Supplements for "A Variational Bayesian Framework for System Degradation State Identification amid Measurement Noise"

This document presents supplementary proof materials specifically for Algorithms 1 and 2 as outlined in the main paper, where the key formulas are presented as follows:

- Variational posterior update formula:

$$q^*(\phi) \propto \exp\left(\mathrm{E}_{q^*(\boldsymbol{\Theta}_{-\phi})}\left[\ln p(\boldsymbol{\Theta}|\boldsymbol{y})\right]\right), \tag{A1}$$

  where $p(\boldsymbol{\Theta}|\boldsymbol{y})$ represents posterior distribution, $\boldsymbol{\Theta}$ is the set of posterior parameters, $\phi \in \boldsymbol{\Theta}$ is an arbitrary element of the set $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}_{-\phi}$ is a subset of $\boldsymbol{\Theta}$ excluding ϕ. $\mathrm{E}_{q^*(\boldsymbol{\Theta}_{-\phi})}[\cdot]$ refers to the expectation with respect to $q^*(\boldsymbol{\Theta}_{-\phi})$.

- Offline posterior distribution:

$$\begin{aligned} p(\boldsymbol{X}, \boldsymbol{\theta} \mid \boldsymbol{y}) \propto \prod_{i=1}^{n} &\mathcal{N}\left(y_i; X_i, \lambda_2^{-1}\right) \mathcal{N}\left(\Delta X_i; \mu\tau_i, \tau_i\lambda_1^{-1}\right) \mathcal{N}\left(\mu; \mu_0, (\kappa_0\lambda_1)^{-1}\right) \times \\ &\mathcal{G}(\lambda_1; \alpha_1, \beta_1)\mathcal{G}(\lambda_2; \alpha_2, \beta_2)\mathcal{N}\left(\gamma; \mu_\gamma, \sigma_\gamma^2\right), \end{aligned} \tag{A2}$$

  where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)'$, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$, $\boldsymbol{\theta} = (\mu, \lambda_1, \lambda_2, \gamma)'$, $\Delta X_i = X_i - X_{i-1}$ for $i \geq 2$, with $\Delta X_1 = X_1$, and $\tau_i = t_i^\gamma - t_{i-1}^\gamma$, with $\tau_1 = t_1^\gamma$.

- Online posterior distribution:

$$\begin{aligned} p(X_n, X_{n+1}, \boldsymbol{\theta} \mid \boldsymbol{y}_{1:(n+1)}) \propto &\mathcal{N}\left(y_{n+1}; X_{n+1}, \lambda_2^{-1}\right) \mathcal{N}\left(X_{n+1}; X_n + \mu\tau_{n+1}, \tau_{n+1}\lambda_1^{-1}\right) \times \\ &\mathcal{N}\left(X_n; \mu_{n|n}, \sigma_{n|n}^2\right) \mathcal{N}\left(\mu; \mu_{0,n}, (\kappa_{0,n}\lambda_1)^{-1}\right) \times \\ &\mathcal{G}(\lambda_1; \alpha_{1,n}, \beta_{1,n})\mathcal{G}(\lambda_2; \alpha_{2,n}, \beta_{2,n})\mathcal{N}\left(\gamma; \mu_{\gamma,n}, \sigma_{\gamma,n}^2\right). \end{aligned} \tag{A3}$$

1 Proof of Algorithm 1

First, the logarithm of the offline posterior distribution (A2) is expanded as

$$\begin{aligned} \ln p(\boldsymbol{X}, \boldsymbol{\theta}|\boldsymbol{y}) \propto &\frac{n}{2}\ln\lambda_2 - \sum_{i=1}^{n}\frac{\lambda_2(y_i - X_i)^2}{2} + \frac{n}{2}\ln\lambda_1 - \sum_{i=1}^{n}\frac{\ln\tau_i}{2} - \sum_{i=1}^{n}\frac{\lambda_1(\Delta X_i - \mu\tau_i)^2}{2\tau_i} \\ &+ \frac{1}{2}\ln\lambda_1 - \frac{\lambda_1\kappa_0(\mu^2 - 2\mu\mu_0)}{2} + (\alpha_1 - 1)\ln\lambda_1 - \beta_1\lambda_1 \\ &+ (\alpha_2 - 1)\ln\lambda_2 - \beta_2\lambda_2 - \frac{\gamma^2 - 2\gamma\mu_\gamma}{2\sigma_\gamma^2}. \end{aligned} \tag{A4}$$

