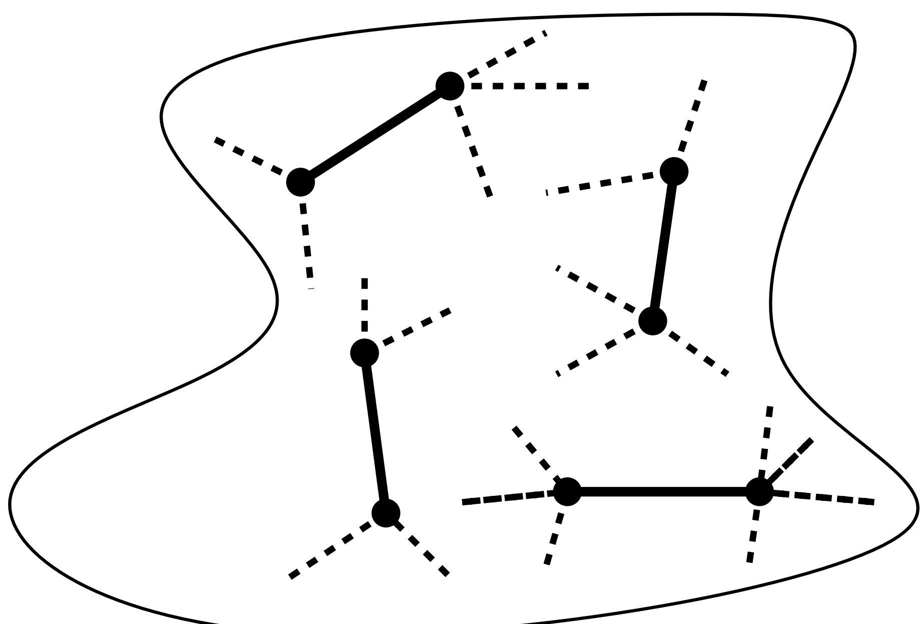
Introduction to Approximation Algorithms, part I

20-12 2022, Mikkel Abrahamsen,
Department of Computer Science

APPROX-VERTEX-COVER(G)
    $C := \emptyset$
    while $E(G) \neq \emptyset$
        choose $uv \in E(G)$
        $C := C \cup \{u, v\}$
        remove all edges incident on u or v from $E(G)$
    return C



The big picture

Last time: Fast exponential algorithms (good for small instances) and parameterized algorithms (good for special cases).

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Today: Approximation algorithms (good when suboptimal solutions are acceptable).

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minimization problem

maximization problem

Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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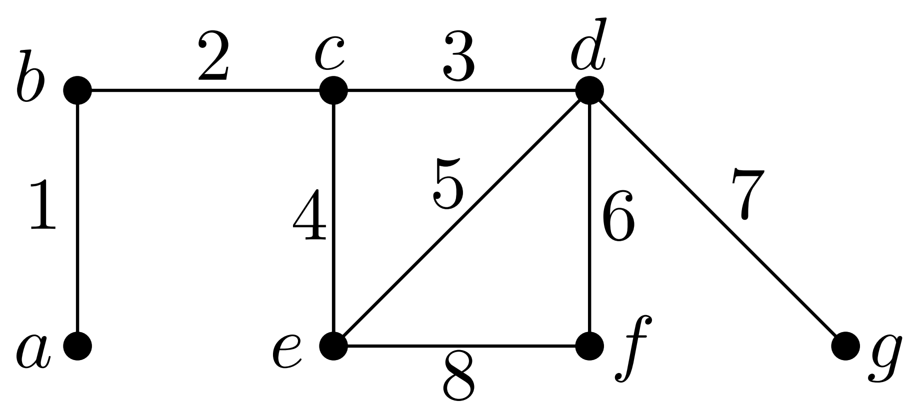
Exercise:



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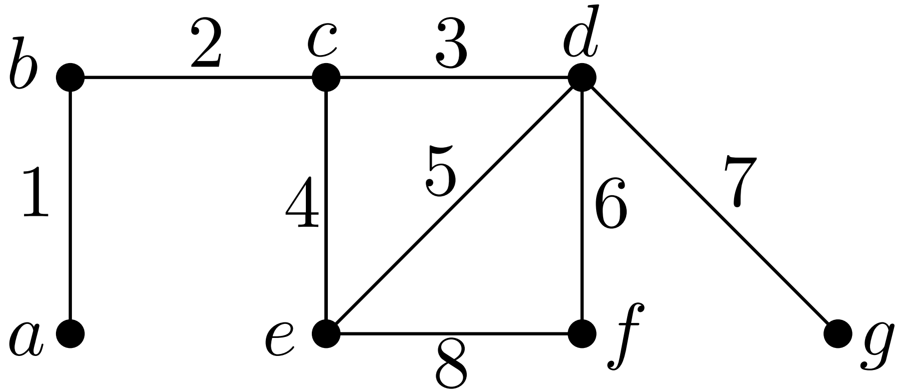

Adjacency lists:

$L[a] = \{b\}$
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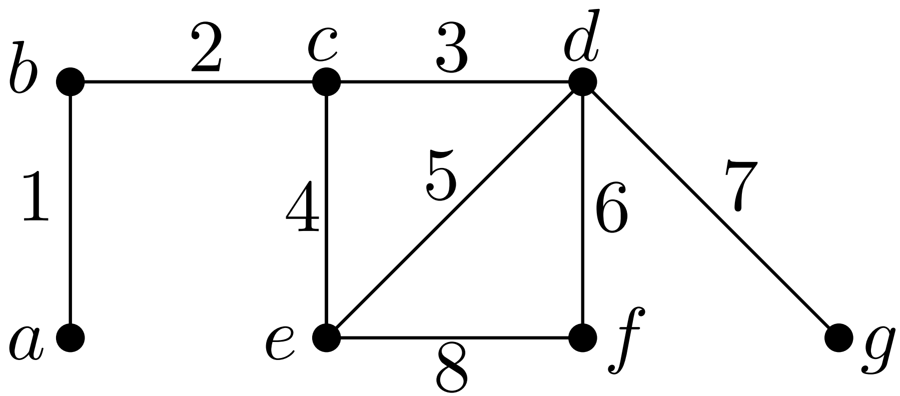

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$L[a] = \{b\}$ $\quad E[a] = [1]$

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Array of edges

$$[(a, b, 1),\ (b, c, 1),\ (c, d, 1),\ (c, e, 1), (d, e, 1),\ \ldots]$$

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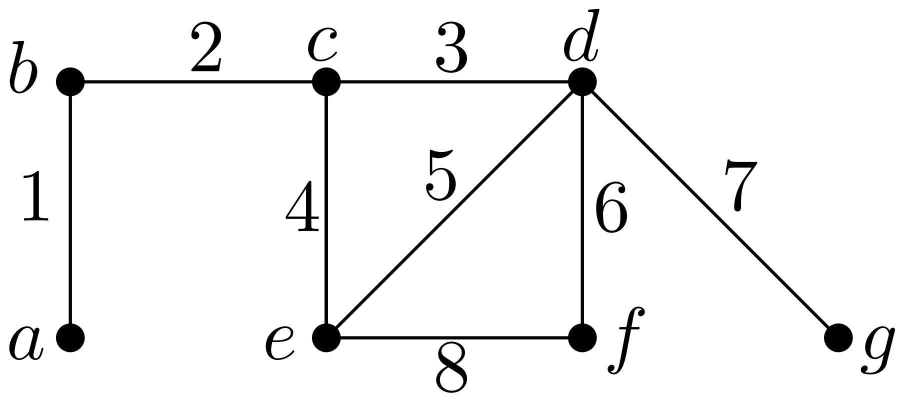

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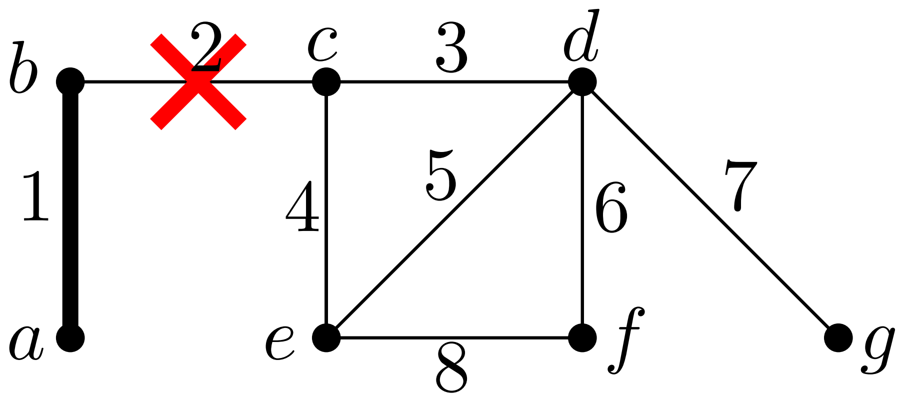

