Untitled

Daniel Czarnievicz

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Ejercicio 11.2

Verificar la siguiente igualdad:

$$\mathrm{E}(\hat{\mathrm{V}}_1) = \mathrm{V}(\hat{\theta}^*) - \frac{1}{A(A-1)} \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \frac{1}{A(A-1)} \sum_{a=1}^{A} \big[\mathrm{E}(\hat{\theta}_a) - \mathrm{E}(\hat{\theta}^*) \big]^2$$

Demostración:

Por un lado:

$$\mathrm{E}(\hat{\mathrm{V}}_1) = \mathrm{E}\Bigg(\frac{1}{A(A-1)} \sum_{a=1}^{A} (\hat{\theta}_a - \hat{\theta}^*)^2 \Bigg) =$$

$$= \frac{1}{A(A-1)} \Big[\sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a^2) - A\mathrm{E}(\hat{\theta}^{*2}) \Big] = \frac{1}{A(A-1)} \sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a^2) - \frac{A}{A(A-1)} \mathrm{E}(\hat{\theta}^{*2})$$

Por otro lado:

$$\mathrm{V}(\hat{\theta}^*) - \frac{1}{A(A-1)} \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \frac{1}{A(A-1)} \sum_{a=1}^{A} \big[\mathrm{E}(\hat{\theta}_a) - \mathrm{E}(\hat{\theta}^*) \big]^2 =$$

$$= \frac{A(A-1)}{A(A-1)} \mathrm{V}(\hat{\theta}^*) - \frac{1}{A(A-1)} \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \frac{1}{A(A-1)} \Big[\sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a)^2 - A\mathrm{E}(\hat{\theta}^*)^2 \Big]$$

Por lo tanto, tenemos que mostrar que se cumple la igualdad:

$$\frac{1}{A(A-1)} \sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a^2) - \frac{A}{A(A-1)} \mathrm{E}(\hat{\theta}^{*2}) =$$

$$= \frac{A(A-1)}{A(A-1)} \mathrm{V}(\hat{\theta}^*) - \frac{1}{A(A-1)} \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \frac{1}{A(A-1)} \Big[\sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a)^2 - A\mathrm{E}(\hat{\theta}^*)^2 \Big]$$

Eliminamos el denominador $A(A-1)$

$$\sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a^2) - A\mathrm{E}(\hat{\theta}^{*2}) = A(A-1)\mathrm{V}(\hat{\theta}^*) - \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a)^2 - A\mathrm{E}(\hat{\theta}^*)^2$$

$$\sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a^2) = A(A-1)\mathrm{V}(\hat{\theta}^*) - \sum \sum_{a \neq b}^{A} \mathrm{C}(\hat{\theta}_a, \hat{\theta}_b) + \sum_{a=1}^{A} \mathrm{E}(\hat{\theta}_a)^2 + A \underbrace{\big[\mathrm{E}(\hat{\theta}^{*2}) - \mathrm{E}(\hat{\theta}^*)^2 \big]}_{\mathrm{V}(\hat{\theta}^*)}$$

$$\sum_{a=1}^{A}\mathrm{E}(\hat{\theta}_a^2)-\sum_{a=1}^{A}\mathrm{E}(\hat{\theta}_a)^2=A(A-1)\mathrm{V}(\hat{\theta}^*)-\sum\sum_{a\neq b}^{A}\mathrm{C}(\hat{\theta}_a,\hat{\theta}_b)+A\mathrm{V}(\hat{\theta}^*)$$

$$\sum_{a=1}^{A}\left[\mathrm{E}(\hat{\theta}_a^2)-\mathrm{E}(\hat{\theta}_a)^2\right]=A^2\mathrm{V}(\hat{\theta}^*)-\sum\sum_{a\neq b}^{A}\mathrm{C}(\hat{\theta}_a,\hat{\theta}_b)$$

Dado que:

$$\star\ \sum_{a=1}^{A}\left[\mathrm{E}(\hat{\theta}_a^2)-\mathrm{E}(\hat{\theta}_a)^2\right]=\sum_{a=1}^{A}\mathrm{V}(\hat{\theta}_a)=\mathrm{V}\left(\sum_{a=1}^{A}\hat{\theta}_a\right)-\sum\sum_{a\neq b}^{A}\mathrm{C}(\hat{\theta}_a,\hat{\theta}_b)$$

Llegamos a que:

$$\underbrace{\mathrm{V}\left(\sum_{a=1}^{A}\hat{\theta}_a\right)}_{A^2V(\hat{\theta}^*)}-\cancel{\sum\sum_{a\neq b}^{A}\mathrm{C}(\hat{\theta}_a,\hat{\theta}_b)}=A^2\mathrm{V}(\hat{\theta}^*)-\cancel{\sum\sum_{a\neq b}^{A}\mathrm{C}(\hat{\theta}_a,\hat{\theta}_b)}$$

Con lo cual, queda demostrada la igualdad.

