## Outline of the chapter 2

- 2.1 Binomial model (repeated experiment with binary outcome)
- 2.2 Posterior as compromise between data and prior information
- 2.3 Posterior summaries
- 2.4 Informative prior distributions (skip exponential families and sufficient statistics)
- 2.5 Gaussian model with known variance
- 2.6 Other single parameter models
  - the normal distribution with known mean but unknown variance is the most important
  - glance through Poisson and exponential
- 2.7 glance through this example, which illustrates benefits of prior information, no need to read all the details (it's quite long example)
- 2.8 Noninformative and weakly informative priors

- Probability of event 1 in trial is $\theta$

# Binomial: known $\theta$

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  $\theta\theta(1 - \theta)\theta(1 - \theta)(1 - \theta)\ldots$

## Binomial: known $\theta$

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- Probability of event 2 in trial is $1 - \theta$
- Probability of several events in independent trials is e.g.
  $\theta\theta(1 - \theta)\theta(1 - \theta)(1 - \theta)\ldots$
- If there are *n* trials and we don't care about the order of the events, then the probability that event 1 happens *y* times is
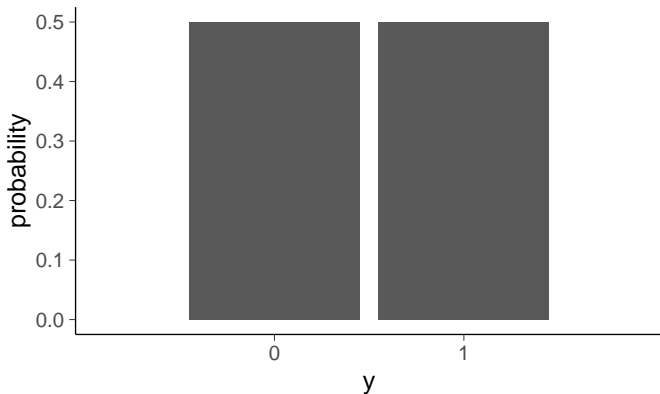
$$p(y|\theta, n) = \binom{n}{y}\theta^y(1 - \theta)^{n-y}$$

## Binomial: known $\theta$

- Observation model (function of $y$, discrete)

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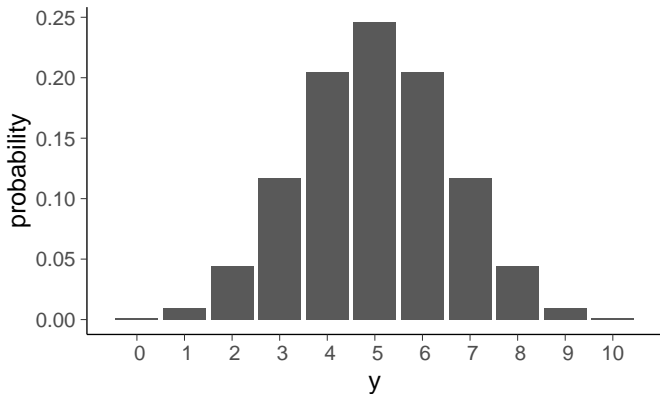$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

Binomial distribution with $\theta$ =0.5, n=1

- Observation model (function of $y$, discrete)

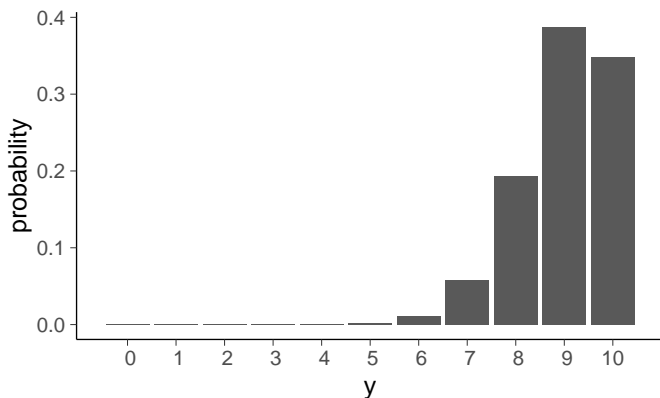$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

Binomial distribution with $\theta$ =0.5, n=10

- Observation model (function of $y$, discrete)

$$p(y|\theta, n) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

Binomial distribution with $\theta$ =0.9, n=10

# Binomial: unknown $\theta$

- Posterior with Bayes rule (function of $\theta$, continuous)

$$p(\theta|y, n, M) = \frac{p(y|\theta, n, M)p(\theta|n, M)}{p(y|n, M)}$$

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- Start with uniform prior

$$p(\theta|n, M) = p(\theta|M) = 1, \text{ kun } 0 \leq \theta \leq 1$$

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$$p(\theta|n, M) = p(\theta|M) = 1, \text{ kun } 0 \le \theta \le 1$$

