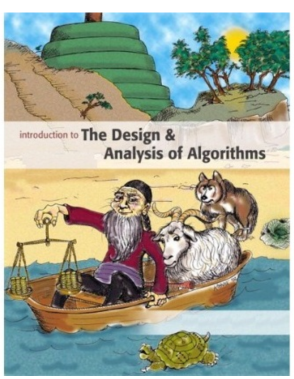
Introduction to

Algorithm Design and Analysis

[12] Directed Acyclic Graph



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In the last class…

- Depth-first and breadth-first search
- Finding connected components

- General DFS/BFS skeleton
- Depth-first search trace

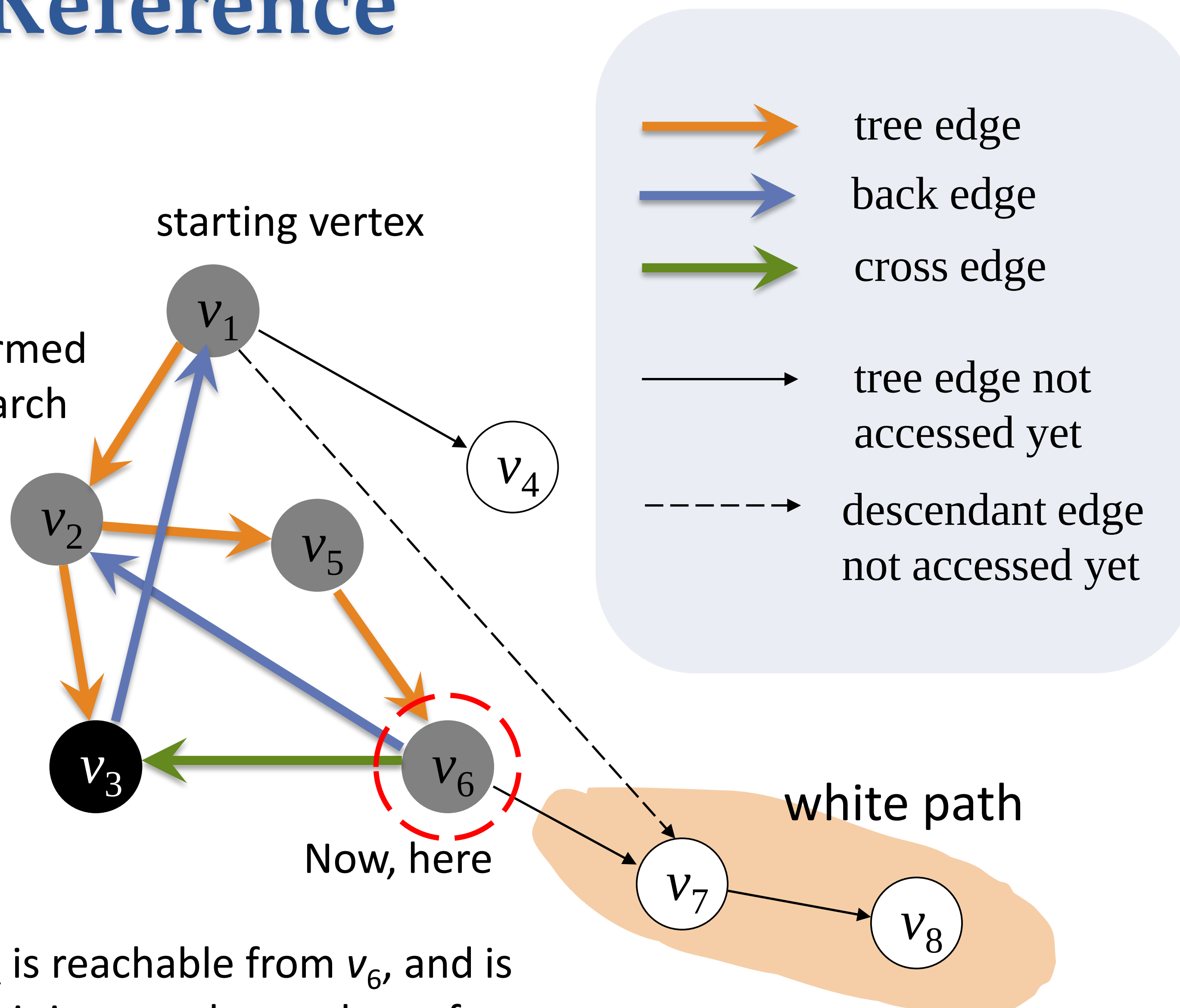
Applications of Graph Decomposition

- **Directed Acyclic Graph**
  - Topological order
  - Critical path analysis

- **Strongly Connected Component (SCC)**
  - Strong connected component and condensation
  - The algorithm
  - Leader of strong connected component

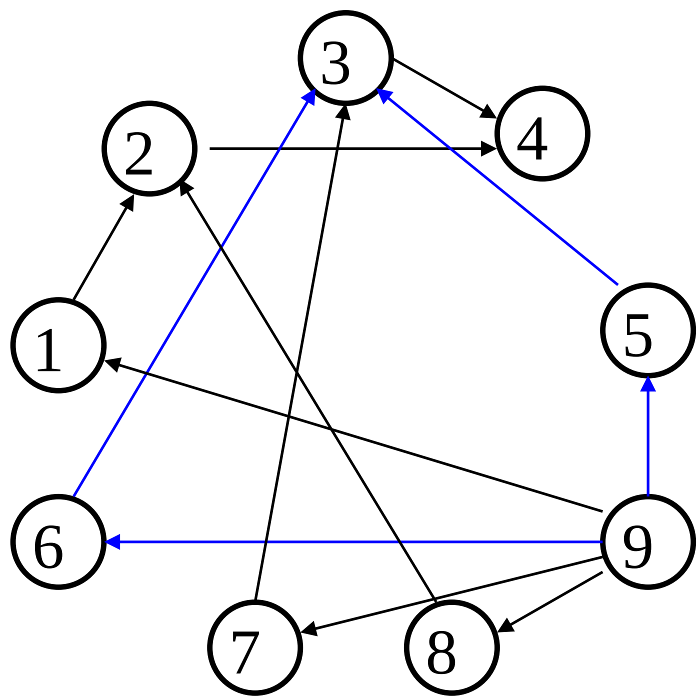
For Your Reference

A DFS tree partially formed at the moment the search checking v_3 from v_6

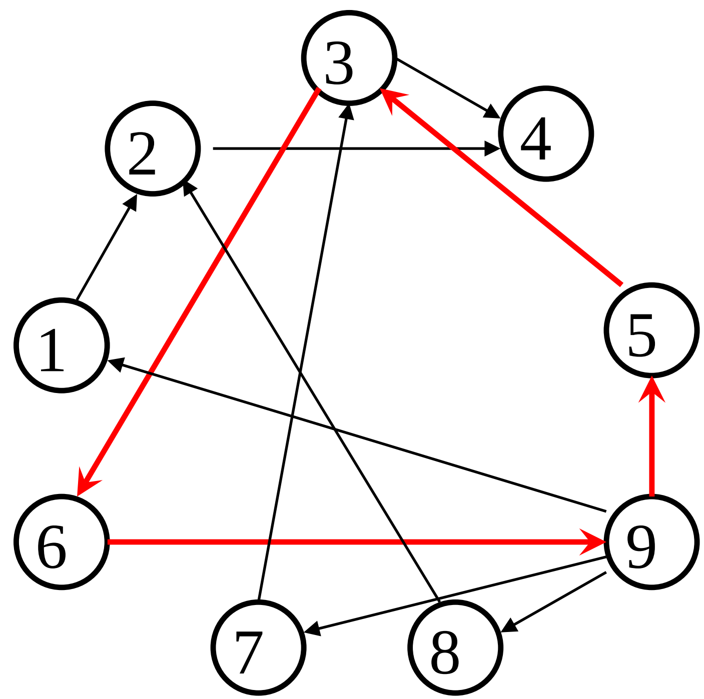


* Note: v_4 is reachable from v_6, and is white, but it is not a descendant of v_6

Directed Acyclic Graph (DAG)



