Fall 2020
EE 236A
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EE236A Linear Programming
Quiz 4 Solutions
Tuesday November 24, 2020

NAME: ____________________ **UID**: ____________________

This quiz has 3 questions, for a total of 20 points.

Open book. The exam is for a total of 1:00 hour. **Please, write your name and UID on the top of each sheet.**

Good luck!

Problem	Mark	Total
P1		6
P2		7
P3		7
Total		20

Problem 1 (6 points)

1) Derive the dual of the following problem using the Lagrangian.

$$\begin{array}{ll} \min\limits_{x,y} & 1^T x + 1^T y \\ \text{subject to} & x \geq 0 \\ & y \geq 2c \\ & y - x = c \end{array} \tag{1}$$

where $x, y \in \mathbf{R}^n$ are variables, $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a given constant vector in $\mathbf{R}^n$, with $c_i > 0$ for $i = 1 \ldots n$.

2) What can you tell about the structure of the optimal solution and form of the vertices? How many vertices does this polyhedron have?

Solution:

1) In order to find the dual of the primal program, we should first write the Lagrangian as follows.

$$L(x, y, \lambda, v, s) = 1^T x + 1^T y - \lambda^T x + v^T(-y + 2c) + s^T(y - x - c) \tag{2}$$

where λ, v and s are dual variables.

As we did in the class the dual objective function is the infimum of the Lagrangian.

$$g(\lambda, v, s) = \inf_{x,y} \; L(x, y, \lambda, v, s) = \inf_{x,y} \; (1 - s - \lambda)^T x + (1 - v + s)^T y + 2v^T c - s^T c \tag{3}$$

$$g(\lambda, v, s) = \begin{cases} 2v^T c - s^T c & \text{if } \; 1 - s - \lambda = 0 \quad \text{and} \quad 1 - v + s = 0 \\ \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, we can write the dual as follows.

$$\begin{array}{ll} \max\limits_{\lambda,v,s} & 2v^T c - s^T c \\ \text{subject to} & 1 - s - \lambda = 0 \\ & 1 - v + s = 0 \\ & \lambda, v \geq 0 \end{array} \tag{4}$$

Since λ is not in the objective function, we can simplify the dual as follows.

$$\begin{array}{ll} \max\limits_{v,s} & 2v^T c - s^T c \\ \text{subject to} & 1 - s \geq 0 \\ & 1 - v + s = 0 \\ & v \geq 0 \end{array} \tag{5}$$

2) Since two of the constraints of the primal problem are $y = x + c$ and $y \geq 2c$, we can find a lower bound on x such that $x \geq c$. There is also a nonnegativity constraint on x and it gives another lower bound on feasible x vectors. Therefore, the optimum x will be the maximum of these two lower bounds.

$$x_i^\star = \max\{0, c_i\} \quad \text{for} \quad i = 1, ..., n \tag{6}$$

Since $c_i > 0$ for $i = 1, ..., n$, the optimum $x^\star = c$ and the optimum $y^\star = x^\star + c = 2c$.

Since there are $2n$ variables in the primal program, vertices should satisfy $2n$ linearly independent constraints with equality to satisfy the rank condition. Every feasible solution will satisfy the equality constraint $y - x = c$, therefore, a vertex should satisfy n linearly independent inequality constraints with equality additionally. Our inequality constraints are $x \geq 0$ and $y \geq 2c$. However, we know that $c_i > 0$ for $i = 1, ..., n$ and therefore, x_i variables cannot be equal to zero because $x_i = 0$ violates $y_i \geq 2c_i$ constraint. Due to this reason, at a vertex, we should have $y = 2c$ and $x = c$ so that we can get a feasible vertex that satisfies the rank condition. Since this the only feasible solution that can satisfy the rank condition, we have a single vertex.

<u>Problem 2</u> (`7 points`)

UCLA organizes the "host a foreign student for Thanksgiving" event; it has 100 foreign students participating and 50 families that volunteered to host 2 students each. UCLA would like to assign families to students, based on geographic location, so that the students do not have to drive far from their residence. Can you formulate an ILP to solve this problem so that the constraint matrix is TUM?