- Then

$$p(\theta|y, n, M) = \frac{p(y|\theta, n, M)}{p(y|n, M)} = \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y}}{\int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}d\theta}$$

$$= \frac{1}{Z}\theta^y(1-\theta)^{n-y}$$

## Binomial: unknown $\theta$

- Normalization term $Z$ (constant given $y$)

$$Z = p(y|n, M) = \int_0^1 \theta^y (1-\theta)^{n-y} d\theta = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

- Normalisation term has Beta function form
    - when integrated over $(0, 1)$ the result can presented with Gamma functions
    - with integers $\Gamma(n) = (n-1)!$
    - for large integers even this is challenging and usually $\log \Gamma(\cdot)$ is computed instead of $\Gamma(\cdot)$

## Binomial: unknown $\theta$

- Posterior is

$$p(\theta|y, n, M) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)} \theta^y (1-\theta)^{n-y},$$
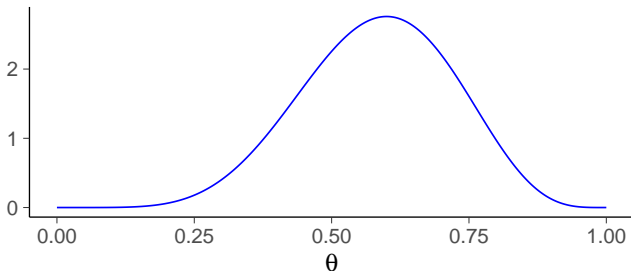
- Posterior is

$$p(\theta|y, n, M) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^y(1-\theta)^{n-y},$$

  which is called Beta distribution

$$\theta|y, n \sim \text{Beta}(y+1, n-y+1)$$

p( $\theta$ | y=6, n=10, M=binom) + unif. prior)

# Binomial: computation

- R
  - density `dbeta`
  - CDF `pbeta`
  - quantile `qbeta`
  - random number `rbeta`
- Python
  - `from scipy.stats import beta`
  - density `beta.pdf`
  - CDF `beta.cdf`
  - prctile `beta.ppf`
  - random number `beta.rvs`

## Binomial: computation*

- Beta CDF not trivial to compute
- For example, `pbeta` in R uses a continued fraction with weighting factors and asymptotic expansion
- Laplace developed normal approximation (Laplace approximation), because he didn't know how to compute Beta CDF

# Placenta previa

- Probability of a girl birth given placenta previa (BDA3 p. 37)
  - 437 girls and 543 boys have been observed
  - is the ratio 0.445 different from the population average 0.485?

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Uniform prior –> Posterior is Beta(438,544)
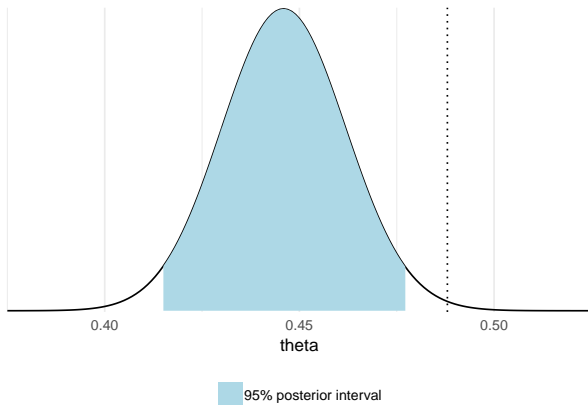


95% posterior interval

- Predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1 | \theta, y, n, M)$$

# Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|y, n, M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta$$

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$$= \int_0^1 \theta p(\theta | y, n, M) d\theta$$
$$= \mathsf{E}[\theta | y]$$

## Predictive distribution – Effect of integration

- Predictive distribution for new $\tilde{y}$ (discrete)

$$
\begin{aligned}
p(\tilde{y} = 1 | y, n, M) &= \int_0^1 p(\tilde{y} = 1 | \theta, y, n, M) p(\theta | y, n, M) d\theta \\
&= \int_0^1 \theta p(\theta | y, n, M) d\theta \\
&= \mathsf{E}[\theta | y]
\end{aligned}
$$

