

# Supplements for "A Variational Bayesian Framework for System Degradation State Identification amid Measurement Noise"

This document presents supplementary proof materials specifically for Algorithms 1 and 2 as outlined in the main paper, where the key formulas are presented as follows:

- Variational posterior update formula:

$$q^*(\phi) \propto \exp \left( \mathbb{E}_{q^*(\Theta_{-\phi})} [\ln p(\Theta|\mathbf{y})] \right), \quad (\text{A1})$$

where  $p(\Theta|\mathbf{y})$  represents posterior distribution,  $\Theta$  is the set of posterior parameters,  $\phi \in \Theta$  is an arbitrary element of the set  $\Theta$  and  $\Theta_{-\phi}$  is a subset of  $\Theta$  excluding  $\phi$ .  $\mathbb{E}_{q^*(\Theta_{-\phi})}[\cdot]$  refers to the expectation with respect to  $q^*(\Theta_{-\phi})$ .

- Offline posterior distribution:

$$p(\mathbf{X}, \boldsymbol{\theta} | \mathbf{y}) \propto \prod_{i=1}^n \mathcal{N}(y_i; X_i, \lambda_2^{-1}) \mathcal{N}(\Delta X_i; \mu\tau_i, \tau_i\lambda_1^{-1}) \mathcal{N}(\mu; \mu_0, (\kappa_0\lambda_1)^{-1}) \times \mathcal{G}(\lambda_1; \alpha_1, \beta_1) \mathcal{G}(\lambda_2; \alpha_2, \beta_2) \mathcal{N}(\gamma; \mu_\gamma, \sigma_\gamma^2), \quad (\text{A2})$$

where  $\mathbf{y} = (y_1, y_2, \dots, y_n)'$ ,  $\mathbf{X} = (X_1, X_2, \dots, X_n)'$ ,  $\boldsymbol{\theta} = (\mu, \lambda_1, \lambda_2, \gamma)'$ ,  $\Delta X_i = X_i - X_{i-1}$  for  $i \geq 2$ , with  $\Delta X_1 = X_1$ , and  $\tau_i = t_i^\gamma - t_{i-1}^\gamma$ , with  $\tau_1 = t_1^\gamma$ .

- Online posterior distribution:

$$p(X_n, X_{n+1}, \boldsymbol{\theta} | \mathbf{y}_{1:(n+1)}) \propto \mathcal{N}(y_{n+1}; X_{n+1}, \lambda_2^{-1}) \mathcal{N}(X_{n+1}; X_n + \mu\tau_{n+1}, \tau_{n+1}\lambda_1^{-1}) \times \mathcal{N}(X_n; \mu_{n|n}, \sigma_{n|n}^2) \mathcal{N}(\mu; \mu_{0,n}, (\kappa_{0,n}\lambda_1)^{-1}) \times \mathcal{G}(\lambda_1; \alpha_{1,n}, \beta_{1,n}) \mathcal{G}(\lambda_2; \alpha_{2,n}, \beta_{2,n}) \mathcal{N}(\gamma; \mu_{\gamma,n}, \sigma_{\gamma,n}^2). \quad (\text{A3})$$

## 1 Proof of Algorithm 1

First, the logarithm of the offline posterior distribution (A2) is expanded as

$$\begin{aligned} \ln p(\mathbf{X}, \boldsymbol{\theta} | \mathbf{y}) &\propto \frac{n}{2} \ln \lambda_2 - \sum_{i=1}^n \frac{\lambda_2(y_i - X_i)^2}{2} + \frac{n}{2} \ln \lambda_1 - \sum_{i=1}^n \frac{\ln \tau_i}{2} - \sum_{i=1}^n \frac{\lambda_1(\Delta X_i - \mu\tau_i)^2}{2\tau_i} \\ &\quad + \frac{1}{2} \ln \lambda_1 - \frac{\lambda_1\kappa_0(\mu^2 - 2\mu\mu_0)}{2} + (\alpha_1 - 1) \ln \lambda_1 - \beta_1\lambda_1 \\ &\quad + (\alpha_2 - 1) \ln \lambda_2 - \beta_2\lambda_2 - \frac{\gamma^2 - 2\gamma\mu_\gamma}{2\sigma_\gamma^2}. \end{aligned} \quad (\text{A4})$$

