Chapter 6

- 6.1 The place of model checking in applied Bayesian statistics
- 6.2 Do the inferences from the model make sense?
- 6.3 Posterior predictive checking
- 6.4 Graphical posterior predictive checks (can be skipped)
- 6.5 Model checking for the educational testing example

Model checking

- demo6_1: Posterior predictive checking - light speed
- demo6_2: Posterior predictive checking - sequential dependence
- demo6_3: Posterior predictive checking - poor test statistic
- demo6_4: Posterior predictive checking - marginal predictive p-value

Model checking – overview

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- Internal validation
  - posterior predictive checking
  - cross-validation predictive checking

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- Newcomb's speed of light measurements
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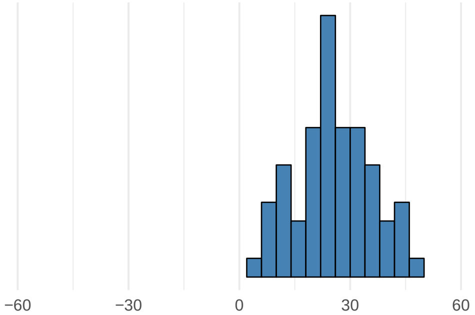

Replicates vs. future observation

- Predictive $\tilde{y}$ is the next not yet observed possible observation. y^{rep} refers to replicating the whole experiment (potentially with same values of x) and obtaining as many replicated observations as in the original data.

Posterior predictive checking – example

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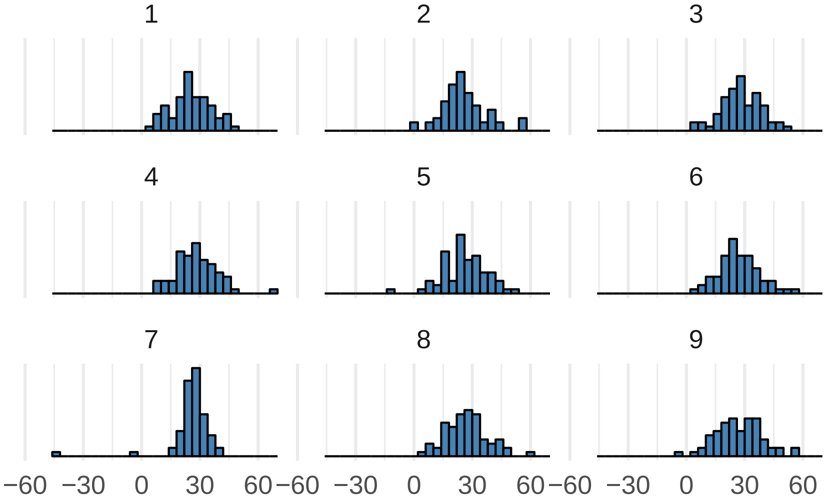

Posterior predictive checking with test statistic

- Replicated data sets y^{rep}
- Test quantity (or discrepancy measure) $T(y, \theta)$
  - summary quantity for the observed data $T(y, \theta)$
  - summary quantity for a replicated data $T(y^{\mathrm{rep}}, \theta)$
  - can be easier to compare summary quantities than data sets

Posterior predictive checking – example

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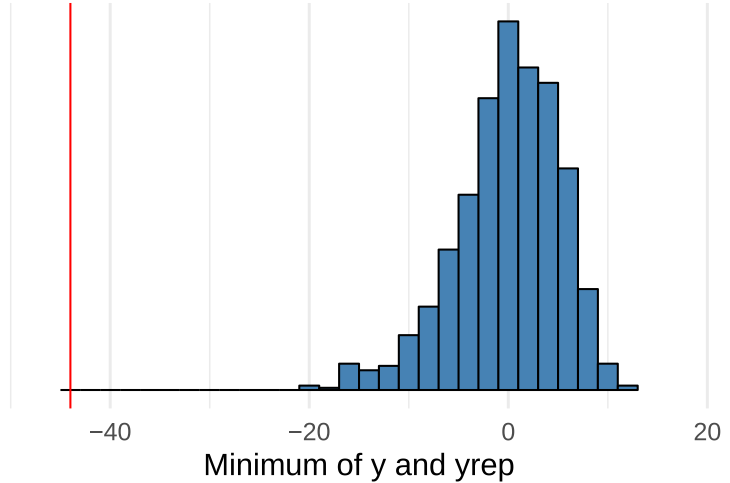