- With uniform prior

$$
\mathsf{E}[\theta | y] = \frac{y + 1}{n + 2}
$$

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\mathsf{E}[\theta|y] = \frac{y+1}{n+2}
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- Extreme cases
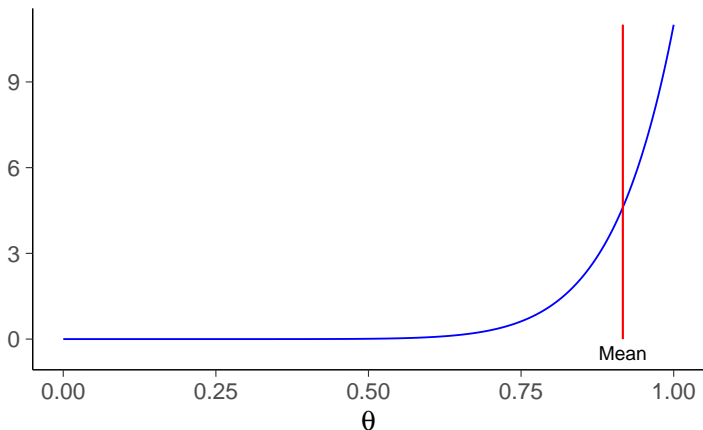
$$
p(\tilde{y} = 1|y = 0, n, M) = \frac{1}{n+2}
$$
$$
p(\tilde{y} = 1|y = n, n, M) = \frac{n+1}{n+2}
$$

- cf. maximum likelihood

Example: $n = 10, y = 10$



Posterior of $\theta$ of Binomial model with y=10, n=

# Predictive distribution

- Prior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|M)d\theta$$

- Posterior predictive distribution for new $\tilde{y}$ (discrete)

$$p(\tilde{y} = 1|y, n, M) = \int_0^1 p(\tilde{y} = 1|\theta, y, n, M)p(\theta|y, n, M)d\theta$$

- $p(\theta|M) = 1$ if

    1) we want the prior predictive distribution to be uniform

    $$p(y|n, M) = \frac{1}{n+1}, \quad y = 0, \ldots, n$$

    - nice justification as it is based on observables $y$ and $n$

# Justification for uniform prior

- $p(\theta|M) = 1$ if
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    $$p(y|n, M) = \frac{1}{n+1}, \quad y = 0, \dots, n$$

    - nice justification as it is based on observables *y* and *n*
    2) we think all values of $\theta$ are equally likely

# Priors

- Conjugate prior (BDA3 p. 35)
- Noninformative prior (BDA3 p. 51)
- Proper and improper prior (BDA3 p. 52)
- Weakly informative prior (BDA3 p. 55)
- Informative prior (BDA3 p. 55)
- Prior sensitivity (BDA3 p. 38)

## Conjugate prior

- Prior and posterior have the same form
  - only for exponential family distributions (plus for some irregular cases)
- Used to be important for computational reasons, and still sometimes used for special models to allow partial analytic marginalization (Ch 3)
  - with Hamiltonian Monte Carlo used e.g. in Stan no any computational benefit

# Beta prior for Binomial model

- Prior

$$\text{Beta}(\theta|\alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- Posterior

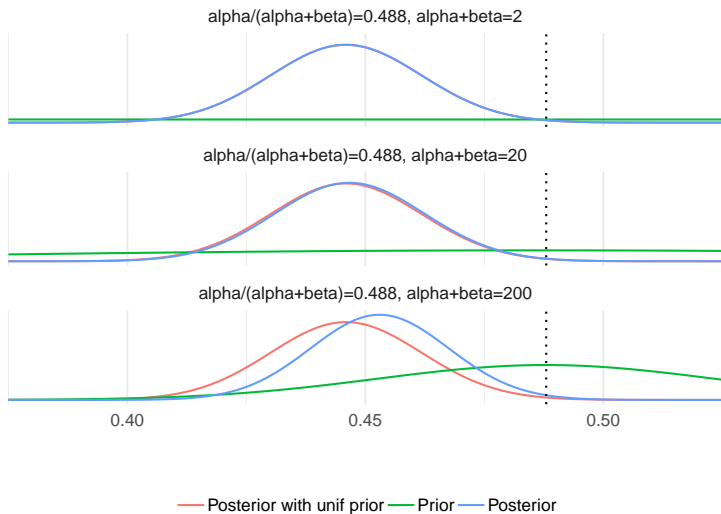$$p(\theta|y, n, M) \propto \theta^y(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

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&= \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&= \text{Beta}(\theta|\alpha+y, \beta+n-y)
\end{aligned}
$$

# Beta prior for Binomial model

- Prior

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$$= \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}$$
$$= \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

- $(\alpha - 1)$ and $(\beta - 1)$ can considered to be number of prior observations
- Uniform prior when $\alpha = 1$ and $\beta = 1$

