Introduction to Approximation Algorithms, part II

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APPROX-SUBSET-SUM(S, t, ε)
    $L'_0 = [0]$
    for $k = 1, \ldots, n$
        L'_k=MERGE-LISTS($L'_{k-1}, L'_{k-1} + x_k$)
        L'_k=TRIM($L'_k, \varepsilon/2n$)
        remove duplicates and elm.s $> t$
    return last(L'_n)

Definition

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$C := \mathbf{E}\left[\text{cost(produced sol.)}\right]$

3-SAT

$$
\begin{aligned}
&(x_1 \lor x_7 \lor \neg x_9) \\
\land&(\neg x_7 \lor x_8 \lor x_9) \\
\land&(\neg x_2 \lor x_3 \lor \neg x_4) \\
&\vdots
\end{aligned}
$$

3-SAT

$$\left.\begin{aligned} &(x_1 \vee x_7 \vee \neg x_9) \\ \wedge&(\neg x_7 \vee x_8 \vee x_9) \\ \wedge&(\neg x_2 \vee x_3 \vee \neg x_4) \\ &\vdots \end{aligned}\right] n \text{ clauses}$$

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Each clause has three *literals* involving three distinct variables.

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Decision version:
NP-complete!

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MAX-3-SAT: Find assignment that maximizes the number of true clauses.

Randomly assigning values

Φ is a MAX-3-SAT instance

RANDOM-ASSIGNMENT(Φ)
  for each variable x_i of Φ
    choose $x_i \in \{0, 1\}$ by flipping fair coin
  return assignment

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Bonus info

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    compute $D := \mathbf{E}[X \mid \text{chosen values of } x_1, \ldots, x_i]$
    if $D < 7n/8$
      $x_i := 1$
  return assignment

By work of Håstad, it is NP-hard to approximate within $8/7 - \varepsilon$ for all $\varepsilon > 0$, so this very simple algorithm is essentially optimal, unless P=NP.

Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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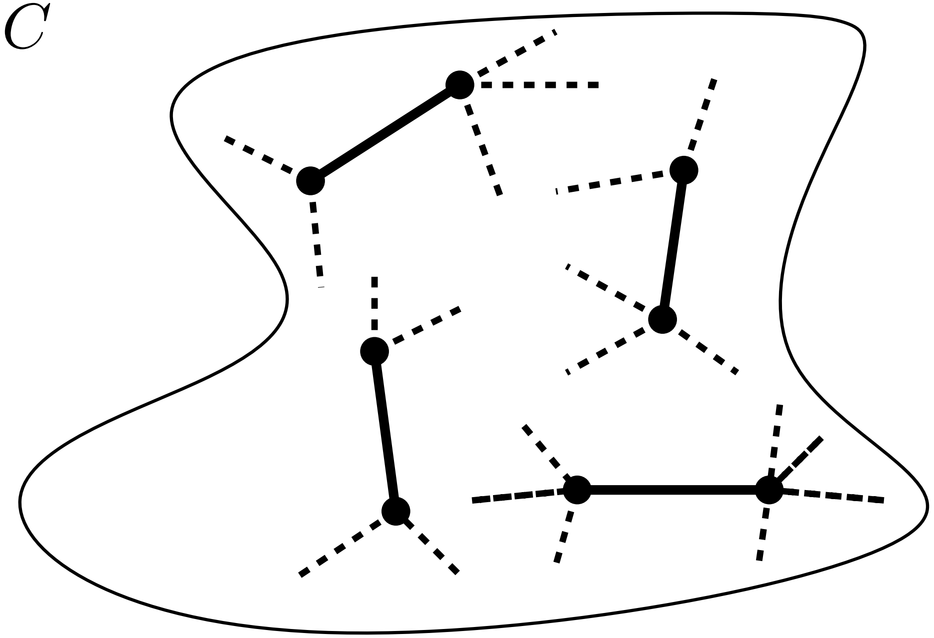
APPROX-VERTEX-COVER(G)
  $C := \emptyset$
  while $E(G) \neq \emptyset$
    choose $uv \in E(G)$
    $C := C \cup \{u, v\}$
    remove all edges incident on u or v from $E(G)$
  return C



$$\frac{C}{C^*} \leq 2$$

Weighted Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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Now: We are given weight $w(v) > 0$ for each $v \in V$.