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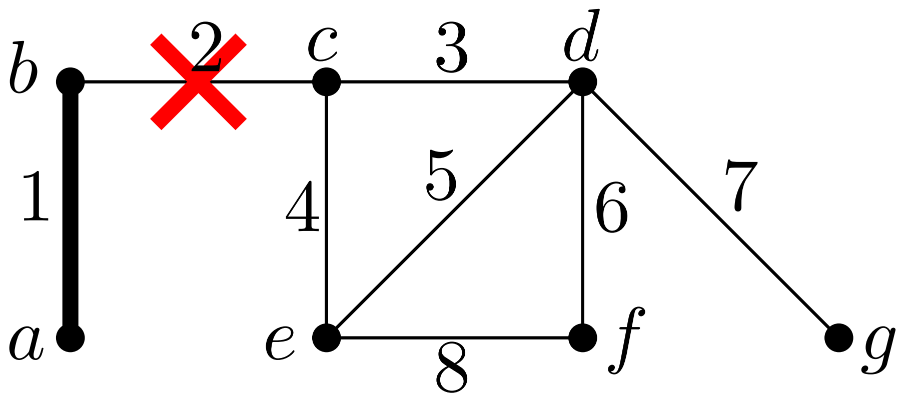

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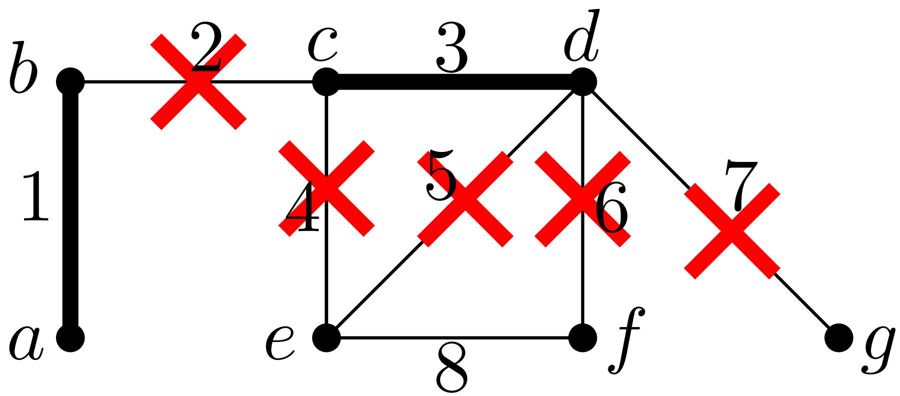

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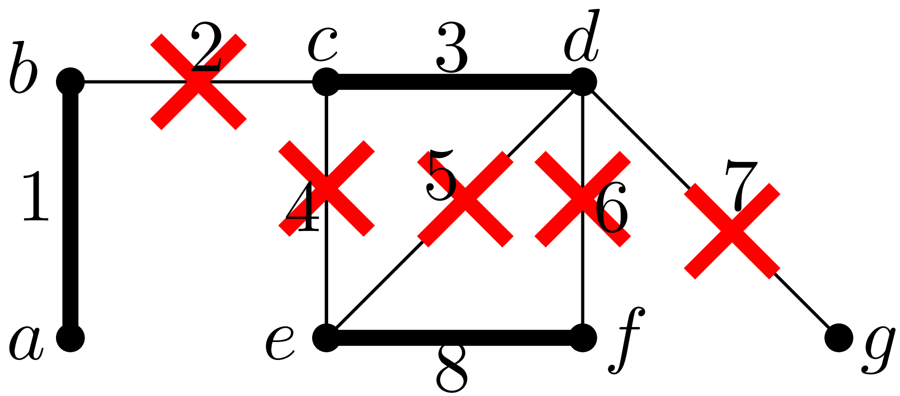

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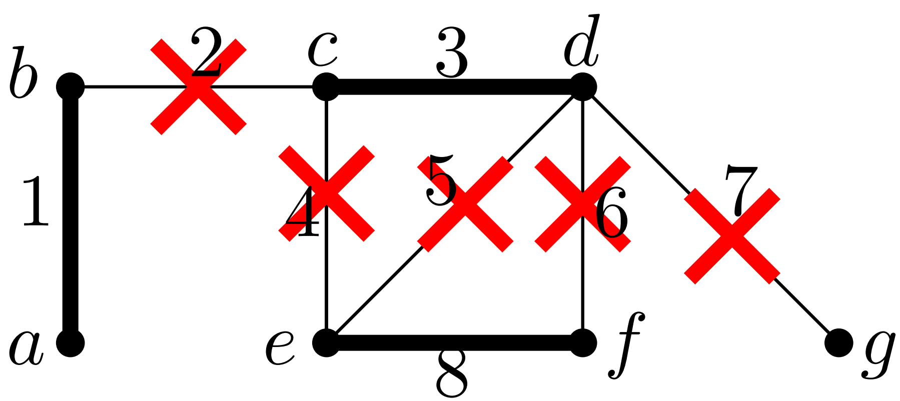

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Running time: $O(|V| + |E|)$

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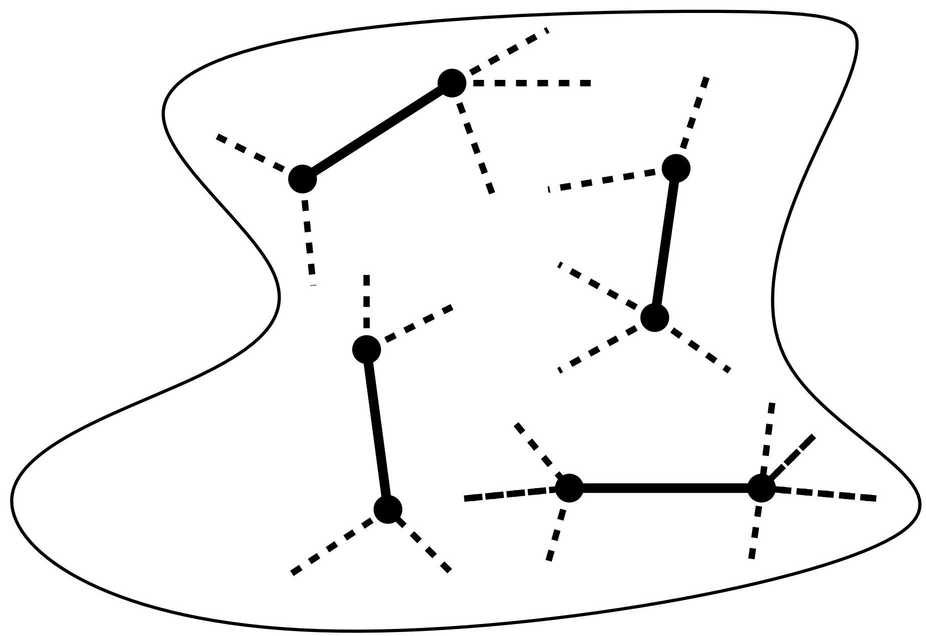
Let $A \subset E$ be the edges chosen by the algorithm.