- Beta prior centered on population average 0.485



alpha/(alpha+beta)=0.488, alpha+beta=2

alpha/(alpha+beta)=0.488, alpha+beta=20

alpha/(alpha+beta)=0.488, alpha+beta=200

0.40          0.45          0.50

Posterior with unif prior —— Prior —— Posterior

Example: $n = 10, y = 10$ - uniform vs Beta(2,2) prior



p( θ | y=10, n=10, M=binom) + unif. prior
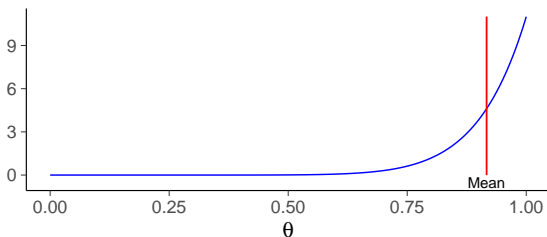


p( θ | y=10, n=10, M=binom) + Beta(2,2) prior
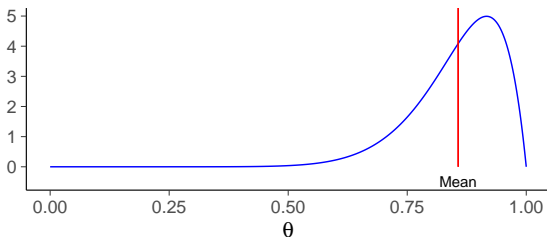
## Beta prior for Binomial model

- Posterior

$$p(\theta|y, n, M) = \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

- Posterior mean

$$\mathsf{E}[\theta|y] = \frac{\alpha + y}{\alpha + \beta + n}$$

  - combination prior and likelihood information
  - when $n \to \infty$, $\mathsf{E}[\theta|y] \to y/n$

# Beta prior for Binomial model

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- Posterior mean

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  - combination prior and likelihood information
  - when $n \to \infty$, $E[\theta|y] \to y/n$

- Posterior variance

$$\text{Var}[\theta|y] = \frac{E[\theta|y](1 - E[\theta|y])}{\alpha + \beta + n + 1}$$

  - decreases when $n$ increases
  - when $n \to \infty$, $\text{Var}[\theta|y] \to 0$

# Noninformative prior, proper and improper prior

- Vague, flat, diffuse of noninformative
  - try to "to let the data speak for themselves"
  - flat is not non-informative
  - flat can be stupid
  - making prior flat somewhere can make it non-flat somewhere else
- Proper prior has $\int p(\theta) = 1$
- Improper prior density doesn't have a finite integral
  - the posterior can still sometimes be proper

# Weakly informative priors

- Weakly informative priors produce computationally better behaving posteriors
  - quite often there's at least some knowledge about the scale
  - useful also if there's more information from previous observations, but not certain how well that information is applicable in a new case uncertainty

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- Construction
  - Start with some version of a noninformative prior distribution and then add enough information so that inferences are constrained to be reasonable.
  - Start with a strong, highly informative prior and broaden it to account for uncertainty in one's prior beliefs and in the applicability of any historically based prior distribution to new data.
- Stan team prior choice recommendations https://github. com/stan-dev/stan/wiki/Prior-Choice-Recommendations

- The percentage of girl births is remarkably stable at about 48.8% girls, rarely varying by more than 0.5% from this rate

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- There was a study on the percentage of girl births among parents in attractiveness categories 1–5 (assessed by interviewers in a face-to-face survey)
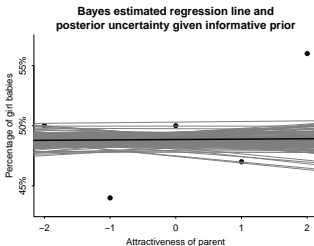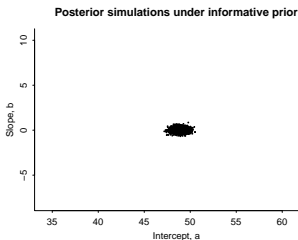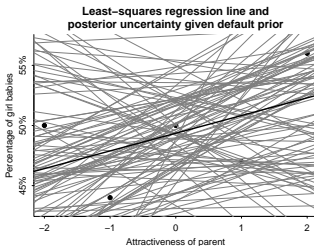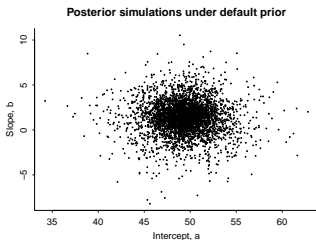
# Example of informative prior

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- There was a study on the percentage of girl births among parents in attractiveness categories 1–5 (assessed by interviewers in a face-to-face survey)



**Data on beauty and sex ratio**

**Data and least−squares regression line**

y = 49.4 + 1.5 x
(Std err of slope is 1.4)

Percentage of girl babies

Attractiveness of parent
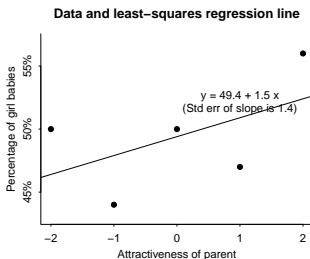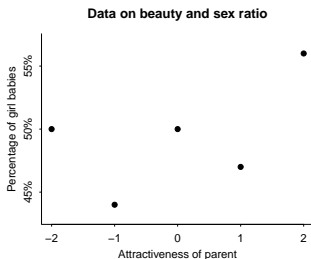
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# Example of weakly informative prior

