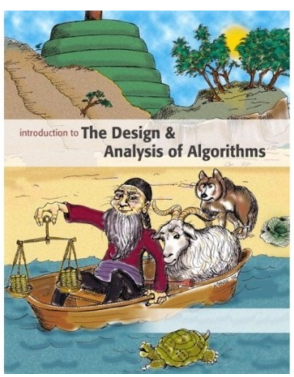
Introduction to

Algorithm Design and Analysis

[16] Dynamic Programming 1



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In the last class…

- **Single-source shortest paths**
  - From BFS to the Dijkstra algorithm
- **All-pairs shortest paths**
  - BF1, BF2, BF3
  - Floyd-Warshall algorithm

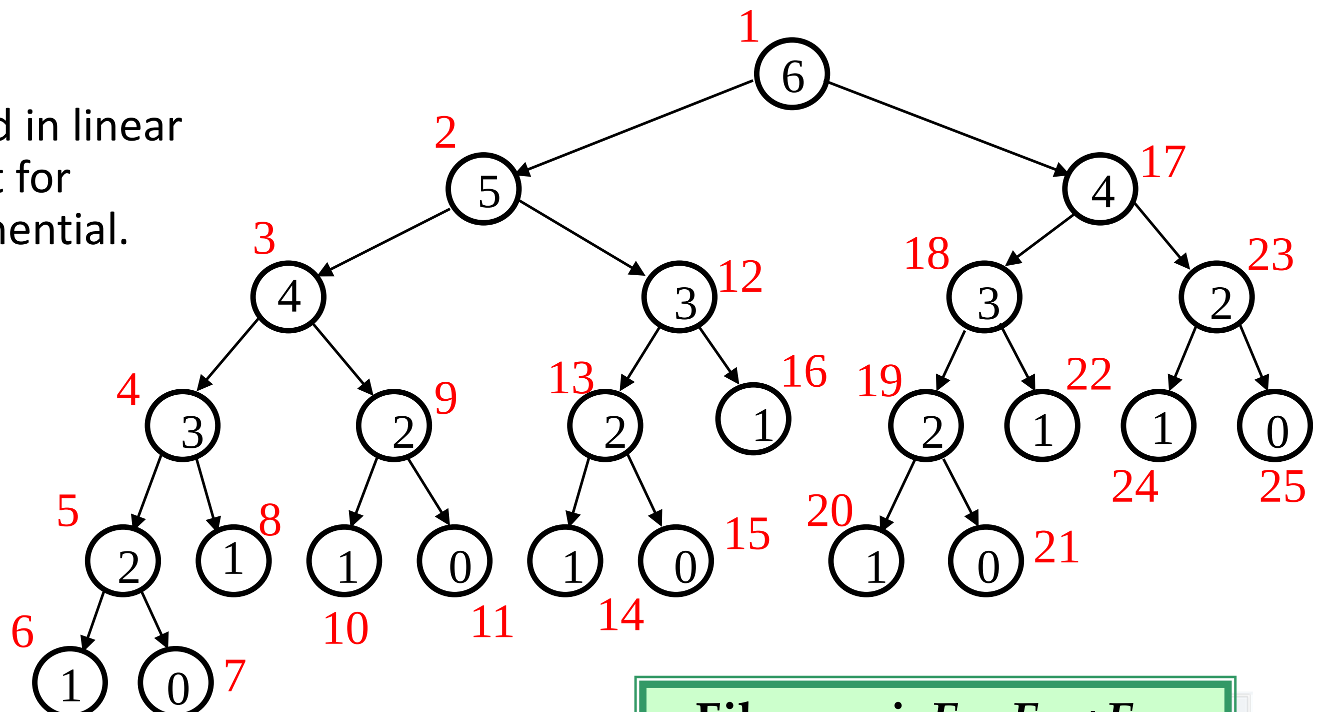
Dynamic Programming

- **Basic Idea of Dynamic Programming (DP)**
  - Smart scheduling of subproblems
- **Minimum Cost Matrix Multiplication**
  - BF1, BF2
  - A DP solution
- **Weighted Binary Search Tree**
  - The "same" DP with matrix multiplication

Brute Force Recursion

The F_n can be computed in linear time easily, but the cost for recursion may be exponential.

The number of activation frames are $2F_{n+1}$-1



For your reference

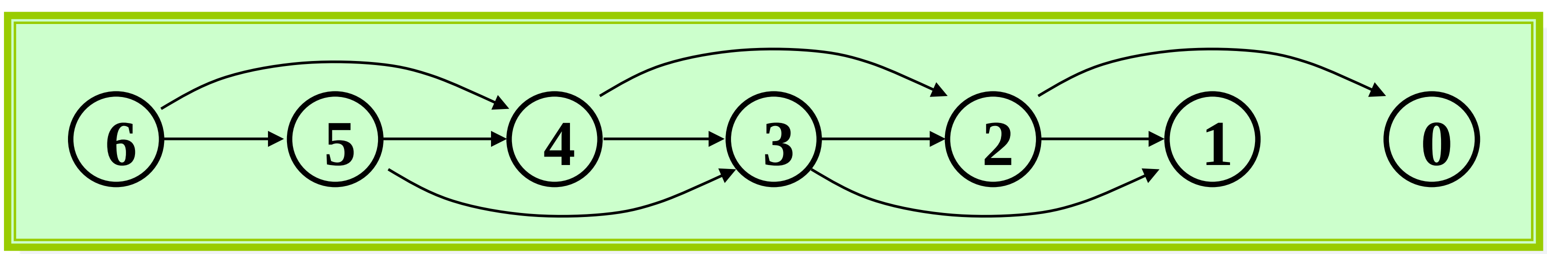
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Fibonacci: $F_n = F_{n-1} + F_{n-2}$

0, 1, 1, 2, 3, 5, 8, 13, 21, 35, ...