- $q^*(\mathbf{X})$

By substituting (A4) into (A1), we derive the following expression:

$$\begin{aligned} \ln q^*(\mathbf{X}) &\propto - \sum_{i=1}^n \mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \frac{X_i^2 - 2y_iX_i}{2} \\ &\quad - \sum_{i=1}^n \mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1] \mathbb{E}_{q^*(\gamma)}[\tau_i^{-1}] \frac{\Delta X_i^2}{2} - \sum_{i=1}^n \mathbb{E}_{q^*(\mu, \lambda_1)}[\mu\lambda_1] \Delta X_i. \end{aligned}$$

The linear and quadratic terms in  $X_i$  indicate that  $q^*(\mathbf{X})$  is the PDF of a multivariate normal distribution. Given the complexity of directly formulating this distribution, we decompose it into the product of two multivariate normal distributions, as follows:

$$\ln q^*(\mathbf{X}) = \ln q_1^*(\mathbf{X}) + \ln q_2^*(\mathbf{X}).$$

The first part is given by

$$\begin{aligned} \ln q_1^*(\mathbf{X}) &\propto - \sum_{i=1}^n \left( \mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \frac{X_i^2 - 2y_i X_i}{2} + \mathbb{E}_{q^*(\mu, \lambda_1)}[\mu \lambda_1] \Delta X_i \right) \\ &\propto \mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \sum_{i=1}^n \frac{X_i^2}{2} + \mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \sum_{i=1}^n y_i X_i + \mathbb{E}_{q^*(\mu, \lambda_1)}[\mu \lambda_1] X_n, \end{aligned}$$

where the mean vector  $\mu_1^*$  and covariance matrix  $\Sigma_1^*$  are

$$\begin{aligned} \mu_1^* &= \left( y_1, y_2, \dots, y_{n-1}, y_n + \frac{\mathbb{E}_{q^*(\mu, \lambda_1)}[\mu \lambda_1]}{\mathbb{E}_{q^*(\lambda_2)}[\lambda_2]} \right)', \\ \Sigma_1^* &= (\mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \text{Diag}((1, \dots, 1))^{-1}. \end{aligned}$$

Here,  $\text{Diag}(\cdot)$  represents the diagonal matrix.

The second multivariate normal PDF is expressed as

$$\ln q_2^*(\mathbf{X}) \propto -\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1] - \sum_{i=1}^n \mathbb{E}_{q^*(\gamma)}[\tau_i^{-1}] \frac{\Delta X_i^2}{2}.$$

In this case, the quadratic form only exists in  $X_i$ , and hence the mean is  $\mathbf{0}$ . The covariance matrix  $\Sigma_2^*$  can be readily computed as

$$\Sigma_2^* = \frac{1}{\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1]} \begin{bmatrix} C_1 + C_2 & -C_2 & \cdots & 0 \\ -C_2 & C_2 + C_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{bmatrix}^{-1}, \text{ where } C_i = \mathbb{E}_{q^*(\gamma)}[\tau_i^{-1}].$$

By applying the multiplication rule for multivariate normal distributions, the mean vector and covariance matrix of the joint distribution are

$$\mu^* = \mu_1^* (\Sigma_1^*)^{-1} \Sigma^* \text{ and } \Sigma^* = [(\Sigma_1^*)^{-1} + (\Sigma_2^*)^{-1}]^{-1}.$$

- $q^*(\mu, \lambda_1)$

Similarly, we have

$$\begin{aligned} \ln q^*(\mu, \lambda_1) &\propto \ln \lambda_1 \left( \frac{n-1}{2} + \alpha_1 \right) + \lambda_1 \mu \left( \kappa_0 \mu_0 + \sum_{i=1}^n \mathbb{E}_{q^*(\mathbf{X})}[\Delta X_i] \right) \\ &\quad - \lambda_1 \left( \beta_1 + \sum_{i=1}^n \frac{\mathbb{E}_{q^*(\mathbf{X})}[\Delta X_i^2]}{2} \mathbb{E}_{q^*(\gamma)}[\tau_i^{-1}] + \frac{\kappa_0 \mu_0^2}{2} \right) \\ &\quad - \lambda_1 \mu^2 \frac{\kappa_0 + \sum_{i=1}^n \mathbb{E}_{q^*(\gamma)}[\tau_i]}{2}. \end{aligned}$$