Goal: Find vertex cover C with minimum

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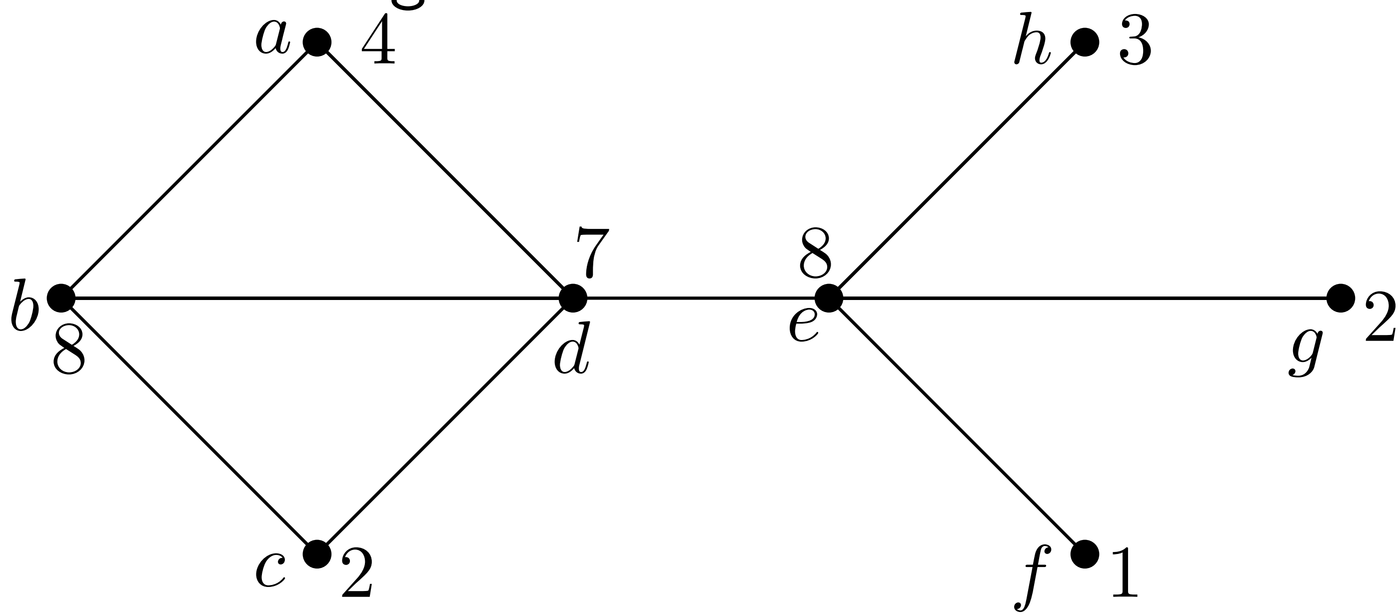
Exercise: Find minimum (unweighted) vertex cover and then minimum weighted vertex cover.



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0-1-integer program (IP):

$x_v \in \{0, 1\}$, $\forall v \in V$ $\quad (x_v = 1 \Leftrightarrow v \in C)$

$x_u + x_v \geq 1$, $\forall uv \in E$ $\quad$ (edge uv covered)

$$\text{minimize} \sum_{v \in V} w(v) x_v$$

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Relaxed solution can be smaller, not larger

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Compute opt. sol. $\bar{x}$ to LP

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2) $uv \in E \Rightarrow \bar{x}_u + \bar{x}_v \ge 1 \Rightarrow \bar{x}_u \ge \frac{1}{2} \vee \bar{x}_v \ge \frac{1}{2} \Rightarrow u \in C \vee v \in C$. ✓

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$$\le 2 \sum_{v \in V} w(v)\bar{x}_v = 2z^* \le 2w(C^*) \Rightarrow \frac{w(C)}{w(C^*)} \le 2. \quad \checkmark$$

Reflection and methodology

How can we prove $w(C)/w(C^*) \leq 2$ when we don't know $w(C^*)$?

Answer: By proving $w(C) \leq 2z^*$ and $|z^*| \leq w(C^*)$.

Reflection and methodology

How can we prove $w(C)/w(C^*) \leq 2$ when we don't know $w(C^*)$?

Answer: By proving $w(C) \leq 2z^*$ and $|z^*| \leq w(C^*)$.

General technique: Find a parameter $\Box$ such that $C \leq \rho \cdot \Box$ and $\Box \leq C^*$.

For weighted vertex cover: $\Box = z^*$ and $\rho = 2$.

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Polynomial-time approximation scheme (PTAS):
Approximation algorithm that takes instance I of an optimization problem P and $\varepsilon > 0$ as input. For any fixed ε works as $(1 + \varepsilon)$-approximation algorithm for P.

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Input: Set $S = \{x_1, \ldots, x_n\} \subset \mathbb{N}$, and $t \in \mathbb{N}$.
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Abstract exact alg.:

for $k = 1, 2, \ldots, n$
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EXACT-SUBSET-SUM(S, t)
  $L_0 = [0]$
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    L_k=MERGE-LISTS$(L_{k-1}, L_{k-1} + x_k)$
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  return last(L_n)

Idea: Trim list $L \subset \{0, 1, \ldots, t\}$ with parameter $\delta > 0$: if we keep $s \in L$, then remove $(s, (1+\delta)s]$.

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TRIM($L = [s_1, \ldots, s_m], \delta$)
  $L' = [s_1]$
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$\boxed{\delta := \varepsilon/2n \Rightarrow 1 + 2n\delta \leq 1 + \varepsilon \checkmark}$

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