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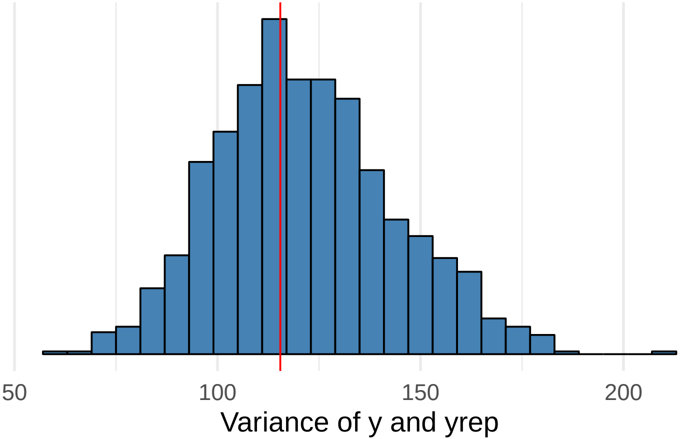

Posterior predictive checking

- *Posterior predictive p-value*

$$
\begin{aligned}
p &= \Pr(T(y^{\mathrm{rep}}, \theta) \geq T(y, \theta) | y) \\
&= \int \int I_{T(y^{\mathrm{rep}}, \theta) \geq T(y, \theta)} p(y^{\mathrm{rep}} | \theta) p(\theta | y) dy^{\mathrm{rep}} d\theta
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where I is an indicator function

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- having $(y^{\mathrm{rep}\,(s)}, \theta^{(s)})$ from the posterior predictive distribution, easy to compute

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- Posterior predictive p-value (ppp-value) estimated whether difference between the model and data could arise by chance
- Not commonly used, since the distribution of test statistic has more information

Marginal and CV predictive checking

- Consider marginal predictive distributions $p(\tilde{y}_i|y)$ and each observation separately
  - marginal posterior p-values

$$p_i = \Pr(T(y_i^{\text{rep}}) \leq T(y_i)|y)$$

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- if $Pr(\tilde{y}_i|y)$ well calibrated, distribution of p_i would be uniform between 0 and 1
  - holds better for cross-validation predictive tests (cross-validation Ch 7)

Marginal predictive checking - Example

- Marginal tail area or Probability integral transform (PIT)

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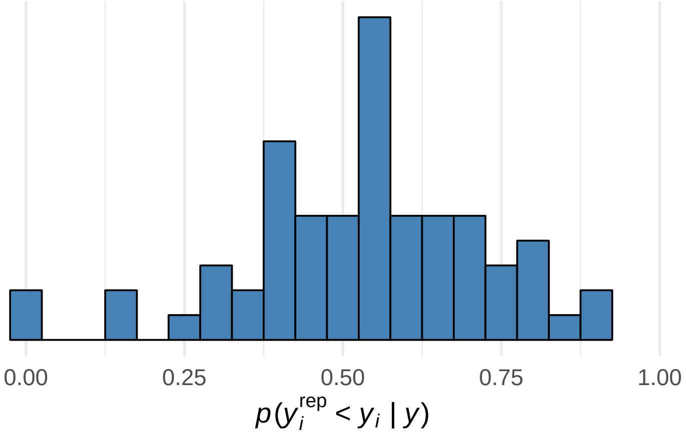

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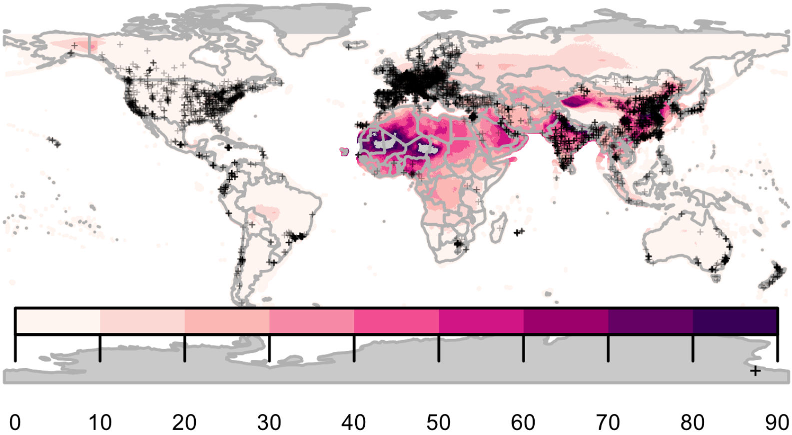
  - robust models are good for testing sensitivity to “outliers”
    - e.g. *t* instead of Gaussian
- Compare sensitivity of essential inference quantities
  - extreme quantiles are more sensitive than means and medians
  - extrapolation is more sensitive than interpolation