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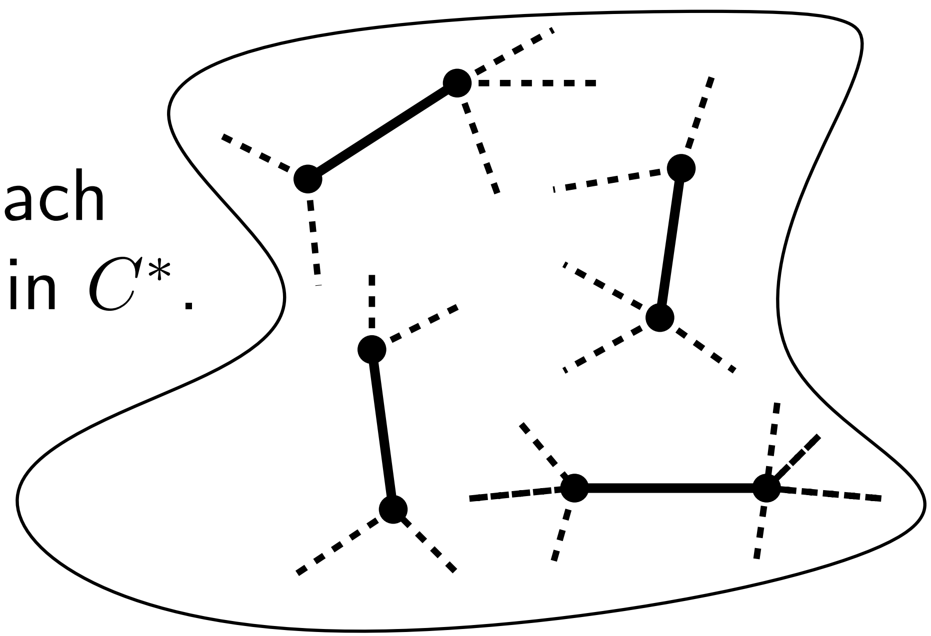

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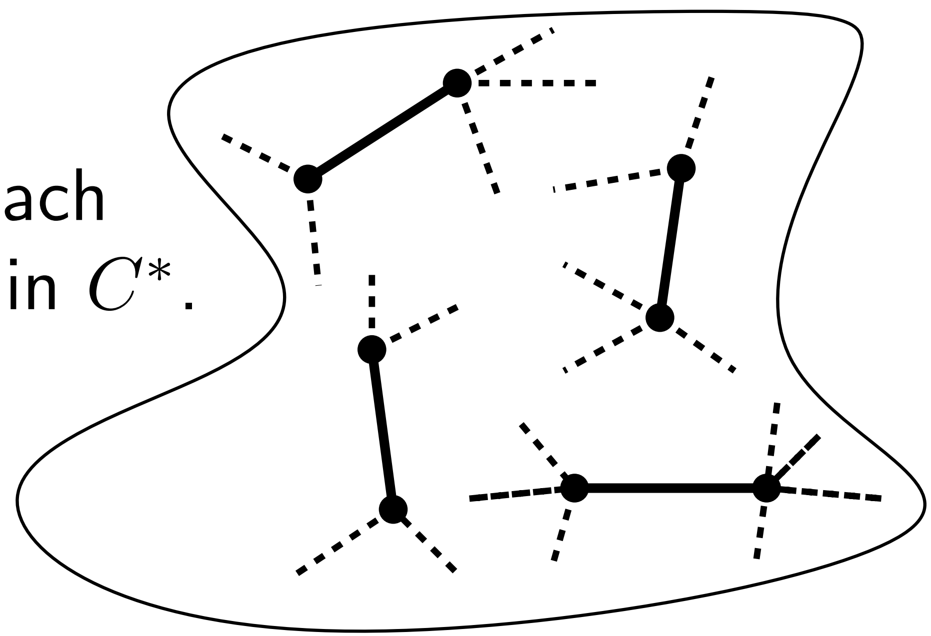

Hence,

$$|C^*| \geq |A| = |C|/2 \Longrightarrow \frac{|C|}{|C^*|} \leq 2.$$

Reflection and methodology

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Answer: By proving $C \leq 2|A|$ and $|A| \leq C^*$.

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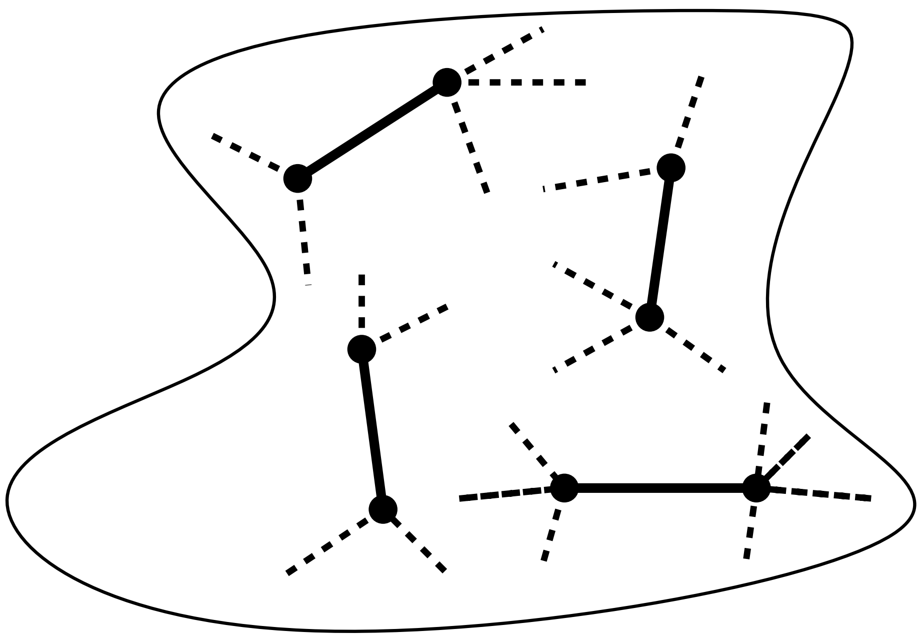
Question

Try to guess: Is there an approximation algorithm with a better approximation ratio?



History

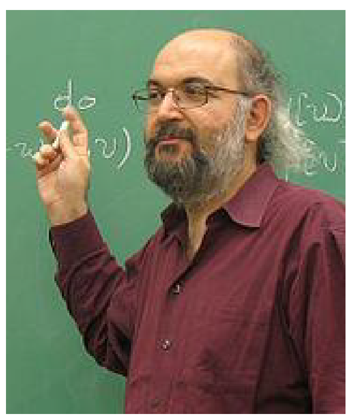
1972: Karp's 21 NP-complete problems
(including vertex cover, set cover, Hamiltonian cycle and subset sum)

Karp

Turing Award

History

19xx: Many $\leq 2 - o(1)$.



...

Gavril Yannakakis

History

Assuming P≠NP:
1999: Håstad, $\geq 7/6$
2005: Dinur & Safra, ≥ 1.38
2018: Khot, Minzer, Safra, ≥ 1.41


Håstad


Dinur


Safra


Khot


Minzer

History

2008: Khot & Regev, $\geq 2 - \varepsilon$ assuming the
Unique Game Conjecture.

Some, but not all people believe it.

Khot

Nevanlinna prize 2016

Regev

Traveling Salesperson

Given a complete undirected graph $G = (V, E)$.

For all $u, v \in V$, we are given $c(uv) \in \{0, 1, \ldots\}$.

Goal: Find minimum weight cycle through all vertices.

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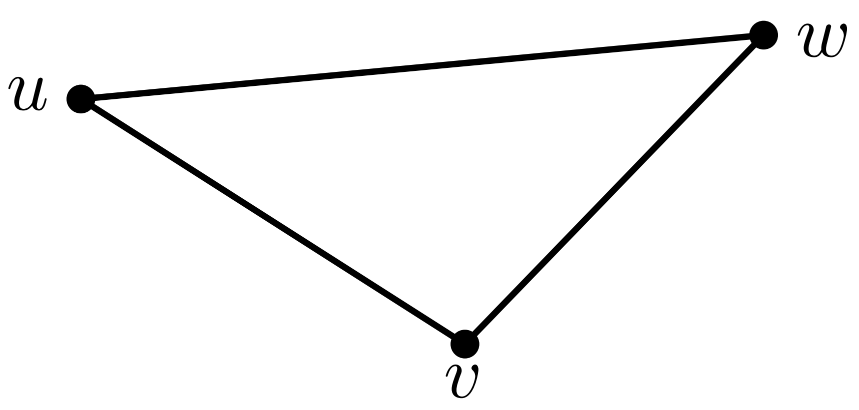

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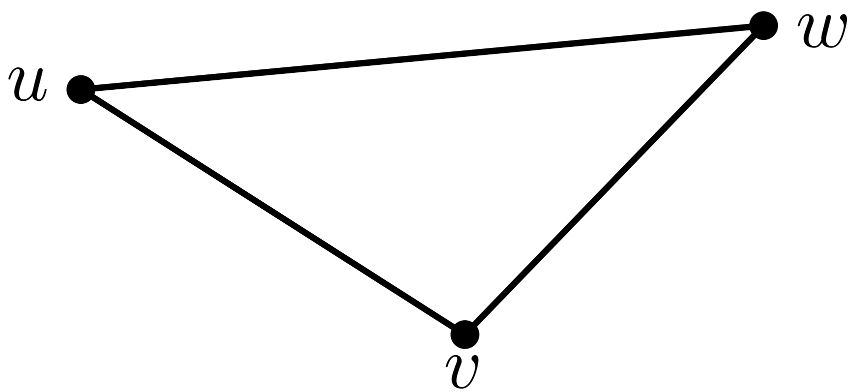
Still NP-hard!