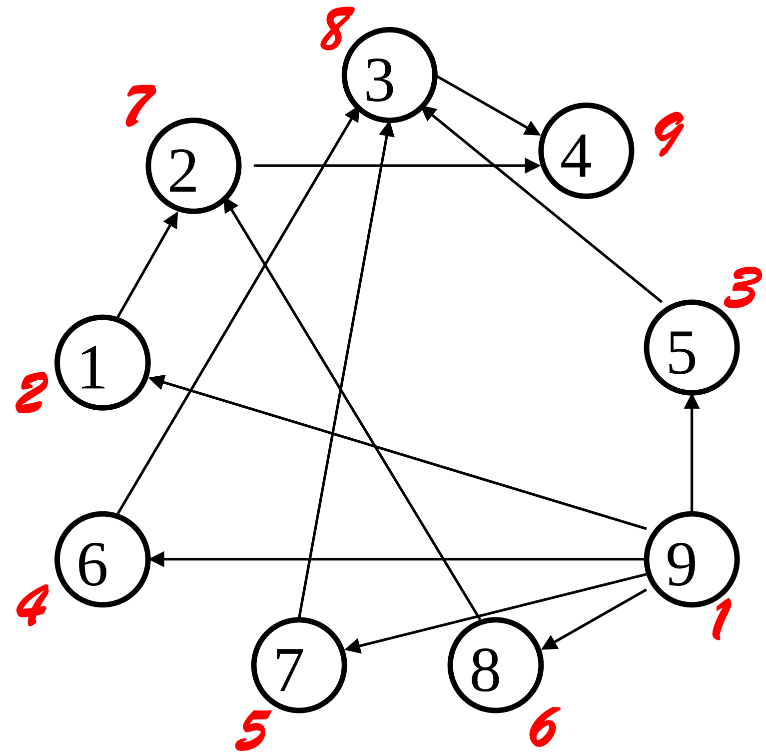
A Directed Acyclic Graph



Not a DAG

Topological Order for G=(V,E)

- **Topological number**
  - An assignment of distinct integer 1,2,…, n to the vertices of V
  - For every $vw \in$E, the topological number of v is less than that of w.
- **Reverse topological order**
  - Defined similarly ("greater than")



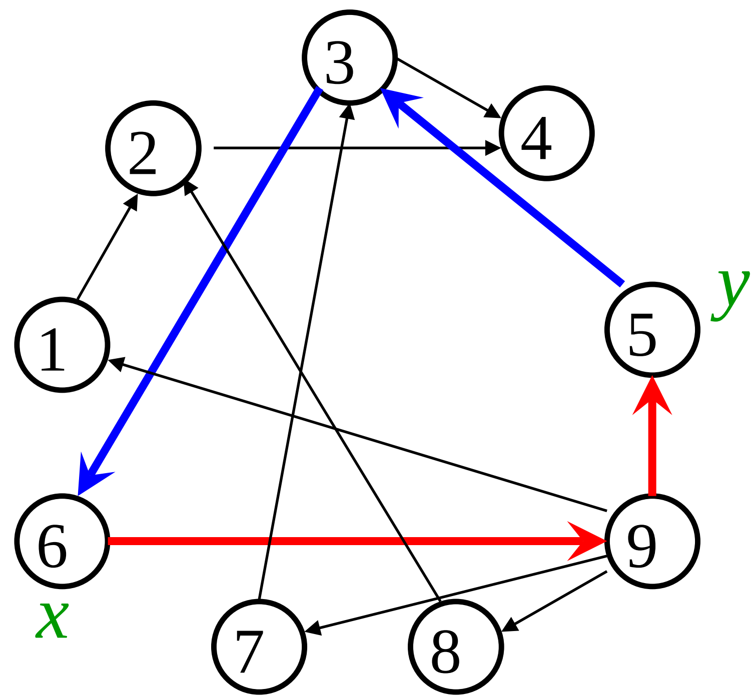
Existence of Topological Order – a Negative Result

- **If a directed graph G has a cycle, then G has no topological order**
- **Proof**
  - [By contradiction]

yx-path

xy-path

For any given topological order, all the vertices on both paths must be in increasing order. Contradiction results for any assignments for *x* and *y*.



Reverse Topological Ordering

- **Specialized parameters**
  - Array *topo*, keeps the topological number assigned to each vertex.
  - Counter *topoNum* to provide the integer to be used for topological number assignments
- **Output**
  - Array *topo* as filled.

Reverse Topological Ordering

- void dfsTopoSweep(IntList[ ] *adjVertices*,int n, int[ ] *topo*)
- int *topoNum*=0
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- dfsTopo(*adjVertices*, color, v, topo, topoNum);
- // *Continue loop*
- return;

For non-reverse topological ordering, initialized as *n*+1

Reverse Topological Ordering

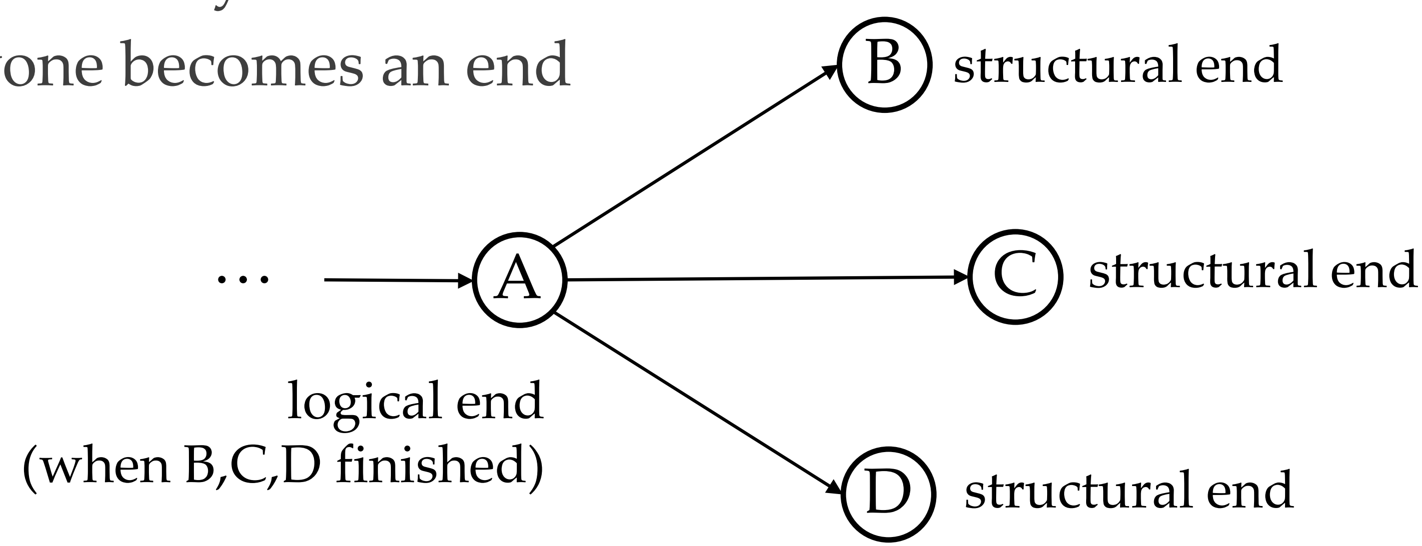
```
void dfsTopo(IntList[] adjVertices, int[] color, int v, int[ ]
    topo, int topoNum)
  int w; IntList remAdj; color[v]=gray;
   remAdj=adjVertices[v];
  while (remAdj≠nil)
    w=first(remAdj);
    if (color[w]==white)
      dfsTopo(adjVertices, color, w, topo, topoNum);
    remAdj=rest(remAdj);
  topoNum++; topo[v]=topoNum
  color[v]=black;
  return;
```

Obviously, in $\Theta(m+n)$

Filling *topo* is a post-order processing, so, the earlier discovered vertex has relatively greater topo number

Reverse Topological Ordering

- **For an "end node"**
  - Easy to decide
- **Acyclic**
  - There is always an end
  - Everyone becomes an end



Correctness of the Algorithm

- **If G is a DAG with *n* vertices, the procedure *dfsTopoSweep* computes a reverse topological order for G in the array *topo*.**
- **Proof**
  - The procedure dfsTopo is called exactly once for a vertex, so, the numbers in *topo* must be distinct in the range 1,2,…*n*.
  - For any edge vw, vw can't be a back edge(otherwise, a cycle is formed). For any other edge types, we have *finishTime*(v)>*finishTime*(w), so, *topo*(w) is assigned earlier than *topo*(v). Note that *topoNum* is incremented monotonically, so, *topo*(v)>*topo*(w).

Existence of Topological Order

- In fact, the proof of correctness of topological ordering has proved that: DAG always has a topological order.
- So, **G has a topological ordering, iff. G is a directed acyclic graph.**

> Partial orders are irreflexive and transitive, which implies acyclic (because any cycle in a transitive relation implies a self-loop). We often visualize partial orders as directed acyclic graphs. Moreover, in such drawings, we usually omit transitively implied edges, to avoid overloading the picture.