Example: Exposure to air pollution

- Example from Jonah Gabry, Daniel Simpson, Aki Vehtari, Michael Betancourt, and Andrew Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378
- Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($\mathrm{PM}_{2.5}$)
  - Exposure to $\mathrm{PM}_{2.5}$ is linked to a number of poor health outcomes and a recent report estimated that $\mathrm{PM}_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al., 2017)
  - In order to estimate the public health effect of ambient $\mathrm{PM}_{2.5}$, we need a good estimate of the $\mathrm{PM}_{2.5}$ concentration at the same spatial resolution as our population estimates.

Example: Exposure to air pollution

- Direct measurements of PM 2.5 from ground monitors at 2980 locations
- High-resolution satellite data of aerosol optical depth



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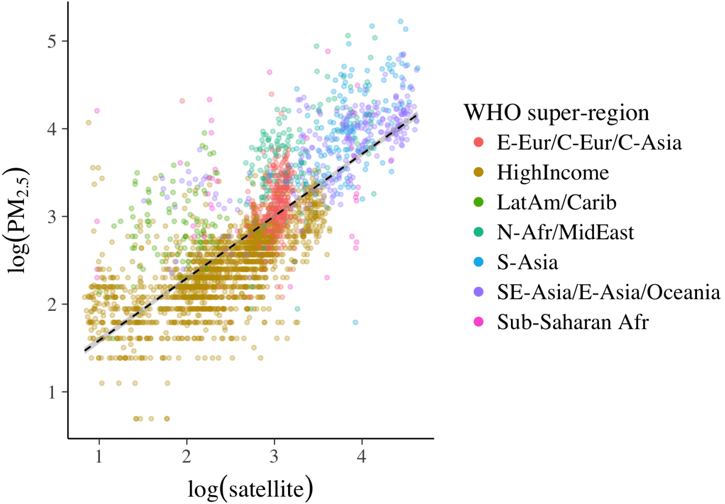

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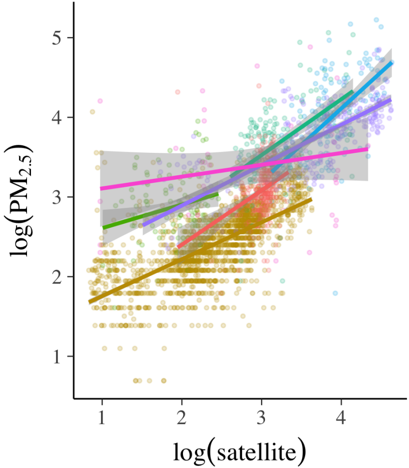

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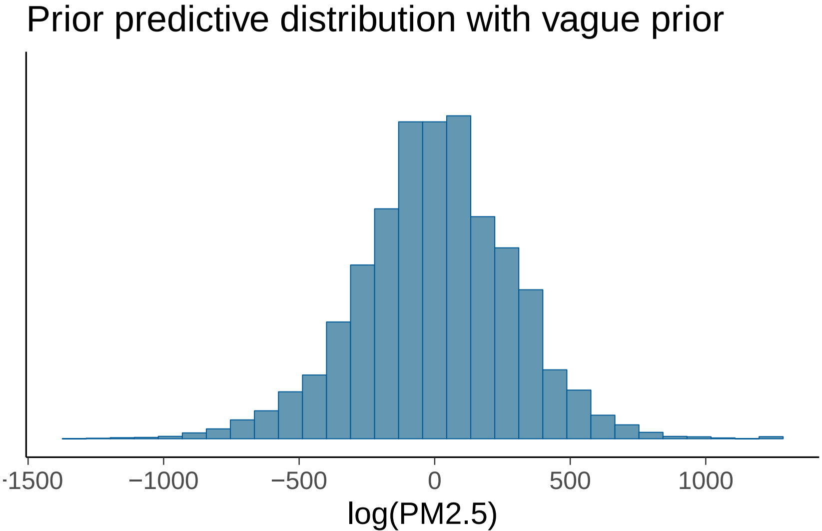
Prior predictive checking



