.

Problem 4 (`3 points`, Exer. 35 (a) in *Linear Programming Exercises*):
Is $\tilde{x} = (1, 1, 1, 1)$ an extreme point of the polyhedron $\mathcal{P}$ defined by the linear inequalities

$$\begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} ?$$

If it is, find a vector c such that $\tilde{x}$ is the unique minimizer of $c^T x$ over $\mathcal{P}$.
Hint: If the objective function is parallel to one of the hyperplanes defining the feasibility region, Do you get an unique minimizer ? Try to think of the solution of this problem graphically.

Problem 5 (`4 points`, Exer. 47 in *Linear Programming Exercises*)

Consider the polyhedron

$$\mathcal{P} = \{x \in \mathbf{R}^4 | Ax \leq b, Cx = d\}$$

where

$$A = \begin{bmatrix} -1 & -1 & -3 & -4 \\ -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 13 & 11 & 12 & 12 \end{bmatrix}, \quad d = 48$$

Prove that $\hat{x} = (1, 1, 1, 1)$ is an extreme point of $\mathcal{P}$.

Problem 6 (`3 points`, Exer. 36 in *Linear Programming Exercises*):

We define the polydedron

$$\mathcal{P} = \{x \in \mathbb{R}^5 \mid \quad Ax \leq b, \ -1 \leq x \leq 1\},$$

with

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & -1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The following three vectors x are in $\mathcal{P}$:

(a) $\hat{x} = (1, -1/2, 0, -1/2, -1)$

(b) $\hat{x} = (0, 0, 1, 0, 0)$

(c) $\hat{x} = (0, 1, 1, -1, 0)$.

Are these vectors extreme points in $\mathcal{P}$? For each $\hat{x}$, if it is an extreme point, give a vector c for which $\hat{x}$ is the unique solution of the optimization problem

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & -1 \leq x \leq 1. \end{array}$$

Problem 7 (`2 points`, Exer. 28 in *Linear Programming Exercises*): Formulate the following problem as an LP. Find the largest ball

$$\mathcal{B}(x_c, R) = \{x \mid \quad \|x - x_c\| \leq R\},$$

enclosed in a given polyhedron

$$\mathcal{P} = \{x | a_i^T x \leq b_i, \; i = 1, \cdots, m\}.$$

In other words, express the problem

$$\begin{array}{ll} \text{maximize} & R \\ \text{subject to} & \mathcal{B}(x_c, R) \subseteq \mathcal{P} \end{array}$$

as an LP. The problem variables are the center $x_c \in \mathbb{R}^n$ and the radius R of the ball.