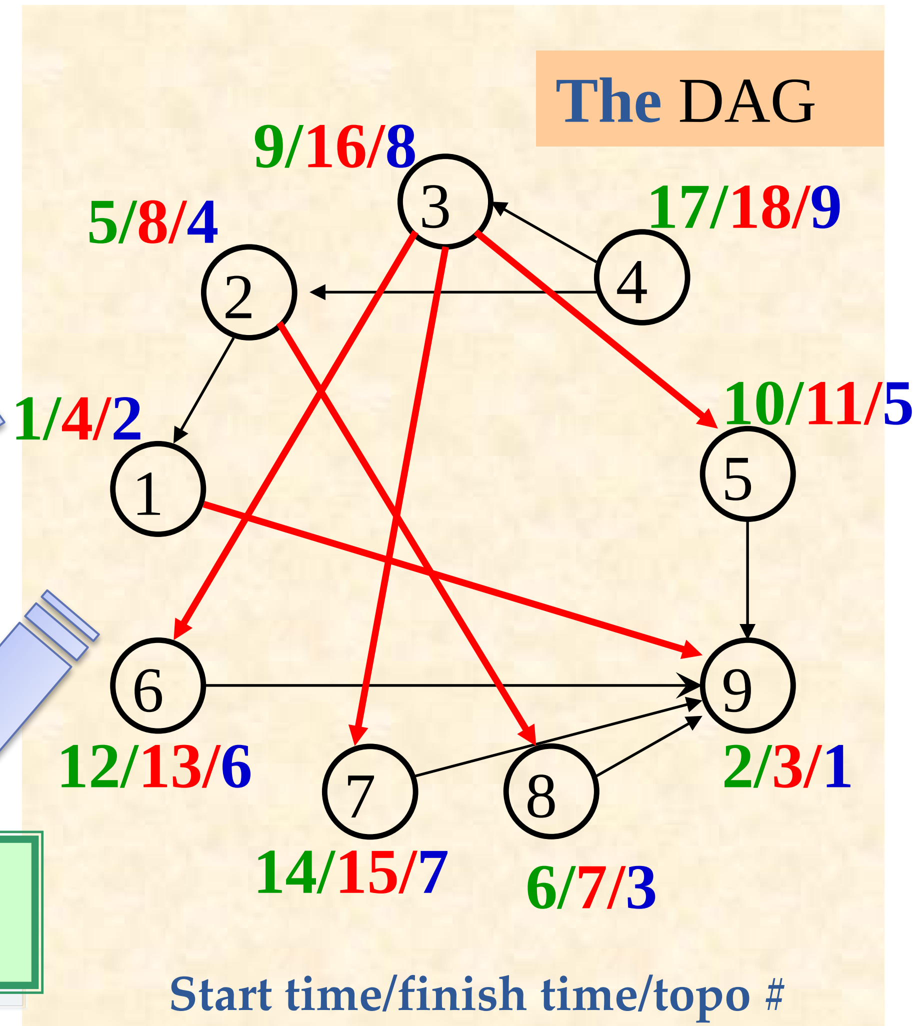
– S. Burckhardt, Principles of Eventual Consistency, § 2.1.3

Task Scheduling

- **Problem:**
  - o Scheduling a project consisting of a set of interdependent tasks to be done by one person.
- **Solution:**
  - o Establishing a dependency graph, the vertices are tasks, and edge vw is included iff. the execution of v depends on the completion of w,
  - o Making task scheduling according to the topological order of the graph(if existing).

Task Scheduling: an Example

Tasks(No.)	Depends on
choose clothes(1)	9
dress(2)	1,8
eat breakfast(3)	5,6,7
leave(4)	2,3
make coffee(5)	9
make toast(6)	9
pour juice(7)	9
shower(8)	9
wake up(9)	-



A reverse topological order
9, 1, 8, 2, 5, 6, 7, 3, 4

Project Optimization Problem

Assuming that **parallel** executions of tasks (v_i) are possible except for prohibited by interdependency.

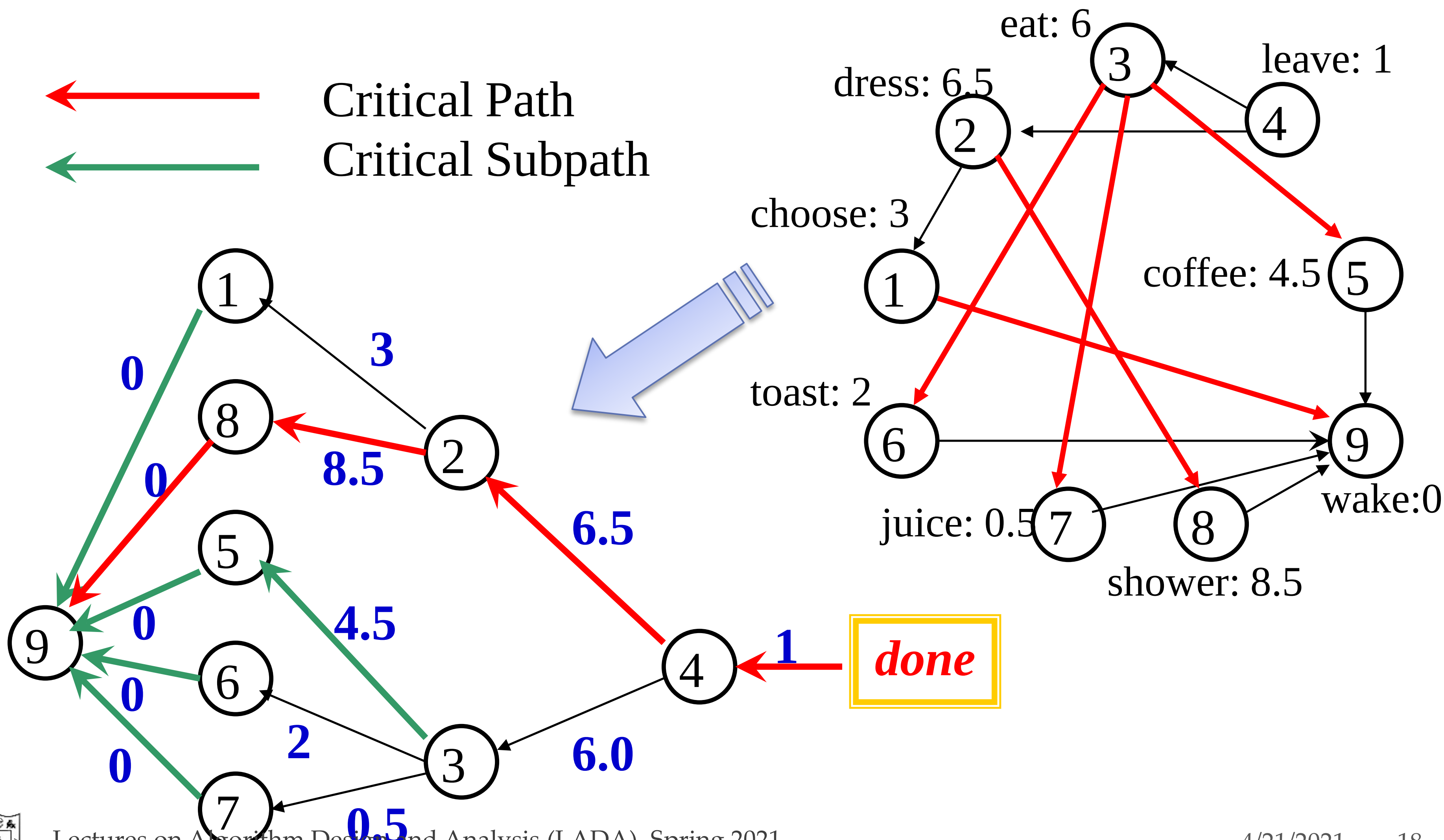
- **Observation**
  - In a critical path, v_{i-1}, is a critical dependency of v_i, i.e. any delay in v_{i-1} will result in delay in v_i.
  - The time for entire project depends on the time for the critical path.
  - Reducing the time of a off-critical-path task is of no help for reducing the total time for the project.
- **The problems**
  - Find the critical path in a **DAG**
  - (Try to reduce the time for the critical path)

This is a precondition.

Critical Path in a Task Graph

- **Earliest start time(est) for a task v**
  - If v has no dependencies, the *est* is 0
  - If v has dependencies, the *est* is the maximum of the earliest finish time of its dependencies.
- **Earliest finish time(eft) for a task v**
  - For any task: *eft* = *est* + *duration*
- **Critical path in a project is a sequence of tasks: v_0, v_1, …, v_k, satisfying:**
  - v_0 has no dependencies;
  - For any v_i(i=1,2,…,k), v_{i-1} is a dependency of v_i, such that *est* of v_i equals *eft* of v_{i-1};
  - *eft* of v_k, is maximum for all tasks in the project.