Consequently, we obtain

$$q^*(\mu, \lambda_1) \propto \lambda_1^{\frac{1}{2}} \exp \left\{ -\frac{\mu^2 - 2\mu \left( \frac{\kappa_0 \mu_0 + \sum_{i=1}^n \mathbb{E}_{q^*}(\mathbf{X}) \Delta[X_i]}{\kappa_0 + \sum_{i=1}^n \mathbb{E}_{q^*}(\gamma) [\tau_i]} \right)}{2 \left[ \lambda_1 \left( \kappa_0 + \sum_{i=1}^n \mathbb{E}_{q^*}(\gamma) [\tau_i] \right) \right]^{-1}} \right\} \times \\ \lambda_1^{\frac{n}{2} + \alpha_1 - 1} \exp \left[ -\lambda_1 \left( \beta_1 + \sum_{i=1}^n \frac{\mathbb{E}_{q^*}(\mathbf{X}) [\Delta X_i^2]}{2} \mathbb{E}_{q^*}(\gamma) [\tau_i^{-1}] + \frac{\kappa_0 \mu_0^2}{2} \right) \right].$$

Therefore,  $q^*(\mu, \lambda_1)$  is the PDF of a normal-inverse-gamma distribution.

- $q^*(\lambda_2)$

Next, we have

$$\ln q^*(\lambda_2) \propto \frac{n}{2} \ln \lambda_2 - \lambda_2 \sum_{i=1}^n \frac{(y_i^2 - 2\mathbb{E}_{q^*}(\mathbf{X})[X_i]y_i + \mathbb{E}_{q^*}(\mathbf{X})[X_i^2])}{2} + (\alpha_2 - 1) \ln \lambda_2 - \beta_2 \lambda_2.$$

Thus, we obtain

$$q^*(\lambda_2) \propto \lambda_2^{\frac{n}{2} + \alpha_2 - 1} \exp \left\{ -\lambda_2 \left[ \sum_{i=1}^n \frac{(y_i^2 - 2\mathbb{E}_{q^*}(\mathbf{X})[X_i]y_i + \mathbb{E}_{q^*}(\mathbf{X})[X_i^2])}{2} + \beta_2 \right] \right\}.$$

Hence,  $q^*(\lambda_2)$  is the PDF of a gamma distribution.

- $q^*(\gamma)$

Lastly, we have

$$\ln q^*(\gamma) \propto -\frac{\gamma^2 - 2\mu_\gamma \gamma}{2\sigma_\gamma^2} - \sum_{i=1}^n \frac{\ln \tau_i}{2} - \sum_{i=1}^n \frac{\mathbb{E}_{q^*}(\mu, \lambda_1) [\lambda_1] \mathbb{E}_{q^*}(\mathbf{X}) [\Delta X_i^2]}{2\tau_i} - \frac{\mathbb{E}_{q^*}(\mu, \lambda_1) [\lambda_1 \mu^2] \tau_n}{2}.$$

We define the function  $f(\gamma)$  as

$$f(\gamma) = -\frac{\gamma^2 - 2\mu_\gamma \gamma}{2\sigma_\gamma^2} - \sum_{i=1}^n \frac{\ln \tau_i}{2} - \sum_{i=1}^n \frac{\mathbb{E}_{q^*}(\mu, \lambda_1) [\lambda_1] \mathbb{E}_{q^*}(\mathbf{X}) [\Delta X_i^2]}{2\tau_i} - \frac{\mathbb{E}_{q^*}(\mu, \lambda_1) [\lambda_1 \mu^2] \tau_n}{2}.$$

Using the Laplace approximation, we approximate  $q^*(\gamma)$  as

$$q^*(\gamma) \approx \mathcal{N}(\gamma; \mu_\gamma^*, \sigma_\gamma^{2*}),$$

where  $\mu_\gamma^* = \arg\max f(\gamma)$  and  $\sigma_\gamma^{2*} = -\frac{d^2 f(\gamma)}{d\gamma^2} \Big|_{\gamma=\mu_\gamma^*}$ .

