



Introduction to

Algorithm Design and Analysis

[5] HeapSort



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In the last class ...

- The *sorting* problem
 - Assumptions
- InsertionSort
 - Design
 - Analysis: inversion
- QuickSort
 - Design
 - Analysis



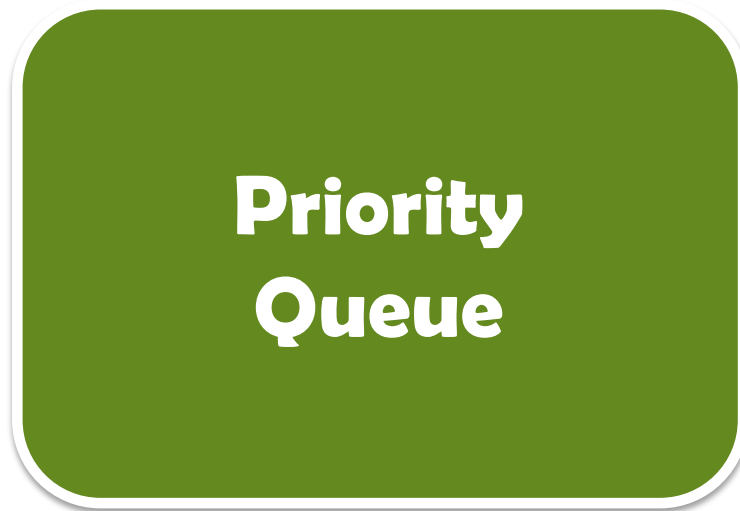
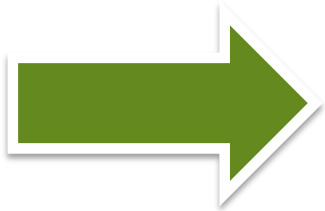
Heapsort

- Heap
- HeapSort
- FixHeap
- ConstructHeap
- Complexity of HeapSort
- Accelerated HeapSort

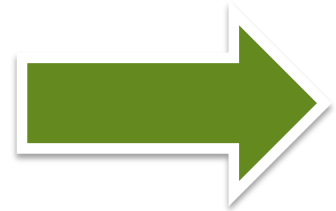


How HeapSort Works

Elements to be sorted



Elements sorted



Implement



Heap

Fibonacci
Heap

Binomial
Heap

Elementary Priority Queue ADT

- “FIFO” in some special sense. The “first” means some kind of “priority”, such as value (largest or smallest)
 - **PriorityQ** **create**()
 - Precondition: none
 - Postconditions: If `pq=create()`, then, `pq` refers to a newly created object and `isEmpty(pq)=true`
 - **boolean** **isEmpty**(PriorityQ `pq`)
 - precondition: none
 - **int** **getMax**(PriorityQ `pq`)
 - precondition: `isEmpty(pq)=false`
 - postconditions: **
 - **void** **insert**(PriorityQ `pq`, **int** `id`, **float** `w`)
 - precondition: none
 - postconditions: `isEmpty(pq)=false`; **
 - **void** **delete**(PriorityQ `pq`)
 - precondition: `isEmpty(pq)=false`
 - postconditions: value of `isEmpty(pq)` updated; **
 - **void** **increaseKey**(PriorityQ `pq`, **int** `id`, **float** `newKey`)

****** `pq` can always be thought as a sequence of pairs (id_i, w_i) , in non-decreasing order of w_i



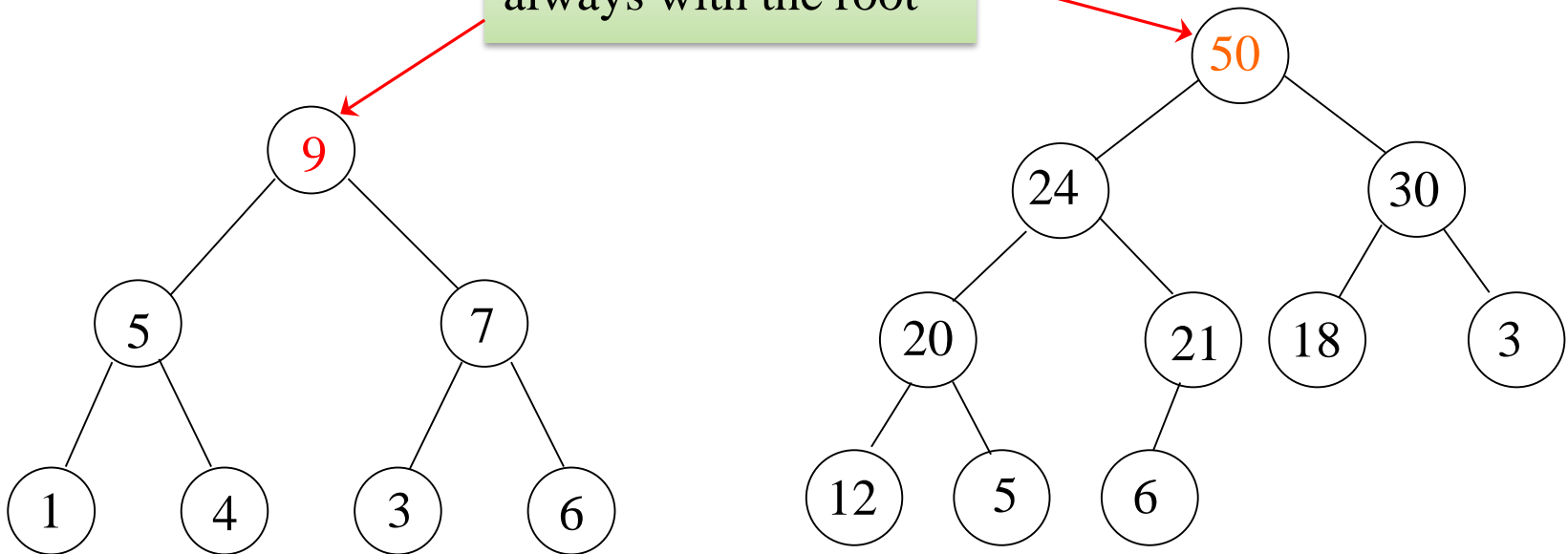
Heap: an Implementation of Priority Queue

- A **binary tree** T is a *heap structure* if:
 - T is complete at least through depth $h-1$
 - All leaves are at depth h or $h-1$
 - All path to a leaf of depth h are to the left of all path to a leaf of depth $h-1$
- Partial order tree *property*
 - A tree T is a (maximizing) partial order tree if and only if the key at any node is greater than or equal to the keys at each of its children (if it has any).