Subproblem Graph

- The **subproblem graph** for a recursive algorithm *A* of some problem is defined as:
  - vertex: the instance of the problem
  - directed edge: I→J if and only if when *A* invoked on I, it makes a recursive call directly on instance J.
- Portion A(*P*) of the subproblem graph for Fibonacci function: here is fib(6)



Properties of Subproblem Graph

- **If A always terminates, the subproblem graph for A is a DAG.**
  - For each path in the tree of activation frames of a particular call of A, A(P), there is a corresponding path in the subproblem graph of A connecting vertex P and a base-case vertex.
  - The subproblem graph can be viewed as a dependency graph of subtasks to be solved.
- **A top-level recursive computation traverse the entire subproblem graph in some memoryless style.**

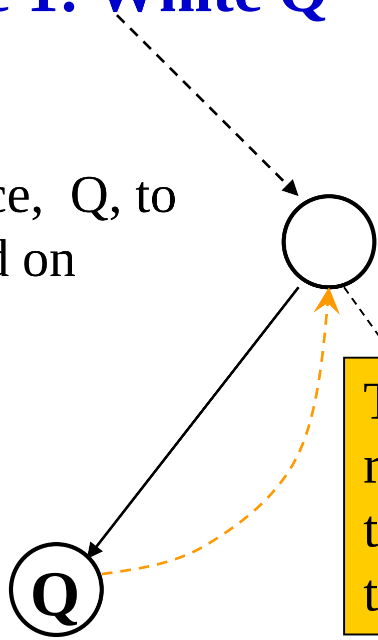
Basic Idea of DP

- **Smart recursion**
  - Compute each subproblem **only once**
- **Basic process of a "smart" recursion**
  - Find a reverse topological order for the subproblem graph
    - In most cases, the order can be determined by particular knowledge of the problem.
    - General method based on DFS is available
  - Scheduling the subproblems according to the reverse topological order
  - Record the subproblem solutions in a dictionary

Recursion by DP

Case 1: White Q

a instance, Q, to be called on

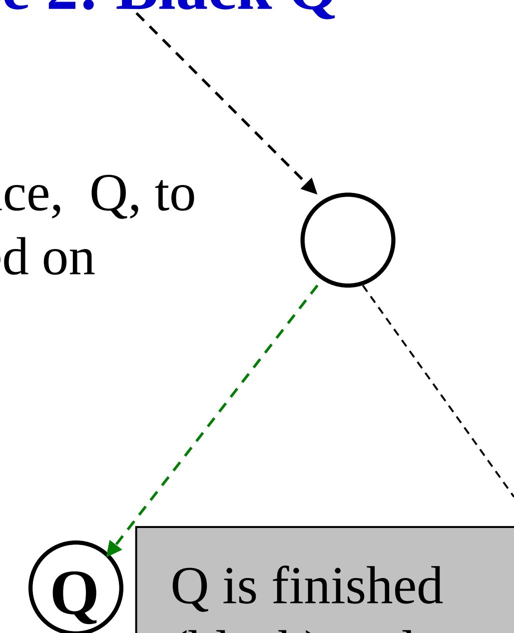


To backtracking, record the result into the dictionary (Q, turned black)

Q is undiscovered (white), go ahead with the recursive call

Case 2: Black Q

a instance, Q, to be called on



Q is finished (black), only "checking" the edge, retrieve the result from the dictionary

Note: for DAG, no gray vertex will be met

Fibonacci by DP

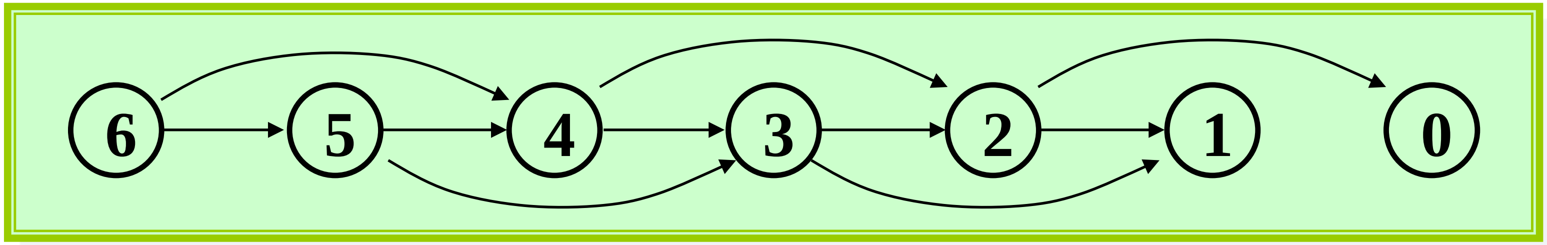
```
fibDPwrap(n)
  Dict soln=create(n);
return fibDP(soln,n)
```

This is the wrapper, which will contain processing existing in original recursive algorithm wrapper.