- Gabry et al (2017). Visualization in Bayesian workflow.
  - Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($PM_{2.5}$)
  - A recent report estimated that $PM_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al, 2017)

# Example of weakly informative prior

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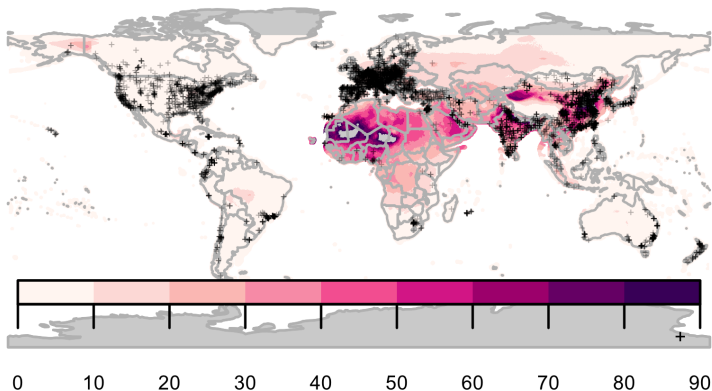


Satellite estimates and ground monitor locations

# Example of weakly informative prior
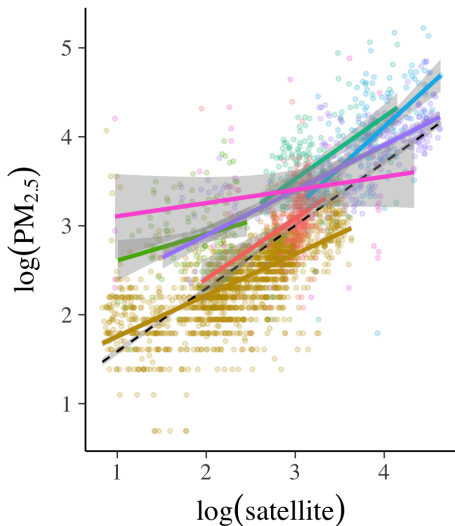
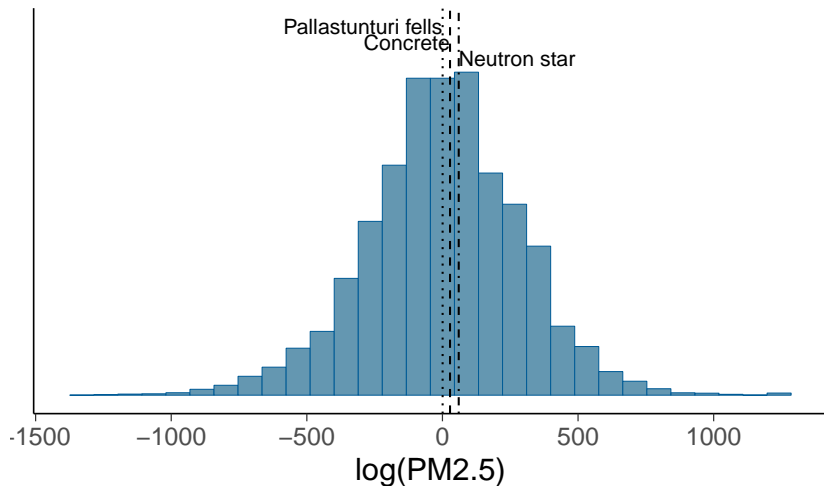- Gabry et al (2017). Visualization in Bayesian workflow.
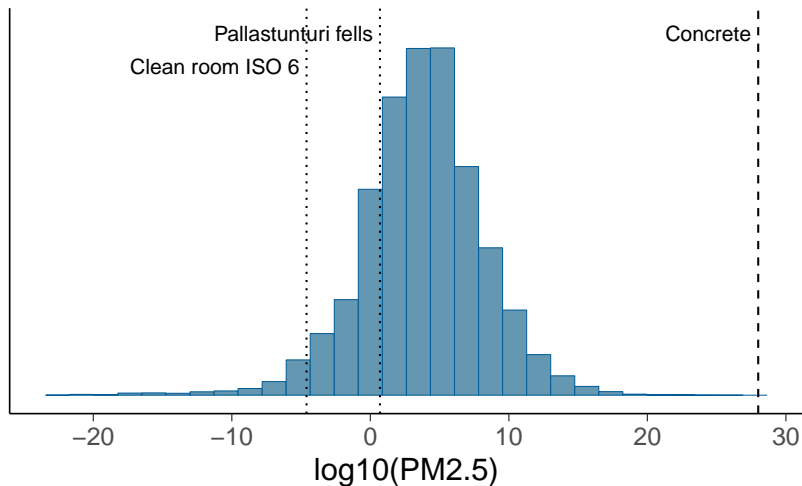
Prior predictive distribution with vague prior



log(PM2.5)