Ejercicio 11.4

Verificar que bajo un diseño arbitrario de tamaño fijo se cumple que:

$$\mathrm{E}(\hat{V}_0)-V=\frac{n}{n-1}(V_0-V),\qquad \text{en donde } \hat{V}_0=\frac{1}{n(n-1)}\sum_s\left(\frac{y_k}{p_k}-\hat{t}_\pi\right)^2$$

Demostración:

$$\mathrm{E}(\hat{V}_0)=\frac{1}{n(n-1)}\mathrm{E}_{p(s)}\left[\sum_s\left(\frac{y_k}{p_k}-\hat{t}_\pi\right)^2\right]=\frac{1}{n(n-1)}\mathrm{E}_{p(s)}\left[\sum_s n^2\left(\frac{y_k}{\pi_k}-\hat{t}_\pi\right)^2\right]=$$

$$=\frac{n}{(n-1)}\mathrm{E}_{p(s)}\left[\sum_s\left(\frac{y_k}{\pi_k}-\hat{t}_\pi\right)^2\right]=\frac{n}{(n-1)}\mathrm{E}_{p(s)}\left[\sum_s\left(\frac{y_k^2}{\pi_k^2}-2\frac{y_k}{\pi_k}\frac{\hat{t}_\pi}{n}+\frac{\hat{t}_\pi^2}{n^2}\right)\right]=$$

$$=\frac{n}{(n-1)}\mathrm{E}_{p(s)}\left[\sum_s\frac{y_k^2}{\pi_k^2}-2\frac{\hat{t}_\pi^2}{n}+\cancel{n}\frac{\hat{t}_\pi^2}{n^{\cancel{2}}}\right]=\frac{n}{(n-1)}\mathrm{E}_{p(s)}\left[\sum_s\frac{y_k^2}{\pi_k^2}-\frac{\hat{t}_\pi^2}{n}\right]=$$

$$=\frac{n}{(n-1)}\left[\mathrm{E}_{p(s)}\left(\sum_s\frac{y_k^2}{\pi_k^2}\right)-\mathrm{E}_{p(s)}\left(\frac{\hat{t}_\pi^2}{n}\right)\right]=\frac{n}{(n-1)}\left[\mathrm{E}_{p(s)}\left(\sum_s\frac{y_k^2}{\pi_k^2}\right)-\frac{1}{n}\left(\mathrm{V}(\hat{t}_\pi)+\underbrace{\mathrm{E}(\hat{t}_\pi)^2}_{t^2}\right)\right]=$$

Por otro lado, tenemos que:

$$\star\ V_0=\frac{1}{n}\sum_U p_k\left(\frac{y_k}{p_k}-t\right)^2=\frac{1}{n}\sum_U\frac{y_k^2}{p_k}-t^2$$

Por lo tanto:

$$\mathrm{E}(\hat{V}_0) - V = \frac{n}{(n-1)}\left[\mathrm{E}_{p(s)}\left(\sum_s \frac{y_k^2}{\pi_k^2}\right) - \frac{1}{n}\left(\mathrm{V}(\hat{t}_\pi) + t^2\right)\right] - \frac{n(n-1)}{n(n-1)}V =$$

$$= \frac{n}{(n-1)}\left[\mathrm{E}_{p(s)}\left(\sum_s \frac{y_k^2}{\pi_k^2}\right) - \cancel{\frac{1}{n}\mathrm{V}(\hat{t}_\pi)} - \frac{t^2}{n} - \frac{n-\cancel{1}}{n}V\right] =$$

$$= \frac{n}{(n-1)}\left[\sum_U \frac{y_k^2}{\pi_k} - \frac{t^2}{n} - V\right] = \frac{n}{(n-1)}\left[\frac{1}{n}\left(\sum_U \frac{y_k^2}{p_k} - t^2\right) - V\right] = \frac{n}{n-1}\left[V_0 - V\right]$$

$$\Rightarrow \boxed{\mathrm{E}(\hat{V}_0) - V = \frac{n}{n-1}(V_0 - V)}$$

Ejercicio 11.6

Mostrar que para el caso de grupos aleatorios dependientes, bajo un diseño SI como el descrito en el ejemplo 11.3.3, la covarianza entre las medias de dos grupos es:

$$C(\bar{y}_{s_a}, \bar{y}_{s_b}) = \mathrm{V}(\bar{y}_s) - \frac{S_{y_U}^2}{n} = -\frac{S_{y_U}^2}{N}$$

$$\mathrm{V}(\bar{y}_s) - \frac{S_{y_U}^2}{n} = \frac{N^2}{N^2 n}(1-f)S_{y_U}^2 - \frac{S_{y_U}^2}{n} = \cancel{\frac{S_{y_U}^2}{n}} - \frac{n}{N}\frac{S_{y_U}^2}{n} - \cancel{\frac{S_{y_U}^2}{n}} = -\frac{S_{y_U}^2}{N}$$

Ejercicio 11.10

Verificar que si el set de medias muestras es balanceado entonces:

$$\hat{V}_{BH} = \hat{V}_0$$

Demostración:

Tenemos que demostrar entonces que:

$$\frac{1}{A}\sum_{a=1}^{A}(\hat{t}_a - \hat{t}_\pi)^2 = \frac{1}{n(n-1)}\sum_s\left(\frac{y_k}{p_k} - \hat{t}_\pi\right)^2$$

Por un lado, tenemos que:

$$\hat{V}_{BH} = \frac{1}{A}\sum_{a=1}^{A}(\hat{t}_a - \hat{t}_\pi)^2 = \frac{1}{A}\sum_{a=1}^{A}(\sum_{h=1}^{H}\left(2\delta_{ah}\frac{y_{h1}}{\pi_{h1}} + 2(1-\delta_{ah})\frac{y_{h2}}{\pi_{h2}} - \sum_{s_h}\frac{y_k}{\pi_k}\right)$$

Observamos que en cada s_h hay solamente 2 elementos:

$$\frac{1}{A}\sum_{a=1}^{A}(\sum_{h=1}^{H}\left(2\delta_{ah}\frac{y_{h1}}{\pi_{h1}} + 2(1-\delta_{ah})\frac{y_{h2}}{\pi_{h2}} - \frac{y_{h1}}{\pi_{h1}} - \frac{y_{h2}}{\pi_{h2}}\right) =$$

$$= \frac{1}{A}\sum_{a=1}^{A}(\sum_{h=1}^{H}\left((2\delta_{ah}-1)\frac{y_{h1}}{\pi_{h1}}+(1-2\delta_{ah})\frac{y_{h2}}{\pi_{h2}}\right)=$$

$$= \frac{1}{A}\sum_{a=1}^{A}\sum_{h=1}^{H}(2\delta_{ah}-1)^2\left(\frac{y_{h1}}{\pi_{h1}}-\frac{y_{h2}}{\pi_{h2}}\right)^2=$$

Desarrollando el primer cuadrado y teniendo en cuenta que los δ_{ah} valen 0 ó 1:

$$= \frac{1}{A}\sum_{a=1}^{A}\sum_{h=1}^{H}(\frac{A}{2}-\frac{A}{2}+A)\left(\frac{y_{h1}}{\pi_{h1}}-\frac{y_{h2}}{\pi_{h2}}\right)^2=\sum_{a=1}^{A}\sum_{h=1}^{H}\left(\frac{y_{h1}}{\pi_{h1}}-\frac{y_{h2}}{\pi_{h2}}\right)^2$$

$$=\sum_{a=1}^{A}\sum_{h=1}^{H}\left(\frac{y_{h1}}{\pi_{h1}}-\hat{t}_{\pi h}+\frac{y_{h1}}{\pi_{h1}}\right)^2=\sum_{a=1}^{A}\sum_{h=1}^{H}\left(2\frac{y_{h1}}{\pi_{h1}}-\hat{t}_{\pi h}\right)^2$$

Y operando y utilizando que : $\sum(w-z)^2=\frac{1}{2}\sum(w-z)^2+(z-w)^2$

$$=\frac{1}{2}\sum_{h=1}^{H}\underbrace{\left(\frac{y_{h1}}{\pi_{h1}}-\frac{y_{h2}}{\pi_{h2}}\right)^2}_{\left(2\frac{y_{h1}}{\pi_{h1}}-\hat{t}_{\pi h}\right)^2}+\underbrace{\left(\frac{y_{h2}}{\pi_{h2}}-\frac{y_{h1}}{\pi_{h1}}\right)^2}_{\left(2\frac{y_{h2}}{\pi_{h2}}-\hat{t}_{\pi h}\right)^2}=\frac{1}{2}\sum_{h=1}^{H}\sum_{s_h}\left(\frac{y_k}{p_k}-\hat{t_{\pi h}}\right)^2=\hat{V}_0$$