Algorithm

APPROX-TSP(G, c)
  Find MST T
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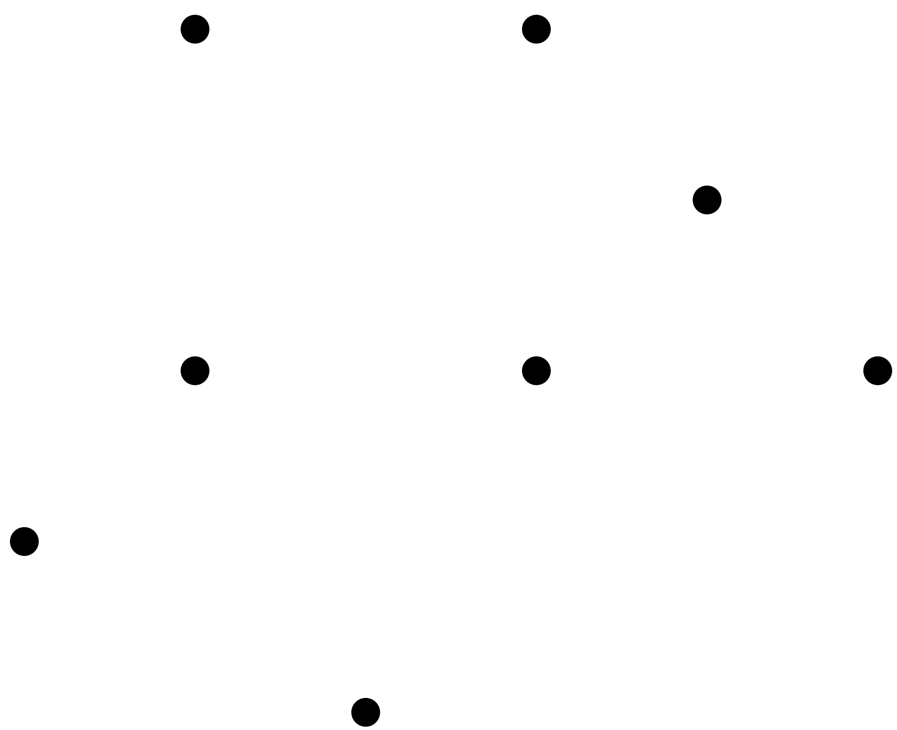

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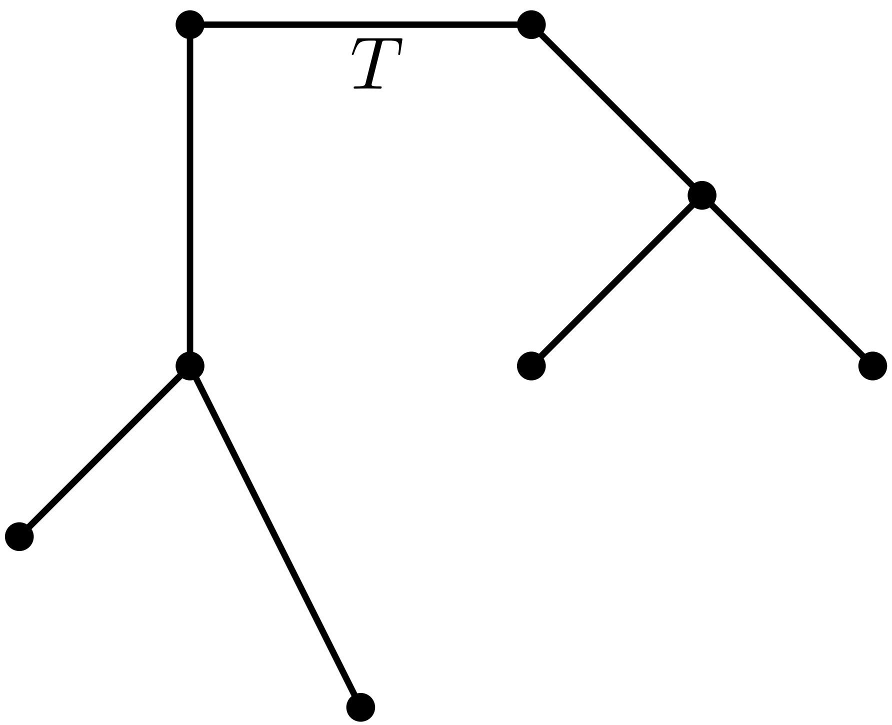

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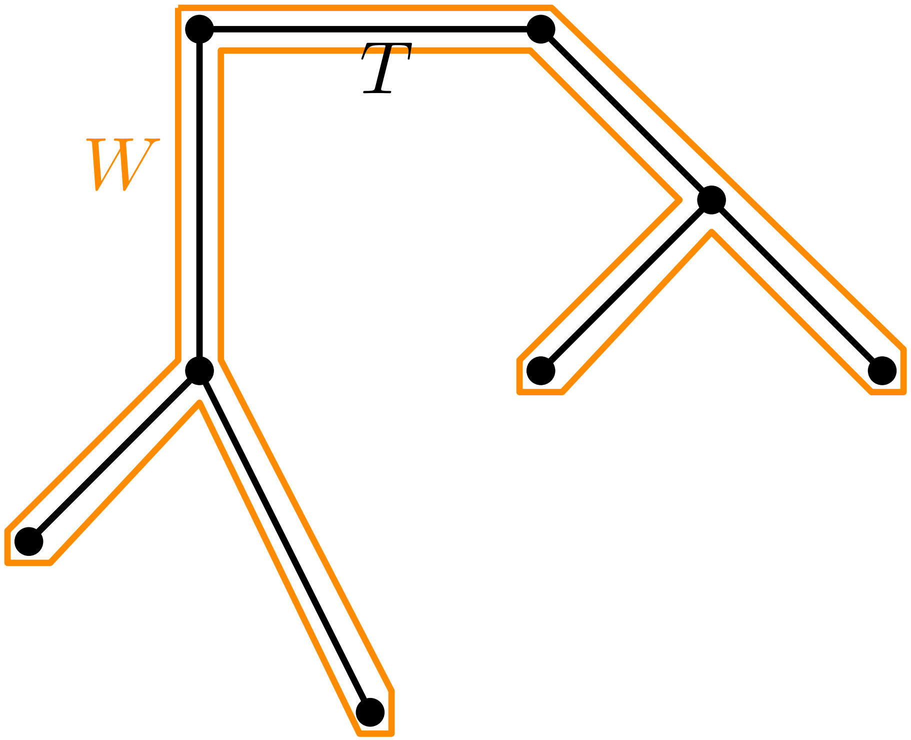

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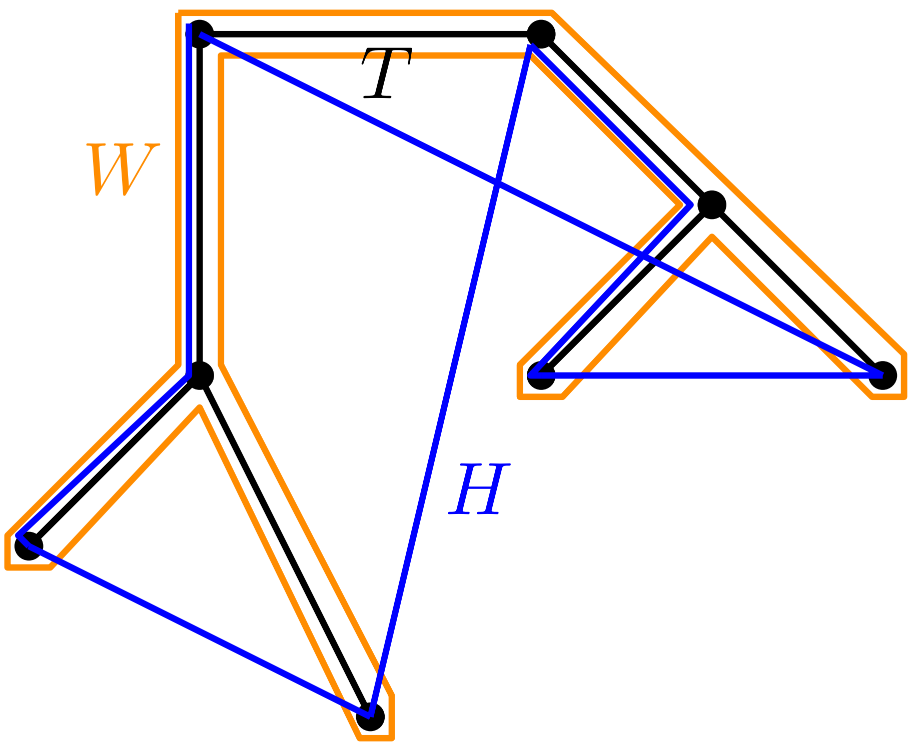

