Advanced algorithms and data structures

Lecture 6: van Emde Boas Trees

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Today's Lecture

van Emde Boas Trees

- Predecessor search/ordered sets
- Naive
- Twolevel
- Recursive
- vEB: worst case $\mathcal{O}(\log\log|U|)$ time
- RS-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n\log\log|U|)$ space
- R^2S-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n)$ space
- Bonus: vEB is optimal for $w = \Theta(\log n)$
- Bonus: Integer sorting in expected $\mathcal{O}(n\log\log|U|)$ time.

Predecessor search/ordered sets

Problem:
Given a universe $U = [u]$ where $u = 2^w$,
maintain subset $S \subseteq U$, $|S| = n$ under:

member(x, S): Return $[x \in S]$.

insert(x, S): Add x to S (assumes $x \notin S$).

delete(x, S): Remove x from S (assumes $x \in S$).

empty(S): Return $[S = \emptyset]$.

min(S): Return $\min S$ (assumes $S \neq \emptyset$).

max(S): Return $\max S$ (assumes $S \neq \emptyset$).

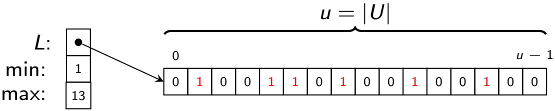
predecessor(x, S): Return $\max\{y \in S \mid y < x\}$
(assumes $\{y \in S \mid y < x\}$ is nonempty,
i.e. $S \neq \emptyset$ and $x > \min(S)$).

successor(x, S): Return $\min\{y \in S \mid y > x\}$
(assumes $\{y \in S \mid y > x\}$ is nonempty,
i.e. $S \neq \emptyset$ and $x < \max(S)$).

Naive

Idea: If we are willing to spend $\mathcal{O}(|U|)$ space...

Store S as a bit-array L of length $|U|$ such that $L[x] = [x \in S]$, and keep track of the min and max values explicitly.

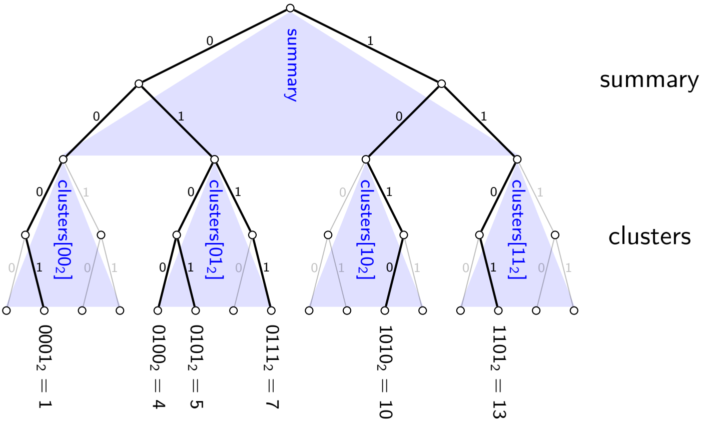


$\{1, 4, 5, 7, 10, 13\}$

How fast is:
empty(S), min(S), and max(S)? worst case $\mathcal{O}(1)$.
member(x, S)? worst case $\mathcal{O}(1)$.
predecessor(x, S) and successor(x, S)? worst case $\Theta(|U|)$.
delete(x, S)? worst case $\mathcal{O}(|U|)$.
insert(x, S)? worst case $\mathcal{O}(1)$.

Bit-Trie

Idea: Think of each key as a w-bit string describing a path in a binary *trie* (= a special kind of tree). The naive structure ignores all intermediate branches and jump directly to the leaves. What if we split each key into a high and a low part?



Twolevel

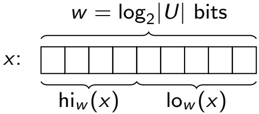
Idea: Split each key into *high* and *low* parts. Use naive for each.