- $q^*(\boldsymbol{X})$

  By substituting (A4) into (A1), we derive the following expression:

$$\begin{aligned} \ln q^*(\boldsymbol{X}) \propto &-\sum_{i=1}^{n}\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\frac{X_i^2 - 2y_iX_i}{2} \\ &- \sum_{i=1}^{n}\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(\gamma)}[\tau_i^{-1}]\frac{\Delta X_i^2}{2} - \sum_{i=1}^{n}\mathrm{E}_{q^*(\mu,\lambda_1)}[\mu\lambda_1]\Delta X_i. \end{aligned}$$

The linear and quadratic terms in X_i indicate that $q^*(\boldsymbol{X})$ is the PDF of a multivariate normal distribution. Given the complexity of directly formulating this distribution, we decompose it into the product of two multivariate normal distributions, as follows:

$$\ln q^*(\boldsymbol{X}) = \ln q_1^*(\boldsymbol{X}) + \ln q_2^*(\boldsymbol{X}).$$

The first part is given by

$$\begin{aligned}\ln q_1^*(\boldsymbol{X}) &\propto -\sum_{i=1}^{n}\left(\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\frac{X_i^2 - 2y_iX_i}{2} + \mathrm{E}_{q^*(\mu,\lambda_1)}[\mu\lambda_1]\Delta X_i\right)\\ &\propto \mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\sum_{i=1}^{n}\frac{X_i^2}{2} + \mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\sum_{i=1}^{n} y_iX_i + \mathrm{E}_{q^*(\mu,\lambda_1)}[\mu\lambda_1]X_n,\end{aligned}$$

where the mean vector μ_1^* and covariance matrix $\boldsymbol{\Sigma}_1^*$ are

$$\begin{aligned}\mu_1^* &= \left(y_1, y_2, \ldots, y_{n-1}, y_n + \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\mu\lambda_1]}{\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]}\right)',\\ \boldsymbol{\Sigma}_1^* &= (\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\mathrm{Diag}(\boldsymbol{\tau}))^{-1}.\end{aligned}$$

Here, $\mathrm{Diag}(\cdot)$ represents the diagonal matrix.

The second multivariate normal PDF is expressed as

$$\ln q_2^*(\boldsymbol{X}) \propto \mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\sum_{i=1}^{n}\mathrm{E}_{q^*(\gamma)}[\tau_i^{-1}]\frac{\Delta X_i^2}{2}.$$

In this case, the quadratic form only exists in X_i, and hence the mean is $\mathbf{0}$. The covariance matrix $\boldsymbol{\Sigma}_2^*$ can be readily computed as

$$\boldsymbol{\Sigma}_2^* = \frac{1}{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]}\begin{bmatrix} C_1 + C_2 & -C_2 & \cdots & 0\\ -C_2 & C_2 + C_3 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & C_n\end{bmatrix}^{-1}, \text{ where } C_i = \mathrm{E}_{q^*(\gamma)}[\tau_i^{-1}].$$

By applying the multiplication rule for multivariate normal distributions, the mean vector and covariance matrix of the joint distribution are

$$\boldsymbol{\mu}^* = \boldsymbol{\mu}_1^*\left(\boldsymbol{\Sigma}_1^*\right)^{-1}\boldsymbol{\Sigma}^* \text{ and } \boldsymbol{\Sigma}^* = \left[(\boldsymbol{\Sigma}_1^*)^{-1} + (\boldsymbol{\Sigma}_2^*)^{-1}\right]^{-1}.$$