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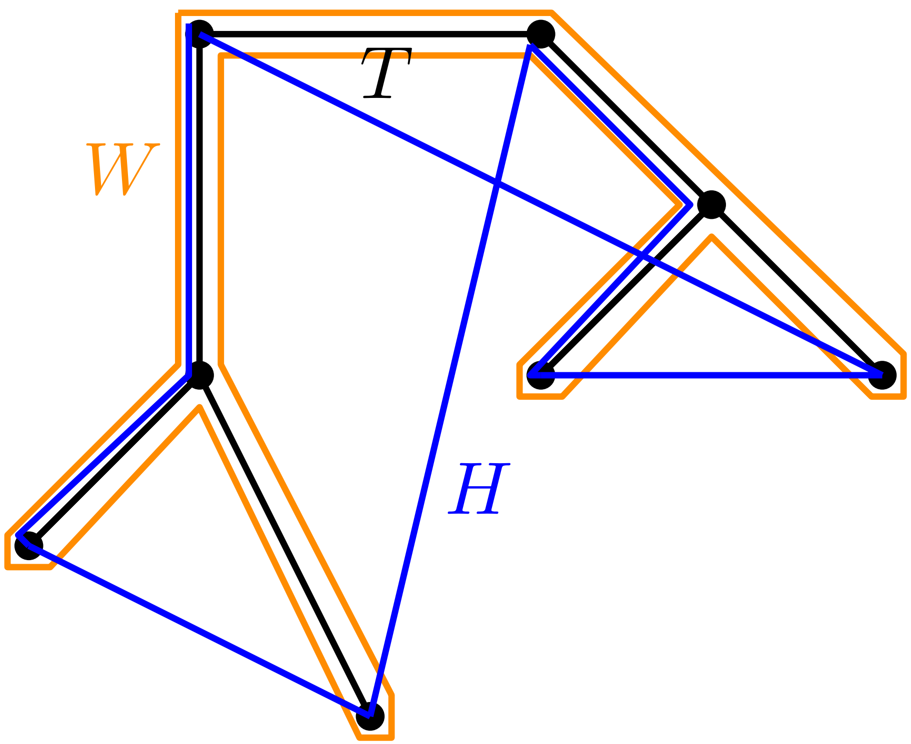
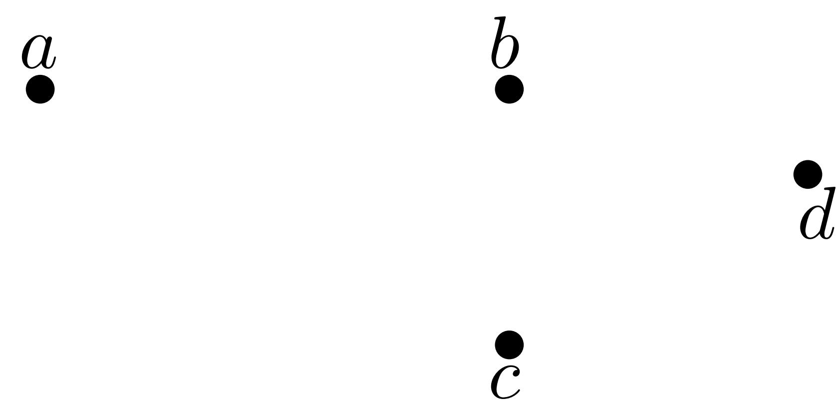


Exercise: Run the algorithm on this instance.



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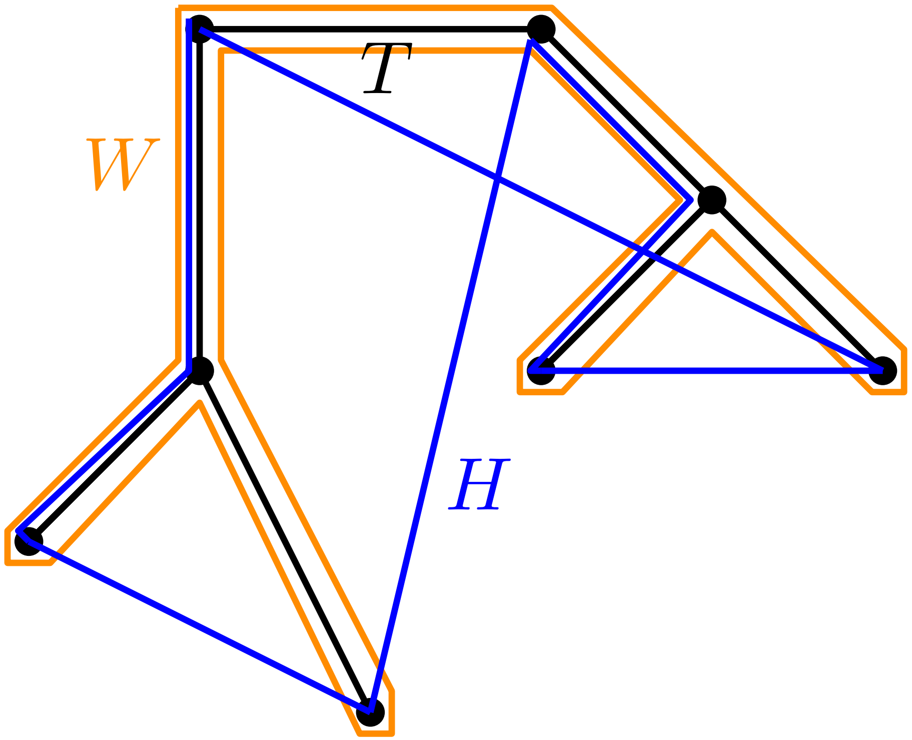

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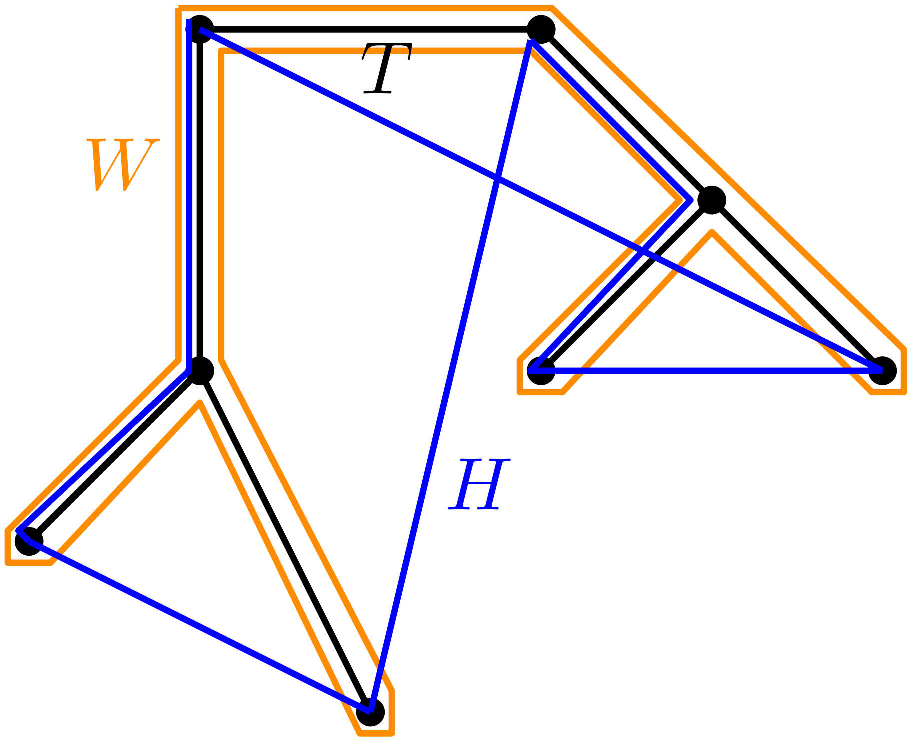