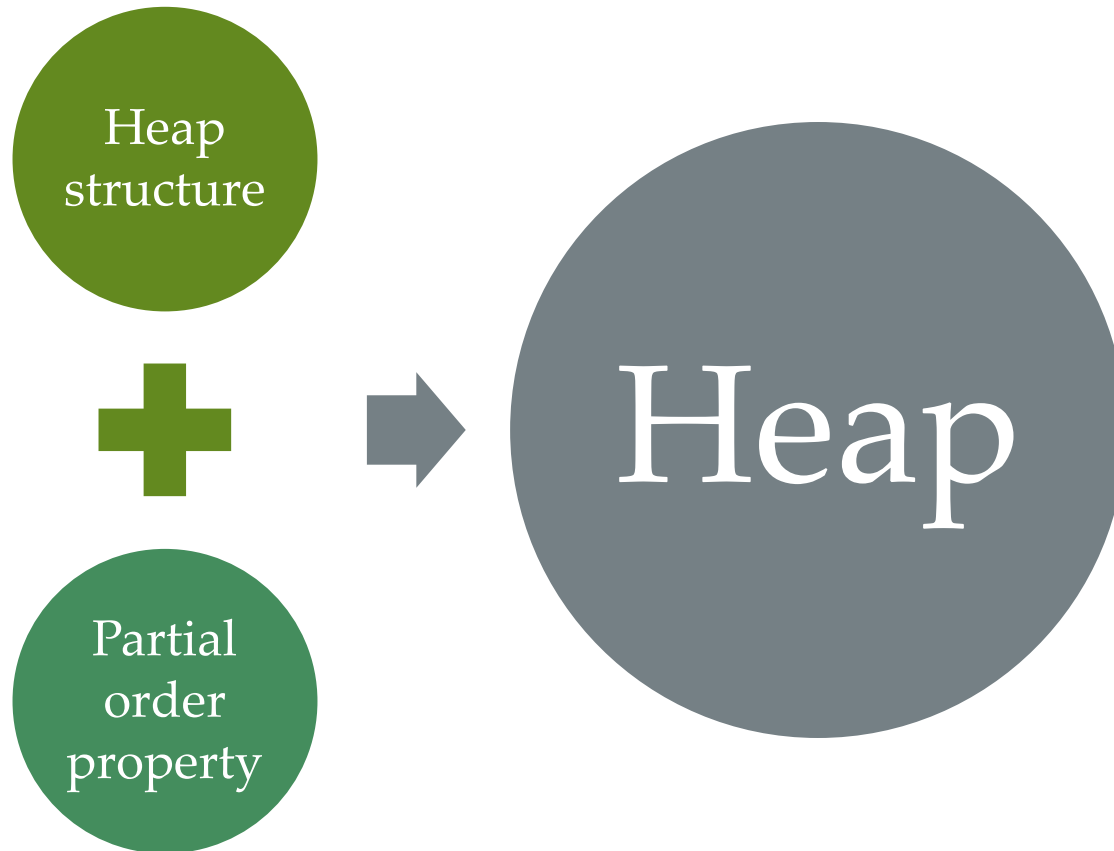


Heap: Examples

The maximal key is always with the root



Heap: an Implementation of Priority Queue



HeapSort: the Strategy

heapSort(E,n)

Construct H from E, the set of n elements to be sorted;

for (i=n;i≥1;i--)

 curMax = getMax(H);

deleteMax(H);

 E[i] = curMax



deleteMax(H)

Copy the rightmost element on the lowest level of H into K;

Delete the rightmost element on the lowest level of H;

fixHeap(H,K)

FixHeap

- **Input:** A nonempty binary tree H with a “vacant” root and its two subtrees in partial order. An element K to be inserted.
- **Output:** H with K inserted and satisfying the partial order tree property.

- **Procedure:**

fixHeap(H, K)

if (H is a leaf) insert K in root(H);

else

Set *largerSubHeap*;

if ($K.\text{key} \geq \text{root}(\text{largerSubHeap}).\text{key}$) insert K in root(H)

else

insert root(*largerSubHeap*) in root(H);

fixHeap(largerSubHeap, K);

return

One comparison:

largerSubHeap is left- or right-Subtree(H), the one with larger key at its root.