- $q^*(\mu, \lambda_1)$
  Similarly, we have

$$\begin{aligned}\ln q^*(\mu,\lambda_1) \propto \ln\lambda_1\left(\frac{n-1}{2} + \alpha_1\right) &+ \lambda_1\mu\left(\kappa_0\mu_0 + \sum_{i=1}^{n}\mathrm{E}_{q^*(\boldsymbol{X})}\Delta[X_i]\right)\\ &- \lambda_1\left(\beta_1 + \sum_{i=1}^{n}\frac{\mathrm{E}_{q^*(\boldsymbol{X})}[\Delta X_i^2]}{2}\mathrm{E}_{q^*(\gamma)}[\tau_i^{-1}] + \frac{\kappa_0\mu_0^2}{2}\right)\\ &- \lambda_1\mu^2\frac{\kappa_0 + \sum_{i=1}^{n}\mathrm{E}_{q^*(\gamma)}[\tau_i]}{2}.\end{aligned}$$

Consequently, we obtain

$$q^*(\mu,\lambda_1) \propto \lambda_1^{\frac{1}{2}} \exp\left\{ -\frac{\mu^2 - 2\mu\left(\frac{\kappa_0\mu_0 + \sum_{i=1}^n \mathrm{E}_{q^*(\boldsymbol{X})}\Delta[X_i]}{\kappa_0 + \sum_{i=1}^n \mathrm{E}_{q^*(\gamma)}[\tau_i]}\right)}{2\left[\lambda_1\left(\kappa_0 + \sum_{i=1}^n \mathrm{E}_{q^*(\gamma)}[\tau_i]\right)\right]^{-1}} \right\} \times$$
$$\lambda_1^{\frac{n}{2}+\alpha_1-1} \exp\left[-\lambda_1\left(\beta_1 + \sum_{i=1}^n \frac{\mathrm{E}_{q^*(\boldsymbol{X})}[\Delta X_i^2]}{2} \mathrm{E}_{q^*(\gamma)}[\tau_i^{-1}] + \frac{\kappa_0\mu_0^2}{2}\right)\right].$$

Therefore, $q^*(\mu,\lambda_1)$ is the PDF of a normal-inverse-gamma distribution.

- $q^*(\lambda_2)$
  Next, we have

$$\ln q^*(\lambda_2) \propto \frac{n}{2}\ln\lambda_2 - \lambda_2 \sum_{i=1}^n \frac{(y_i^2 - 2\mathrm{E}_{q^*(\boldsymbol{X})}[X_i]y_i + \mathrm{E}_{q^*(\boldsymbol{X})}[X_i^2])}{2} + (\alpha_2 - 1)\ln\lambda_2 - \beta_2\lambda_2.$$

Thus, we obtain

$$q^*(\lambda_2) \propto \lambda_2^{\frac{n}{2}+\alpha_2-1} \exp\left\{ -\lambda_2\left[\sum_{i=1}^n \frac{(y_i^2 - 2\mathrm{E}_{q^*(\boldsymbol{X})}[X_i]y_i + \mathrm{E}_{q^*(\boldsymbol{X})}[X_i^2])}{2} + \beta_2\right]\right\}.$$

Hence, $q^*(\lambda_2)$ is the PDF of a gamma distribution.

- $q^*(\gamma)$
  Lastly, we have

$$\ln q^*(\gamma) \propto -\frac{\gamma^2 - 2\mu_\gamma\gamma}{2\sigma_\gamma^2} - \sum_{i=1}^n \frac{\ln\tau_i}{2} - \sum_{i=1}^n \frac{\mathrm{E}_{q^*(\mu.\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(\boldsymbol{X})}[\Delta X_i^2]}{2\tau_i} - \frac{\mathrm{E}_{q^*(\mu.\lambda_1)}[\lambda_1\mu^2]\tau_n}{2}.$$

We define the function $f(\gamma)$ as

$$f(\gamma) = -\frac{\gamma^2 - 2\mu_\gamma\gamma}{2\sigma_\gamma^2} - \sum_{i=1}^n \frac{\ln\tau_i}{2} - \sum_{i=1}^n \frac{\mathrm{E}_{q^*(\mu.\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(\boldsymbol{X})}[\Delta X_i^2]}{2\tau_i} - \frac{\mathrm{E}_{q^*(\mu.\lambda_1)}[\lambda_1\mu^2]\tau_n}{2}.$$

Using the Laplace approximation, we approximate $q^*(\gamma)$ as

$$q^*(\gamma) \approx \mathcal{N}(\gamma; \mu_\gamma^*, \sigma_\gamma^{2^*}),$$

where $\mu_\gamma^* = \mathrm{argmax}\ f(\gamma)$ and $\sigma_\gamma^{2^*} = -\left.\frac{d^2 f(\gamma)}{d\gamma^2}\right|_{\gamma=\mu_\gamma^*}$.