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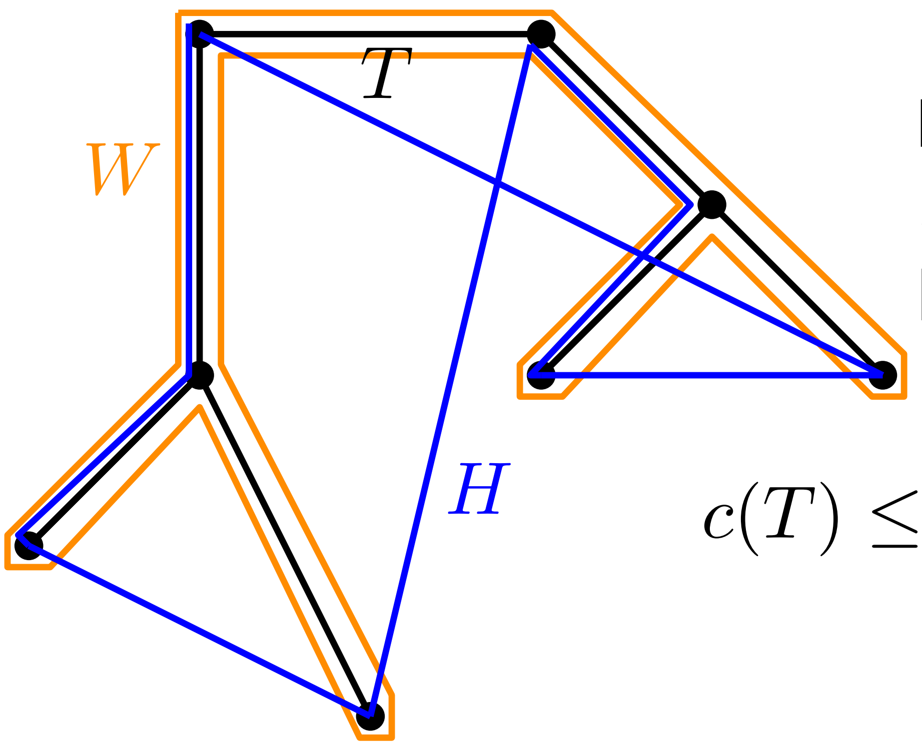

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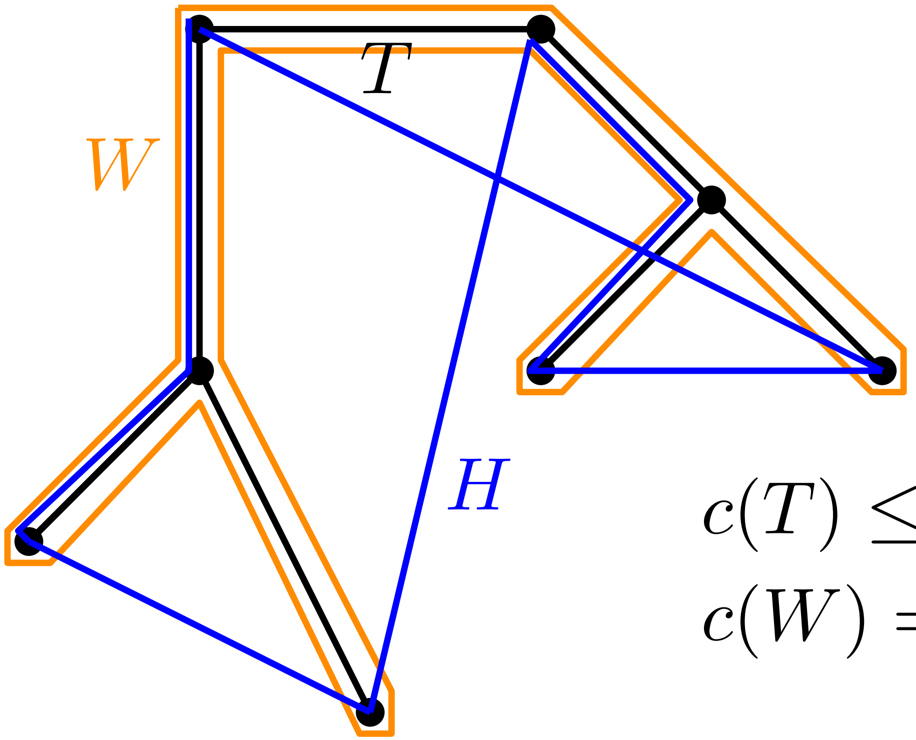

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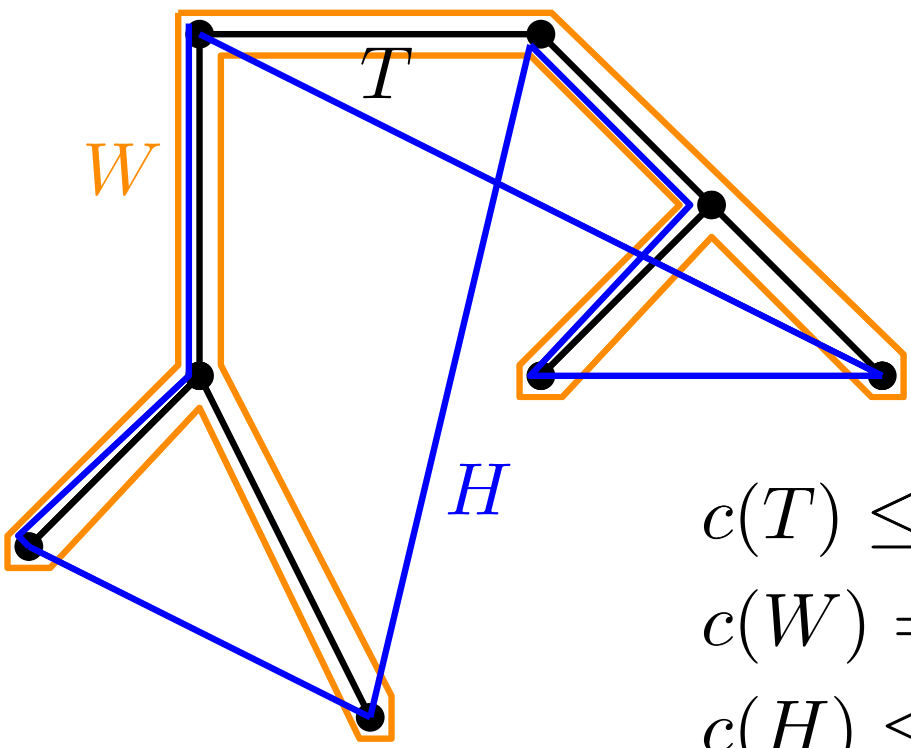

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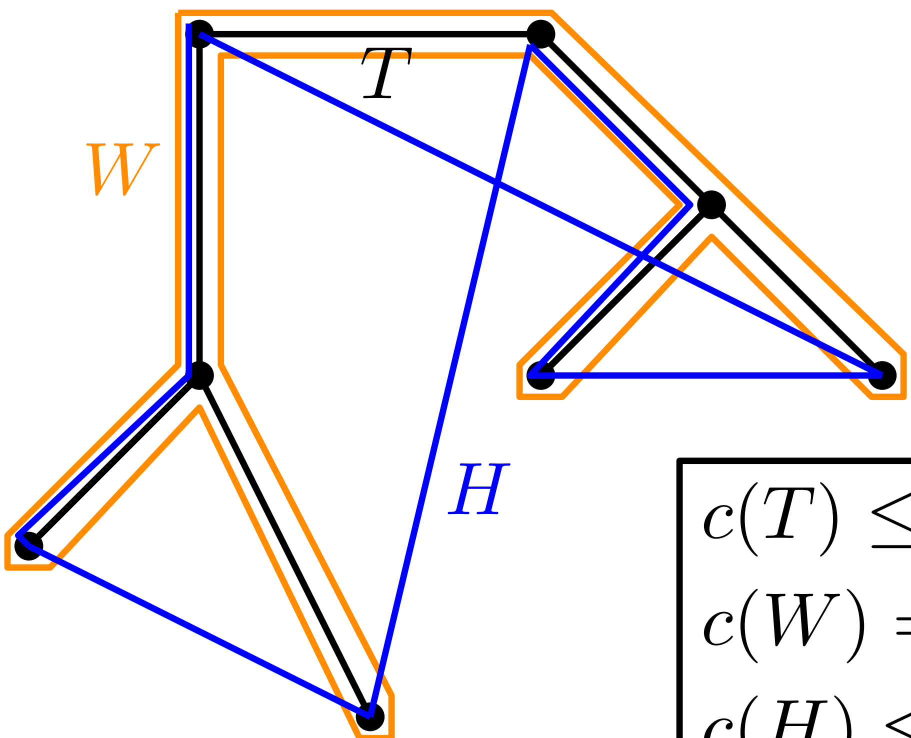

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$$\boxed{\begin{array}{l} c(T) \le c(H^*) \\ c(W) = 2c(T) \\ c(H) \le c(W) \end{array}} \implies c(H) \le 2c(H^*)$$

Reflection and methodology

How can we prove $c(H)/c(H^*) \leq 2$ when we don't know H^*?