Recall that $U = [2^w]$, and define:

$$\mathrm{hi}_w(x) := \left\lfloor \frac{x}{2^{\lceil w/2 \rceil}} \right\rfloor$$

$$\mathrm{lo}_w(x) := x \bmod 2^{\lceil w/2 \rceil}$$

$$\mathrm{index}_w(h, \ell) := h \cdot 2^{\lceil w/2 \rceil} + \ell$$



note that $x = \mathrm{index}_w(\mathrm{hi}_w(x), \mathrm{lo}_w(x))$.

Now let the structure directly store the values:

$$\min := \min(S) \text{ if } S \neq \emptyset, \text{ else } 1$$

$$\max := \max(S) \text{ if } S \neq \emptyset, \text{ else } 0$$

and use the naive structure to store

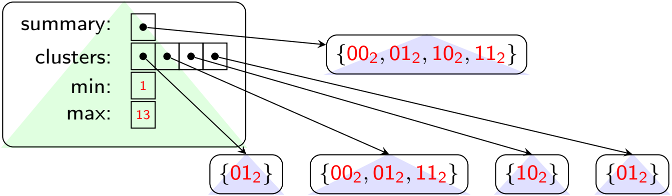
$$\mathrm{summary} := \{\mathrm{hi}_w(x) \mid w \in S\}$$

$$\mathrm{clusters}[h] := \{\ell \in [2^{\lceil w/2 \rceil}] \mid \mathrm{index}_w(h, \ell) \in S\} \qquad \forall h \in [2^{\lfloor w/2 \rfloor}]$$

note that $S = \bigcup_{h \in [2^{\lfloor w/2 \rfloor}]} \{\mathrm{index}_w(h, \ell) \mid \ell \in \mathrm{clusters}[h]\}$.

Twolevel

We can draw the structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



How fast (worst case) is:

empty(S), min(S), max(S)?
$\mathcal{O}(1)$
member(x, S)?
$\mathcal{O}(1) + 1\times$naive $= \mathcal{O}(1)$
predecessor(x, S), successor(x, S)?
$\mathcal{O}(1) + 1\times$naive
$= \Theta(2^{\lceil w/2 \rceil}) = \Theta(\sqrt{|U|})$
delete(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \Theta(\sqrt{|U|})$
insert(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \mathcal{O}(1)$

function $\text{EMPTY}(S)$
 return S.min $>$ S.max

function $\text{MIN}(S)$
 ▷ Assumes $S \neq \emptyset$
 return S.min

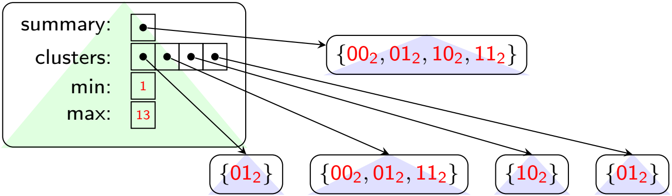
function $\text{MAX}(S)$
 ▷ Assumes $S \neq \emptyset$
 return S.max

function $\text{MEMBER}_w(x, S)$
 return $\text{MEMBER}_{\lceil w/2 \rceil}(\text{lo}_w(x), S.\text{clusters}[\text{hi}_w(x)])$

How can we improve the time further?

Twolevel

We can draw the structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



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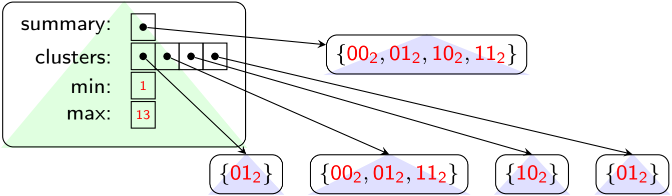
$= \Theta(2^{\lceil w/2 \rceil}) = \Theta(\sqrt{|U|})$
delete(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \Theta(\sqrt{|U|})$
insert(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \mathcal{O}(1)$

function $\textsc{Predecessor}_w(x, S)$
    $\triangleright$ Assumes $S \neq \emptyset$ and $\min(S) < x$
    if $x > S.\text{max}$ **then**
        return $S.\text{max}$
    $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$, $C \leftarrow S.\text{clusters}[p]$
    if not $\textsc{Empty}(C)$ and $C.\text{min} < s$ **then**
        return $\text{index}_w(p, \textsc{Predecessor}_{\lceil w/2 \rceil}(s, C))$
    $p \leftarrow \textsc{Predecessor}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
    return $\text{index}_w(p, S.\text{clusters}[p].\text{max})$

How can we improve the time further?

