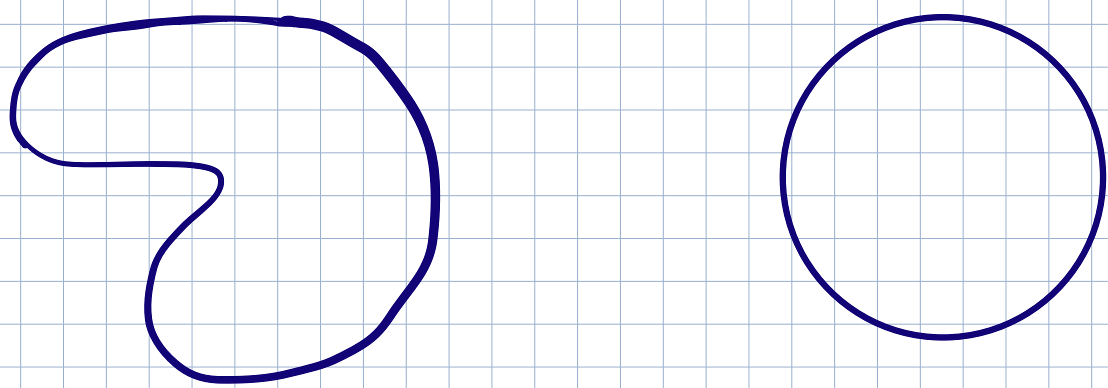
Lecture 5

Geometry of LPs

Convex set $\mathcal{C}$



for all
$x_1, x_2 \in \mathcal{C}$

Example

$k = 3 \qquad \theta_i \geqslant 0$

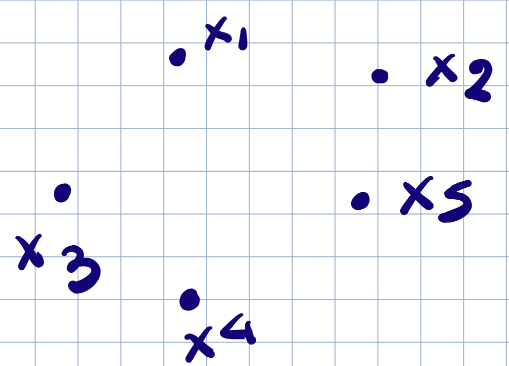
$\theta_1 + \theta_2 + \theta_3 = 1 \quad , \quad x_1, x_2, x_3 \in C$

$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 =$

Definition

Given N points $x_1, x_2, \dots, x_N$, we define the <u>convex hull</u> of these points to be the set:

$$S = \{ \qquad \qquad \qquad \qquad \}$$



Examples: are the following sets convex?

1) $\mathbb{R}^+ = \{ x \in \mathbb{R}^n \mid x \geq 0 \}$

2) Sphere $S = \{ x \mid \| x - x_c \| \leq r \}$

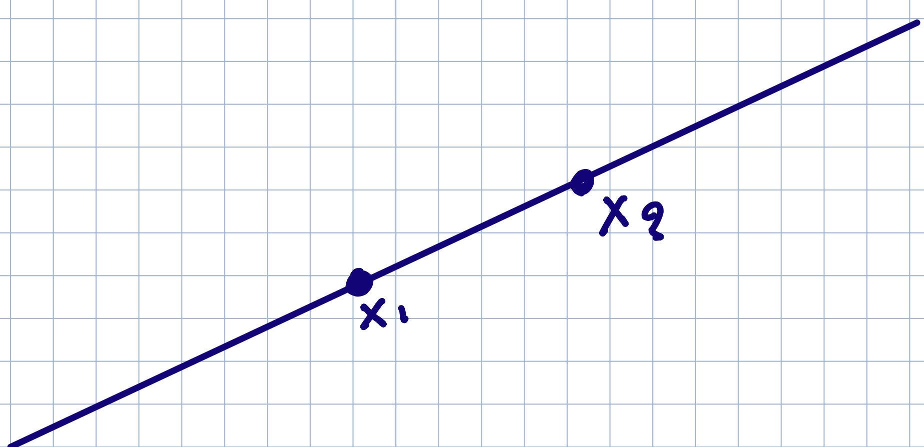
3) Halfspace $\{x \mid a^T x \leq b\}$

Hyperplane $\{x \mid a^T x = b\}$

4) Intersection of convex sets

5) Polyhedron:

Affine Sets



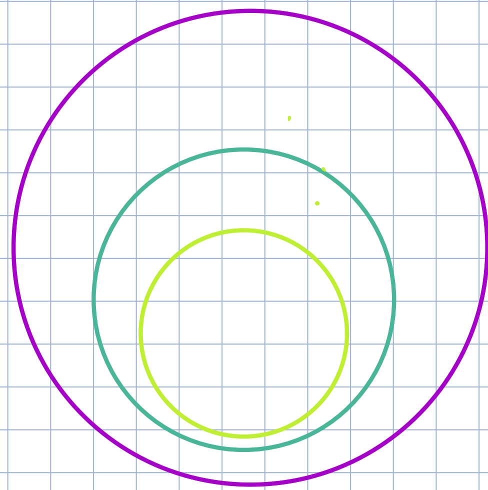
Extends recursively

Affine hull

Example 1) hyperplane $\{x \mid a^T x = b\}$

2) halfspace $\{x \mid a^T x \leq b\}$

Subspace



Example Let A be an $m \times n$ matrix.