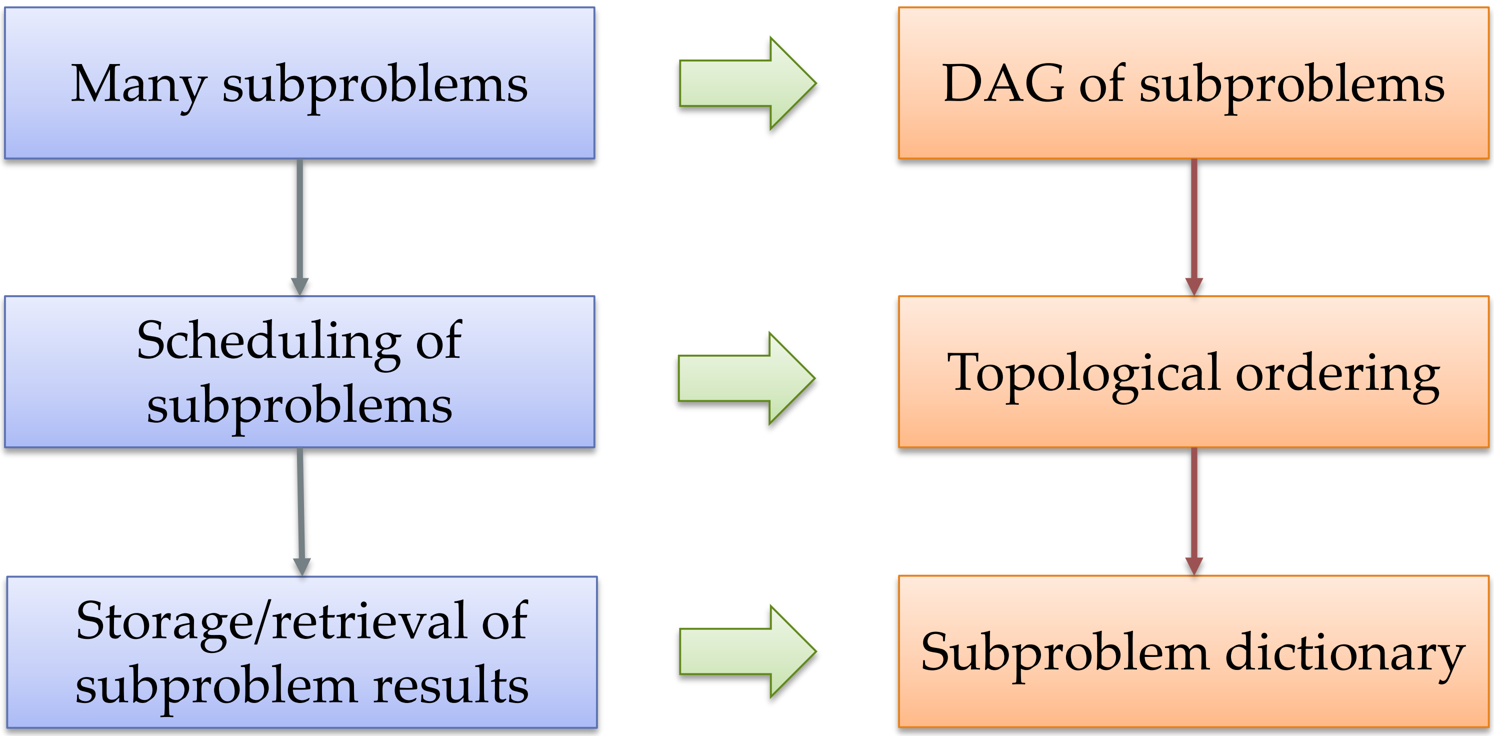
```
fibDP(soln,k)
   int fib, f1, f2;
   if (k<2) fib=k;
   else
      if (member(soln, k-1)==false)
         f1=fibDP(soln, k-1);
      else
         f1= retrieve(soln, k-1);
      if (member(soln, k-2)==false)
         f2=fibDP(soln, k-2);
      else
         f2= retrieve(soln, k-2);
      fib=f1+f2;
   store(soln, k, fib);
return fib
```

Fibonacci by DP

- **Subproblems**
  - o soln[0], soln[1], …, soln[n]
- **Topology order to the subproblems**
  - o soln[0] = 0; soln[1] = 1;
  - o for{i=2; i≤n; i++}{
    soln[i] = soln[i-1] + soln[i-2];
    } // end for



DP: New Concept Recursion



Matrix Multiplication Order Problem

- **The task:**

  Find the product: $A_1 \times A_2 \times \ldots \times A_{n-1} \times A_n$

  A_i is 2-dimentional array of different legal sizes

- **The issues:**
  - Matrix multiplication is associative
  - Different computing order results in great difference in the number of operations
- **The problem:**
  - Which is the best computing order

Cost for Matrix Multiplication

Let $C = A_{p\times q}\times$

An example: $A_1 \times A_2 \times A_3 \times A_4$

$30\times1 \quad 1\times40 \quad 40\times10 \quad 10\times25$

$((A_1\times A_2)\times A_3)\times A_4$: 20700 multiplications

$A_1\times(A_2\times(A_3\times A_4))$: 11750

$(A_1\times A_2)\times(A_3\times A_4)$: 41200

$A_1\times((A_2\times A_3)\times A_4)$: 1400

$$c_{i,j} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

There are q multiplication

C has $p\times r$ elements as $c_{i,j}$

So, pqr multiplications altogether

Looking for a Greedy Solution

- **Strategy 1: “cheapest multiplication first”**
  - o Success: $A_{30\times1}\times((A_{1\times40}\times A_{40\times10})\times A_{10\times25}$
  - o Fail: $(A_{4\times1}\times A_{1\times100})\times A_{100\times5}$
- **Strategy 2: “largest dimension first”**
  - o Correct for the second example above
  - o Fail: $A_{1\times10}\times A_{10\times10}\times A_{10\times2}$

Intuitive Solution

- **Matrices:** $A_1, A_2, \ldots, A_n$
- **Dimension: dim:** $d_0, d_1, d_2, \ldots, d_{n-1}, d_n$, **for** A_i **is** $d_{i-1} \times d_i$
- **Sub-problem: seq:** $s_0, s_1, s_2, \ldots, s_{len-1}, s_{len}$, **which means the multiplication of** k **matrices, with the dimensions:** $d_{s0} \times d_{s1}, d_{s1} \times d_{s2}, \ldots$
  - Note: the original problem is: seq=$(0,1,2,\ldots,n)$

Intuitive Solution

```
mmTry1(dim, len, seq)
  if (len<3) bestCost=0
  else
    bestCost=∞;
    for (i=1; i≤len-1; i++)
      c=cost of multiplication at position seq[i];
      newSeq=seq with ith element deleted;
      b=mmTry1(Dim, len-1, newSeq);
      bestCost=min(bestCost, b+c);
  return bestCost
```

Recursion on index sequence:
(seq): 0, 1, 2, …, *n* (len=n)
with the kth matrix is A_k ($k \neq 0$) of the size $d_{k-1} \times d_k$,
and the kth(k<n) multiplication is $A_k \times A_{k+1}$.