Twolevel

We can draw the structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



How fast (worst case) is:

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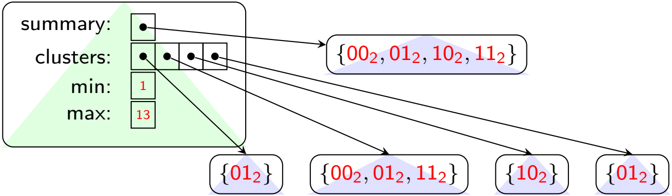
predecessor(x, S), successor(x, S)?
$\mathcal{O}(1) + 1\times$naive
$= \Theta(2^{\lceil w/2 \rceil}) = \Theta(\sqrt{|U|})$
delete(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \Theta(\sqrt{|U|})$
insert(x, S)?
$\mathcal{O}(1) + 2\times$naive $= \mathcal{O}(1)$

function $\textsc{Delete}_w(x, S)$
    ▷ Assumes $x \in S$
    $p \leftarrow \mathrm{hi}_w(x)$, $s \leftarrow \mathrm{lo}_w(x)$, $C \leftarrow S.\mathrm{clusters}[p]$
    $\textsc{Delete}_{\lceil w/2 \rceil}(s, C)$
    if $\textsc{Empty}(C)$ **then**
        $\textsc{Delete}_{\lfloor w/2 \rfloor}(p, S.\mathrm{summary})$
    if $S.\mathrm{min} = S.\mathrm{max}$ **then**
        $S.\mathrm{min} \leftarrow 1$, $S.\mathrm{max} \leftarrow 0$
    else if $x = S.\mathrm{min}$ **then**
        $p \leftarrow S.\mathrm{summary.min}$, $S.\mathrm{min} \leftarrow \mathrm{index}_w(p, S.\mathit{clusters}[p].\mathrm{min})$
    else if $x = S.\mathrm{max}$ **then**
        $p \leftarrow S.\mathrm{summary.max}$, $S.\mathrm{max} \leftarrow \mathrm{index}_w(p, S.\mathit{clusters}[p].\mathrm{max})$

How can we improve the time further?

Twolevel

We can draw the structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



How fast (worst case) is:

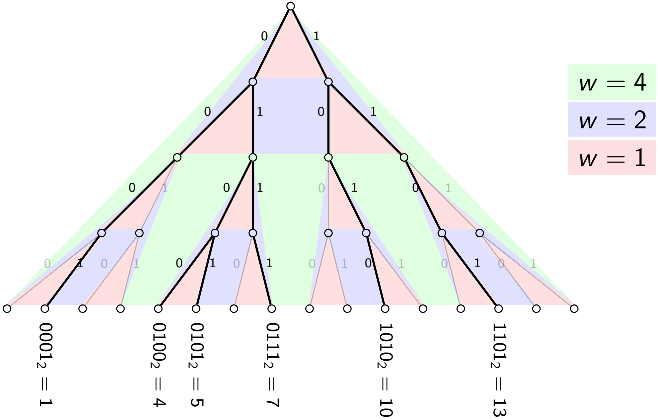
empty(S), min(S), max(S)?
$\mathcal{O}(1)$
member(x, S)?
$\mathcal{O}(1) + 1\times\text{naive} = \mathcal{O}(1)$
predecessor(x, S), successor(x, S)?
$\mathcal{O}(1) + 1\times\text{naive}$
$= \Theta(2^{\lceil w/2 \rceil}) = \Theta(\sqrt{|U|})$
delete(x, S)?
$\mathcal{O}(1) + 2\times\text{naive} = \Theta(\sqrt{|U|})$
insert(x, S)?
$\mathcal{O}(1) + 2\times\text{naive} = \mathcal{O}(1)$

