Fall 2020
EE 236A
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EE236A Linear Programming
Quiz 2 Solutions
Tuesday October 27, 2020

NAME: ____________________ **UID**: ____________________

This quiz has 3 questions, for a total of 20 points.

Open book.
The exam is for a total of 1:00 hour. **Please, write your name and UID on the top of each sheet.**

Good luck!

Problem	Mark	Total
P1		6
P2		7
P3		7
Total		20

Problem 1 (`6 points`) The following two questions are not related.

1) (3 points) Consider the set

$$C = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq n\} \tag{1}$$

Argue that this set is a pointed polyhedron in $\mathbb{R}^n$, by expressing it through a set of linear inequalities in $\mathbb{R}^n$. Find what form the vertices of this polyhedron have. How many vertices are there?

2) (3 points) Is the point $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ a vertex in the polyhedron P described next?

$$P = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_i \geq 0, \quad x_1 + x_2 \geq 2, \quad x_1 + x_3 \geq 2, \quad 3x_1 + 2x_2 + x_3 \geq 6 \right\} \tag{2}$$

Solution:

1) Since $\|x\|_\infty = \max_i \{|x_i|\}$ and $\|x\|_\infty \leq n$, we see that $|x_i| \leq n \quad \text{for } i = 1, .., n$. Therefore, $-n \leq x_i \leq n \quad \text{for } i = 1, ..., n$ and the set C can be expressed as in the following format.

$$C = \{x \in \mathbb{R}^n \mid Ax \leq n\} \tag{3}$$

where $A = \begin{pmatrix} I \\ -I \end{pmatrix}$. To prove that the set C is a pointed polyhedron, we need to check its linearity space $\mathcal{L}$ which is the nullspace of matrix A. Since matrix A is full rank, its nullspace constains only the zero vector. Therefore, set C is indeed a pointed polyhedron.

Each vertex of the set C should satisfy n linearly independent constraints with equality due to the rank condition. If we look at the constraints that define the set C, each x_i should be between $-n$ and n. Since x_i cannot be equal to $-n$ and n at the same time, each x_i will be $-n$ or n if x is a vertex so that it can satisfy n linearly independent constraints. Since each element x_i can be $-n$ or n, the set C has 2^n vertices.

2) First we note that this is a pointed polyhedron, as the linearity space is empty, and thus has vertices. We then need to check the rank condition to understand if x_0 is a vertex. Therefore, we first need to determine the active constraints. x_0 satisfies the last three constraints with equality. Therefore, these constraints are active and we will use them to create the A_J matrix. According to rank condition, if a point is a vertex, the rank of the matrix A_J should be equal to n which is equal to 3 in this question. However, the rank of matrix A_J in (4) is 2, therefore, x_0 is not a vertex.

$$A_J = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ -3 & -2 & -1 \end{pmatrix} \tag{4}$$

Problem 2 (`7 points`): Which of the following statements are true and which are false? Give a brief justification (answers without justification do not get points).

- (i) (2 points) Consider m points in $\mathbb{R}^n$ with $m > n$. The convex hull of these points is a polyhedron with exactly m vertices.
- (ii) (2 points) The minimal face of a polyhedron is always a single point.
- (iii) (3 points) Let $P \subseteq \mathbb{R}^{\mathrm{d}}$ and $Q \subset \mathbb{R}^e$ be polytopes. Then the following set $\mathbb{S} \subseteq \mathbb{R}^{d+e+1}$ is also a polytope, where:

$$\mathbb{S} = \left\{ z \in \mathbb{R}^{d+e+1} | z = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix}, \text{with } x \in P \text{ and } y \in Q \right\} \tag{5}$$

Solution:

- The statement is FALSE because some of these points can belong in the convex hull of others, and will not be vertices in this case (eg consider three points on the same line).
- The statement is FALSE. The polyhedron should be a pointed polyhedron in order for the minimal face to be a single point. Consider the following polyhedron.

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid a^T x \leq b \right\} \tag{6}$$

  For this polyhedron, the minimal face will be a hyperplane.