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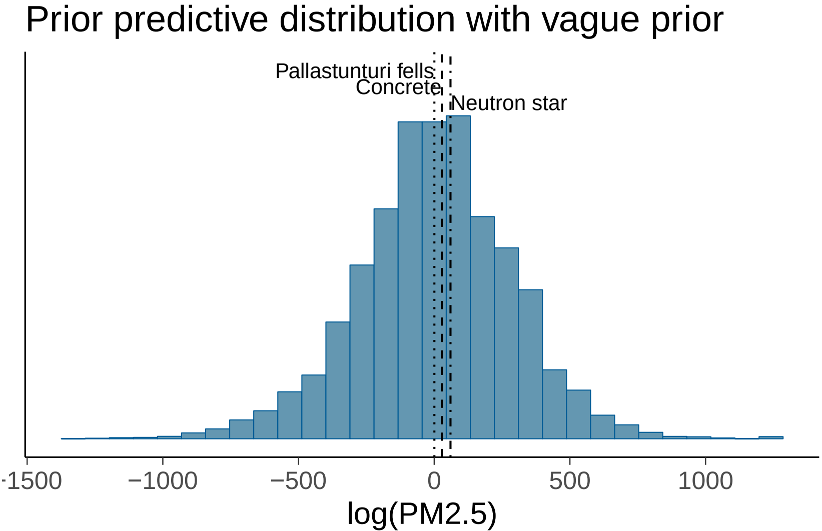

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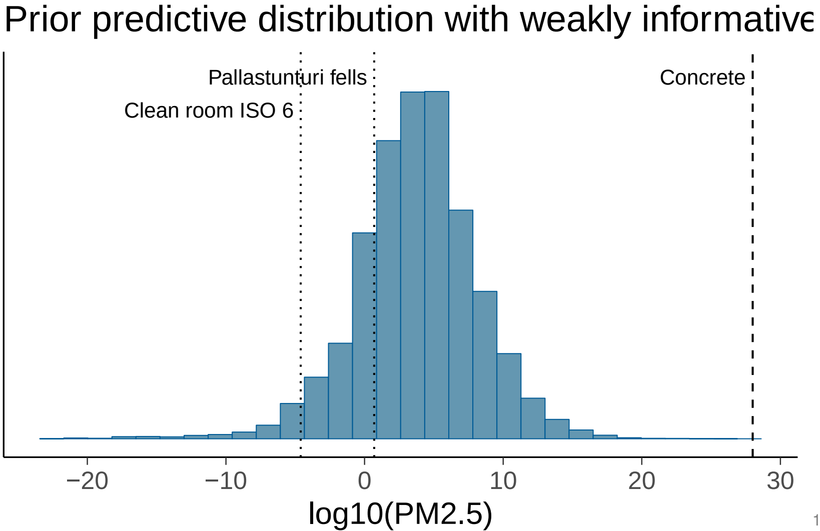
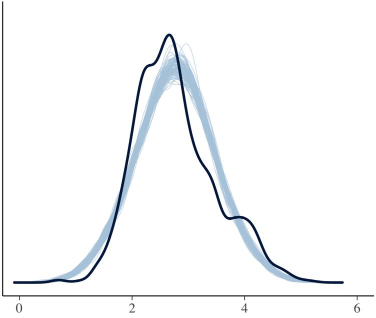
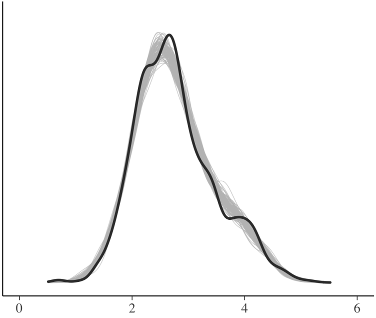