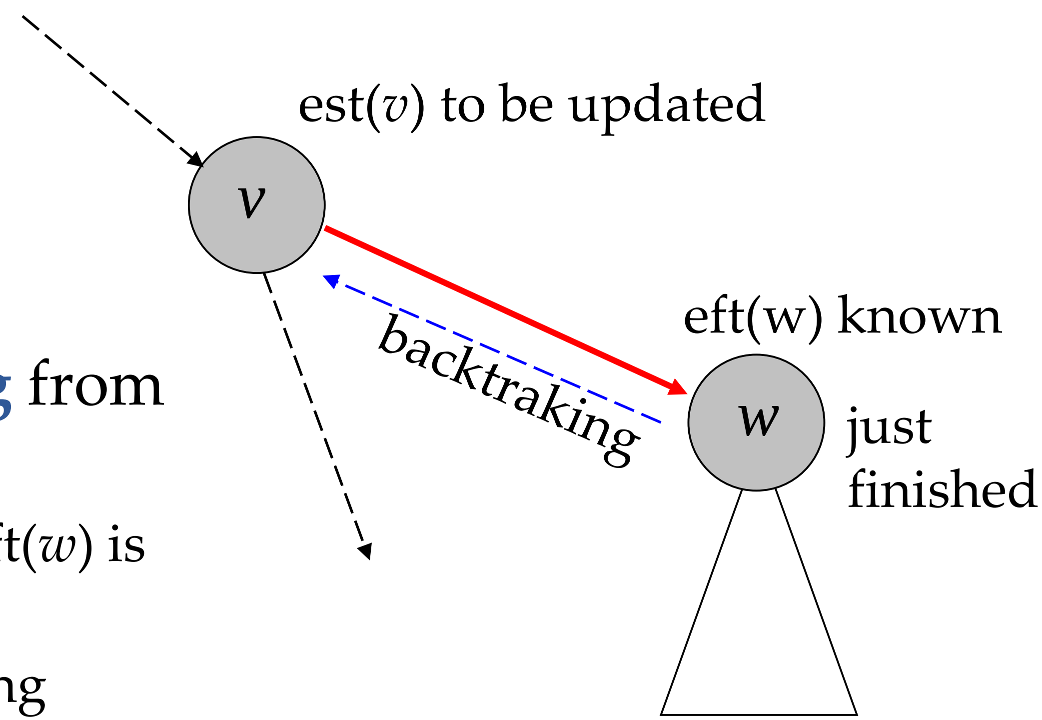
DAG with Weights



Critical Path Finding - DFS

- **Specialized parameters**
  - Array *duration*, keeps the execution time of each vertex.
  - Array *critDep*, keeps the critical dependency of each vertex.
  - Array *eft*, keeps the earliest finished time of each vertex.
- **Output**
  - Array *topo*, *critDep*, *eft* as filled.
- **Critical path is built by tracing the output.**

Critical Path – Case 1



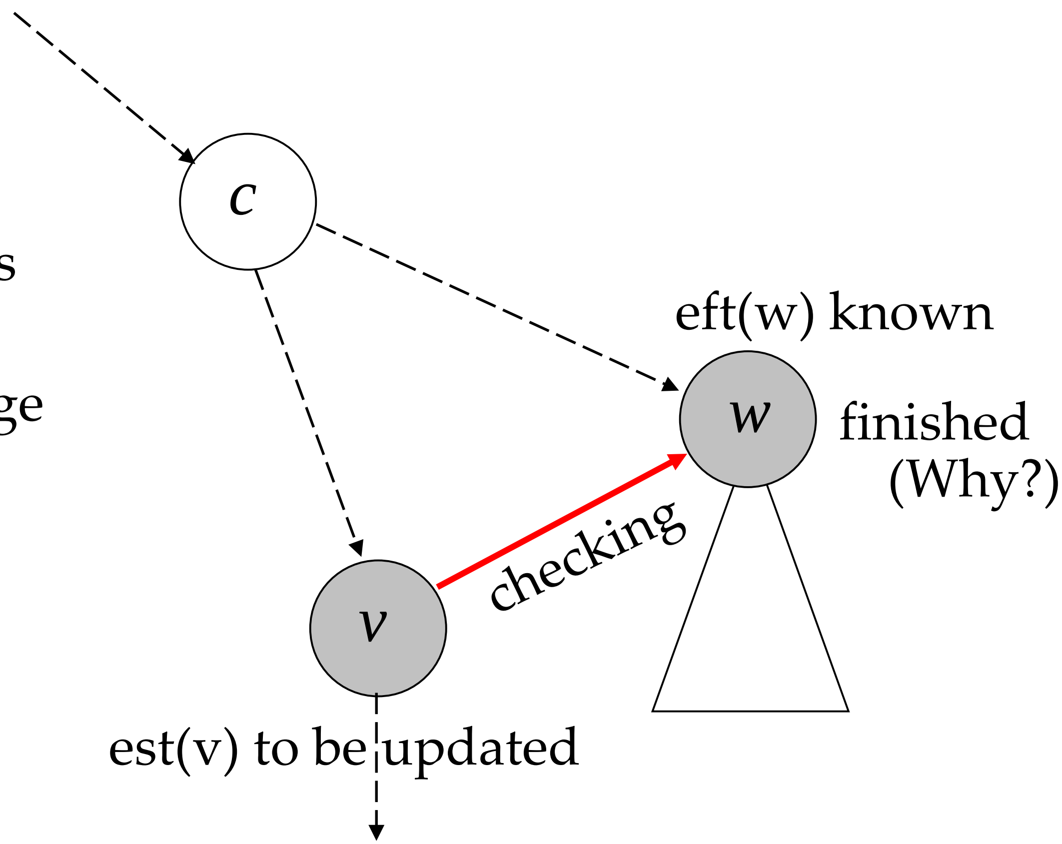
Upon backtracking from w:

- est(v) is updated if eft(w) is larger than est(v)
- and the path including edge vw is recognized as the critical path for tast v
- and the eft(v) is updated accordingly

Critical Path – Case 2

Checking w:

- est(v) is updated if eft(w) is larger than est(v)
- and the path including edge vw is recognized as the critical path for task v
- and the eft(v) is updated accordingly



Critical Path by DFS

- **void dfsCritSweep(IntList[ ] *adjVertices*,int n, int[ ] *duration*, int[ ] *critDep*, int[ ] *eft*)**
- **<Allocate color array and initialize to white>**
- **For each vertex *v* of G, in some order**
- **if (color[v]==white)**
- **dfsCrit(*adjVertices*, color, v, duration, critDep, eft);**
- **// *Continue loop***
- **return;**

Critical Path by DFS

- void dfsCrit(.. *adjVertices*, .. color, .. v, int[ ] *duration*, int[ ] *critDep*, int[ ] *eft*)
- int w; IntList remAdj; int est=0;
- color[v]=gray; critDep[v]=-1; remAdj=*adjVertices*[v];
- while (remAdj≠nil) w=first(remAdj);
- if (color[w]==white)
- dfsCrit(*adjVertices*, color, w, duration, critDep, efs);
- if (eft[w]≥est) est=eft[w]; critDep[v]=w
- else//checking for nontree edge
- if (eft[w]≥est) est=eft[w]; critDep[v]=w
- remAdj=rest(remAdj);
- eft[v]=est+duration[v]; color[v]=black;
- return;

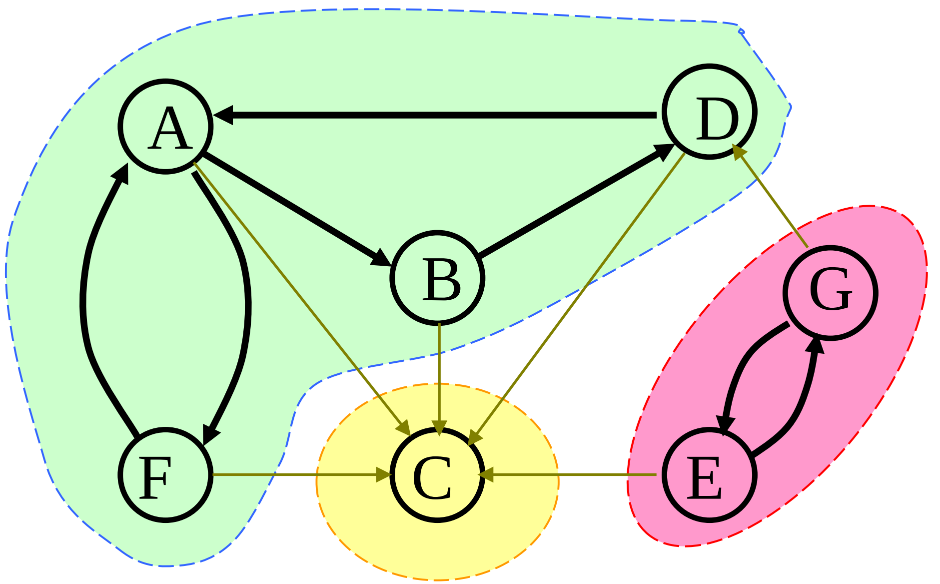
When is the eft[w] initialized?