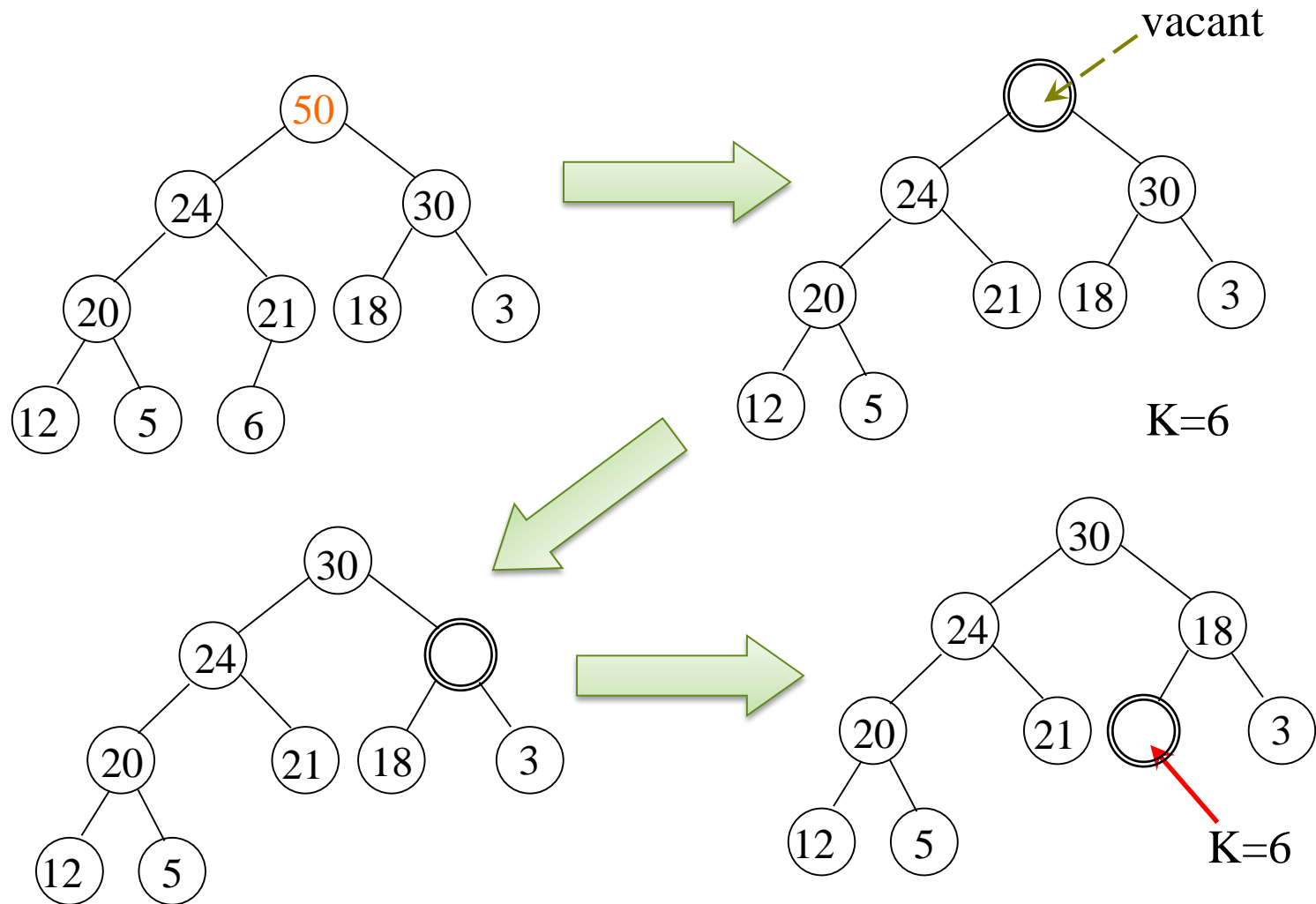
Special case: rightSubtree is empty

Recursion

“Vacant” moving down



FixHeap: an Example



Worst Case Analysis for fixHeap

- *2 comparisons* at most in one activation of the procedure
- The tree *height decreases by one* in the recursive call
- So, *$2h$ comparisons are needed in the worst case*, where h is the height of the tree

- **Procesure:**

fixHeap(H,K)

if (H is a leaf) insert K in root(H)

else

Set *largerSubHeap*;

if (K.key \geq root(*largerSubHeap*).key) insert K in root(H)

else

insert root(*largerSubHeap*) in root(H);

fixHeap(largerSubHeap, K);

return

One comparison:

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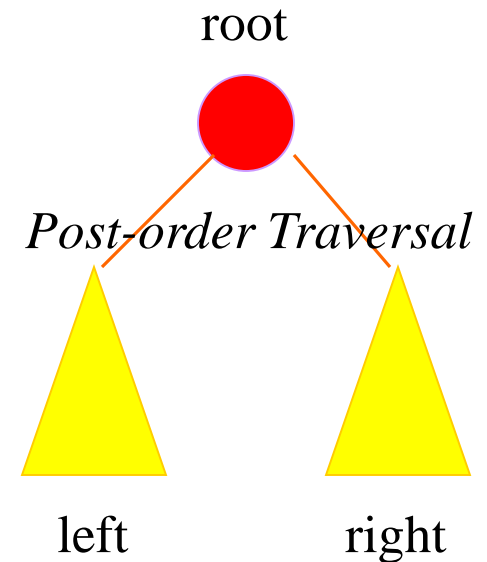
“Vacant” moving down



Heap Construction

- **Note:** *if* left subtree and right subtree **both satisfy** the partial order tree property, then **fixHeap(H,root(H))** gets the thing done.
- We **begin from a Heap Structure H**:

```
void constructHeap(H)
  if (H is not a leaf)
    constructHeap(left subtree of H);
    constructHeap(right subtree of H);
    Element K=root(H);
    fixHeap(H,K)
  return
```



Correctness of *constructHeap*

- **Specification**

- Input: A heap structure H , not necessarily having the partial order tree property.
- Output: H with the same nodes rearranged to satisfy the partial order tree property.

```
void constructHeap(H)
```

```
  if (H is not a leaf)
```

```
    constructHeap(left subtree of H);
```

```
    constructHeap(right subtree of H);
```

```
    Element K=root(H);
```

```
    fixHeap(H,K)
```

```
  return
```

H is a leaf: base case, satisfied trivially.

Preconditions hold respectively?

Postcondition of *constructHeap* satisfied?

Linear Time Heap Construction!

- The recursion equation: $W(n) = W(n-r-1) + W(r) + 2\log(n)$
 - Number of nodes in right subheap (points to $W(r)$)
 - Cost of fixHeap (points to $2\log(n)$)
- A special case: H is a complete binary tree:
 - The size $N = 2^d - 1$,
(then, for arbitrary n , $N/2 < n \leq N \leq 2n$, so $W(n) \leq W(N) \leq W(2n)$)
 - Note: $W(N) = 2W((N-1)/2) + 2\log(N)$
 - The **Master Theorem** applies, with $b=c=2$, and the critical exponent $E=1$, $f(N) = 2\log(N)$
 - Note:
$$\lim_{N \rightarrow \infty} \frac{2\log(N)}{N^{1-\varepsilon}} = \lim_{N \rightarrow \infty} \frac{2\log N}{N^{1-\varepsilon} \log 2} = \lim_{N \rightarrow \infty} \frac{2N^\varepsilon}{((1-\varepsilon)\log 2)N}$$
 - When $0 < \varepsilon < 1$, this limit is equal to zero
 - So, $2\log(N) \in O(N^{E-\varepsilon})$, case 1 satisfied, we have $W(n) \in \Theta(n)$



Direct Analysis of Heap Construction

- **Heap construction**

- From *recursion* to *iteration*
- Sum of rowsums

$$\text{Cost} = \sum_{h=0}^{\lfloor \log n \rfloor} n \frac{O(h)}{2^{h+1}} = O(n)$$

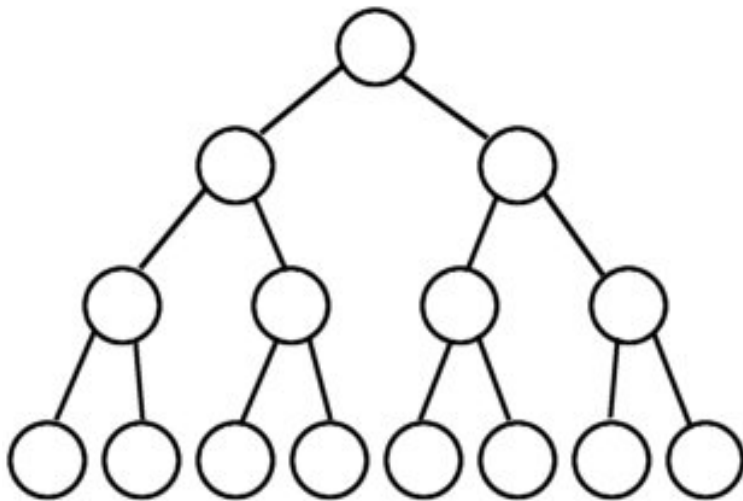
$c = \log n$ fix; $h = \log n$; $\# = 1$