Example: Exposure to air pollution

Posterior predictive checking – marginal predictive distributions



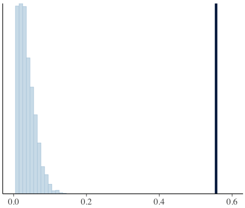
(a) Model 1



(b) Model 2

Example: Exposure to air pollution

Posterior predictive checking – test statistic (skewness)



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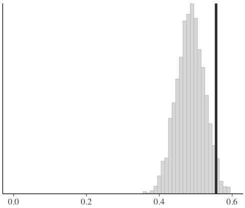
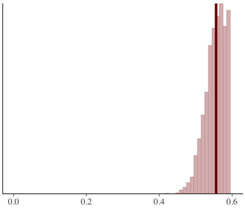


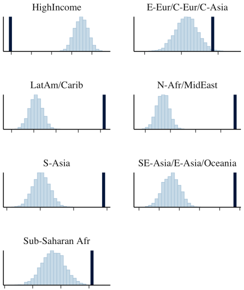
(b) Model 2



(c) Model 3

Example: Exposure to air pollution

Posterior predictive checking – test statistic (median for groups)



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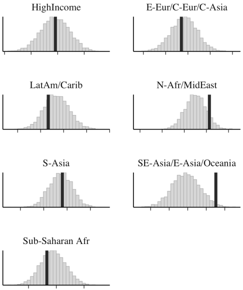

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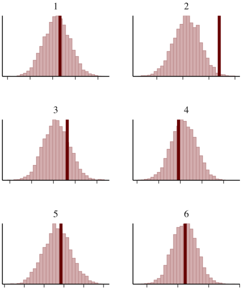

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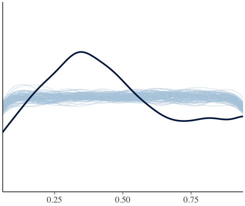
LOO predictive checking – LOO-PIT



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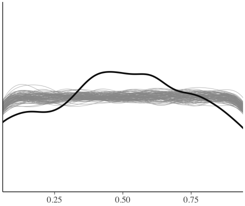

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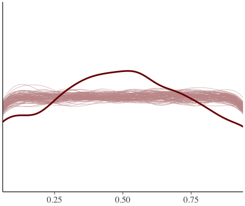

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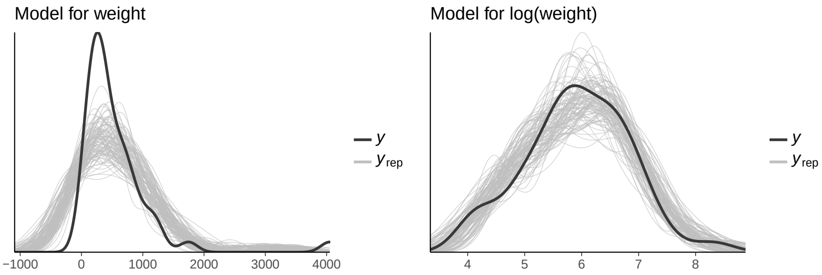
Example of posterior predictive checking

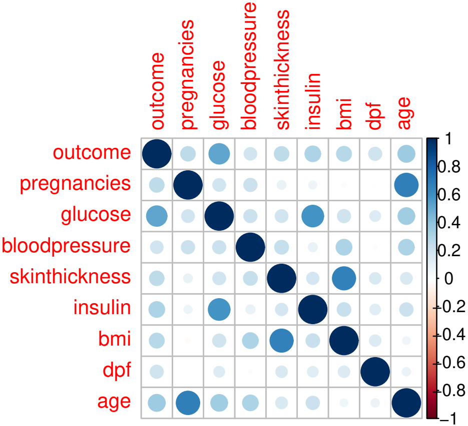


Predicting the yields of mesquite bushes.

Gelman, Hill & Vehtari (2020): Regression and Other Stories, Chapter 11.