function $\text{INSERT}_w(x, S)$
 ▷ Assumes $x \notin S$
 if $\text{EMPTY}(S)$ **then**
  $S.\text{min} \leftarrow x$, $S.\text{max} \leftarrow x$
 if $x < S.\text{min}$ **then** $S.\text{min} \leftarrow x$
 if $x > S.\text{max}$ **then** $S.\text{max} \leftarrow x$
 $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$
 if $\text{EMPTY}(S.\text{clusters}[p])$ **then**
  $\text{INSERT}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
 $\text{INSERT}_{\lceil w/2 \rceil}(s, S.\text{clusters}[p])$

How can we improve the time further?

Recursive

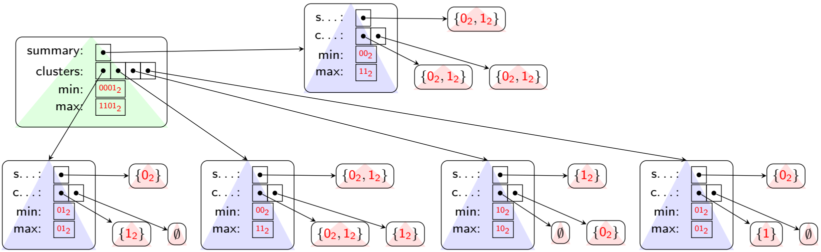
Idea: Instead of using the naive structure for summary and clusters[h], recursively use the same type of structure (stop recursion when $w = 1$).



Recursive

Idea: Instead of using the naive structure for summary and clusters[h], recursively use the same type of structure (stop recursion when $w = 1$).

We can draw the recursive structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



This structure is (essentially) what the book calls "proto-vEB". Note that the naive structure we started with is completely gone.

Recursive

Theorem

The recursion depth of this structure, when used on the universe $U = [2^w]$ *is* $\lceil \log_2 w \rceil = \mathcal{O}(\log \log |U|)$*.*

Proof.

Let $d(w)$ be the recursion depth when working with a universe of size 2^w. We will prove by induction that $d(w) = \lceil \log_2 w \rceil$.
The base case is $w = 1$ where there is no recursion, so $d(1) = 0 = \lceil \log_2(1) \rceil$.
For the induction case, suppose $w > 1$ and that $d(w') = \lceil \log_2(w') \rceil$ for all $w' \in [w]_+$. Now let $w' = \lceil w/2 \rceil \in [w]_+$, then the largest universe size used in the recursion is $2^{w'}$, and (by induction) $d(w) = 1 + d(w') = 1 + \lceil \log_2(w') \rceil = \lceil \log_2(2w') \rceil$. What remains is to show $\lceil \log_2(2w') \rceil = \lceil \log_2(w) \rceil$.

If w is even, $2w' = w$ and we are done.
Otherwise w is odd and $w \geq 3$ so the smallest integer $k = \lceil \log_2(w) \rceil$ such that $2^k \geq w$ satisfies $k \geq 1$. Thus 2^k is even and so must satisfy $2^k \geq w + 1 = 2w'$ and therefore also $k = \lceil \log_2(2w') \rceil = \lceil \log_2(w) \rceil$. □

Recursive

Using the recursive structure, how fast is:
empty(S), min(S), and max(S)? worst case $\mathcal{O}(1)$.
member(x, S)? $\mathcal{O}(1) + 1\times\text{recursion} = \mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$.
predecessor(x, S) and successor(x, S)? $O(1) + 1\times\text{recursion}$
$= \mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$.
insert(x, S) and delete(x, S)? $\mathcal{O}(1) + 2\times\text{recursion}$
$= \Theta(2^{d(w)}) = \Theta(w) = \Theta(\log|U|)$.