- The statement is TRUE. If a set is a polytope, it means it is a bounded polyhedron. Therefore, the elements of a vector z in the polytope cannot go to infinity. In (5), the first d elements of a vector z are elements of a vector x from polytope P, therefore, they won't go to infinity. The next e elements of the vector z are elements of a vector y from polytope Q and they won't go to infinity as well. Finally, the last element of the vector z is a scalar constant and it won't go to infinity. Hence, the set of points z is from a bounded set defined with hyperplanes i.e. (5) is indeed a polytope. To see that the region is indeed a polyhedron, let us define polyhedrons P and Q:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^d \mid A_p x \leq b_p \right\} \tag{7}$$

$$\mathcal{Q} = \{ y \in \mathbb{R}^e \mid A_q y \leq b_q \} \tag{8}$$

  Then we can write the polyhedron for z as follows:

$$\mathcal{Z} = \left\{ z \in \mathbb{R}^{d+e+1} \mid A_z z \leq b_z \right\} \tag{9}$$

  where:

$$A_z = \begin{pmatrix} A_p & 0\dots & 0 \\ 0\dots & A_q & 0 \\ 0\dots & 0\dots & 1 \\ 0\dots & 0\dots & -1 \end{pmatrix} \quad \textit{and} \quad b_z = \begin{pmatrix} b_p \\ b_q \\ 1 \\ -1 \end{pmatrix} \tag{10}$$

Problem 3 (7 points) Assume $x_1, x_2, y_1, y_2, z \in \mathbb{R}^n$, A is a given $m \times n$ matrix, λ is a constant and c is a given column vector of dimension $n \times 1$. Prove that the following two programs, (11) and (12), are equivalent.

$$\begin{aligned} \min_{x_1,x_2,y_1,y_2} \quad & -c^T(x_1 - x_2) \\ \text{subject to} \quad & A(x_1 - x_2) + y_1 - y_2 = 0 \\ & y_1 + y_2 = \lambda \mathbf{1} \\ & x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0 \end{aligned} \tag{11}$$

$$\begin{aligned} \max_z \quad & c^T z \\ \text{subject to} \quad & \|Az\|_\infty \leq \lambda \end{aligned} \tag{12}$$

Solution: We will start from the first problem (11), and show that a feasible solution allows to construct a feasible solution for the program in (12) that achieves the same objective function value. For this, we will select $z = x_1 - x_2$.

Consider x_1, x_2, y_1, y_2 are feasible and set $z = x_1 - x_2$, then we have that $Az = y_2 - y_1$. Because y_1, y_2 are positive, then $-y_1 - y_2 \leq y_2 - y_1 \leq y_1 + y_2$. Moreover $y_1 + y_2 = \lambda 1$ which also implies that $-y_1 - y_2 = -\lambda 1$. Then we have $-\lambda 1 \leq y_2 - y_1 \leq \lambda 1$. This gives us $-\lambda 1 \leq A^T z \leq \lambda 1$, and thus $\|A^T z\|_\infty \leq \lambda$. That is, z is feasible in (12). Moreover a minimization problem can be converted to a maximization problem by negating the objective function, and the objective function value is the same. This concludes the first direction.

Second, we start from a feasible solution in (12) and argue that we can achieve a feasible solution in (11) that achieves the same objective function value. Again, a maximization problem can be converted to a minimization problem by negating the objective function.

Let y_1 be a nonnegative variable y_1 such that $A^T z + 2y_1 = \lambda \cdot 1$ and let y_2 be a nonnegative variable y_2 such that $A^T z - 2y_2 = -\lambda \cdot 1$. By summing up those two constraints we get that $2(A^T z + y_1 - y_2) = 0$; by subtracting them we get that $y_1 + y_2 = \lambda 1$.

Next, we select nonnegative variables x_1 and x_2, such that $x_1 : x_{1i} = \max\{z_i, 0\}$, and $x_2 : x_{2i} = \max\{-z_i, 0\}$. Clearly $z = x_1 - x_2$ and thus $A(x_1 - x_2) + y_1 - y_2 = 0$. From construction, we also have that $x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0$. Thus the selected variables are feasible in (11) and achieve the same objective function value.