Only black vertex

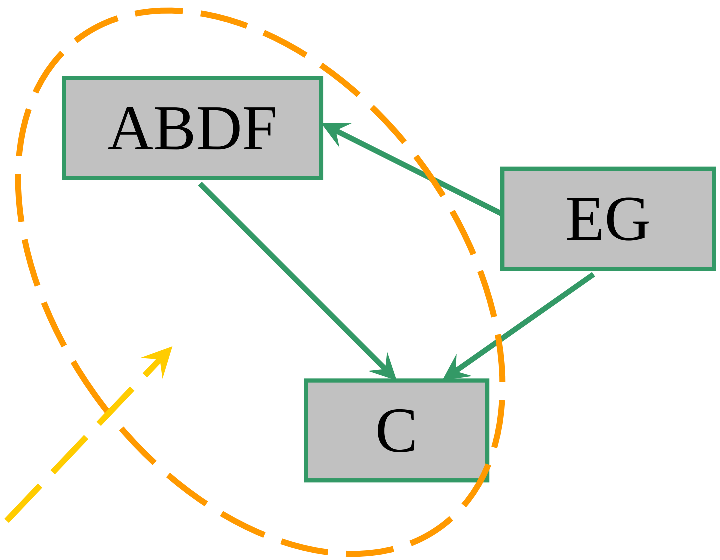
Analysis of Critical Path Algorithm

- **Correctness:**
  - When *eft*[*w*] is accessed in the while-loop, the w must not be gray(otherwise, there is a cycle), so, it must be black, with *eft* initialized.
  - According to DFS, each entry in the *eft* array is assigned a value exactly once. The value satisfies the definition of *eft*.
- **Complexity**
  - Simply same as DFS, that is **Θ(n+m)**.

SCC: Strongly Connected Component



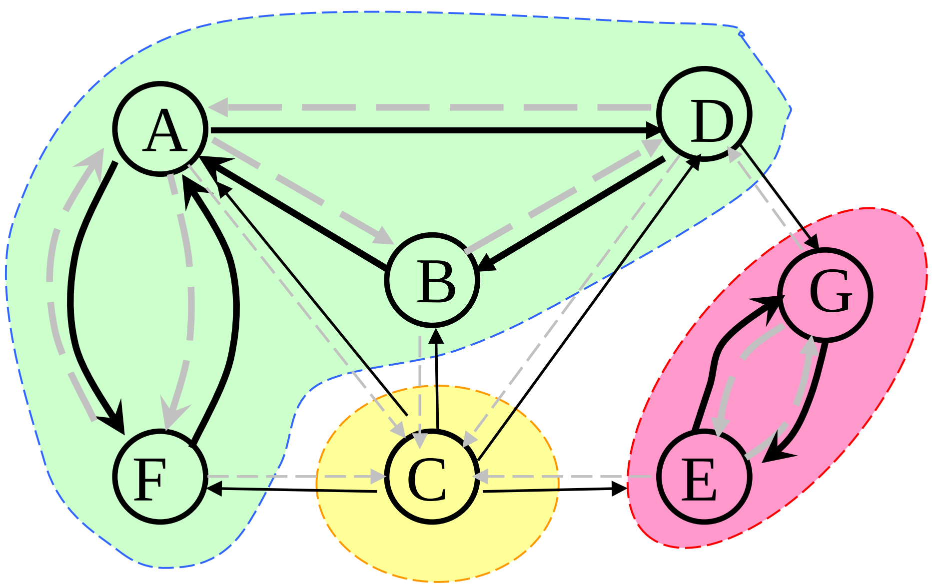
Graph G
3 Strongly Connected Components



It's acyclic, ***Why?***

Note: two SCC in one DFS tree

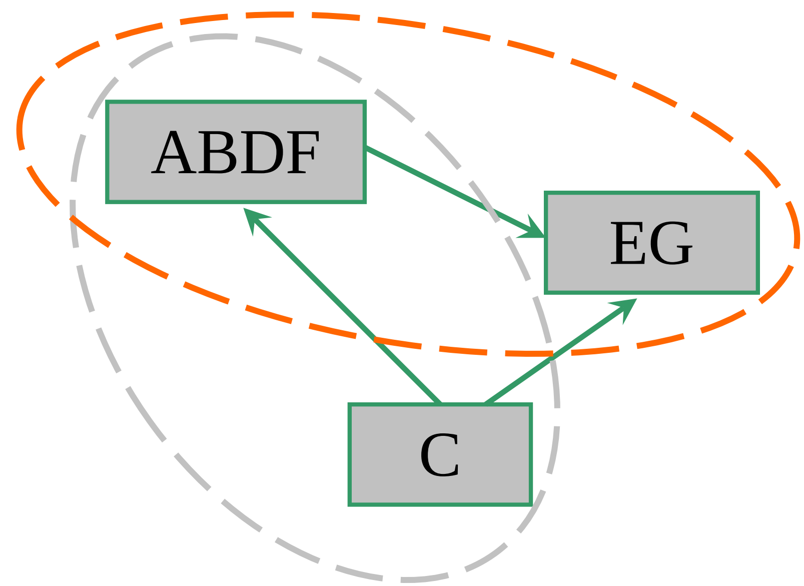
Transpose Graph



Tranpose Graph G^T

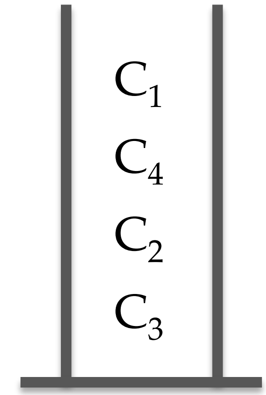
Connected Components unchanged according to vertices

Condensation Graph G↓

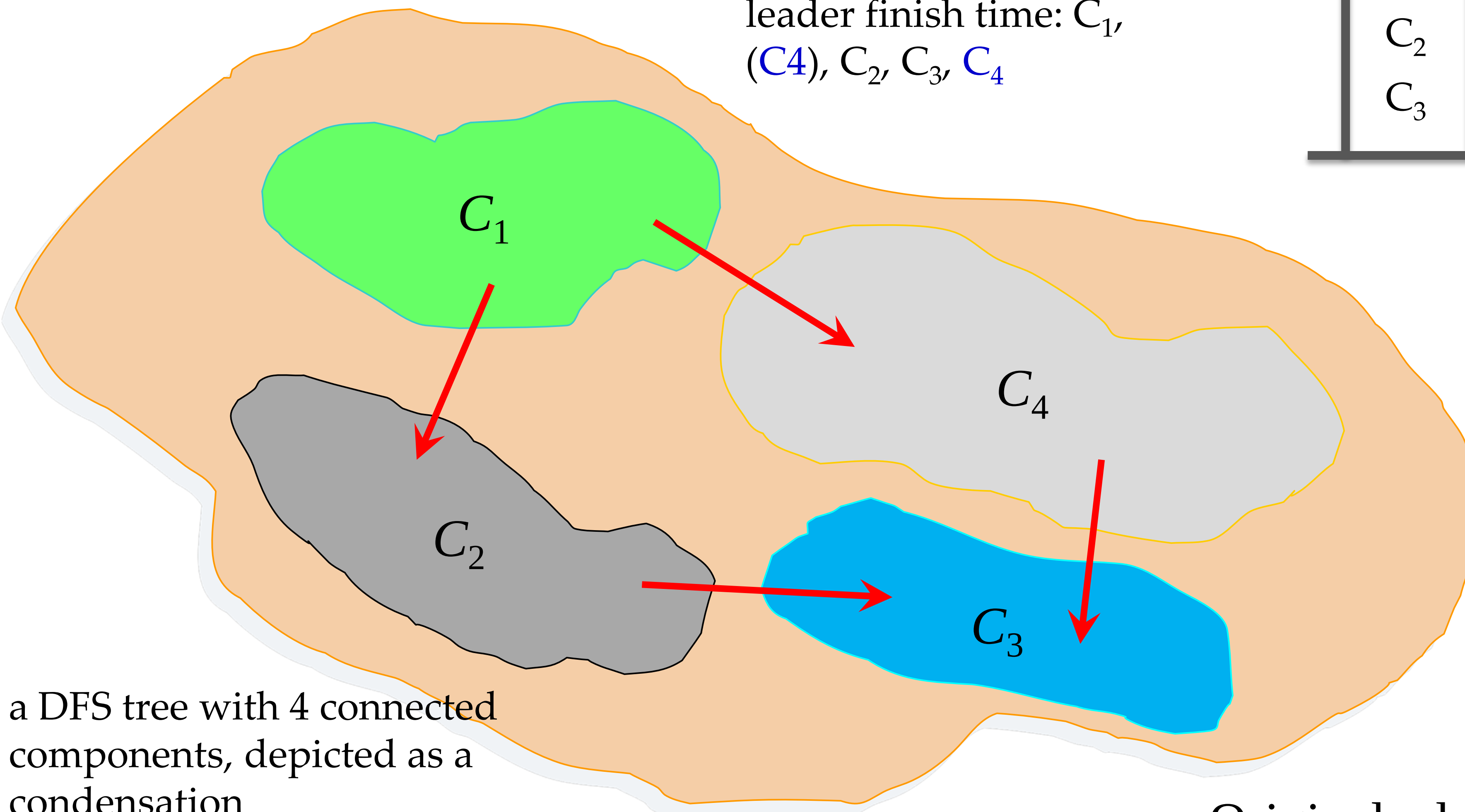


But, DFS tree changed

Basic Idea - G



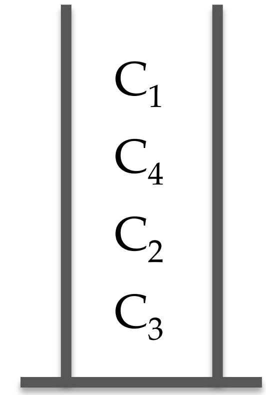
Reverse topo order for leader finish time: C_1, ($C4$), C_2, C_3, C_4



