1) Big omega Notation: Prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

Solution

1) To prove that there exist positive constants c and n_0 such that for all $n \geq n_0$ to show $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

$g(n) \geq c \cdot n^3$

Given $g(n) = n^3 + 2n^2 + 4n$, we can choose $c = 1$ and $n_0 = 1$. these for all $n \geq 1$:

$$n^3 + 2n^2 + 4n \geq 1 \cdot n^3$$

thus $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

2) Big theta Notation: Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not.

solution:

To determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$, we need to check if there exist positive constants c_1, c_2 and n_0 such that for all $n \geq n_0$:

$$c_1 \cdot n^2 \leq h(n) \leq c_2 \cdot n^2$$

Given $h(n) = 4n^2 + 3n$:

1) for the upper bound

$$h(n) = 4n^2 + 3n \leq 4n^2 + 3n^2 = 7n^2$$

Choose $c_2 = 7$.

2) for the lower bound

$$h(n) = 4n^2 + 3n \geq 4n^2.$$

Choose $c_1 = 4$

Thus, we can choose $c_1=4$, $c_2=7$ and $n_0=1$, showing that

$4n^2 \le 4n^2+3n \le 7n^2$.

Therefore, $h(n)=4n^2+3n$ is $\theta(n^2)$

3) Let $f(n)=n^3-2n^2+n$ and $g(n)=n^2$ show that $f(n)=\Omega g(n)$ is true or false Justify your answer.?

Solution

$$f(n) \ge c \cdot g(n)$$

substituting $f(n)$ and $g(n)$ into this inequality we get

$$n^3-2n^2+n \ge c \cdot (-n^2)$$

find c and n_0 holds $n \ge n_0$

$$n^3-2n^2+n \ge -cn^2$$

$$n^3-2n^2+n+cn^2 \ge 0$$

$$n^3-(c-2)n^2+n \ge 0$$

$$n^3+(c-2)n^2+n \ge 0 \quad (n^3 \ge 0)$$

$$n^3+(1-2)n^2+n = n^3-n^2+n \ge 0$$

$$f(n)=n^3-2n^2+n \text{ is } \Omega(g(n)=\Omega(-n^2))$$

Therefore the statement $f(n)=\Omega g(n)$ is true.

4) Determine whether $h(n)=n\log n+n$ is $\theta(n\log n)$ Prove a rigorous proof for your conclusion.

Solution

$$c_1 n\log n \le h(n) \le c_2 n \log n.$$

Upper bound

$$h(n) \le c_2 n\log n$$

$$h(n) \le n\log n + n$$

$$n\log n + n \le c_2\, n\log n$$

Divide both sides by $n\log n$

$$1 + \frac{n}{n\log n} \le 2$$

$$1 + \frac{1}{\log n} \le 2$$

Then $h(n)$ is $O(n\log n)$

Lower bound:

$$h(n) \ge c_1\, n\log n$$

$$h(n) = n\log n + n$$

$$n\log n + n \ge c_1 n\log n$$

Divide both side by $n\log n$

$$1 + \frac{n}{n\log n} \ge c_1$$

$$1 + \frac{1}{\log n} \ge c_1$$

$$1 + \frac{1}{\log n} \ge 1$$

$$\frac{1}{\log n} \ge 0$$

$h(n)$ is $\Omega(n\log n)$ $(c_1 = 1, n_0 = 1)$

$h(n) = n\log n + n$ is $\Theta(n\log n)$

5) Solve the following recurrence relations and find the order of growth of solutions, $T(n) = 4T(n/2) + n^2$, $T(1) = 1$

$T(n) = 4T(n/2) + n^2, \; T(1) = 1$

$T(n) = aT(n/b) + f(n)$

$a = 4, b = 2, \; f(n) = n^2$

applying master theorem

$T(n) = aT(n/b) + f(n)$

$f(n) = O(n^{\log_b^a - \epsilon})$

$f(n) = \Theta(n^{\log_b^a})$, then $T(n) = \Theta(n^{\log_b^a} \log n)$

$f(n) = \Omega(n^{\log_b^a + \epsilon})$, then $T(n) = f(n)$

Calculating $\log_b^a$:

$\log_b^a = \log_2 4 = 2$

$f(n) = n^2 = \Theta(n^2)$

$f(n) = \Theta(n^2) = \Theta(n^{\log_b^a})$

$T(n) = 4T(n/2) + n^2$

$T(n) = \Theta(n^{\log_b^a} \log n) = \Theta(n^2 \log n)$

Order of growth

$T(n) = 4T(n/2) + n^2$ with $T(1) = 1$

is $\Theta(n^2 \log n)$