Prior predictive distribution with vague prior

Pallastunturi fells
Concrete
Neutron star

log(PM2.5)
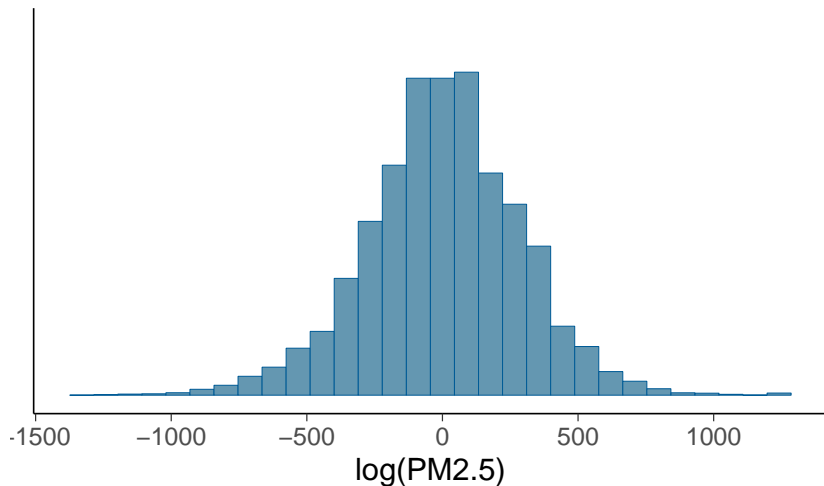
Prior predictive distribution with weakly informative
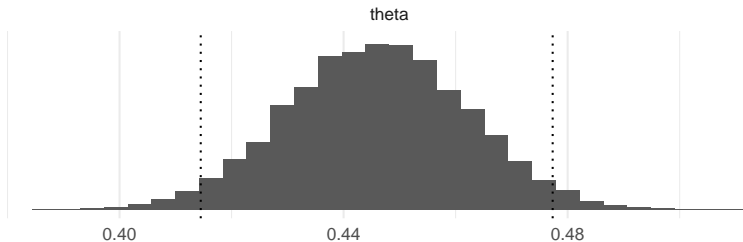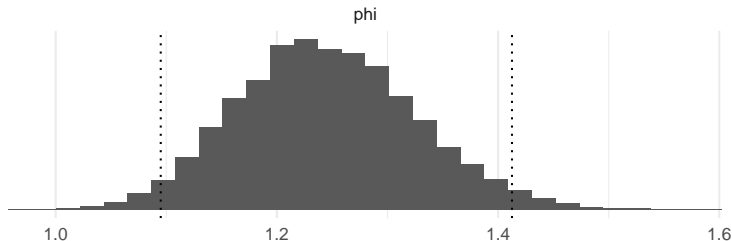


log10(PM2.5)

- Introduce bias, but often still produce smaller estimation error because the variance is reduced
  - bias-variance tradeoff

# Sufficient statistics*

- The quantity $t(y)$ is said to be a *sufficient statistic* for $\theta$, because the likelihood for $\theta$ depends on the data *y* only through the value of $t(y)$.
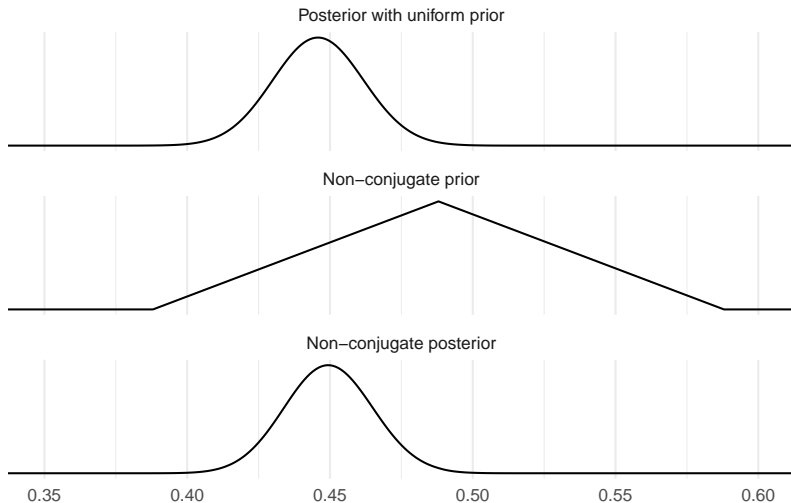
- demo2_3: Simulate samples from Beta(438,544), and draw a histogram of $\theta$ with quantiles.

- demo2_4: Compute posterior distribution in a grid.