2 Proof of Algorithm 2

Firstly, the logarithm of the online posterior distribution (A3) is expanded as

$$\begin{aligned}\ln p(X_n, X_{n+1}, \boldsymbol{\theta}|\boldsymbol{y}_{1:(n+1)}) \propto & \frac{\ln\lambda_2}{2} - \frac{\lambda_2(y_{n+1} - X_{n+1})^2}{2} + \frac{\ln\lambda_1}{2} - \frac{\ln\tau_{n+1}}{2} - \frac{\lambda_1(\Delta X_{n+1} - \mu\tau_{n+1})^2}{2\tau_{n+1}} \\ & - \frac{X_n^2 - 2X_n\mu_{n|n}}{2\sigma_{n|n}^2} + \frac{1}{2}\ln\lambda_1 - \frac{\lambda_1\kappa_{0,n}(\mu^2 - 2\mu\mu_{0,n})}{2} + (\alpha_{1,n} - 1)\ln\lambda_1 \\ & - \beta_{1,n}\lambda_1 + (\alpha_{2,n} - 1)\ln\lambda_2 - \beta_{2,n}\lambda_2 - \frac{\gamma^2 - 2\gamma\mu_{\gamma,n}}{2\sigma_{\gamma,n}^2}.\end{aligned} \tag{A5}$$

- $q^*(X_{n+1}, X_n)$
  By substituting (A5) into (A1), we have the following expression:

$$\ln q^*(X_n, X_{n+1}) \propto -\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\frac{2y_{n+1}X_{n+1} - X_{n+1}^2}{2} - \frac{\left(\Delta X_{n+1} - \Delta\mu_{n+1}^*\right)^2}{2\Delta\sigma_{n+1}^{2^*}} - \frac{X_n^2 - 2X_n\mu_{n|n}}{2\sigma_{n|n}^2},$$

where $\Delta\mu_{n+1}^* = \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\mu\lambda_1]}{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}]}$ and $\Delta\sigma_{n+1}^{2^*} = \frac{1}{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}]}$.
This expression can be rewritten as

$$\begin{aligned} q^*(X_n, X_{n+1}) \propto &\mathcal{N}\left(y_{n+1}; X_{n+1}, \left(\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\right)^{-1}\right) \\ &\times \mathcal{N}\left(X_{n+1}; X_n + \Delta\mu_{n+1}^*, \Delta\sigma_{n+1}^{2^*}\right)\mathcal{N}\left(X_n; \mu_{n|n}, \sigma_{n|n}^2\right). \end{aligned} \tag{A6}$$

For clarity, we further define the following:

$$\begin{aligned} p(X_n|\boldsymbol{y}_{1:n}) &= \mathcal{N}\left(X_n; \mu_{n|n}, \sigma_{n|n}^2\right), \\ p(X_{n+1}|X_n) &= \mathcal{N}\left(X_{n+1}; X_n + \Delta\mu_{n+1}^*, \Delta\sigma_{n+1}^{2^*}\right), \\ p(y_{n+1}|X_{n+1}) &= \mathcal{N}\left(y_{n+1}; X_{n+1}, \left(\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\right)^{-1}\right). \end{aligned}$$

To obtain $q^*(X_{n+1})$, we integrate over X_n in (A6):

$$\begin{aligned} q^*(X_{n+1}) &\propto \int p(y_{n+1}|X_{n+1})p(X_{n+1}|X_n)p(X_n|\boldsymbol{y}_{1:n})\,dX_n \\ &\propto p(y_{n+1}|X_{n+1})p(X_{n+1}|\boldsymbol{y}_{1:n}). \end{aligned}$$