$T(n)=(n\text{-}1)T(n\text{-}1)+n$, in $\Theta((n\text{-}1)!)$

Subproblem Graph

- **key issue**
  - How can a subproblem be denoted using a **concise identifier**?
  - For mmTry1, the difficulty originates from the varied intervals in each newSeq.
- **If we look at the last (contrast to the first) multiplication, the two (not one) resulted subproblems are both contiguous subsequences, which can be uniquely determined by the pair:**

<head-index, tail-index>

Improved Recursion

```
mmTry2(dim, low, high)
    if (high-low==1) bestCost=0
    else
        bestCost=∞;
        for (k=low+1; k≤high-1; k++)
            a=mmTry2(dim, low, k);
            b=mmTry2(dim, k, high);
            c=cost of multiplication at position k;
            bestCost=min(bestCost, a+b+c);
    return bestCost
```

Only one matrix

with dimensions: dim[low], dim[k], and dim[high]

Still in $\Omega(2^n)$!

Smart Recursion by DP

- **DFS can traverse the subproblem graph in time O(n^3)**
  - At most $n^2/2$ vertices, as <i,j>, $0 \le i < j \le n$.
  - At most $2n$ edges leaving a vertex

```
mmTry2DP(dim, low, high, cost)
  ……
  for (k=low+1; k≤high-1; k++)
    if (member(low,k)==false) a=mmTry2(dim, low, k);
      else a=retrieve(cost, low, k);
    if (member(k,high)==false) b=mmTry2(dim, k, high);
      else b=retrieve(cost, k, high);
    ……
  store(cost, low, high, bestCost);
  return bestCost
```

Corresponding to the recursive procedure of DFS

Order of Computation

- Dependency between subproblems

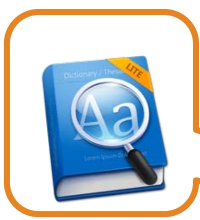
matrixOrder(n, cost, last)

- for (low=n-1; low≥1; low--)
- for (high=low+1; high≤n; high++)

Compute solution of subproblem (low, high) and store it in cost[low][high] and last[low][high]

- return cost[0][n]

DP dict



Multiplication Order

- Input: array **dim** =(d_0, d_1, …, d_n), the dimension of the matrices.
- Output: array **multOrder**, of which the ith entry is the index of the ith multiplication in an optimum sequence.

Using the stored results

```
float matrixOrder(int[] dim, int n, int[]
    multOrder)
  <initialization of last,cost,bestcost,bestlast…>
   for (low=n-1; low≥1; low--)
     for (high=low+1; high≤n; high++)
       if (high-low==1) <base case>
       else bestcost=∞;
       for (k=low+1; k≤high-1; k++)
         a=cost[low][k];
         b=cost[k][high]
         c=multCost(dim[low], dim[k],
   dim[high]);
         if (a+b+c<bestCost)
           bestCost=a+b+c; bestLast=k;
       cost[low][high]=bestCost;
       last[low][high]=bestLast;
  extrctOrderWrap(n, last, multOrder)
  return cost[0][n]
```

An Example

- **Input:** d_0=30, d_1=1, d_2=40, d_3=10, d_4=25

***cost* as finished**

$$\begin{bmatrix} - & 0 & 1200 & 700 & 1400 \\ - & - & 0 & 400 & 650 \\ - & - & - & 0 & 10000 \\ - & - & - & - & 0 \\ - & - & - & - & - \end{bmatrix}$$

First entry filled

Last entry filled

Note: *cost*[i][j] is the least cost of $A_{i+1} \times A_{i+2} \times \ldots A_j$.

For each selected *k*, retrieving:

- least cost of $A_{i+1} \times \ldots \times A_k$.
- least cost of $A_{k+1} \times \ldots \times A_j$.

and computing:

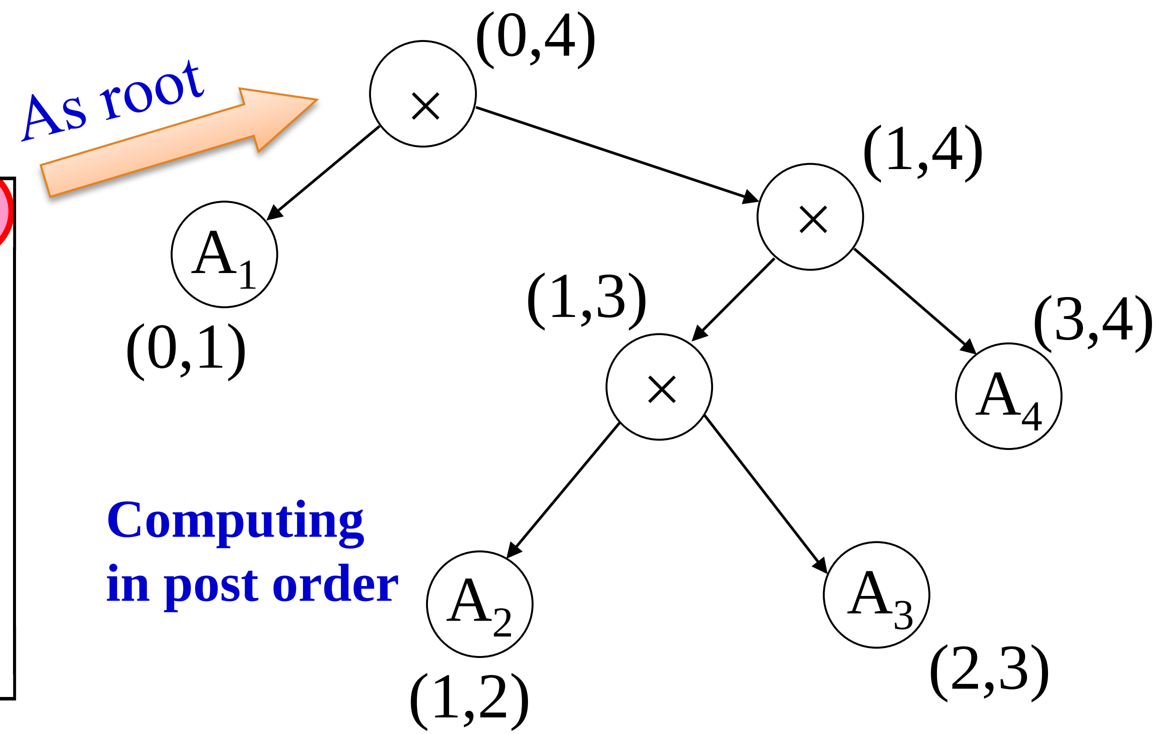
- cost of the last multiplication

Arithmetic Expression Tree

- **Example input:** $d_0=30,\ d_1=1,\ d_2=40,\ d_3=10,\ d_4=25$

***last* as finished**

$$\begin{bmatrix} - & -1 & 1 & 1 & 1 \\ - & - & -1 & 2 & 3 \\ - & - & - & -1 & 3 \\ - & - & - & - & -1 \\ - & - & - & - & - \end{bmatrix}$$



Getting the Optimal Order

- The core procedure is extractOrder, which fills the multiOrder array for subproblem (low,high), using informations in *last* array.

```
extractOrder(low, high, last, multOrder)
  int k;
  if (high-low>1)
    k=last[low][high];
    extractOrder(low, k, last, multOrder);
    extractOrder(k, high, last, multOrder);
    multOrder[multOrderNext]=k;
    multOrderNext++;
```

Just a post-order traversal

initialized in the wrapper

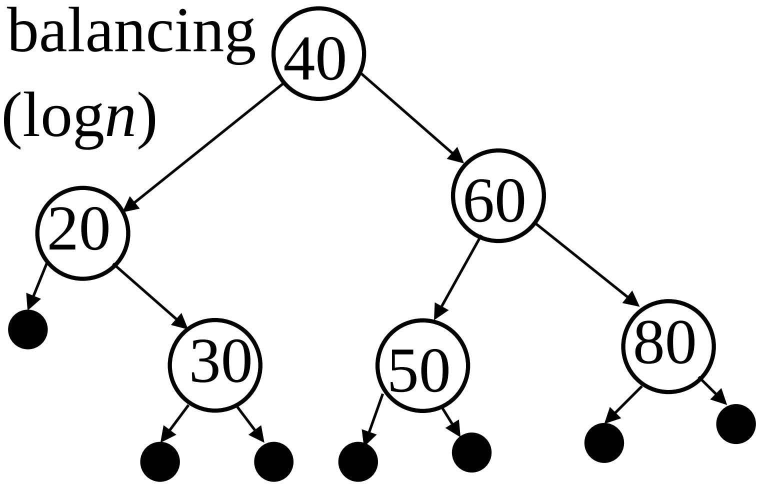
Analysis of matrixOrder

- **Main body: 3 layer of loops**
  - Time: the innermost processing costs constant, which is executed $\Theta(n^3)$ times.
  - Space: extra space for *cost* and *last*, both in $\Theta(n^2)$
- **Order extracting**
  - There are $2n$-1 nodes in the arithmetic-expression tree. For each node, extractOrder is called once. Since non-recursive cost for extractOrder is constant, so, the complexity of extractOrder is in $\Theta(n)$