Posterior with uniform prior

Non−conjugate prior

Non−conjugate posterior

0.35          0.40          0.45          0.50          0.55          0.60

- demo2_4: Sample using the inverse-cdf method.
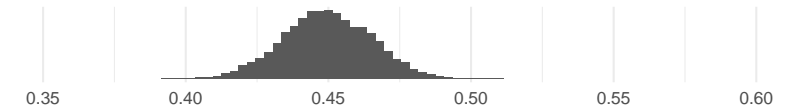
Non−conjugate posterior
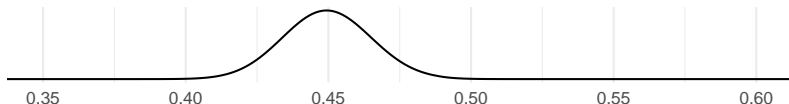


Posterior−cdf
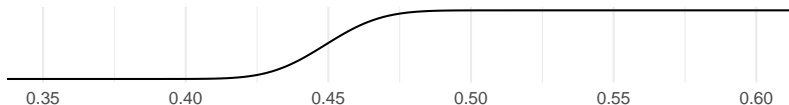


Histogram of posterior samples

Algae status is monitored in 274 sites at Finnish lakes and rivers. The observations for the 2008 algae status at each site are presented in file algae.mat ('0': no algae, '1': algae present). Let $\pi$ be the probability of a monitoring site having detectable blue-green algae levels.

- Use a binomial model for observations and a beta(2,10) prior.
- What can you say about the value of the unknown $\pi$?
- Experiment how the result changes if you change the prior.

## Normal / Gaussian

- Observations *y* real valued
- Mean $\theta$ and variance $\sigma^2$ (or deviation $\sigma$)
  (first assume $\sigma^2$ known)

$$p(y|\theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$
$$y \sim \mathrm{N}(\theta, \sigma^2)$$
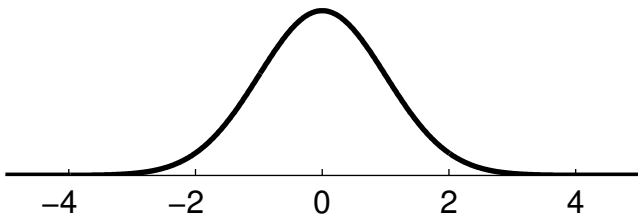
# Reasons to use Normal distribution

- Normal distribution often justified based on central limit theorem
- More often used due to the computational convenience or tradition

# Central limit theorem*

- De Moivre, Laplace, Gauss, Chebysev, Liapounov, Markov, et al.
- Given certain conditions sum (and mean) of random variables approach Gaussian distribution as d $n \to \infty$
- Problems
  - does not hold for all distributions, e.g., Cauchy
  - may require large *n*,
    e.g. Binomial, when $\theta$ close to 0 or 1
  - does not hold if one the variables has much larger scale

## Normal distribution - conjugate prior for $\theta$

- Assume $\sigma^2$ known

  Likelihood
  $$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$$

  Prior
  $$p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

- Assume $\sigma^2$ known

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  $$\exp(a)\exp(b) = \exp(a+b)$$

# Normal distribution - conjugate prior for $\theta$

- Assume $\sigma^2$ known

  Likelihood $\qquad p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right)$

  Prior $\qquad\qquad p(\theta) \propto \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$

  $$\exp(a)\exp(b) = \exp(a+b)$$

  Posterior $\qquad p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$

- Posterior (see ex 2.14a)

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$

$$\propto \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)$$

$$\theta|y \sim \mathsf{N}(\mu_1, \tau_1^2), \quad \text{where} \quad \mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2}$$
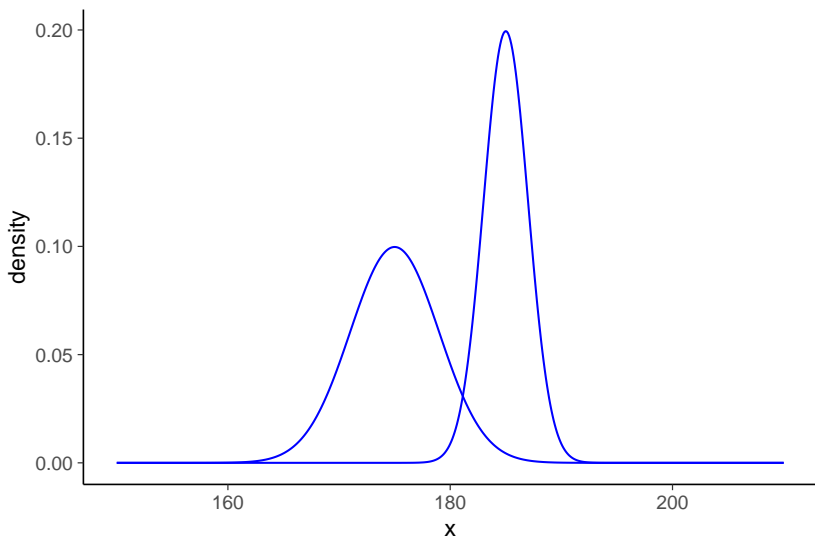
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- Posterior (see ex 2.14a)