a DFS tree with 4 connected components, depicted as a condensation

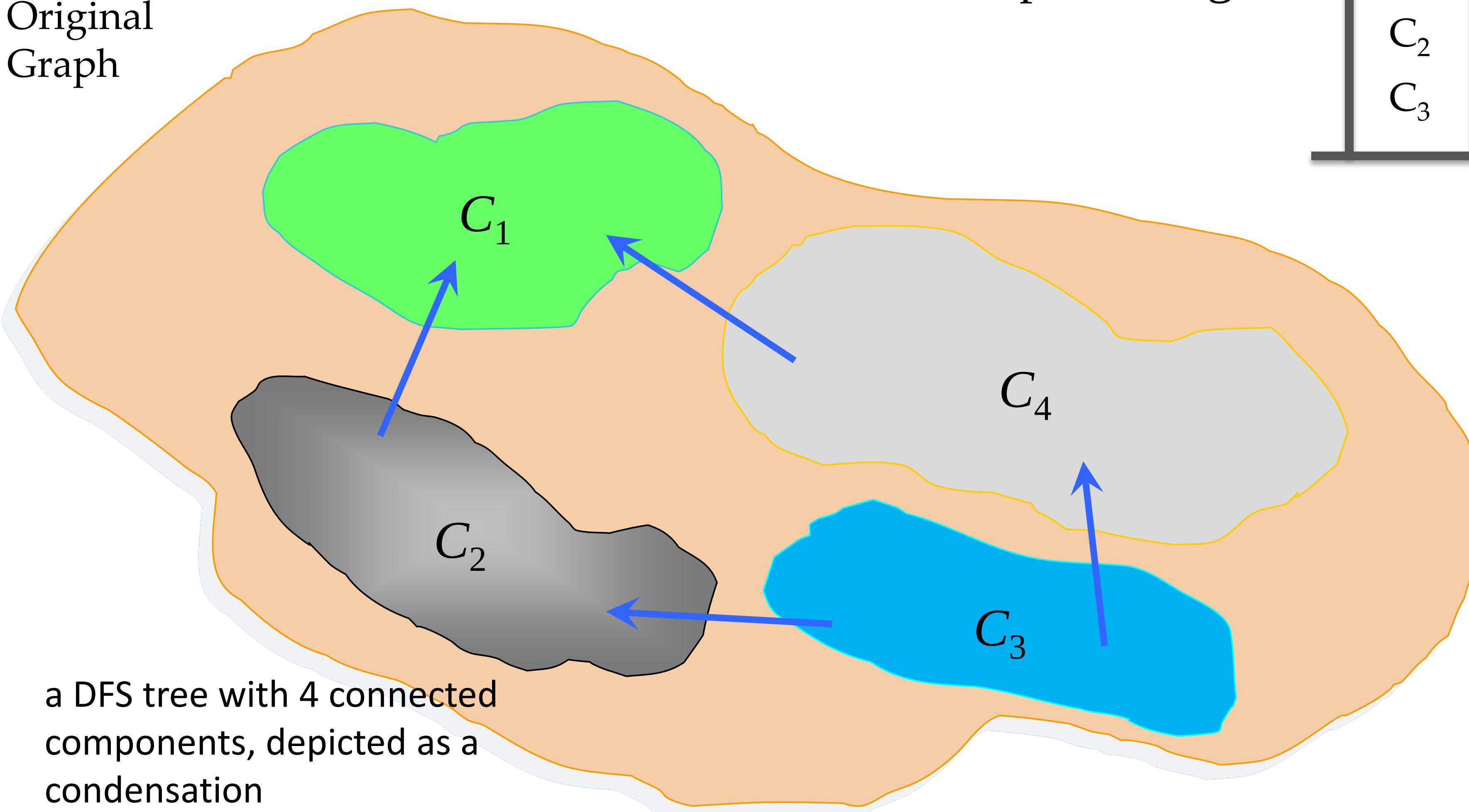
Original edge

Basic Idea - G^T



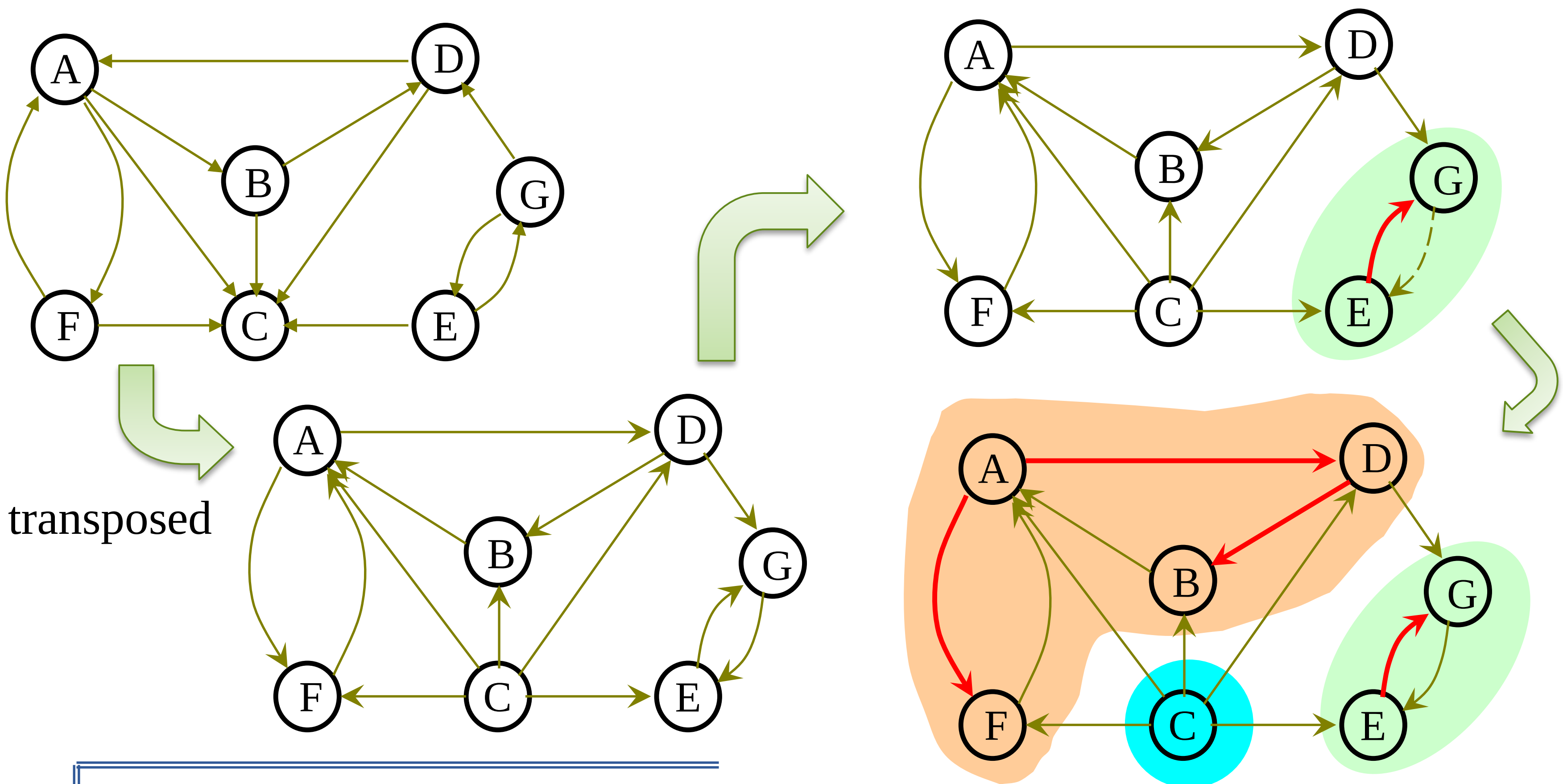
Transposed edge

Original
Graph



a DFS tree with 4 connected
components, depicted as a
condensation

SCC - An Example



C D B F A G E

push/pop

Strong Component Algorithm: Outline

- void strongComponents(IntList[] *adjVertices*, int *n*, int[] *scc*)
- //*Phase* 1
- 1. IntStack *finishStack*=create(*n*);
- 2. Perform a depth-first search on *G*, using the DFS skeleton. At postorder processing for vertex *v*, insert the statement: **push(*finishStack*, *v*)**
- //*Phase* 2
- 3. Compute G^T, the transpose graph, represented as array *adjTrans* of adjacency list.
- 4. **dfsTsweep(*adjTrans*, *n*, *finishStack*, *scc*)**;
- return

Note: *G* and G^T have the same SCC sets

Strong Component Algorithm: Core

- void dfsTsweep(IntList[] *adjTrans*, int *n*, IntStack *finishStack*, int[] *scc*)
-     <Allocate *color* array and initialize to white>
-     while (finishStack is not empty)
-         int v=top(*finishStack*);
-         pop(*finishStack*);
-         if (color[*v*]==white)
-             dfsT(*adjTrans*, *color*, *v*, *v*, *scc*);
-     return;
- void dfsT(IntList[] *adjTrans*, int[] *color*, int *v*, int *leader*, int[] *scc*)
-     Use the standard depth-first search skeleton. At postorder processing for vertex v insert the statement:
-         scc[v]=leader;
-         Pass leader and scc into recursive calls.