How can we improve the update time? Somehow make insert and delete recurse in only one substructure.

vEB: worst case $\mathcal{O}(\log\log|U|)$ time

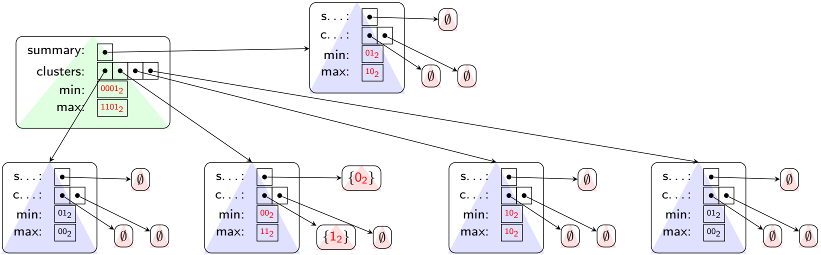
Idea: Exclude $\min(S)$ and/or $\max(S)$ from the set of keys stored in summary and clusters. (CLRS excludes only $\min(S)$, I exclude both).

Specifically, redefine summary and clusters to recursively store the sets:

$$\text{summary} := \{\text{hi}_w(x) \mid w \in S \setminus \{\min, \max\}\}$$

$$\text{clusters}[h] := \{\ell \in [2^{\lceil w/2 \rceil}] \mid \text{index}_w(h, \ell) \in S \setminus \{\min, \max\}\} \quad \forall h \in [2^{\lfloor w/2 \rfloor}]$$

We can draw the van Emde Boas tree for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



vEB: worst case $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(|U|)$ space

function $\textsc{Predecessor}_w(x, S)$ ▷ Assumes $S \neq \emptyset$ and $x > S.\mathrm{min}$
    if $x > S.\mathrm{max}$ **then**
        return $S.\mathrm{max}$
    if $w = 1$ **then**
        return $S.\mathrm{min}$
    $p \leftarrow \mathrm{hi}_w(x),\ s \leftarrow \mathrm{lo}_w(x),\ C \leftarrow S.\mathrm{clusters}[p]$
    if not $\textsc{Empty}(C)$ and $C.\mathrm{min} < s$ **then**
        return $\mathrm{index}_w(p, \textsc{Predecessor}_{\lceil w/2 \rceil}(s, C))$
    if $\textsc{Empty}(S.\mathrm{summary})$ or $p \leq S.\mathrm{summary.min}$ **then**
        return $S.\mathrm{min}$
    $p \leftarrow \textsc{Predecessor}_{\lfloor w/2 \rfloor}(p, S.\mathrm{summary})$
    return $\mathrm{index}_w(p, S.\mathrm{clusters}[p].\mathrm{max})$

Theorem

$\textsc{Predecessor}_w(x, S)$ *takes worst case* $\mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$ *time.*

Proof.

It makes at most one recursive call. □

vEB: worst case $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(|U|)$ space

function $\text{INSERT}_w(x, S)$ ▷ Assumes $x \notin S$
  if $\text{EMPTY}(S)$ **then**
    $S.\text{min} \leftarrow x$, $S.\text{max} \leftarrow x$, **return**
  if $S.\text{min} = S.\text{max}$ **then**
    if $x < S.\text{min}$ **then** $S.\text{min} \leftarrow x$
    if $x > S.\text{max}$ **then** $S.\text{max} \leftarrow x$
    return
  if $x < S.\text{min}$ **then** $S.\text{min} \leftrightarrow x$
  if $x > S.\text{max}$ **then** $S.\text{max} \leftrightarrow x$
  $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$
  if $\text{EMPTY}(S.\text{clusters}[p])$ **then**
    $\text{INSERT}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
  $\text{INSERT}_{\lceil w/2 \rceil}(s, S.\text{clusters}[p])$

Theorem

$\text{INSERT}_w(x, S)$ *takes worst case* $\mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$ *time.*

Proof.

It makes at most one recursive call on a non-empty substructure, and inserting in an empty substructure takes constant time. □

RS-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n\log\log|U|)$ space

Idea: Use a hash table instead of an array to store clusters[$\cdot$], and don't store empty substructures.