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right)$$

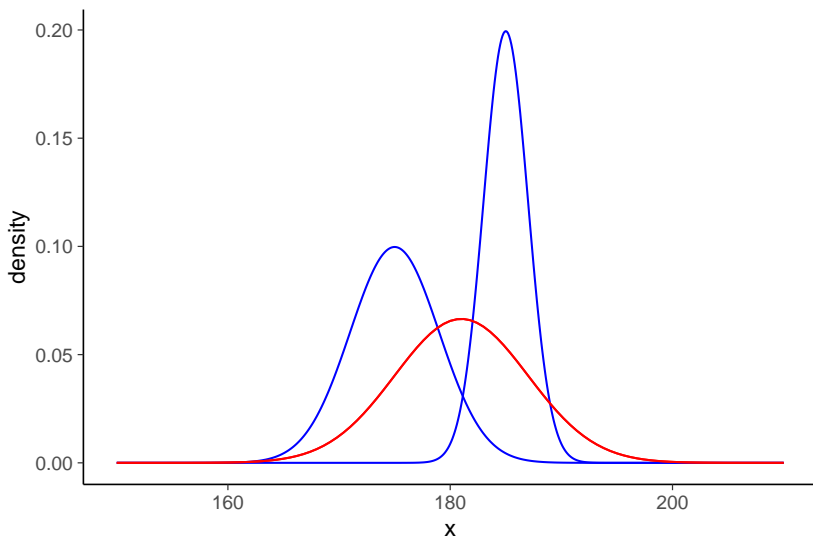$$\propto \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right)$$

$$\theta|y \sim \mathsf{N}(\mu_1, \tau_1^2), \quad \text{where} \quad \mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2}$$

- 1/variance = precision
- Posterior precision = prior precision + data precision
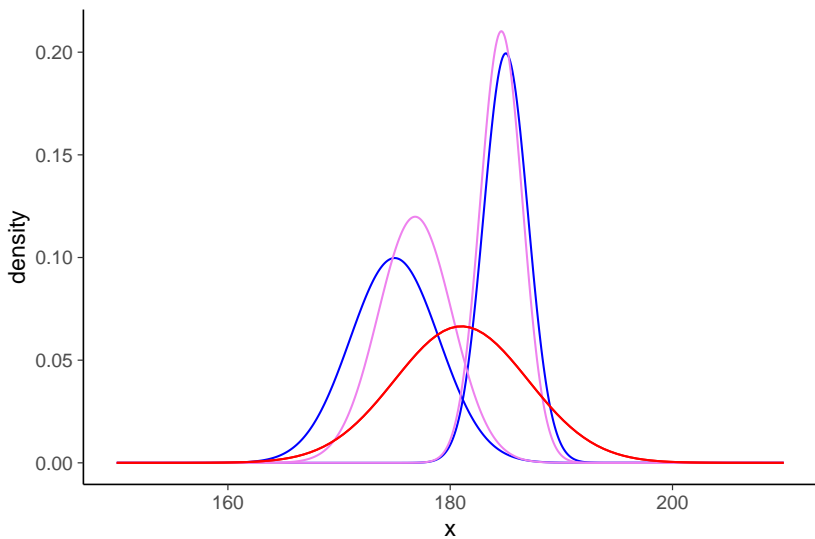- Posterior mean is precision weighted mean

- Several observations – use chain rule

## Normal distribution - conjugate prior for $\theta$

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = N(\theta|\mu_n, \tau_n^2)$$

where $\quad \mu_n = \dfrac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad$ ja $\quad \dfrac{1}{\tau_n^2} = \dfrac{1}{\tau_0^2} + \dfrac{n}{\sigma^2}$

- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value $\mu_0$

# Normal distribution - conjugate prior for $\theta$

- Several observations $y = (y_1, \ldots, y_n)$

$$p(\theta|y) = \mathsf{N}(\theta|\mu_n, \tau_n^2)$$

$$\text{where} \quad \mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{ja} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

- If $\tau_0^2 = \sigma^2$, prior corresponds to one virtual observation with value $\mu_0$
- If $\tau_0 \to \infty$ when $n$ fixed
  or if $n \to \infty$ when $\tau_0$ fixed

$$p(\theta|y) \approx \mathsf{N}(\theta|\bar{y}, \sigma^2/n)$$

# Normal distribution - conjugate prior for $\theta$

- Posterior predictive distribution

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$

$$p(\tilde{y}|y) \propto \int \exp\left(-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right) \exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right) d\theta$$

$$\tilde{y}|y \sim \mathsf{N}(\mu_1, \sigma^2 + \tau_1^2)$$

- Predictive variance = observation model variance $\sigma^2$ + posterior variance $\tau_1^2$