$c = 2$ fix; $h = 2$; $\# = n/8$

$c = 1$ fix; $h = 1$; $\# = n/4$

$c = 0$ fix; $h = 0$; $\# = n/2$

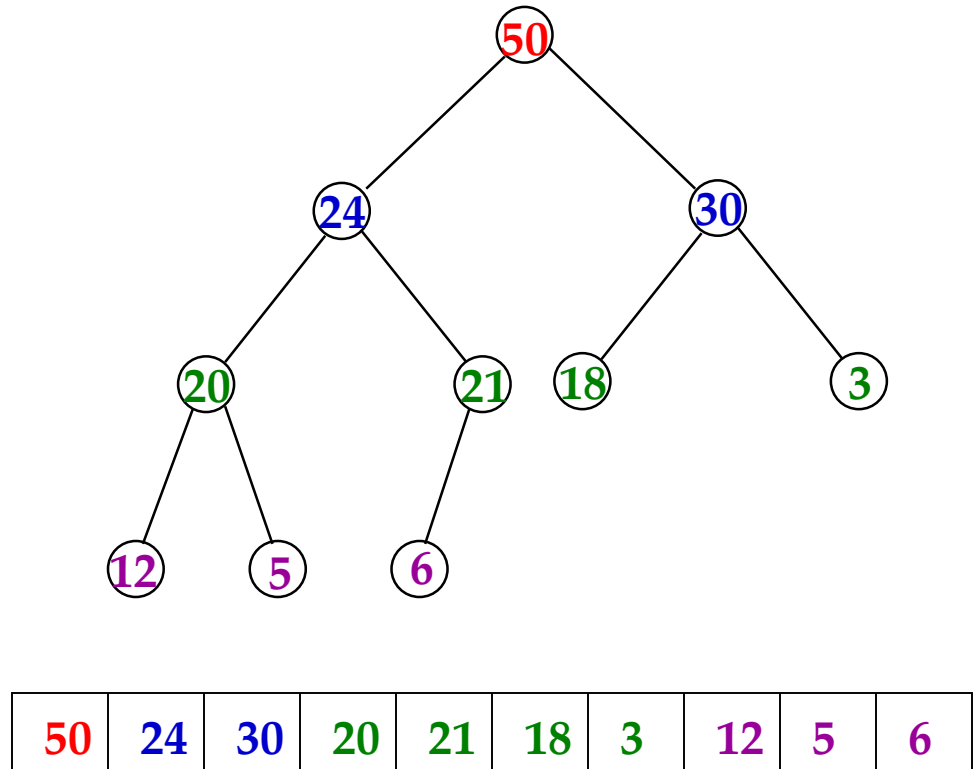
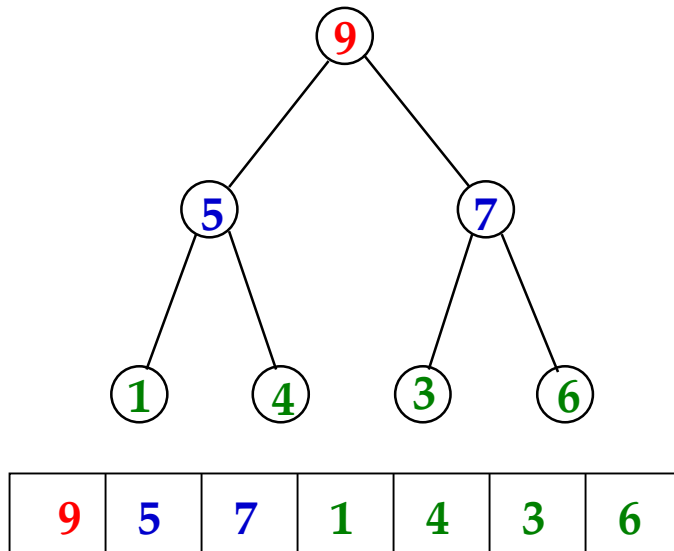
$1 \text{ fix} = 2 \text{ comparisons}$



Understanding the Heap

- **Where is the k^{th} element in the heap?**
 - 1st? 2nd ? 3rd?
 - k^{th} ? at what cost?
- **Sum of heights**
 - At most $n-1$
 - When the sum reaches $n-1$?

Implementing Heap Using Array



Looking for the Children Quickly

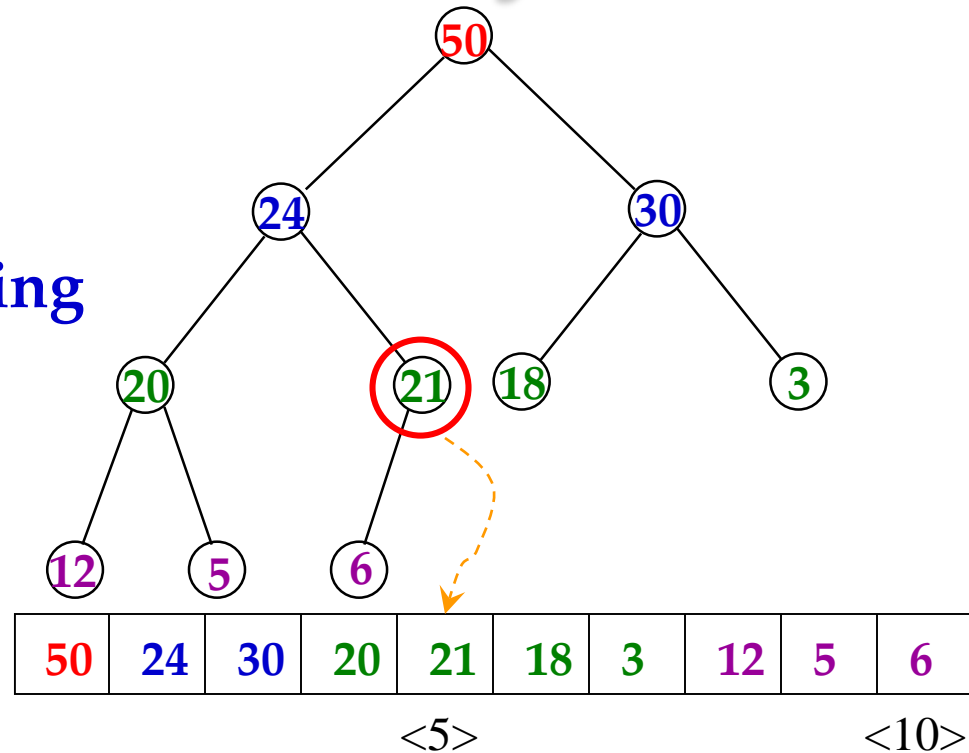
Starting from 1, not *zero*, then the j th level has 2^{j-1} elements. and there are $2^{j-1}-1$ elements in the proceeding $j-1$ levels altogether.