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General technique: Find a parameter $\Box$ such that $C \leq \rho \cdot \Box$ and $\Box \leq C^*$.

For TSP: $\Box = c(T)$ and $\rho = 2$.

Question

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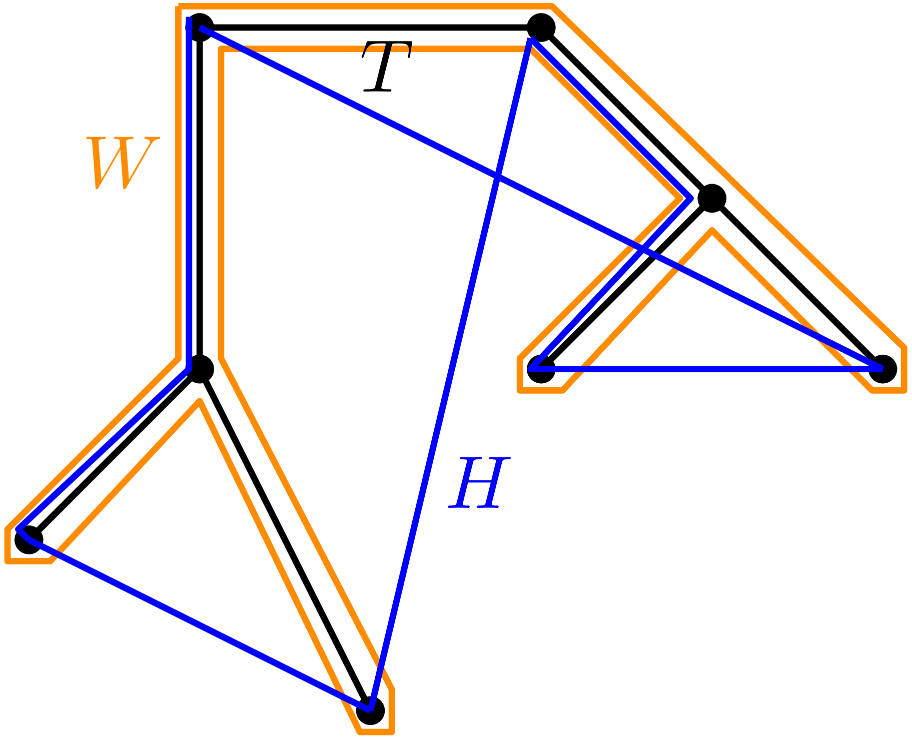

History

1976: Christofides, Serdyukov, 1.5-apx algorithm
It's simple! See, e.g., Wikipedia. No improvement for decades

2021: Karlin, Klein, Gharan, $(1.5 - \varepsilon)$-apx algorithm for some $\varepsilon > 10^{-36}$

Quanta magazine

Computer Scientists Break Traveling Salesperson Record

24

After 44 years, there's finally a better way to find approximate solutions to the notoriously difficult traveling salesperson problem.

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Input: Pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is a family of subsets of X.

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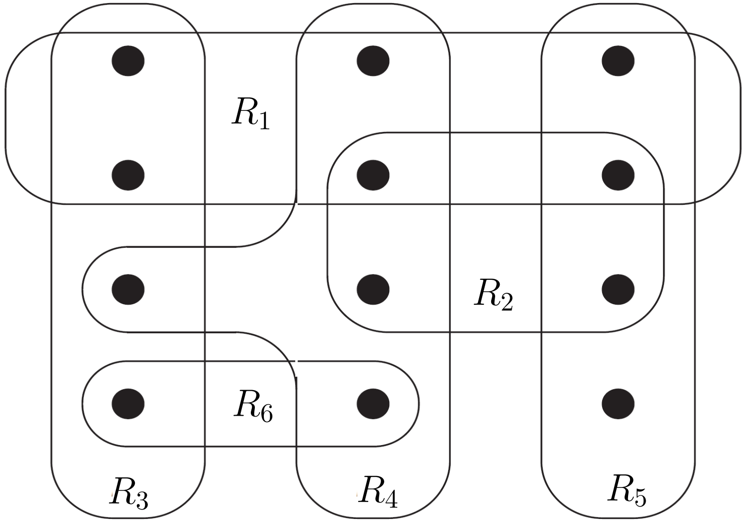

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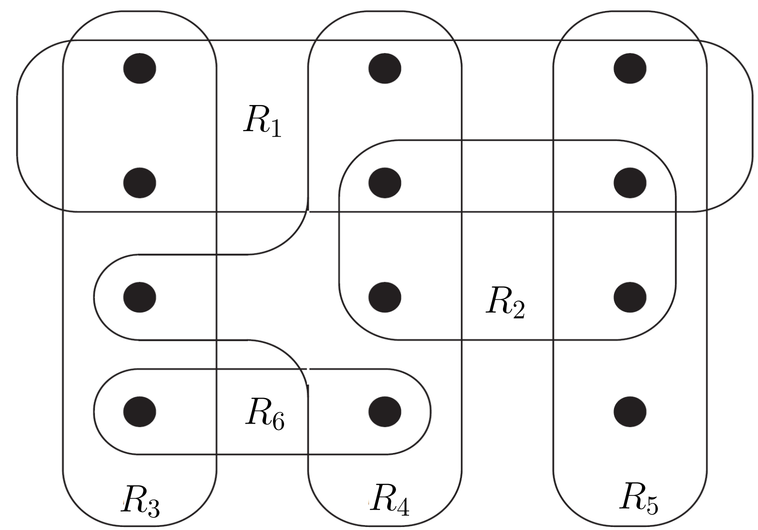
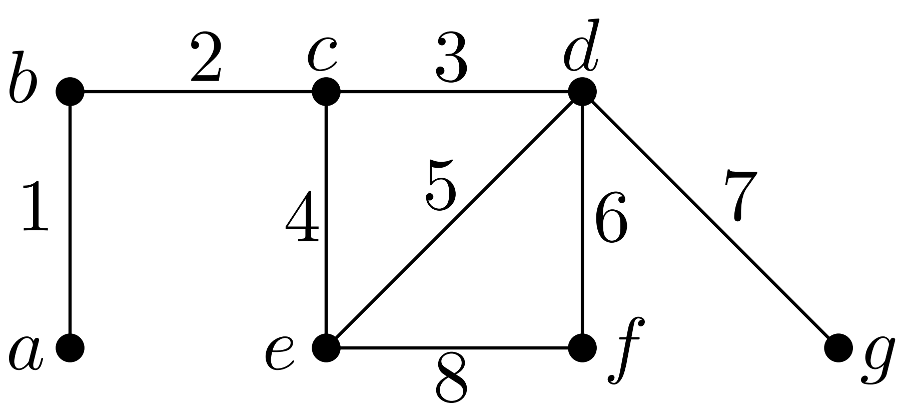
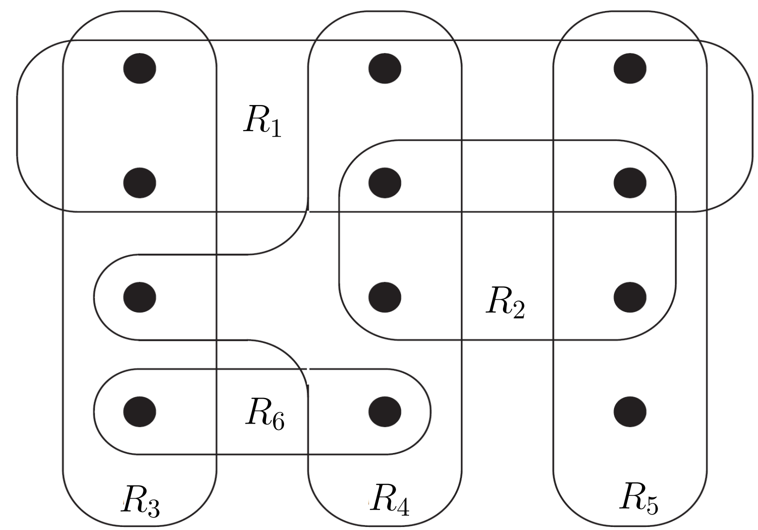


Exercise: Show that vertex cover is a special case.