$$\Rightarrow \boxed{\hat{V}_{BH}=\hat{V}_0}$$

Ejercicio 11.11

Verificar que si el set de muestras es full-ortogonal balance:

$$\hat{t}_{BH}=\hat{t}_\pi$$

$$\hat{t}_{BH}=\frac{1}{A}\sum_{a=1}^{A}\hat{t}_a=\frac{1}{A}\sum_{a=1}^{A}\sum_{h=1}^{H}\left[\delta_{ah}\frac{y_{h1}}{p_{h1}}+(1-\delta_{ah})\frac{y_{h2}}{p_{h2}}\right]=\frac{1}{A}\sum_{h=1}^{H}\sum_{a=1}^{A}\delta_{ah}\frac{y_{h1}}{p_{h1}}+\frac{1}{A}\sum_{h=1}^{H}\sum_{a=1}^{A}(1-\delta_{ah})\frac{y_{h2}}{p_{h2}}=$$

$$\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{p_{h1}}\sum_{a=1}^{A}\delta_{ah}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{p_{h2}}\sum_{a=1}^{A}(1-\delta_{ah})=\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{p_{h1}}\sum_{a=1}^{A}\frac{1+\varepsilon_{ah}}{2}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{p_{h2}}\sum_{a=1}^{A}\frac{1-\varepsilon_{ah}}{2}=$$

$$\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{p_{h1}}\sum_{a=1}^{A}\frac{1+\varepsilon_{ah}}{2}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{p_{h2}}\sum_{a=1}^{A}\frac{1-\varepsilon_{ah}}{2}=$$

$$\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{p_{h1}}\sum_{a=1}^{A}\frac{1}{2}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{2p_{h1}}\underbrace{\sum_{a=1}^{A}\varepsilon_{ah}}_{=0}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{p_{h2}}\sum_{a=1}^{A}\frac{1}{2}-\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{2p_{h2}}\underbrace{\sum_{a=1}^{A}\varepsilon_{ah}}_{=0}=$$

$$\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h1}}{p_{h1}}\frac{A}{2}+\frac{1}{A}\sum_{h=1}^{H}\frac{y_{h2}}{p_{h2}}\frac{A}{2}=\frac{2}{2}\sum_{h=1}^{H}\frac{y_{h1}}{\pi_{h1}}+\frac{2}{2}\sum_{h=1}^{H}\frac{y_{h2}}{\pi_{h2}}=\sum_{h=1}^{H}\frac{y_{h1}}{\pi_{h1}}+\sum_{h=1}^{H}\frac{y_{h2}}{\pi_{h2}}=\sum_{h=1}^{H}\sum_{s_h}\check{y}_k=\hat{t}_\pi$$

$$\boxed{\hat{t}_{BH}=\frac{1}{A}\sum_{a=1}^{A}\hat{t}_a=\hat{t}_\pi}$$

Ejercicio 11.17

Probar que bajo los supuestos del ejemplo 11.5.1 se cumple que:

Parte a

$$\mathrm{E}[\hat{\theta}_{(a)}|S]=\sum_{s}\frac{y_k}{\pi_k}\text{ y que por lo tanto }\mathrm{E}[\hat{\theta}_{(a)}]=t$$

$$\mathrm{E}[\hat{\theta}_{(a)}|S]=\mathrm{E}\left(\frac{A}{A-1}\sum_{S-S_a}\frac{y_k}{\pi_k}\,|\,S\right)=\frac{A}{A-1}\mathrm{E}\left(\sum_{S}\frac{y_k}{\pi_k}\,|\,S\right)-\frac{A}{A-1}\mathrm{E}\left(\sum_{S_a}\frac{y_k}{\pi_k}\,|\,S\right)=$$

$$=\frac{A}{A-1}\sum_{S}\frac{y_k}{\pi_k}-\frac{1}{A-1}\mathrm{E}\left(\sum_{S_a}\frac{y_k}{\frac{\pi_k}{A}}\,|\,S\right)=\frac{A}{A-1}\sum_{S}\frac{y_k}{\pi_k}-\frac{1}{A-1}\sum_{S}\frac{y_k}{\pi_k}=\frac{A-1}{A-1}\sum_{S}\frac{y_k}{\pi_k}=\hat{t}_\pi$$