Binary Search Tree

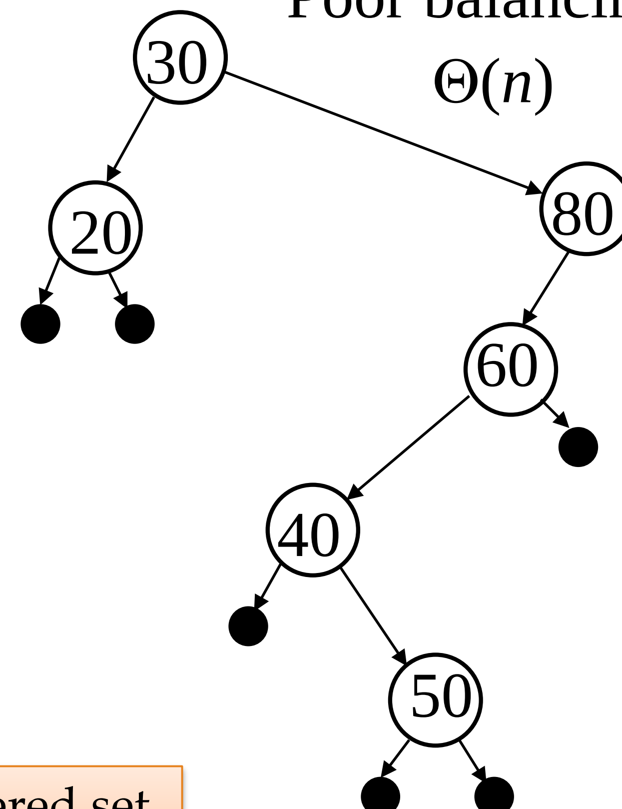
Good balancing

$\Theta(\log n)$



Poor balancing

$\Theta(n)$

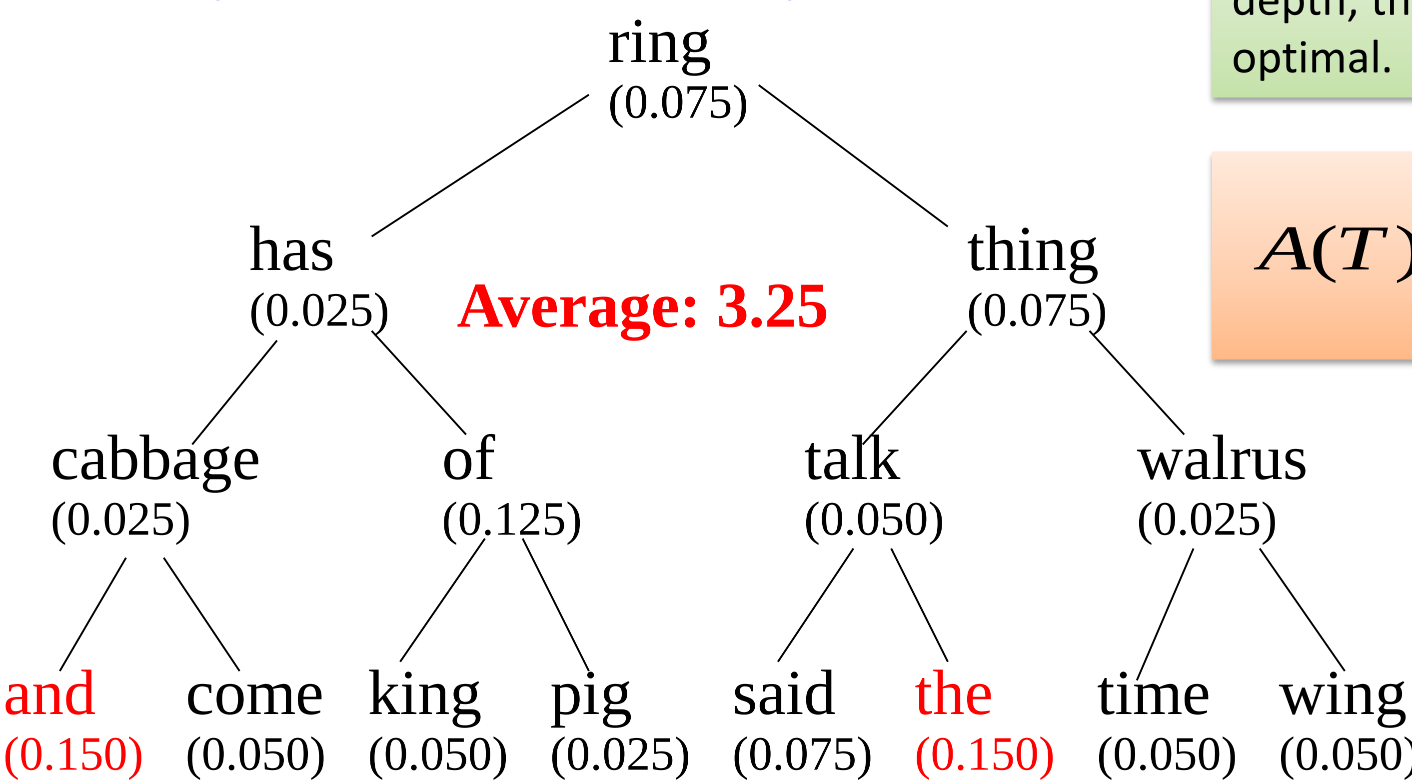


In a properly drawn tree, pushing forward to get the ordered list.

- Each node has a key, belonging to a linear ordered set
- An inorder traversal produces a sorted list of the keys