## 2 Proof of Algorithm 2

Firstly, the logarithm of the online posterior distribution (A3) is expanded as

$$\ln p(X_n, X_{n+1}, \boldsymbol{\theta} | \mathbf{y}_{1:(n+1)}) \propto \frac{\ln \lambda_2}{2} - \frac{\lambda_2 (y_{n+1} - X_{n+1})^2}{2} + \frac{\ln \lambda_1}{2} - \frac{\ln \tau_{n+1}}{2} - \frac{\lambda_1 (\Delta X_{n+1} - \mu \tau_{n+1})^2}{2\tau_{n+1}} \\ - \frac{X_n^2 - 2X_n \mu_{n|n}}{2\sigma_{n|n}^2} + \frac{1}{2} \ln \lambda_1 - \frac{\lambda_1 \kappa_{0,n} (\mu^2 - 2\mu \mu_{0,n})}{2} + (\alpha_{1,n} - 1) \ln \lambda_1 \\ - \beta_{1,n} \lambda_1 + (\alpha_{2,n} - 1) \ln \lambda_2 - \beta_{2,n} \lambda_2 - \frac{\gamma^2 - 2\gamma \mu_{\gamma,n}}{2\sigma_{\gamma,n}^2}. \quad (\text{A5})$$

- $q^*(X_{n+1}, X_n)$

By substituting (A5) into (A1), we have the following expression:

$$\ln q^*(X_n, X_{n+1}) \propto -\mathbb{E}_{q^*(\lambda_2)}[\lambda_2] \frac{2y_{n+1}X_{n+1} - X_{n+1}^2}{2} - \frac{(\Delta X_{n+1} - \Delta\mu_{n+1}^*)^2}{2\Delta\sigma_{n+1}^{2*}} - \frac{X_n^2 - 2X_n\mu_{n|n}}{2\sigma_{n|n}^2},$$

$$\text{where } \Delta\mu_{n+1}^* = \frac{\mathbb{E}_{q^*(\mu, \lambda_1)}[\mu\lambda_1]}{\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1]\mathbb{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}]} \text{ and } \Delta\sigma_{n+1}^{2*} = \frac{1}{\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1]\mathbb{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}]}.$$

This expression can be rewritten as

$$\begin{aligned} q^*(X_n, X_{n+1}) &\propto \mathcal{N}\left(y_{n+1}; X_{n+1}, (\mathbb{E}_{q^*(\lambda_2)}[\lambda_2])^{-1}\right) \\ &\quad \times \mathcal{N}\left(X_{n+1}; X_n + \Delta\mu_{n+1}^*, \Delta\sigma_{n+1}^{2*}\right) \mathcal{N}\left(X_n; \mu_{n|n}, \sigma_{n|n}^2\right). \end{aligned} \quad (\text{A6})$$

For clarity, we further define the following:

$$\begin{aligned} p(X_n | \mathbf{y}_{1:n}) &= \mathcal{N}\left(X_n; \mu_{n|n}, \sigma_{n|n}^2\right), \\ p(X_{n+1} | X_n) &= \mathcal{N}\left(X_{n+1}; X_n + \Delta\mu_{n+1}^*, \Delta\sigma_{n+1}^{2*}\right), \\ p(y_{n+1} | X_{n+1}) &= \mathcal{N}\left(y_{n+1}; X_{n+1}, (\mathbb{E}_{q^*(\lambda_2)}[\lambda_2])^{-1}\right). \end{aligned}$$

To obtain  $q^*(X_{n+1})$ , we integrate over  $X_n$  in (A6):

$$\begin{aligned} q^*(X_{n+1}) &\propto \int p(y_{n+1} | X_{n+1}) p(X_{n+1} | X_n) p(X_n | \mathbf{y}_{1:n}) dX_n \\ &\propto p(y_{n+1} | X_{n+1}) p(X_{n+1} | \mathbf{y}_{1:n}). \end{aligned}$$