So, If $E[i]$ is the k^{th} element at level j , then $i=(2^{j-1}-1)+k$, and the index of its left child (if existing) is

$$i+(2^{j-1}-k)+2(k-1)+1=2i$$

The number of node on the right of $E[i]$ on level j

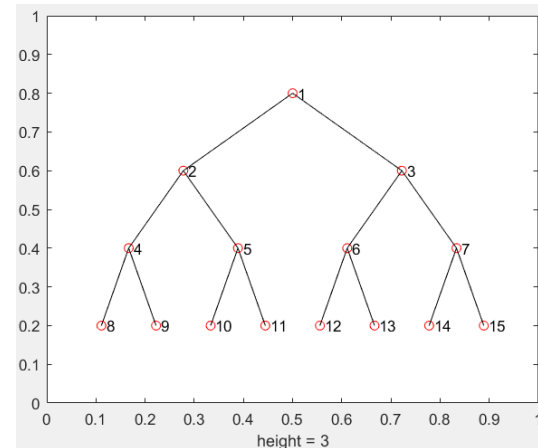
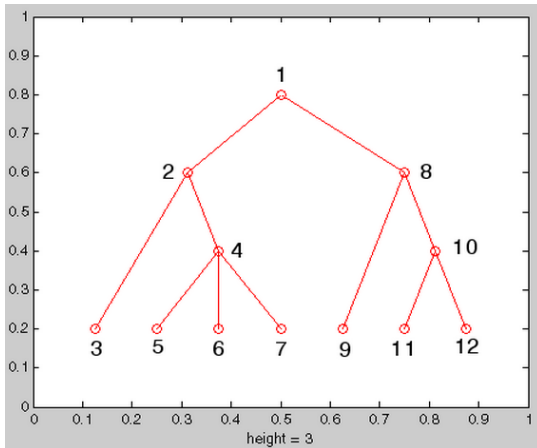
The number of children of the nodes on level j on the left of $E[i]$



For $E[i]$:
Left subheap: $E[2i]$
right subheap: $E[2i+1]$

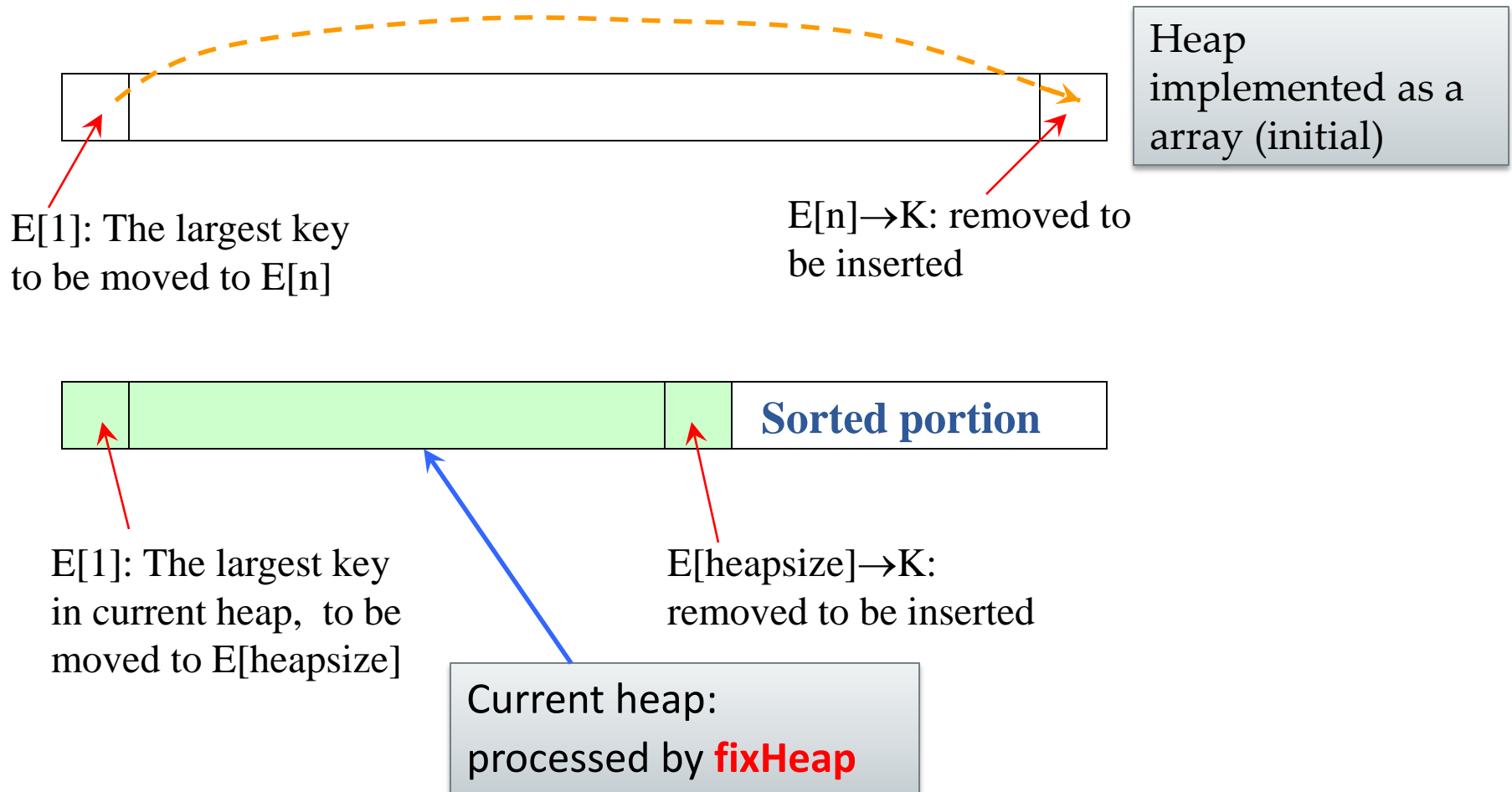
An Example from Matlab

nodes = [0 1 2 2 4 4 4 1 8 8 10 10]



```
for i=2:1:treeSize
    nodeArray(i)=floor(i/2);
end
```

HeapSort: In-Space Implementation



FixHeap: Using Array

- `Void fixHeap(Element[] E, int heapSize, int root, Element K)`
- `int left=2*root; right=2*root+1;`
- `if (left>heapSize) E[root]=K; //Root is a leaf.`
- `else`
- `int largerSubHeap; //Right or Left to filter down.`
- `if (left==heapSize) largerSubHeap=left; // No right SubHeap.`
- `else if (E[left].key>E[right].key) largerSubHeap=left;`
- `else largerSubHeap=right;`
- `if (K.key≥E[largerSubHeap].key) E[root]=K;`
- `else E[root]=E[largerSubHeap]; //vacant filtering down one level.`
- `fixHeap(E, heapSize, largerSubHeap, K);`
- `return`