This expression represents the filtering distribution of X_{n+1}. Then, we further derive the following closed recursive formulas [1]:

$$\begin{aligned} \mu_{n+1|n}^* &= \mu_{n|n} + \Delta\mu_{n+1}^*, \\ \sigma_{n+1|n}^{2^*} &= \sigma_{n|n}^2 + \Delta\sigma_{n+1}^{2^*}, \\ S^* &= \sigma_{n+1|n}^{2^*} + \left(\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\right)^{-1}, \\ K^* &= \sigma_{n+1|n}^{2^*}\left(S^*\right)^{-1}, \\ \mu_{n+1|n+1}^* &= \mu_{n+1|n}^* + K^*\left(y_{n+1} - \mu_{n+1|n}^*\right), \\ \sigma_{n+1|n+1}^{2^*} &= \sigma_{n+1|n}^{2^*} - \left(K^*\right)^2 S^*. \end{aligned}$$

Similarly, to obtain $q^*(X_n)$, we integrate over X_{n+1} in (A6):

$$\begin{aligned} q^*(X_n) &\propto \int p(y_{n+1}|X_{n+1})p(X_{n+1}|X_n)p(X_n|\boldsymbol{y}_{1:n})\,dX_{n+1} \\ &\propto p(X_n|\boldsymbol{y}_{1:n})\int p(y_{n+1}|X_{n+1})p(X_{n+1}|\boldsymbol{y}_{1:n})\,dX_{n+1}. \end{aligned}$$

This expression represents the smoothing distribution of X_n given $\boldsymbol{y}_{1:(n+1)}$ and the filtered distribution $q^*(X_{n+1})$. Again, we derive the recursive formulas for the mean and variance of the smoothed distribution:

$$\begin{aligned} G^* &= \sigma_{n|n}^2\left(\sigma_{n+1|n}^{2^*}\right)^{-1}, \\ \mu_{n|n+1}^* &= \mu_{n|n} + G^*\left(\mu_{n+1|n+1}^* - \mu_{n+1|n}^*\right), \\ \sigma_{n|n+1}^{2^*} &= \sigma_{n|n}^2 + \left(G^*\right)^2\left[\sigma_{n+1|n+1}^{2^*} - \sigma_{n+1|n}^{2^*}\right]. \end{aligned}$$

Furthermore, since the covariance between X_n and X_{n+1} is required during the variational iteration, the quadratic term in (A6) yields the following expression for the inverse covariance matrix:

$$\left(\mathbf{\Sigma}^*_{n:n+1}\right)^{-1} = \begin{pmatrix} \mathrm{E}_{q^*(\lambda_2)}[\lambda_2] + \left(\Delta\sigma^{2^*}_{n+1}\right)^{-1} & -\left(\Delta\sigma^{2^*}_{n+1}\right)^{-1} \\ -\left(\Delta\sigma^{2^*}_{n+1}\right)^{-1} & \sigma^2_{n|n} + \left(\Delta\sigma^{2^*}_{n+1}\right)^{-1} \end{pmatrix}.$$

Inverting this covariance matrix, the covariance between X_n and X_{n+1} is given by

$$\mathrm{Cov}^*(X_n, X_{n+1}) = \sigma^2_{n|n}\left(S^*\mathrm{E}_{q^*(\lambda_2)}[\lambda_2]\right)^{-1}.$$

- $q^*(\mu, \lambda_1)$
  Similarly, we have

$$\begin{aligned}\ln q^*(\mu, \lambda_1) \propto &\left(\frac{1}{2} + \alpha_{1,n}\right)\ln\lambda_1 + \lambda_1\mu\left(\kappa_{0,n}\mu_{0,n} + \mathrm{E}_{q^*(X_n,X_{n+1})}[X_{n+1}]\right) \\ &- \lambda_1\left(\beta_{1,n} + \frac{\mathrm{E}_{q^*(X_n,X_{n+1})}[\Delta X^2_{n+1}]}{2}\mathrm{E}_{q^*(\gamma)}[\tau^{-1}_{n+1}] + \frac{\kappa_{0,n}\mu^2_{0,n}}{2}\right) \\ &- \lambda_1\mu^2\frac{\kappa_{0,n} + \mathrm{E}_{q^*(\gamma)}[\tau_{n+1}]}{2}.\end{aligned}$$