This expression represents the filtering distribution of  $X_{n+1}$ . Then, we further derive the following closed recursive formulas [1]:

$$\begin{aligned} \mu_{n+1|n}^* &= \mu_{n|n} + \Delta\mu_{n+1}^*, \\ \sigma_{n+1|n}^{2*} &= \sigma_{n|n}^2 + \Delta\sigma_{n+1}^{2*}, \\ S^* &= \sigma_{n+1|n}^{2*} + (\mathbb{E}_{q^*(\lambda_2)}[\lambda_2])^{-1}, \\ K^* &= \sigma_{n+1|n}^{2*} (S^*)^{-1}, \\ \mu_{n+1|n+1}^* &= \mu_{n+1|n}^* + K^* (y_{n+1} - \mu_{n+1|n}^*), \\ \sigma_{n+1|n+1}^{2*} &= \sigma_{n+1|n}^{2*} - (K^*)^2 S^*. \end{aligned}$$

Similarly, to obtain  $q^*(X_n)$ , we integrate over  $X_{n+1}$  in (A6):

$$\begin{aligned} q^*(X_n) &\propto \int p(y_{n+1} | X_{n+1}) p(X_{n+1} | X_n) p(X_n | \mathbf{y}_{1:n}) dX_{n+1} \\ &\propto p(X_n | \mathbf{y}_{1:n}) \int p(y_{n+1} | X_{n+1}) p(X_{n+1} | \mathbf{y}_{1:n}) dX_{n+1}. \end{aligned}$$

This expression represents the smoothing distribution of  $X_n$  given  $\mathbf{y}_{1:(n+1)}$  and the filtered distribution  $q^*(X_{n+1})$ . Again, we derive the recursive formulas for the mean and variance of the smoothed distribution:

$$\begin{aligned} G^* &= \sigma_{n|n}^2 \left( \sigma_{n+1|n}^{2*} \right)^{-1}, \\ \mu_{n|n+1}^* &= \mu_{n|n} + G^* \left( \mu_{n+1|n+1}^* - \mu_{n+1|n}^* \right), \\ \sigma_{n|n+1}^{2*} &= \sigma_{n|n}^2 + (G^*)^2 \left[ \sigma_{n+1|n+1}^{2*} - \sigma_{n+1|n}^{2*} \right]. \end{aligned}$$

Furthermore, since the covariance between  $X_n$  and  $X_{n+1}$  is required during the variational iteration, the quadratic term in (A6) yields the following expression for the inverse covariance matrix:

$$(\Sigma_{n:n+1}^*)^{-1} = \begin{pmatrix} \mathbb{E}_{q^*(\lambda_2)}[\lambda_2] + (\Delta\sigma_{n+1}^{2*})^{-1} & -(\Delta\sigma_{n+1}^{2*})^{-1} \\ -(\Delta\sigma_{n+1}^{2*})^{-1} & \sigma_{n|n}^2 + (\Delta\sigma_{n+1}^{2*})^{-1} \end{pmatrix}.$$

Inverting this covariance matrix, the covariance between  $X_n$  and  $X_{n+1}$  is given by

$$\text{Cov}^*(X_n, X_{n+1}) = \sigma_{n|n}^2 (S^* \mathbb{E}_{q^*(\lambda_2)}[\lambda_2])^{-1}.$$

- $q^*(\mu, \lambda_1)$

Similarly, we have

$$\begin{aligned} \ln q^*(\mu, \lambda_1) &\propto \left(\frac{1}{2} + \alpha_{1,n}\right) \ln \lambda_1 + \lambda_1 \mu (\kappa_{0,n} \mu_{0,n} + \mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}]) \\ &\quad - \lambda_1 \left( \beta_{1,n} + \frac{\mathbb{E}_{q^*(X_n, X_{n+1})}[\Delta X_{n+1}^2]}{2} \mathbb{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}] + \frac{\kappa_{0,n} \mu_{0,n}^2}{2} \right) \\ &\quad - \lambda_1 \mu^2 \frac{\kappa_{0,n} + \mathbb{E}_{q^*(\gamma)}[\tau_{n+1}]}{2}. \end{aligned}$$