Heapsort: the Algorithm

- Input: E, an unsorted array with $n(>0)$ elements, indexed from 1
- Sorted E, in nondecreasing order
- Procedure:

```
void heapSort(Element[] E, int n)
    int heapsize
    constructHeap(E,n,root)
    for (heapsize=n; heapsize≥2; heapsize--;)
        Element curMax=E[1];
        Element K=E[heapsize];
        fixHeap(E,heapsize-1,1,K);
        E[heapsize]=curMax;
    return;
```

"array version"

Worst Case Analysis of Heapsort

- We have: $W(n) = W_{cons}(n) + \sum_{k=1}^{n-1} W_{fix}(k)$
- It has been shown that: $W_{cons}(n) \in \Theta(n)$ and $W_{fix}(k) \leq 2 \log k$
- Recall that:

$$2 \sum_{k=1}^{n-1} \lceil \log k \rceil \leq 2 \int_1^n \log e \ln x dx = 2 \log e (n \ln n - n) = 2(n \log n - 1.443n)$$

- So, $W(n) \leq 2n \lg n + \Theta(n)$, that is $W(n) \in \Theta(n \log n)$

Coefficient doubles that of mergeSort approximately

HeapSort: the Right Choice

- For heapSort, $W(n) \in \Theta(n \log n)$
- Of course, $A(n) \in \Theta(n \log n)$
- More good news: HeapSort is an in-space algorithm (using iteration instead of recursion)
- It will be more competitive *if only* the coefficient of the leading term can be decreased to 1



Number of Comparisons in fixHeap

Procedure:

fixHeap(H,K)

if (H is a leaf) insert K in root(H);

else

Set *largerSubHeap*;

if (K.key \geq root(*largerSubHeap*).key) insert K in root(H)

else

insert root(*largerSubHeap*) in root(H);

fixHeap(largerSubHeap, K);

return

2 comparisons are
done in filtering
down for one level.



A One-Comparison-per-Level Fixing

- **Bubble-Up Heap Algorithm:**

```
void bubbleUpHeap(Element []E, int root, Element K,  
    int vacant)
```

```
    if (vacant==root) E[vacant]=K;
```

```
    else
```

```
        int parent=vacant/2;
```

```
        if (K.key≤E[parent].key)
```


```
            E[vacant]=K
```

```
        else
```

```
            E[vacant]=E[parent];
```

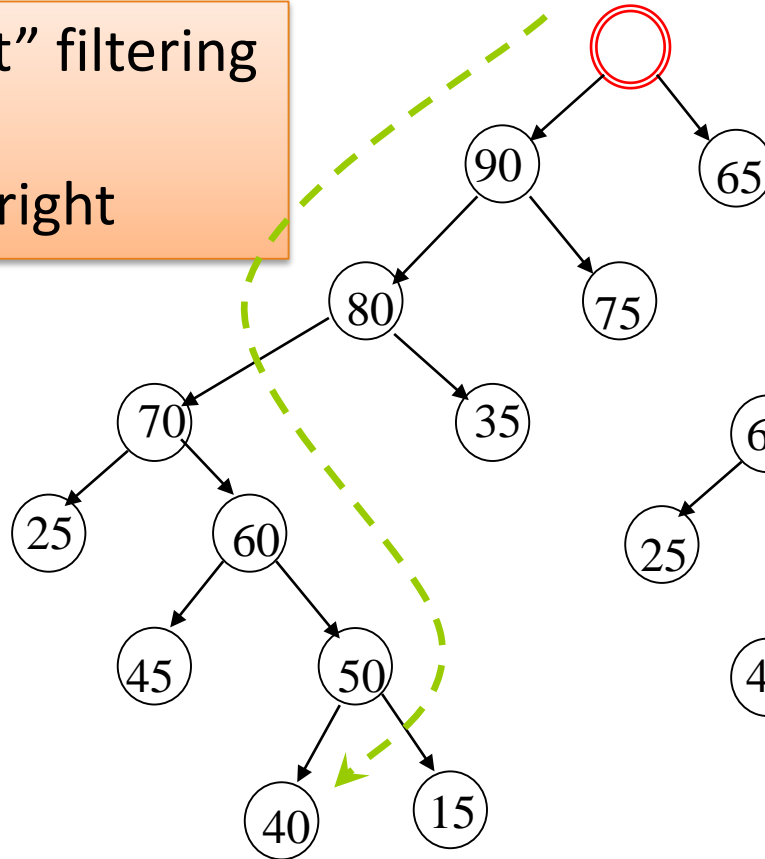
```
            bubbleUpHeap(E,root,K,parent);
```