  From this, the distribution $q^*(\mu, \lambda_1)$ can be expressed as:

$$\begin{aligned}q^*(\mu, \lambda_1) \propto &\lambda_1^{\frac{1}{2}}\exp\left\{-\frac{\mu^2 - 2\mu\left(\frac{\kappa_{0,n}\mu_{0,n} + \mathrm{E}_{q^*(X_n,X_{n+1})}[X_{n+1}]}{\kappa_{0,n} + \mathrm{E}_{q^*(\gamma)}[\tau_{n+1}]}\right)}{2\left[\lambda_1\left(\kappa_{0,n} + \mathrm{E}_{q^*(\gamma)}[\tau_{n+1}]\right)\right]^{-1}}\right\} \times \\ &\lambda_1^{\alpha_{1,n}-\frac{1}{2}}\exp\left[-\lambda_1\left(\beta_{1,n} + \frac{\mathrm{E}_{q^*(X_n,X_{n+1})}[\Delta X^2_{n+1}]}{2}\mathrm{E}_{q^*(\gamma)}[\tau^{-1}_{n+1}] + \frac{\kappa_{0,n}\mu^2_{0,n}}{2}\right)\right].\end{aligned}$$

  Therefore, $q^*(\mu, \lambda_1)$ is the PDF of a normal-inverse-gamma distribution.

- $q^*(\lambda_2)$
  Next, we have

$$\begin{aligned}\ln q^*(\lambda_2) \propto &\frac{1}{2}\ln\lambda_2 - \lambda_2\frac{\left(y^2_{n+1} - 2y_{n+1}\mathrm{E}_{q^*(X_n,X_{n+1})}[X_{n+1}] + \mathrm{E}_{q^*(X_n,X_{n+1})}[X^2_{n+1}]\right)}{2} \\ &+ (\alpha_{2,n} - 1)\ln\lambda_2 - \beta_{2,n}\lambda_2.\end{aligned}$$

  Then, the variational distribution $q^*(\lambda_2)$ is given by

$$q^*(\lambda_2) \propto \lambda_2^{\alpha_{2,n}-\frac{1}{2}}\exp\left\{-\lambda_2\left[\frac{\left(y^2_{n+1} - 2\mathrm{E}_{q^*(X_n,X_{n+1})}[X_{n+1}]y_{n+1} + \mathrm{E}_{q^*(X_n,X_{n+1})}[X^2_{n+1}]\right)}{2} + \beta_{2,n}\right]\right\}.$$

  Hence, $q^*(\lambda_2)$ is the PDF of a gamma distribution.

- $q^*(\gamma)$
  Lastly, we have

$$\ln q^*(\gamma) \propto -\frac{\gamma^2 - 2\mu_{\gamma,n}\gamma}{2\sigma^2_{\gamma,n}} - \frac{\ln\tau_{n+1}}{2} - \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(X_n,X_{n+1})}[\Delta X^2_{n+1}]}{2\tau_{n+1}} - \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1\mu^2]\tau_{n+1}}{2}.$$

Define $f(\gamma)$ as

$$f(\gamma) = -\frac{\gamma^2 - 2\mu_{\gamma,n}\gamma}{2\sigma_{\gamma,n}^2} - \frac{\ln \tau_{n+1}}{2} - \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathrm{E}_{q^*(X_n,X_{n+1})}[\Delta X_{n+1}^2]}{2\tau_{n+1}} - \frac{\mathrm{E}_{q^*(\mu,\lambda_1)}[\lambda_1\mu^2]\tau_{n+1}}{2}.$$

Using the Laplace approximation, we approximate $q^*(\gamma)$ as

$$q^*(\gamma) \approx \mathcal{N}(\gamma; \mu_{\gamma,n+1}^*, \sigma_{\gamma,n+1}^{2^*}),$$

where $\mu_{\gamma,n+1}^* = \text{argmax } f(\gamma)$ and $\sigma_{\gamma,n+1}^{2^*} = -\frac{d^2 f(\gamma)}{d\gamma^2}\Big|_{\gamma=\mu_\gamma^*}$.

References

[1] S. Särkkä and L. Svensson, *Bayesian Filtering And Smoothing.* Cambridge University Press, 2023, vol. 17.