Example of posterior predictive checking

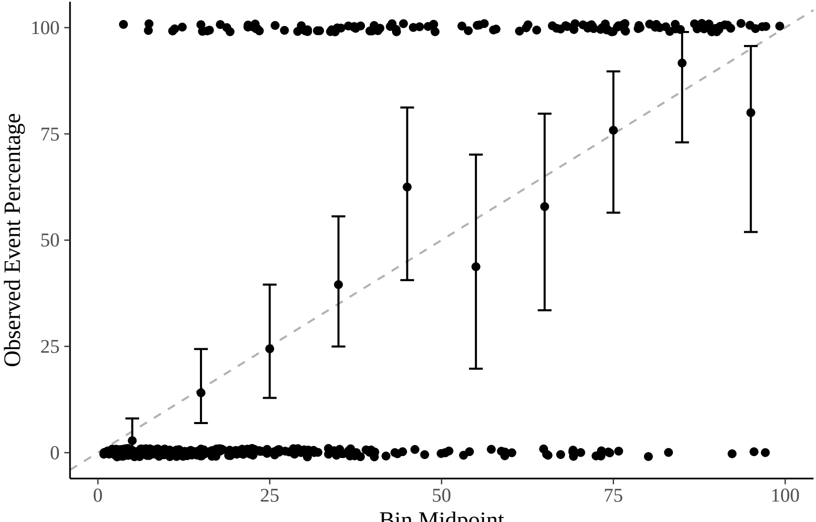
Diabetes prediction with logistic regression - diabetes demo



Example of posterior predictive checking

Diabetes prediction with logistic regression - diabetes demo

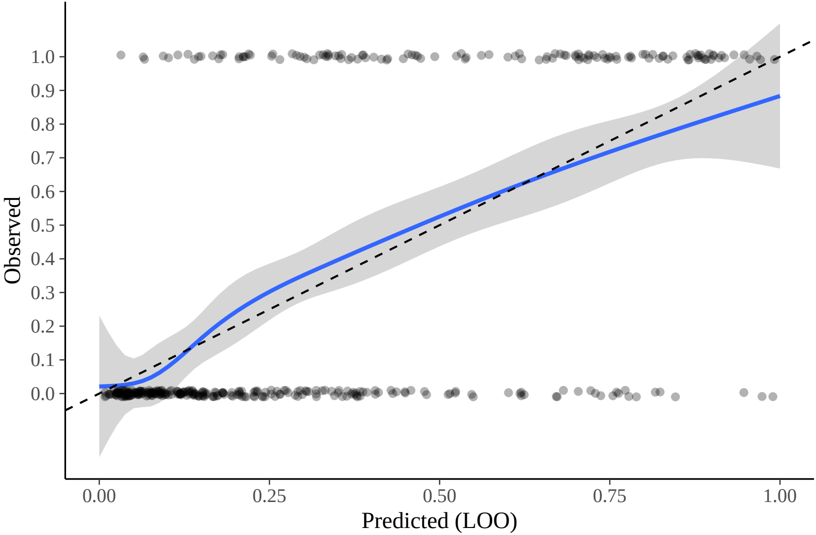
PPC with binning for binary data



Example of posterior predictive checking

Diabetes prediction with logistic regression - diabetes demo

PPC with non-linear regression for binary data



Posterior predictive checking

- demo demos_rstan/ppc/poisson-ppc.Rmd

```
data {
  int<lower=1> N;
  int<lower=0> y[N];
}
parameters {
  real<lower=0> lambda;
}
model {
  lambda ~ exponential(0.2);
  y ~ poisson(lambda);
}
generated quantities {
  real log_lik[N];
  int y_rep[N];
  for (n in 1:N) {
    y_rep[n] = poisson_rng(lambda);
    log_lik[n] = poisson_lpmf(y[n] | lambda);
    }
}
```

Further reading and examples

- Jonah Gabry, Daniel Simpson, Aki Vehtari, Michael Betancourt, and Andrew Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378.
- Graphical posterior predictive checks using the bayesplot package
  http://mc-stan.org/bayesplot/articles/graphical-ppcs.html
- Another demo demos_rstan/ppc/poisson-ppc.Rmd
- Michael Betancourt's workflow case study with prior and posterior predictive checking
  - for RStan https://betanalpha.github.io/assets/case_studies/principled_bayesian_workflow.html
  - for PyStan
    https://github.com/betanalpha/jupyter_case_studies/blob/master/principled_bayesian_workflow/principled_bayesian_workflow.ipynb