Leader of a Strong Component

- **For a DFS, the first vertex discovered in a strong component S_i is called the leader of S_i .**
- **Each DFS tree of a digraph G contains only complete strong components of G, one or more.**
    - Proof: Applying White Path Theorem whenever the leader of S_i (i=1,2,…p) is discovered, starting with all vertices being white.
- **The leader of S_i is the last vertex to finish among all vertices of S_i. (since all of them in the same DFS tree)**

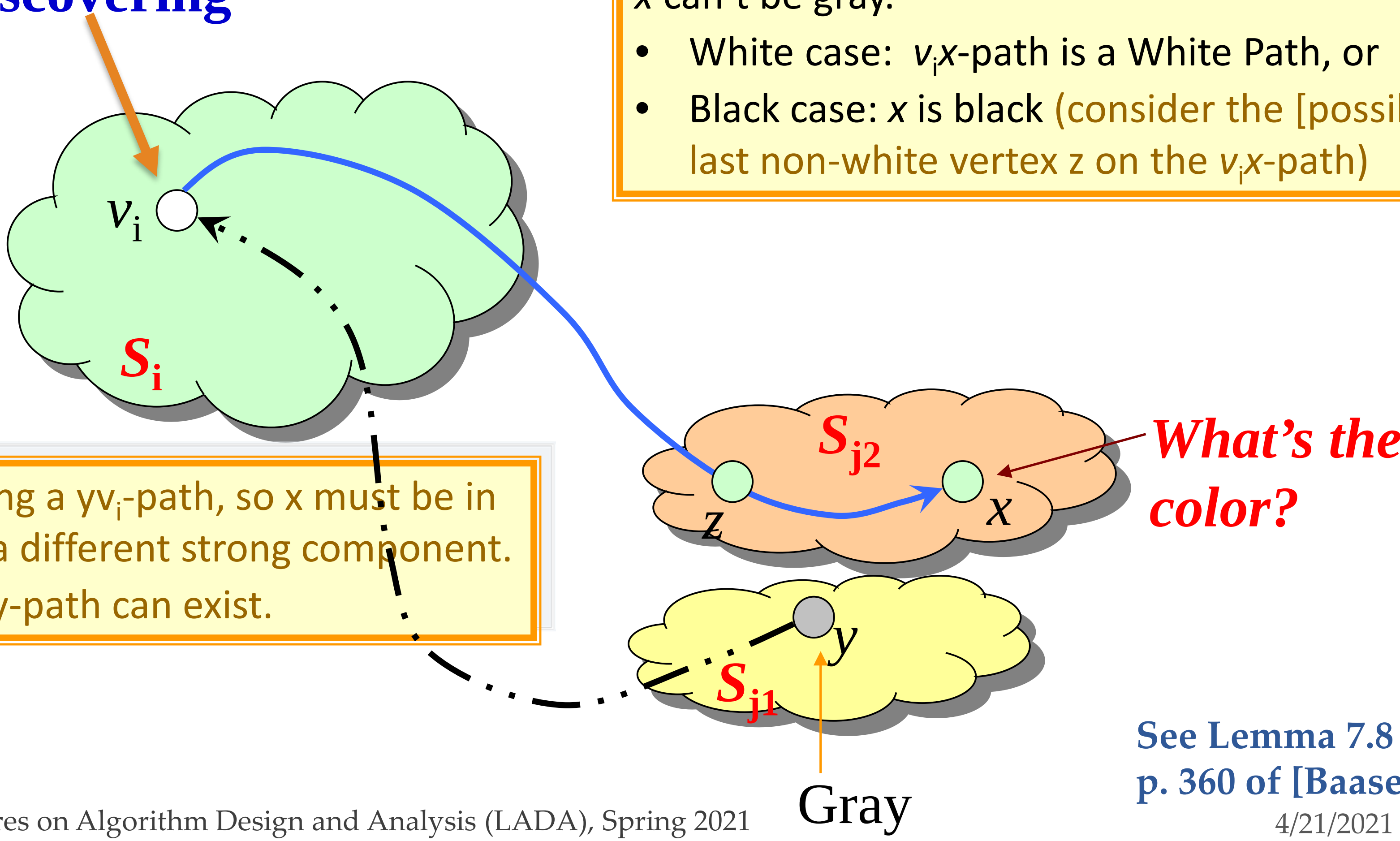
Path between SCCs

The leader of S_i

At discovering

x can't be gray.

- White case: $v_i x$-path is a White Path, or
- Black case: *x* is black (consider the [possible] last non-white vertex z on the $v_i x$-path)

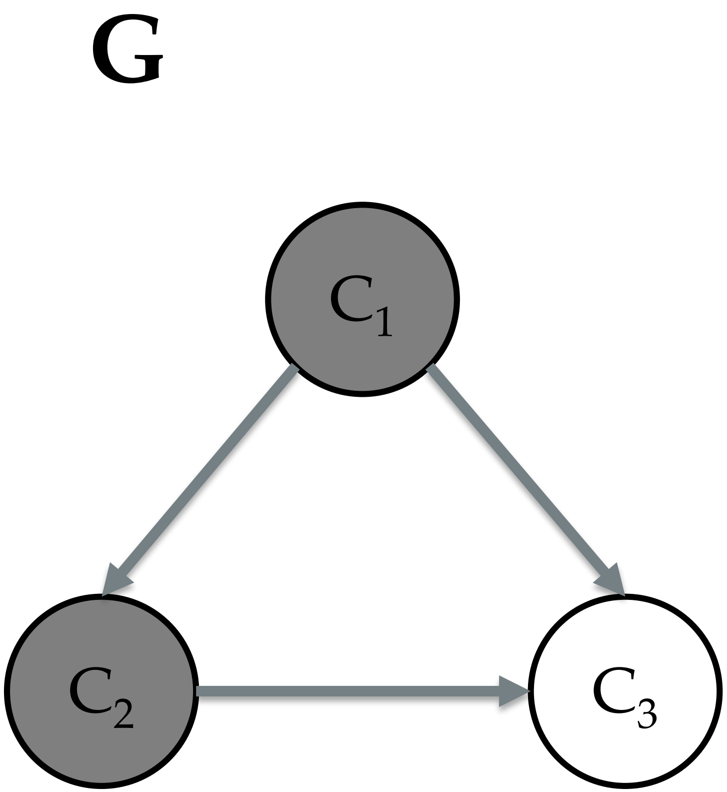


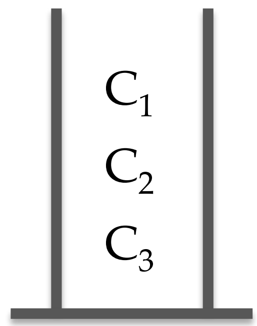
Existing a yv_i-path, so x must be in a different strong component.
No $v_i y$-path can exist.

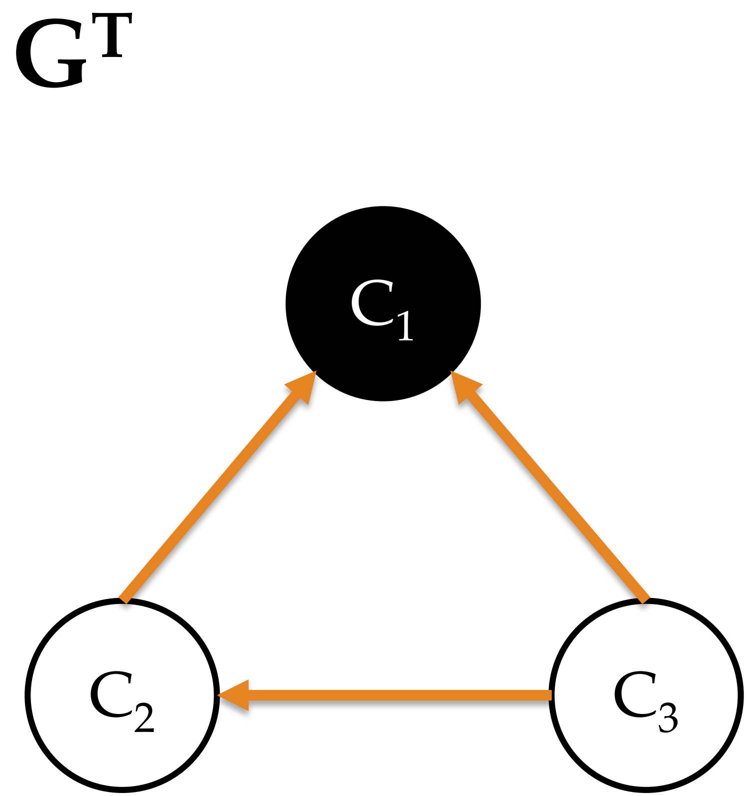
See Lemma 7.8 & 7.9 p. 360 of [Baase01]