<u>Solution</u>:

The goal is to minimize the distance that the students will travel. Define set of students as S and set of families as F. We can further define a set called F' in which all the families are duplicated i.e. $|F'| = 100$,whereas $|F| = 50$. In this form we can cast our problem as a perfect matching problem. We can define variables x_e such that $x_e = 1$ if the edge e, which connects a particular student from S to a particular family in F', is in the set of selected edges; and $x_e = 0$ otherwise. We can also define w_e, the distance between the student and family which are connected with edge e. Then our ILP is the following:

$$\begin{array}{ll} \min\limits_{x} & \sum_{e \in \xi} w_e x_e \\ \text{subject to} & \sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in S \cup F' \\ & x_e \in \{0, 1\} \quad \forall e \in \xi \end{array} \tag{7}$$

where $\delta(v)$ denotes the edges that are intact with vertex v. Here the first constraint is due to the fact that each student will go to only one house, and the second constraint is due to the definition of $x_e s$. The following is the relaxed LP:

$$\begin{array}{ll} \min\limits_{x} & \sum_{e \in \xi} w_e x_e \\ \text{subject to} & \sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in S \cup F' \\ & 0 \leq x_e \quad \forall e \in \xi \end{array} \tag{8}$$

Note that, from the lectures we know that the last constraint has no effect on whether the overall constraint matrix is TUM or not. Hence, we need to check if the matrix constraint $Mx = \mathbf{1}$ has a TUM matrix M. This matrix is exactly in the same form of the matrix M we saw in Bipartite graphs (Class 13) and we proved that M was TUM in the class.

Problem 3 (`7 points`): Write an ILP that takes as input a graph and identifies whether the graph is bipartite or not. If you relax you ILP constraints, explain how the solutions of the resulting LP compare with those of your ILP - in particular, are the optimal solutions the same? are the feasible solutions the same? discuss why.

Solution:

1) As we talked in the class, a graph $G = (V, \xi)$ is called bipartite if we can partition the vertices in two sets V_1 and V_2, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ so that each edge has one end at V_1 and one end at V_2. We can understand if a given graph is bipartite by using graph coloring, particularly vertex coloring. Bipartite graphs are two-colorable graphs, i.e., we can use two colors to color the vertices such that no two adjacent vertices are of the same color. Therefore, we can solve this problem by using the following feasibility program that checks if a given graph is two-colorable.

$$\begin{aligned} \min_{x} \quad & 0 \\ \text{subject to} \quad & x_i + x_j = 1 \quad \forall (i,j) \in \xi \\ & x_i \in \{0,1\} \quad \forall i \in V \end{aligned} \tag{9}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_{|V|} \end{bmatrix}$.

Here, x_i is an indicator variable for vertex i and we can define it as follows.

$$x_i = \begin{cases} 1 & \text{if vertex } i \text{ is colored with RED} \\ \\ 0 & \text{if vertex } i \text{ is colored with BLUE} \end{cases}$$

where $i \in V$.

For feasibility problems, every feasible solution is also optimal. In (9), the first constraint is put to make sure that vertices of each edge will have different colors. Therefore, if the ILP in (9) finds a solution, this means that the input graph is two-colorable, i.e., the graph is bipartite.

We can relax the ILP in (9) into the following LP.

$$\begin{aligned} \min_{x} \quad & 0 \\ \text{subject to} \quad & x_i + x_j = 1 \quad \forall (i,j) \in \xi \\ & 0 \leq x \leq 1 \end{aligned} \tag{10}$$

Since this is the relaxed version of the ILP in (9), feasible solutions are not the same. Elements of a feasible x vector in (9) can only take values 0 and 1, however, elements of a feasible x vector in

(10) can take real values between 0 and 1. Note that any feasible solution of ILP is also feasible for its relaxed version.

Moreover, in feasibility problems, any feasible solution is also optimal, therefore, the set of optimal solutions of LP in (10) is not the same as the set of optimal solutions of ILP in (9). An optimal solution of LP in (10) is one of the feasible solutions whose elements are not necessarily integer but an optimal solution of ILP in (9) needs to have integer elements. Note that any optimal solution of ILP is an optimal solution of LP in (10).