Why does this change updates from worst case $\mathcal{O}(\log\log|U|)$ to expected $\mathcal{O}(\log\log|U|)$ time? Because updates to a hash table take expected $\mathcal{O}(1)$ time rather than worst case.

Why does this only use $\mathcal{O}(n \cdot d(w)) = \mathcal{O}(n\log\log|U|)$ space? Because the empty structure uses $\mathcal{O}(1)$ space and INSERT_w only creates or updates $\mathcal{O}(d(w))$ substructures in the worst case. Each of these costs at most an additional $\mathcal{O}(1)$ space

R^2S-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n)$ space

Idea: Partition S into "chunks" of size $\Theta(\min\{n, \log\log|U|\})$, and store only one element representing each chunk in the RS-vEB structure. I.e. RS-vEB stores only $\mathcal{O}(\max\{1, n/\log\log|U|\})$ elements, using $\mathcal{O}(n)$ space.

We can store each chunk as a sorted linked list, and keep a hash table mapping each representative to its chunk. This also takes $\mathcal{O}(n)$ space.

PREDECESSOR and SUCCESSOR uses the RS-vEB structure to find the nearest two representatives in $\mathcal{O}(\log\log|U|)$ time, and can then spend linear time in the size of the two chunks to find the result.

INSERT may have to split a chunk that becomes too large and insert a new representative in the RS-vEB structure. Splitting the chunk can take linear time in the size of the chunk, and together with inserting the new representative into the RS-vEB structure, this still only takes expected $\mathcal{O}(\log\log|U|)$ time.

Similarly, DELETE may have to join two chunks and delete a representative, but again this only takes expected $\mathcal{O}(\log\log|U|)$ time.

Bonus: vEB is optimal for $w = \Theta(\log n)$

In fact, $\mathcal{O}(\log w)$ is the optimal query time when $w \in \Theta(\log n)$, or equivalently when $n \in 2^{\Theta(w)}$. This was shown in:

Time-space trade-offs for predecessor search,
Mihai Pătrașcu and Mikkel Thorup,
STOC'06: Proceedings of the thirty-eighth annual ACM symposium on Theory of Computing, pages 232–240.
`https://doi.org/10.1145/1132516.1132551`

Bonus: Integer sorting in expected $\mathcal{O}(n \log \log |U|)$ time.

Idea: Insert all elements in R^{2}S-vEB structure, use $\textsc{min}(S)$ then repeatedly $\textsc{successor}(x, S)$.

This takes expected $\mathcal{O}(n \log \log |U|) = \mathcal{O}(n \log w)$ time.

Compare this to other famous sorting methods:

$\textsc{QuickSort}$ Takes expected $\mathcal{O}(n \log n)$ time.

$\textsc{CountingSort}$ Takes $\mathcal{O}(n + |U|)$ time.

$\textsc{RadixSort}$ Takes $\mathcal{O}((n + r) \log_r |U|) = \mathcal{O}((n + r) \frac{w}{\log r})$ time for radix r, e.g. $\mathcal{O}(\frac{n}{\log n} \log |U|) = \mathcal{O}(\frac{nw}{\log n})$ for $r = n$.

We observe that for $\log n \in \Omega(\log w) \cap \mathcal{O}(\frac{w}{\log w})$, or equivalently $n \in w^{\Omega(1)} \cap 2^{\mathcal{O}(\frac{w}{\log w})}$, sorting using vEB should be better than these alternatives.

Summary

Todays topic was van Emde Boas Trees, which is a data structure for sets of bounded integers. We have covered

- A naive implementation of ordered sets.
- A slightly less naive two-level implementation.
- A recursive implementation with good query time (proto-vEB).
- The actual van Emde Boas Tree (vEB).
- An extension to vEB using hashing to save space (RS-vEB).
- A further space-saving extension using “indirection” (R^2S-vEB).
- Some notes about the optimality of vEB and on integer sorting.
- Next time: NP-completeness