$$\Rightarrow\boxed{\mathrm{E}[\hat{\theta}_{(a)}|S]=\hat{t}_\pi}$$

Por otra parte:

$$\mathrm{E}[\hat{\theta}_{(a)}]=\mathrm{E}\left(\frac{A}{A-1}\sum_{S-S_a}\frac{y_k}{\pi_k}\right)=\frac{A}{A-1}\mathrm{E}\left(\sum_{S}\frac{y_k}{\pi_k}\right)-\frac{A}{A-1}\mathrm{E}\left(\sum_{S_a}\frac{y_k}{\pi_k}\right)=$$

$$=\frac{A}{A-1}\mathrm{E}\left(\sum_{S}\frac{y_k}{\pi_k}\right)-\frac{1}{A-1}\mathrm{E}\left(A\sum_{S_a}\frac{y_k}{\pi_k}\right)$$

π- expandiendo tenemos que:

$$=\frac{A}{A-1}\sum_{U}y_k-\frac{1}{A-1}\sum_{U}\mathrm{E}(I_k)\frac{y_k}{\pi_k}=\frac{A-1}{A-1}\sum_{U}y_k=\sum_{U}y_k=t$$

$$\Rightarrow\boxed{\mathrm{E}[\hat{\theta}_{(a)}]=t}$$

Parte b

Demostrar que $\hat{\theta}_a = A\sum_{S_a}\frac{y_k}{\pi_k}$

Del ejemplo, tenemos que:

$$\color{red}{\star}\ \hat{\theta} = \sum_{S}\frac{y_k}{\pi_k} \qquad \color{red}{\star}\ \hat{\theta}_{(a)} = \frac{A}{A-1}\sum_{S-S_a}\frac{y_k}{\pi_k}$$

Entonces:

$$\hat{\theta}_a = A\hat{\theta} - (A-1)\hat{\theta}_{(a)} = A\sum_{S}\frac{y_k}{\pi_k} - (A-1)\frac{A}{A-1}\sum_{S-S_a}\frac{y_k}{\pi_k}$$

$$\cancel{A\sum_{S}\frac{y_k}{\pi_k}} - \cancel{A\sum_{S}\frac{y_k}{\pi_k}} + A\sum_{S_a}\frac{y_k}{\pi_k} = A\sum_{S_a}\frac{y_k}{\pi_k}$$

$$\Rightarrow \boxed{\hat{\theta}_a = A\sum_{S_a}\frac{y_k}{\pi_k}}$$

Parte c

Probar que $\hat{\theta}_{JK} = \sum_{S}\frac{y_k}{\pi_k}$

$$\hat{\theta}_{JK} = \frac{1}{A}\sum_{a=1}^{A} A\sum_{S_a}\frac{y_k}{\pi_k} = \sum_{a=1}^{A}\sum_{S_a}\frac{y_k}{\pi_k} = \sum_{S}\frac{y_k}{\pi_K} = \hat{t}_\pi$$

$$\Rightarrow \boxed{\hat{\theta}_{JK} = \hat{t}_\pi}$$

Parte d

Si $A = n$, es decir $m = 1$, demostrar que entonces $\hat{V}_{JK} = \hat{V}_0$

$$\hat{V}_{JK} = \frac{1}{A(A-1)}\sum_{a=1}^{A}(\hat{\theta}_a - \hat{\theta}_{JK})^2 = \frac{1}{A(A-1)}\sum_{a=1}^{A}(A\sum_{S_a}\frac{y_k}{\pi_k} - \hat{t}_\pi)^2 =$$

Usando que $A = n$

$$= \frac{1}{n(n-1)}\sum_{a=1}^{n}(n\sum_{S_a}\frac{y_k}{\pi_k} - \hat{t}_\pi)^2 = \frac{1}{n(n-1)}\sum_{a=1}^{n}(\sum_{S_a}\frac{y_k}{\frac{\pi_k}{n}} - \hat{t}_\pi)^2 = \frac{1}{n(n-1)}\sum_{a=1}^{n}(\sum_{S_a}\frac{y_k}{p_k} - \hat{t}_\pi)^2 =$$

Y dado que $m = 1$ la suma en S_a implica sumar un solo término, con lo cual:

$$= \frac{1}{n(n-1)}\sum_{a=1}^{n}(\frac{y_k}{p_k} - \hat{t}_\pi)^2 = \frac{1}{n(n-1)}\sum_{S}(\frac{y_k}{p_k} - \hat{t}_\pi)^2 = \hat{V}_0 \Rightarrow \boxed{\hat{V}_{JK} = \hat{V}_0}$$