From this, the distribution  $q^*(\mu, \lambda_1)$  can be expressed as:

$$\begin{aligned} q^*(\mu, \lambda_1) &\propto \lambda_1^{\frac{1}{2}} \exp \left\{ -\frac{\mu^2 - 2\mu \left( \frac{\kappa_{0,n} \mu_{0,n} + \mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}]}{\kappa_{0,n} + \mathbb{E}_{q^*(\gamma)}[\tau_{n+1}]} \right)}{2 [\lambda_1 (\kappa_{0,n} + \mathbb{E}_{q^*(\gamma)}[\tau_{n+1}])]^{-1}} \right\} \times \\ &\quad \lambda_1^{\alpha_{1,n} - \frac{1}{2}} \exp \left[ -\lambda_1 \left( \beta_{1,n} + \frac{\mathbb{E}_{q^*(X_n, X_{n+1})}[\Delta X_{n+1}^2]}{2} \mathbb{E}_{q^*(\gamma)}[\tau_{n+1}^{-1}] + \frac{\kappa_{0,n} \mu_{0,n}^2}{2} \right) \right]. \end{aligned}$$

Therefore,  $q^*(\mu, \lambda_1)$  is the PDF of a normal-inverse-gamma distribution.

- $q^*(\lambda_2)$

Next, we have

$$\begin{aligned} \ln q^*(\lambda_2) &\propto \frac{1}{2} \ln \lambda_2 - \lambda_2 \frac{(y_{n+1}^2 - 2y_{n+1} \mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}] + \mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}^2])}{2} \\ &\quad + (\alpha_{2,n} - 1) \ln \lambda_2 - \beta_{2,n} \lambda_2. \end{aligned}$$

Then, the variational distribution  $q^*(\lambda_2)$  is given by

$$q^*(\lambda_2) \propto \lambda_2^{\alpha_{2,n} - \frac{1}{2}} \exp \left\{ -\lambda_2 \left[ \frac{(y_{n+1}^2 - 2\mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}]y_{n+1} + \mathbb{E}_{q^*(X_n, X_{n+1})}[X_{n+1}^2])}{2} + \beta_{2,n} \right] \right\}.$$

Hence,  $q^*(\lambda_2)$  is the PDF of a gamma distribution.

- $q^*(\gamma)$

Lastly, we have

$$\ln q^*(\gamma) \propto -\frac{\gamma^2 - 2\mu_{\gamma,n}\gamma}{2\sigma_{\gamma,n}^2} - \frac{\ln \tau_{n+1}}{2} - \frac{\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1] \mathbb{E}_{q^*(X_n, X_{n+1})}[\Delta X_{n+1}^2]}{2\tau_{n+1}} - \frac{\mathbb{E}_{q^*(\mu, \lambda_1)}[\lambda_1 \mu^2] \tau_{n+1}}{2}.$$

Define  $f(\gamma)$  as

$$f(\gamma) = -\frac{\gamma^2 - 2\mu_{\gamma,n}\gamma}{2\sigma_{\gamma,n}^2} - \frac{\ln \tau_{n+1}}{2} - \frac{\mathbb{E}_{q^*(\mu,\lambda_1)}[\lambda_1]\mathbb{E}_{q^*(X_n,X_{n+1})}[\Delta X_{n+1}^2]}{2\tau_{n+1}} - \frac{\mathbb{E}_{q^*(\mu,\lambda_1)}[\lambda_1\mu^2]\tau_{n+1}}{2}.$$

Using the Laplace approximation, we approximate  $q^*(\gamma)$  as

$$q^*(\gamma) \approx \mathcal{N}(\gamma; \mu_{\gamma,n+1}^*, \sigma_{\gamma,n+1}^{2*}),$$

$$\text{where } \mu_{\gamma,n+1}^* = \operatorname{argmax} f(\gamma) \text{ and } \sigma_{\gamma,n+1}^{2*} = -\frac{d^2 f(\gamma)}{d\gamma^2} \Big|_{\gamma=\mu_{\gamma}^*}.$$

## References

- [1] S. Särkkä and L. Svensson, *Bayesian Filtering And Smoothing*. Cambridge University Press, 2023, vol. 17.