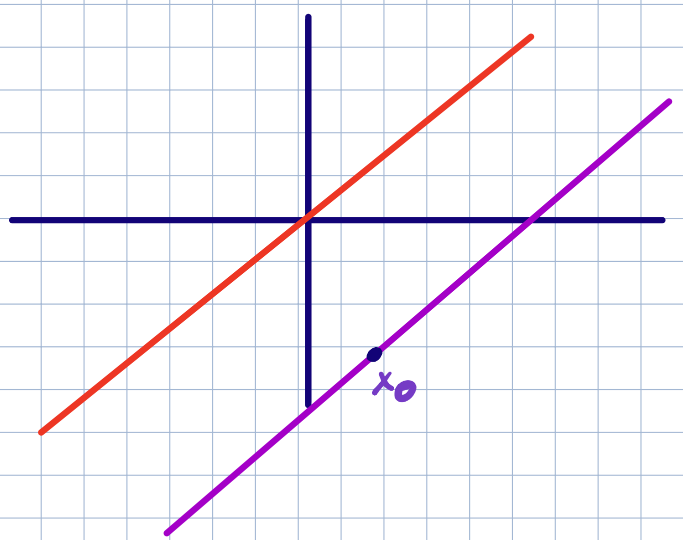
1) $R(A) = \{ x \mid x = Ay, \; y \in \mathbb{R}^n \}$

2) $N(A) = \{ x \mid Ax = 0 \}$

3) Set of solutions of linear equations

$$C = \{ x \mid Ax = b \}$$

Why are we interested in affine sets?



Parallel subspace

Let C be an affine set
and let $x_0 \in C$

Proof (that V is a subspace)

Assume $v_1, v_2 \in V$, we need to show that

$$a v_1 + b v_2 \in V, \quad \forall a, b.$$

Summary

For every affine set C } parallel subspace $V = \{x - x_0,\ x, x_0 \in C\}$

If C is affine, $C - x_0$ is a subspace.
If V is a subspace, $V + x_0$ is an affine set.

Set of solutions of linear equations

$$C = \{x \mid \underset{m \times n}{A} x = b\}$$

Polyhedra

Polyhedra

$$P = \{ x \mid a_i^T x \leq b_i \,,\; c_i^T x = d_i \}$$
$$\qquad\qquad i = 1, \dots, m \qquad\qquad i = 1, \dots, k$$

$$P = \{ x \mid A x \leq b,\; C x = d \}$$

Linearity Space

Pointed Polyhedron

Examples

Examples

1) $\{ x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0 \}$

2) Halfspace $\{ x \mid a^T x \leq b \}$

3) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid |x| \leq 1, |y| \leq 1, |z| \leq 1 \right\}$

Face of a polyhedron :

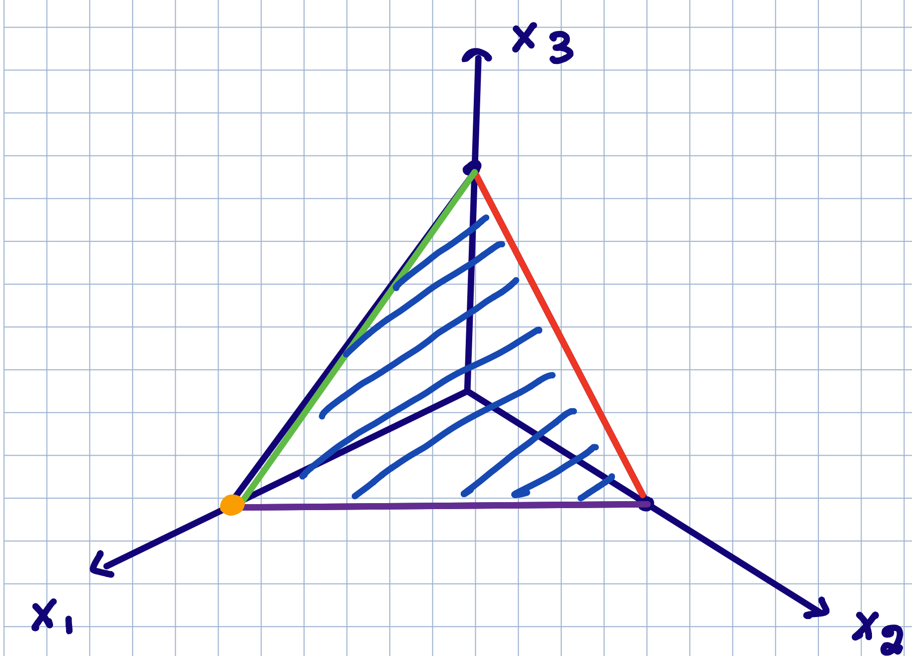
$$P = \{ x \mid a_i^T x \le b_i \,,\; c_i^T x = d_i \}$$
$$\qquad i = 1, \dots, m \qquad\qquad i = 1, \dots, K$$

$$\mathbb{F}_J = \{ \qquad\qquad\qquad\qquad \}$$

if this set is <u>not empty</u> it is called a face of P.

Example

$$\{ x \in \mathbb{R}^3 \mid x_i \ge 0 \,,\; \textstyle\sum x_i = 1 \} \quad i = 1, 2, 3$$



Minimal face :

Extreme point (or vertex) :

Extreme point (or vertex):

Properties