Keys with Different Frequencies

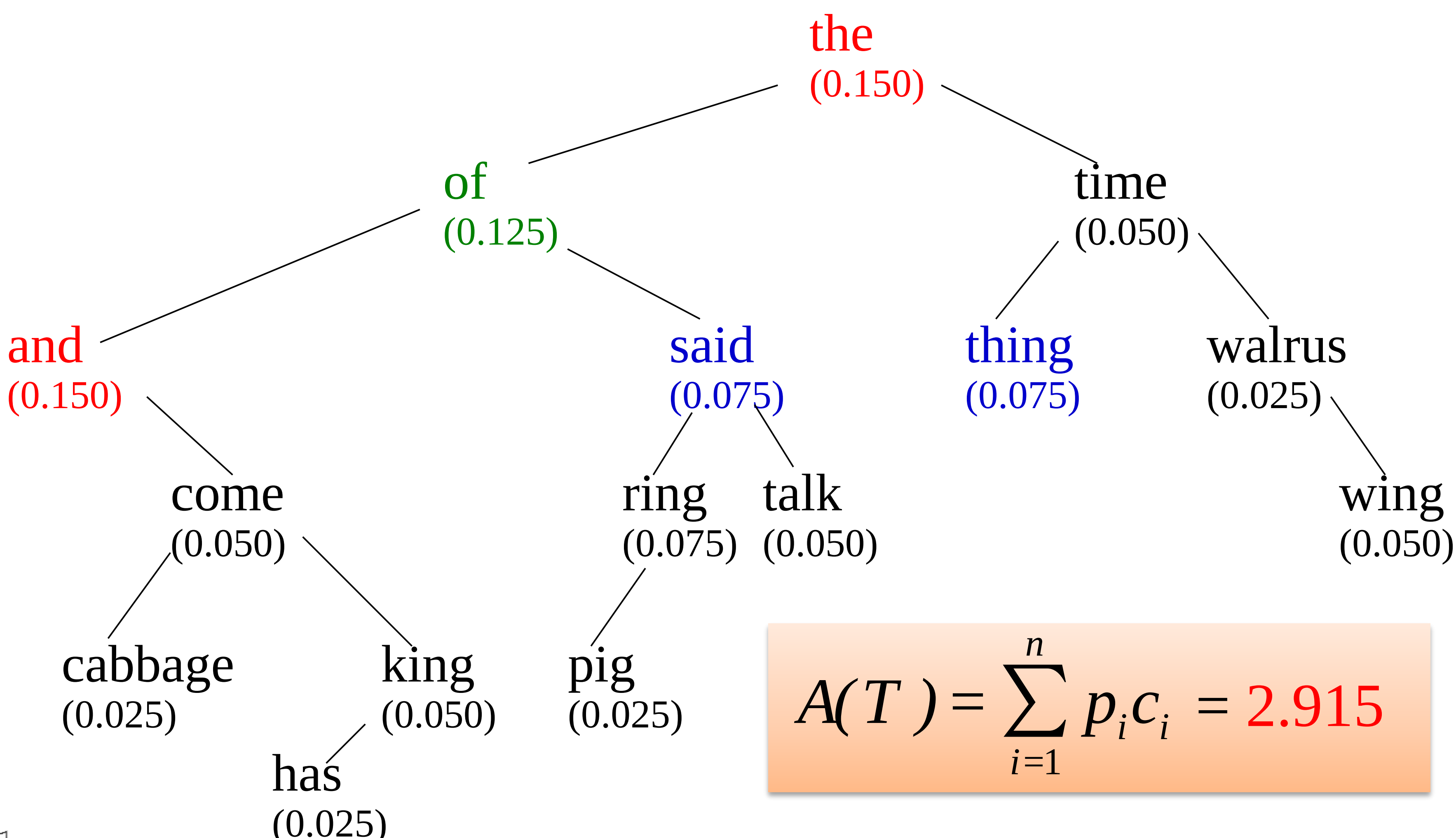
A binary search tree perfectly balanced



Since the keys with larger frequencies have larger depth, this tree is not optimal.

$$A(T) = \sum_{i=1}^{n} p_i c_i$$

Unbalanced but Improved

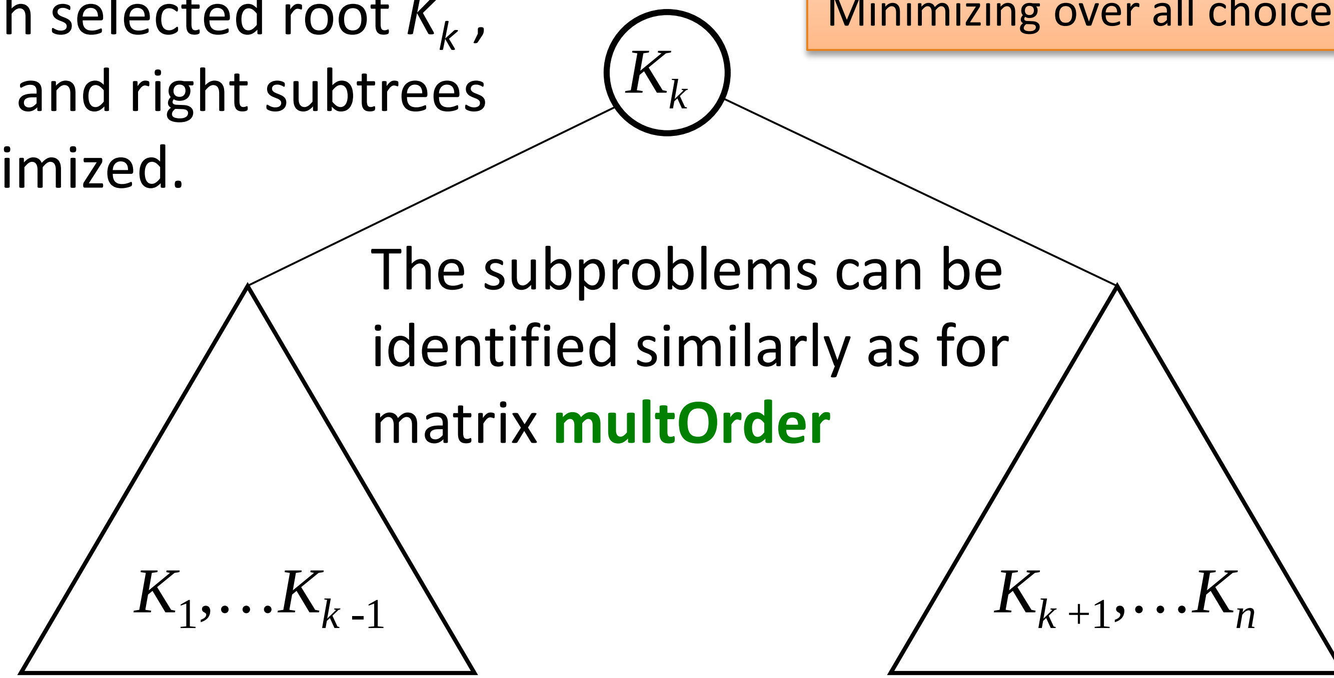


$$A(T) = \sum_{i=1}^{n} p_i c_i = 2.915$$

Optimal Binary Tree

The problem is decomposes by the choices of the root.
Minimizing over all choices

For each selected root K_k , the left and right subtrees are optimized.



The subproblems can be identified similarly as for matrix **multOrder**

Subproblems as left and right subtrees

Problem Rephrased

- **Subproblem identification**
  - The keys are in sorted order.
  - Each subproblem can be identified as a pair of index (low, high)
- **Expected solution of the subproblem**
  - For each key K_i, a weight p_i is associated.

    Note: p_i is the probability that the key is searched for.
  - The subproblem (low, high) is to find the binary search tree with *minimum weighted retrieval cost*.

Minimum Weighted Retrieval Cost

- ***A*(low, high, *r*)** is the minimum weighted retrieval cost for subproblem (low, high) when K_r is chosen as the root of its binary search tree.
- ***A*(low, high)** is the minimum weighted retrieval cost for subproblem (low, high) over all choices of the root key.
- ***p*(low, high)**, equal to $p_{\text{low}}+p_{\text{low+1}}+\ldots+p_{\text{high}}$, is the weight of the subproblem (low, high).

  Note: *p*(low, high) is the probability that the key searched for is in this interval.