Bubbling up from
vacant through to the
root, recursively

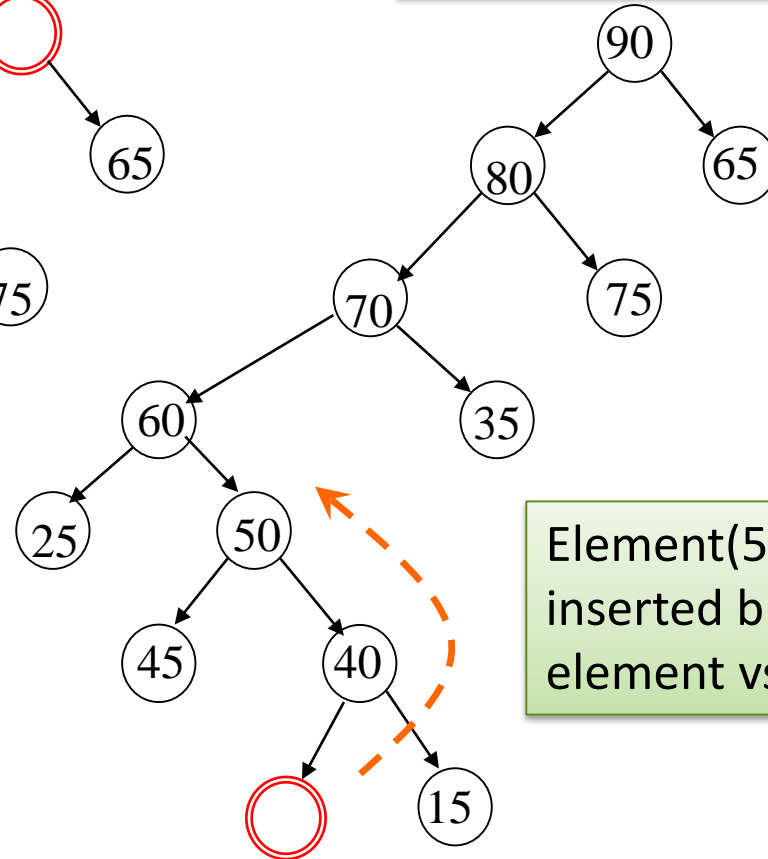


Risky FixHeap

“Vacant” filtering
down:
left vs. right



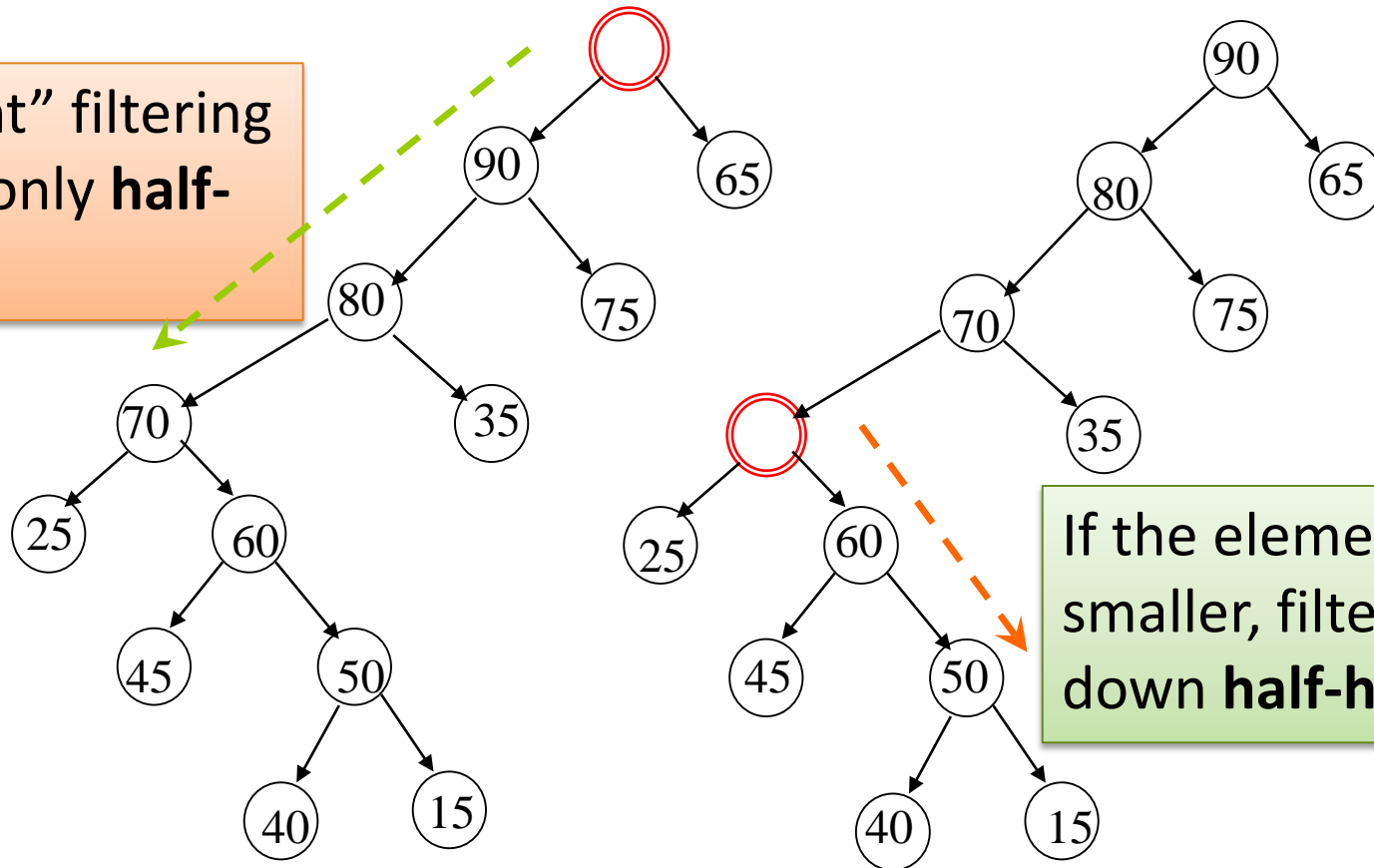
In fact, the “risk” is no
more than “no
improvement”



Element(55) to be
inserted bubbling up:
element vs. parent

Improvement by Divide-and-Conquer

“Vacant” filtering down only **half-way**




If the element is smaller, filtering down **half-half-way**

*The bubbling up will not
beyond last vacStop*

Depth Bounded Filtering Down

```
int promote(Element [ ] E, int hStop, int vacant, int h)
    int vacStop;
    if (h ≤ hStop) vacStop=vacant;
    else if (E[2*vacant].key ≤ E[2*vacant+1].key)
        E[vacant]=E[2*vacant+1];
        vacStop=promote(E, hStop, 2*vacant+1, h-1);
    else
        E[vacant]=E[2*vacant];
        vacStop=promote(E, hStop, 2*vacant, h-1);
    return vacStop
```

Depth Bound



FixHeap Using Divide-and-Conquer

```
void fixHeapFast(Element [ ] E, Element K, int vacant, int h)
    //  $h = \lceil \lg(n+1)/2 \rceil$  in uppermost call
    if (h ≤ 1) Process heap of height 0 or 1;
    else
        int hStop = h/2;
        int vacStop = promote(E, hStop, vacant, h);
        int vacParent = vacStop/2;
        if (E[vacParent].key ≤ K.key)
            E[vacStop] = E[vacParent];
            bubbleUpHeap(E, vacant, K, vacParent);
        else
            fixHeapFast(E, K, vacStop, hStop)
```



Number of Comparisons in Accelerated FixHeap

- Moving the vacant one level up or down need one comparison exactly in promote or bubbleUpHeap.
- In a cycle, t calls of promote and 1 call of bubbleUpHeap are executed at most. So, the number of comparisons in promote and bubbleUpHeap calls are:

$$\sum_{k=1}^t \left\lceil \frac{h}{2^k} \right\rceil + \left\lceil \frac{h}{2^t} \right\rceil = h = \log(n + 1)$$

- At most, $\lg(h)$ checks for reverse direction are executed. So, the number of comparisons in a cycle is at most $h + \log(h)$
- So, for accelerated heapSort: **$W(n) = n \log n + \Theta(n \log \log n)$**