C_1 : The End Case

Looking at C_2, C_3 from C_1

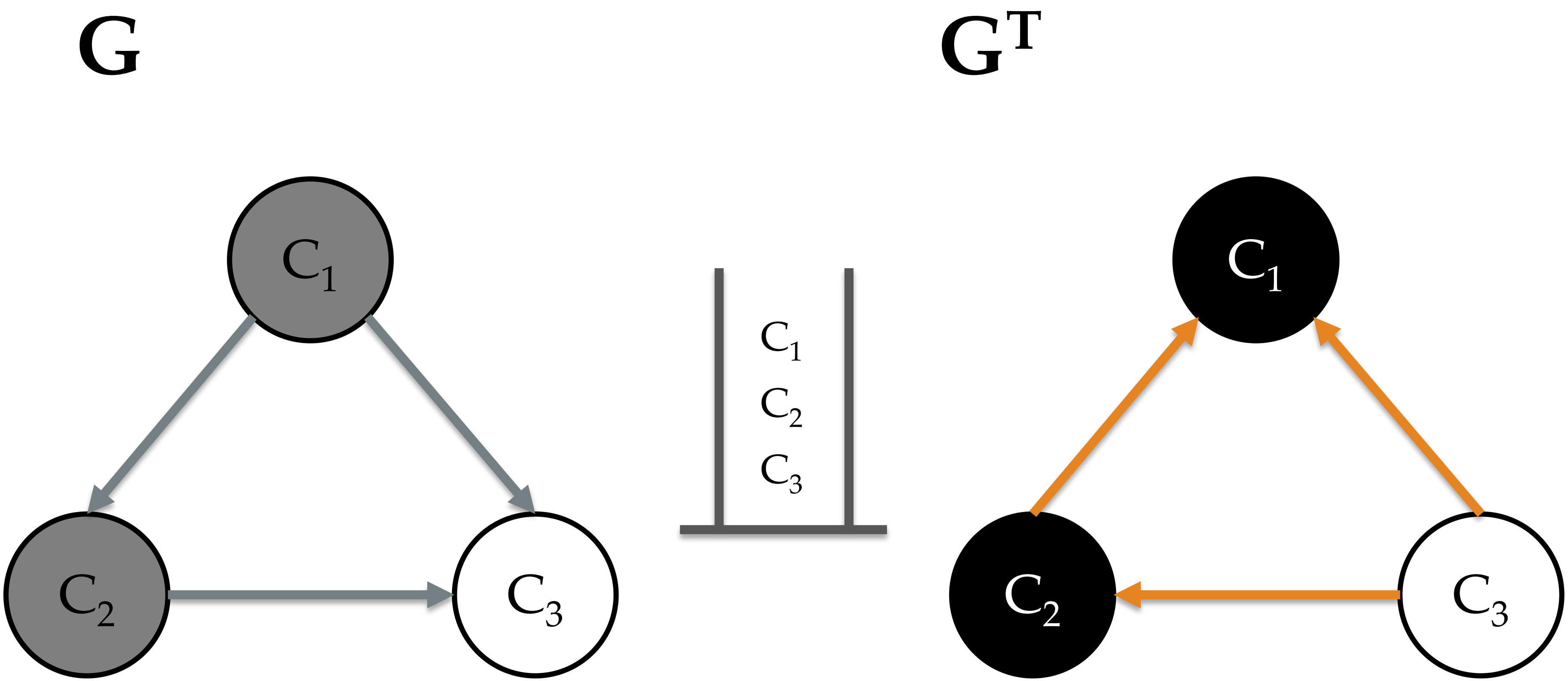






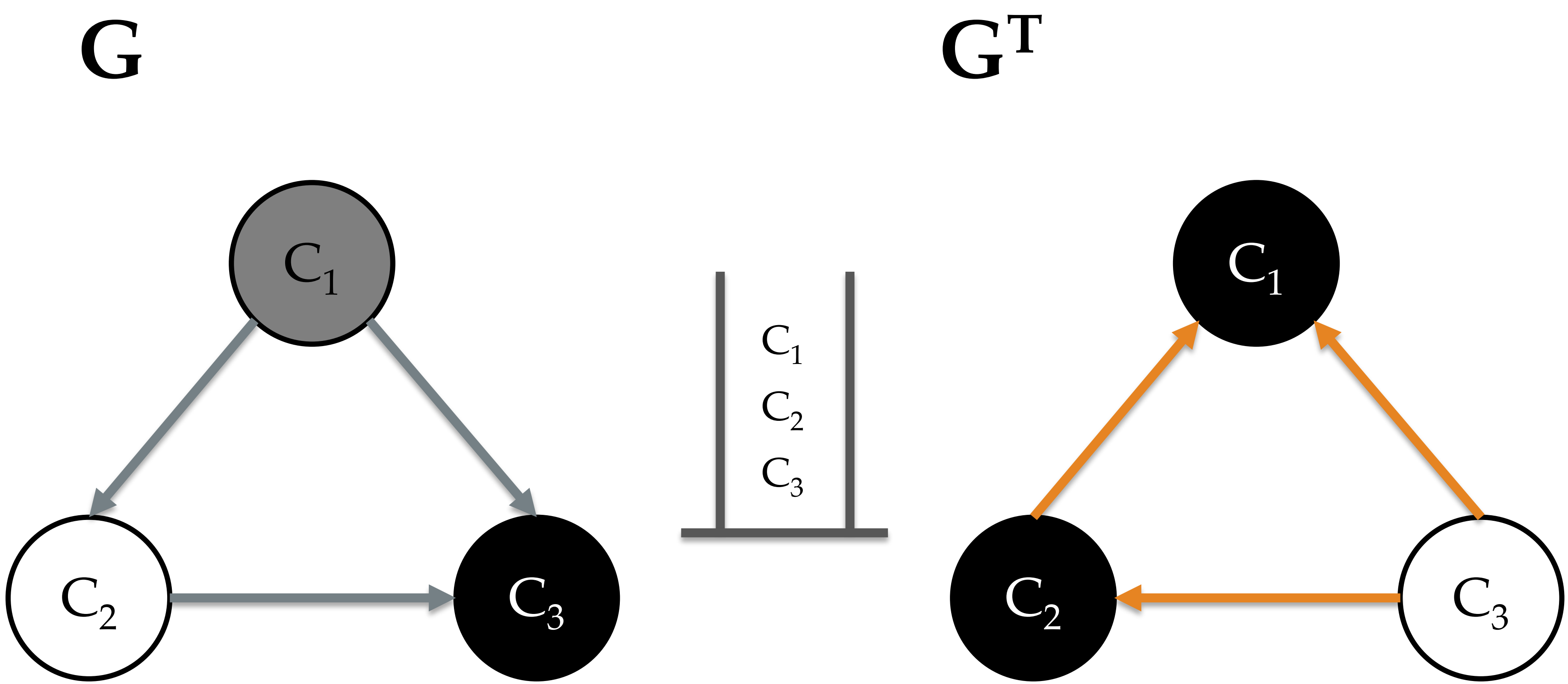
C_2 : The White Case

Looking at C_3 from C_2



C_2 : The Black Case

Looking at C_3 from C_2



Active Intervals

- **If there is an edge from S_i to S_j, then it is impossible that the active interval of v_j is entirely after that of v_i. (Note: for leader v_i only)**
  - There is no path from a leader of a strong component to any gray vertex.
  - If there is a path from the leader v of a strong component to any x in a different strong component, v finishes later than x.

Correctness of Strong Component Algorithm (1)

- **In phase 2, each time a white vertex is popped from *finishStack*, that vertex is the Phase 1 leader of a strong component.**
    - The later finished, the earlier popped
    - The leader is the first to get popped in the strong component it belongs to
    - If x popped is not a leader, then some other vertex in **the** strong component has been visited previously. But not a partial strong component can be in a DFS tree, so, x must be in a completed DFS tree, and is not white.

Correctness of Strong Component Algorithm (2)

- **In phase 2, each depth-first search tree contains exactly one strong component of vertices**
  - Only "exactly one" need to be proved
  - Assume that v_i, a phase 1 leader is popped. If another component S_j is reachable from v_i in G^T, there is a path in G from v_j to v_i. So, in phase 1, v_j finished later than v_i, and popped earlier than v_i in phase 2. So, when v_i popped, all vertices in S_j are black. So, S_j are not contained in DFS tree containing $v_i(S_i)$.

Thank you!

Q & A

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