Subproblem Solutions

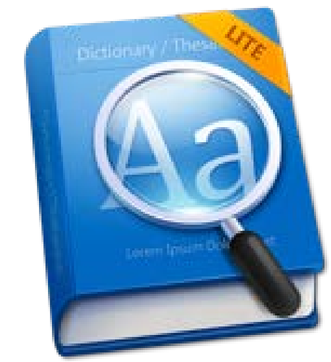
- **Weighted retrieval cost of a subtree**
  - T contains $K_{low}, \ldots, K_{high}$, and the weighted retrieval cost of T is W, with T being a whole tree.
  - As a subtree with the root at level 1, the weighted retrieval cost of T will be: **$W+p$(low, high)**
- **So, the recursive relations are:**
  - $A(\text{low, high}, r)$

    $= p_r+p(\text{low}, r\text{-}1)+A(\text{low}, r\text{-}1)+p(r+1, \text{high})+A(r+1, \text{high})$

    $= p(\text{low, high})+A(\text{low}, r\text{-}1)+A(r+1, \text{high})$
  - $A(\text{low, high}) = \min\{A(\text{low, high}, r) \mid \text{low} \leq r \leq \text{high}\}$

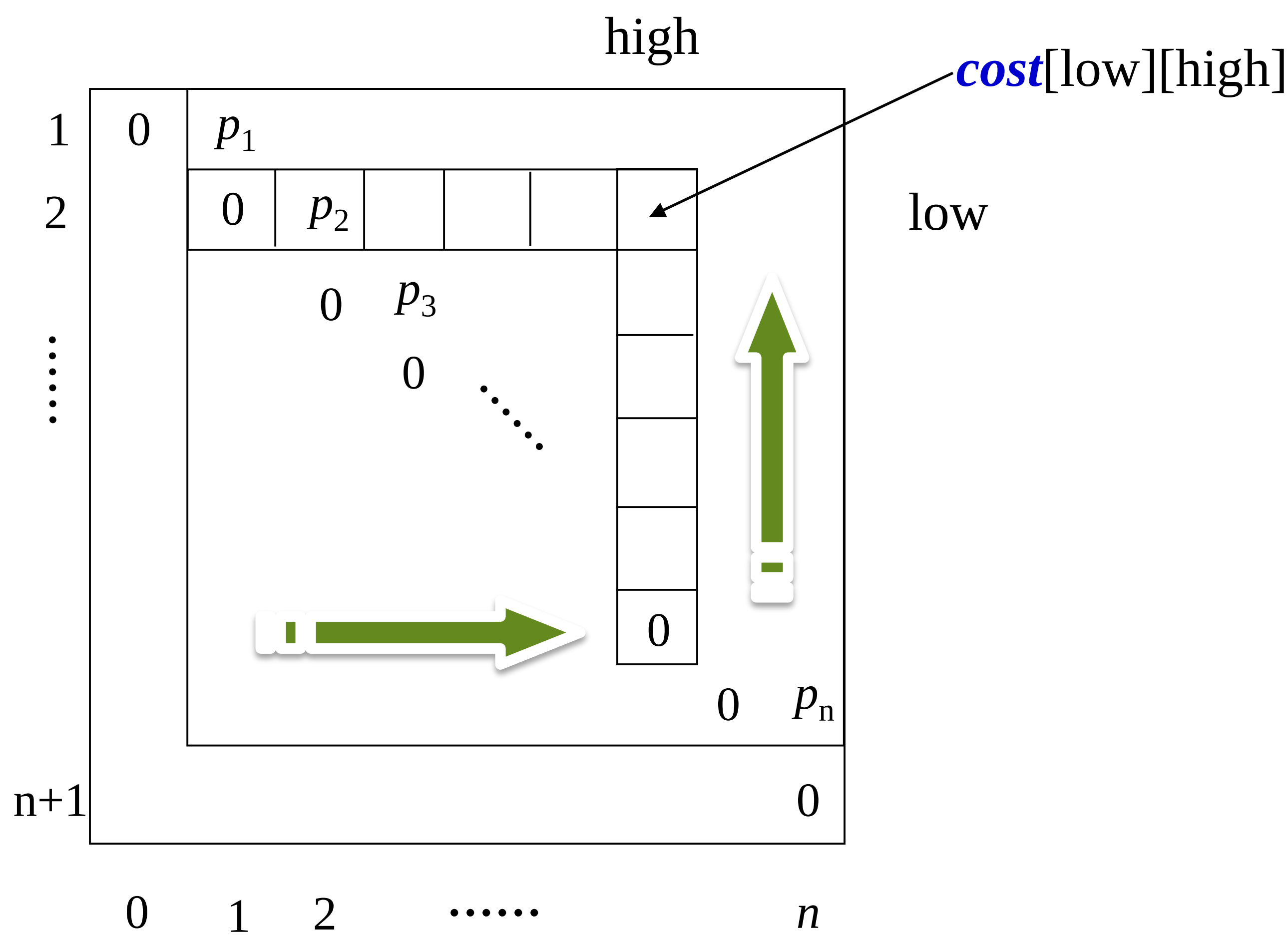
Using DP

DP dictionary



- **Array** ***cost***
  - *Cost*[low][high] gives the minimum weighted search cost of subproblem (low,high).
  - The *cost*[low][high] depends upon subproblems with higher first index (row number) and lower second index (column number)
- **Array** ***root***
  - *root*[low][high] gives the best choice of root for subproblem (low,high)

Array cost[]



Optimal BST by DP

```
bestChoice(prob, cost, root, low, high)
    if (high<low)
        bestCost=0;
        bestRoot=-1;
    else
        bestCost=∞;
    for (r=low; r≤high; r++)
        rCost=p(low,high)+cost[low][r-1]+cost[r+1][high];
        if (rCost<bestCost)
            bestCost=rCost;
            bestRoot=r;
        cost[low][high]=bestCost;
        root[low][high]=bestRoot;
    return
```

```
optimalBST(prob,n,cost,root)
    for (low=n+1; low≥1; low--)
        for (high=low-1; high≤n; high++)
            bestChoice(prob,cost,root,low,high)
    return cost
```

in $\Theta(n^3)$

Thank you!

Q & A

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