Recursion Equation of Accelerated heapSort

- The recurrence equation about h , which is about $\log(n+1)$

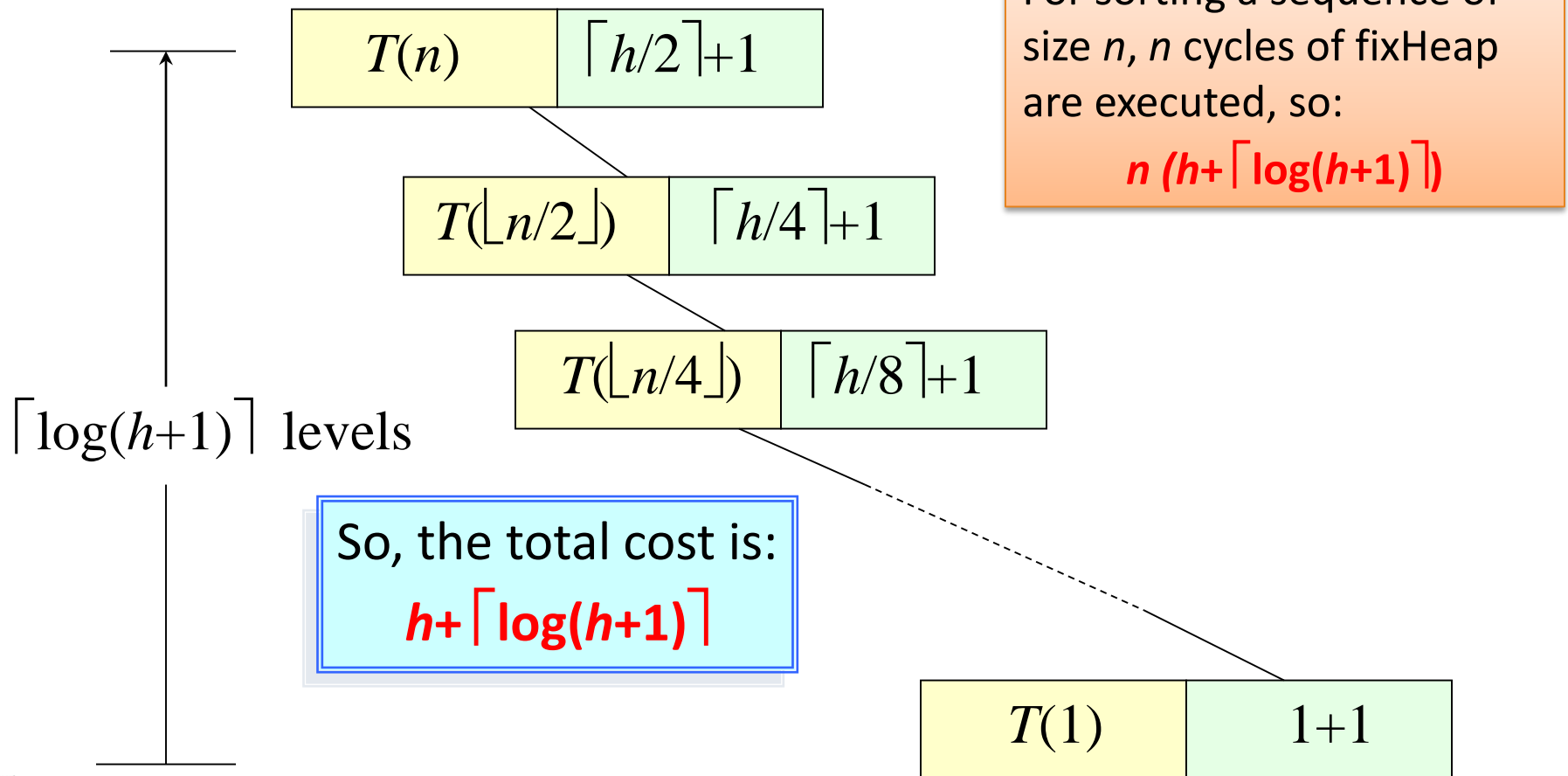
$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + \max\left(\left\lceil \frac{h}{2} \right\rceil, 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor\right)\right) \end{cases}$$

- Assuming $T(h) \geq h$, then:

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor\right) \end{cases}$$



Solving the Recurrence Equation by Recursive Tree



Inductive Proof

$$\begin{cases} T(1) = 2 \\ T(h) = \left\lceil \frac{h}{2} \right\rceil + 1 + T\left(\left\lfloor \frac{h}{2} \right\rfloor\right) \end{cases}$$

- The recurrence equation for fixHeapFast:
- Proving the following solution by induction:

$$T(h) = h + \lceil \log(h+1) \rceil$$

- According to the recurrence equation:

$$T(h+1) = \left\lceil (h+1)/2 \right\rceil + 1 + T(\lfloor (h+1)/2 \rfloor)$$

- Applying the inductive assumption to the last term:

$$T(h+1) = \left\lceil (h+1)/2 \right\rceil + 1 + \lfloor (h+1)/2 \rfloor + \lceil \log(\lfloor (h+1)/2 \rfloor + 1) \rceil$$

(It can be proved that for any positive integer:

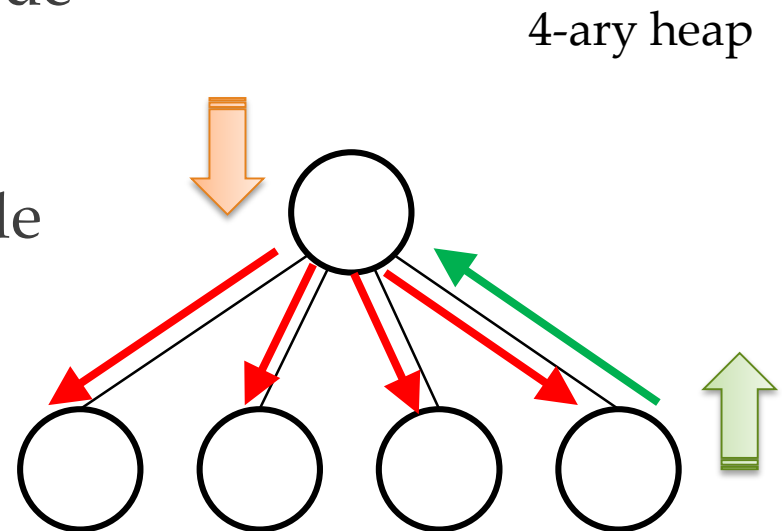
$$\lceil \log(\lfloor h/2 \rfloor + 1) \rceil + 1 = \lceil \log(h+1) \rceil)$$

The sum is $h+1$

$W(n) = n \log n + \Theta(n \log \log n)$ for Accelerated HeapSort

Generalization of a Heap

- **d-ary heap**
 - Structure / partial order
- **How to choose “d”?**
 - Top-down: fix the parent node
 - Cost: d comparisons in the worst case
 - Bottom-up: fix the child node
 - Cost: always 1



Not only for Sorting

- **Eg1: how to find the k^{th} max element?**
 - The cost should be $f(k)$
- **Eg2: how to find the first k elements**
 - In sorted order?
- **Eg3: how to merge k sorted lists?**
- **Eg4: how to find the median dynamically?**
- ...

Thank you!

Q & A

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