Fall 2020
ECE 236A
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Linear Programming

Homework 4
Due: 9 am, Friday Nov. 20th

Problem 1 (`3 points`): Explain why the following proof for strong duality is flawed:

From week duality we have

$$d^* \leq p^*. \tag{1}$$

Since the dual of the dual problem is the primal problem, applying weak duality again gives

$$p^* \leq d^*. \tag{2}$$

From (1), (2), $p^* = d^*$.

Problem 2 (`3 points`):

The following are the max-flow LP and the min-cut ILP formulations as we've seen in class and the provided chapter notes (the variables are as described there):

$$\begin{array}{ll} \text{maximize} & f_{ts} \\ \text{subject to} & f_{ij} \leq c_{ij},\ \forall (i,j) \in \mathcal{E} \\ & \sum\limits_{j:(j,i)\in\mathcal{E}} f_{ji} - \sum\limits_{k:(i,k)\in\mathcal{E}} f_{ik} \leq 0,\ \forall i \in \mathcal{V} \\ & f_{ij} \geq 0,\ (i,j) \in \mathcal{E} \end{array}$$

$$\begin{array}{ll} \text{minimize} & c^T d \\ \text{subject to} & p_s - p_t \geq 1, \\ & d_{ij} - p_i + p_j \geq 0, \\ & d_{ij} \in \{0,1\}, \\ & p_i \in \{0,1\} \end{array}$$

(a) Prove that the constraint matrix of the max-flow LP is TUM.

(b) Prove that the dual of the max-flow LP gives the same solution as the ILP (we will derive the dual LP in class, you do not need to re-derive it).

Problem 3 (`3 points`):

Hospital H is going through a hiring phase. With the staff of surgeons that they already have, the hospital knows that there are some surgeries which they are not equipped to handle. These surgeries are enumerated in the list $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$. After advertising for surgeon positions, they received a list of surgeon applicants $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$. Each applicant in their resume indicated which of the surgeries in $\mathcal{S}$ they are able to perform, and the salaries they wish to get if they are hired. The hospital would like to hire a number of surgeons which can handle all the surgeries in $\mathcal{S}$ while paying the minimum amount in salaries. Write an ILP that solves the hospital's problem.

Problem 4 (3 points):

Coach P wants to organize some one-on-one training basketball matches between the players of her team. She wants to set up those matches so that a match is beneficial to both the players in the match. Based on her experiences with the players, she now has a list of possible match-ups between her players: each of these match-ups will help the two players develop their game and overcome their playing deficiencies. The list looks like this

Names of players	**Notes on the match**
James and Kevin	Will improve their defensive plays
David and Mike	Will improve their speed plays
Kevin and Shaun	Learn how to play a game with size mismatches
$\vdots$	$\vdots$

In the mind of Coach P, all these match-ups are equally beneficial. However, she notices that it is not the case that each player is a part of a "single" possible match-up (e.g., Kevin in the table above). Moreover, she notices that some players are part of possible match-ups more than the others. Out of this list of possible match-ups, she would like to set up the maximum number of games, to run in parallel.

a) Write an ILP that would help Coach P. Can the constraints of the ILP be expressed using a TUM matrix?

b) Suppose that Coach P wants to devote three separate Sundays to these 1-on-1 matches. Coach P wants to arrange as many matches as possible on the three days, without repeating any matchups. Thus, each player can play in up to three different matches over the three days. Write an ILP that Coach P can use.

Problem 5 (4 points): Bob just started his new job as a bike delivery boy in a famous restaurant. As his first task, he was assigned to deliver meals to a set of houses $\mathcal{H}$. Bob has a very accurate map, so he knows exactly how far it is to go from any given house directly to the other. He wants to find the best route that gets him to do his deliveries with the minimum amount of cycling.

But there is a catch! The bad thing about his job is that he knows that the food at the restaurant he works is really bad. So he wants to make sure that he doesn't go by a house that he has already delivered food to, lest the customers there catch him on his way and complain. Can you help him find such a route using the LP techniques that you learned? *Hint:* transform this problem into a graph problem, and see what properties your graph solution needs to have. It is okay to use an ILP instead of an LP.

Problem 6 (`4 points`): As we discussed in class, the following LP

$$
\begin{array}{ll}
\text{maximize} & \sum f_i \\
\text{subject to} & \sum_{i:\, e \in P_i} f_i \leq c_e, \ \forall \text{ edges } e \\
& f_i \geq 0
\end{array}
\tag{3}
$$

is the max-flow problem formulation with flows associated with paths, where $P = \{P_i\}$ is the set of all paths from source to destination, and f_i is the flow associated with the ith path.

- Prove that the LP in (3) is equivalent to the max-flow problem formulation with flows associated to each edge (provided in Problem 1).
- In a graph with unit capacity edges (i.e., $c_e = 1, \ \forall$ edges e), prove that if the the solution of the max-flow problem is k then we can find k edge-disjoint paths from s to t in the graph. (Two paths are said to be edge-disjoint if they do not share a common edge.)
- Derive the dual of (3) and give an interpretation to the dual variables and the objective function.