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1. Derive the dual of the following problem

$$\min \quad 1^T x + 1^T y$$

$$\text{s.t} \quad x \geq 0$$

$$y \geq 2c$$

$$y - x = c$$

$\Rightarrow$ $y - x = c$ $\quad y = x + c$

$\therefore$ $\min \quad 1^T x + 1^T (x + c)$

$$\text{s.t} \quad x \geq 0$$

$$x + c \geq 2c \rightarrow x \geq c$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad c_i > 0 \qquad \therefore \begin{array}{l} x \geq 0 \\ x \geq c \end{array} \Rightarrow x \geq c$$

$\therefore$ The primal becomes

$$\min \quad 2^T x + 1^T c$$

$$\text{s.t} \quad x \geq c$$

derive the dual :

$$L(x, \lambda) = 2^T x + \lambda^T (c - x) + 1^T c$$

$$g(\lambda) = \inf L(x, \lambda) = 2^T x + \lambda^T c - \lambda^T x + 1^T c = (2^T - \lambda^T) x + \lambda^T c + 1^T c$$

$$= \begin{cases} \lambda^T c + 1^T c & \text{if } 2^T - \lambda^T = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$\therefore$ The dual is:

$$\max_{\lambda} \quad \lambda^T c + 1^T c$$
$$\text{s.t.} \quad \lambda - 2 = 0$$

=) From the dual we know that

$\lambda = 2$ is the only solution

$\therefore$ for dual problem, the optimal value is $3^T c = 3 \cdot \sum_{i=1}^{n} c_i$

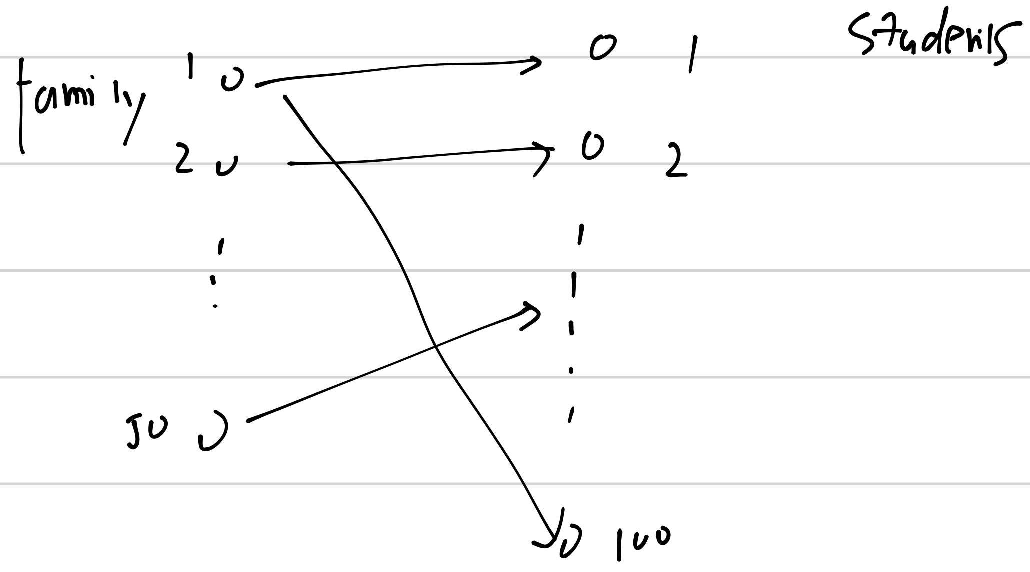
due to the strong duality

the optimal solution for primal is $3^T c = 3 \cdot \sum_{i=1}^{n} c_i$

The polyhedron has only 1 vertices $\begin{cases} x = c \\ y = 2c \end{cases}$

2. 100 students and 50 families

each family host 2 students



each family can take any 2 students from the 100 students

we need to select the matching

let the indicator $x(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$

And we assign the weights $w(e)$ for each matching

$w(e)$ equals the distance between family u and student v

$u \xrightarrow{w(e)} v$

we want to minimized the cost $\sum_{e \in E} x(e) \cdot w(e)$

we can use the Matrix M (TUM) to describe the problem.

$$M = \begin{array}{rc} & \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{array}{r} \text{family} \left\{ \begin{array}{c} 1 \\ 2 \\ \vdots \\ 50 \end{array} \right. \\ \text{students} \left\{ \begin{array}{c} 1 \\ 2 \\ \vdots \\ 100 \end{array} \right. \end{array} & \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & & \\ 0 & \vdots & & \\ 1 & 0 & & \\ 0 & 1 & & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{bmatrix} \end{array} \begin{bmatrix} x(e_1) \\ x(e_2) \\ \vdots \\ x(e_i) \\ \vdots \\ x(e_n) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} 50 \\ \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} 100 \end{array}$$

M $\qquad$ x $\qquad$ b

each matching match a point from students and a point from families

each families will have 2 students, each student would be assigned to exactly 1 family

$\therefore$ we need the first 50 elements in b to be 2, and last 100 to be 1

$\therefore$ ILP

$$\min \sum_{e \in E} x(e) \cdot w(e)$$

$$\text{s.t } Mx = b$$

$$x(e) \geq 0 \qquad x(e) \in \{0,1\}$$

consider M.

(i) if there exists an all zero column $\rightarrow \det M = 0$

(ii) if exists a column that has only one 1, expand the det along this column

(iii) all columns have exactly 2 one $\rightarrow \det = 0$

sum the rows corresponding to family

sum the rows corresponding to students

we get the same rows $\rightarrow$ rows linearly dependent

$\Rightarrow M$ is TUM matrix

Problem 3.

Write an ILP that takes as input a graph and identifies whether the graph is bipartite or not

$$x(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

ILP is

$$\max_{e \in E} \sum x(e)$$

$\delta(v)$ = set of edges incident to v

$$\text{s.t} \quad \sum_{e \in \delta(v)} x(e) \le 1 \quad \text{for all vertices } v$$

$$x(e) \ge 0 \quad x(e) \in \mathbb{Z}$$

The relaxation LP is

$$\max_{e \in E} \sum x(e)$$

$$\text{s.t} \quad \sum_{e \in \delta(v)} x(e) \le 1$$

$$x(e) \ge 0 \quad x(e) \in \mathbb{R}$$

From the class, we know that we can formulate it using a Matrix M

$$\Rightarrow \max \; 1^T x$$

$$\text{s.t} \quad Mx \le 1$$

$$x \ge 0$$

Since M is TUM matrix, the relaxation LP and ILP achieves the same optimal solutions

The feasible solutions are not the same.