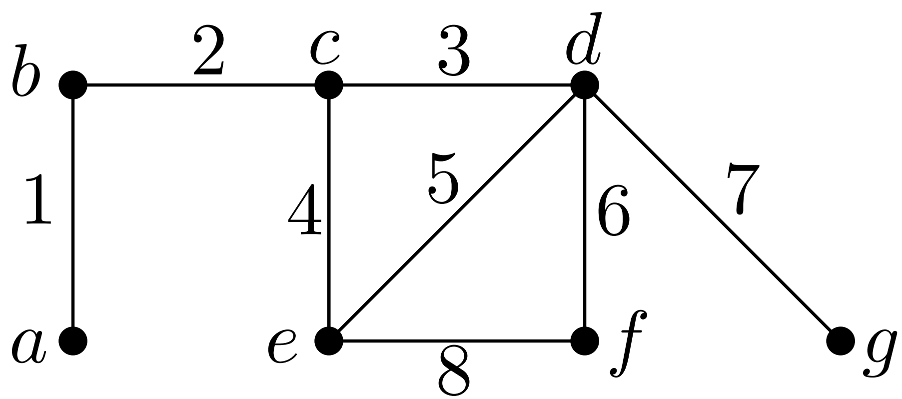
Set Cover

Input: Pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is a family of subsets of X.
Goal: Find $\mathcal{C} \subseteq \mathcal{F}$ covering X, i.e., $\bigcup_{S \in \mathcal{C}} S = X$, with $|\mathcal{C}|$ minimum.



Exercise: Show that vertex cover is a special case.

$X := \{1, 2, \ldots, 8\}$
$\mathcal{F} := \{\{1\}, \{1, 2\}, \{2, 3, 4\}, \{3, 5, 6\}, \{4, 5, 8\}, \{6, 8\}, \{7\}\}$



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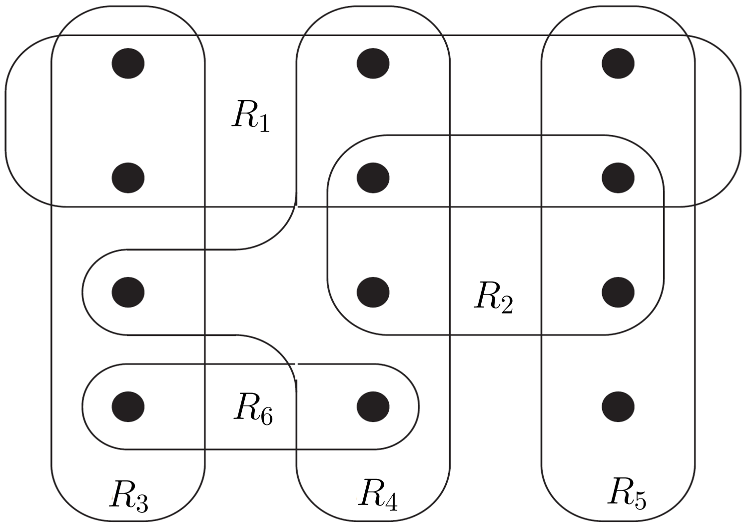

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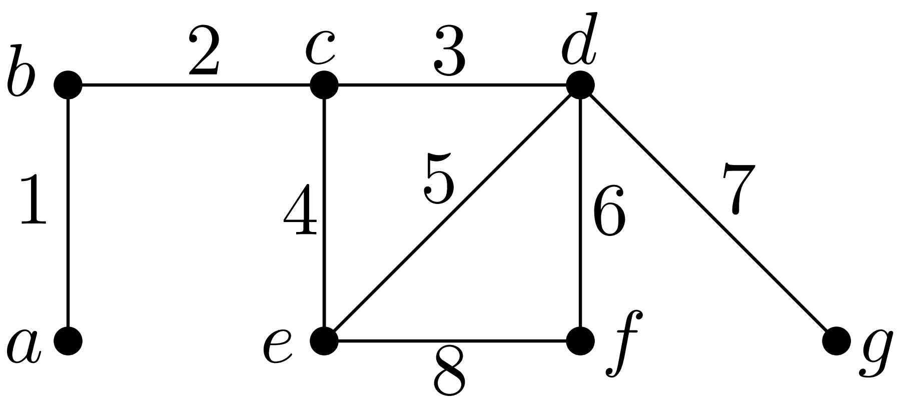

$$X := E$$
$$\mathcal{F} := \{E(v) \mid v \in V\}$$
$$E(v) := \{uv \in E \mid u \in V\}$$

Greedy Algorithm

GREEDY-SET-COVER($X, \mathcal{F}$)
- $i := 0$
- while $X \setminus S_{<i+1} \neq \emptyset$
  - $i := i + 1$
  - Pick $S_i \in \mathcal{F}$ with max $|S_i \setminus S_{<i}|$
- Return $\mathcal{C} := \{S_1, \ldots, S_i\}$

Here, $S_{<i} := \bigcup_{j=1}^{i-1} S_j$.

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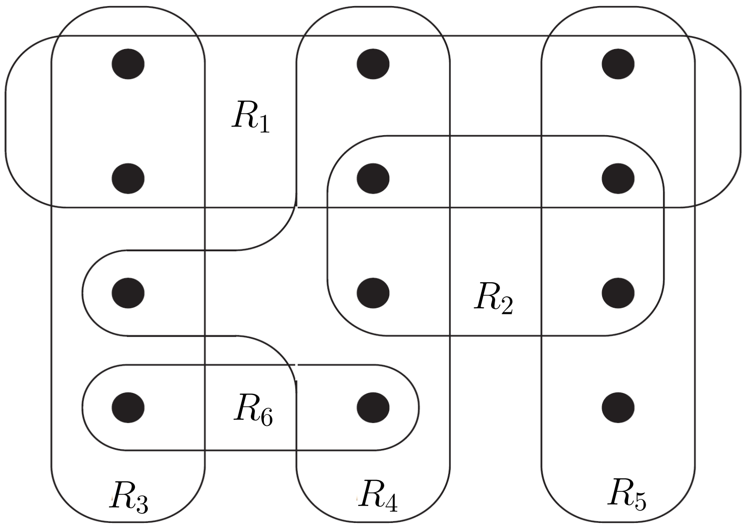
Exercise: Run the algorithm on this instance.



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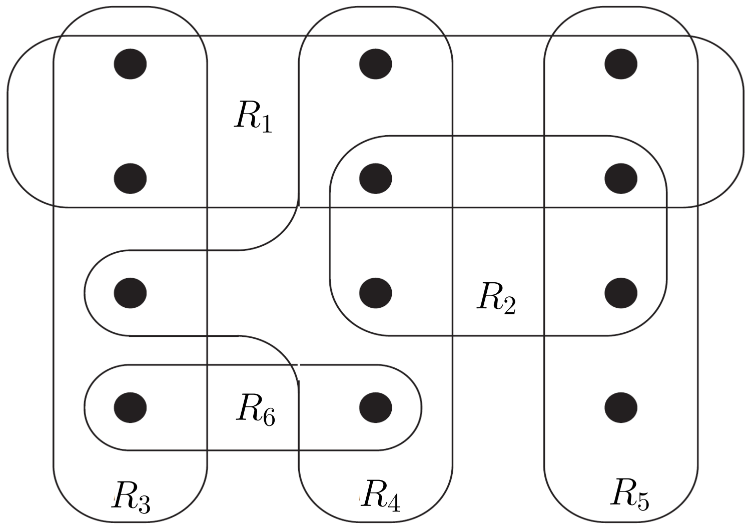
Return $\mathcal{C} := \{S_1, \ldots, S_i\}$

Here, $S_{<i} := \bigcup_{j=1}^{i-1} S_j$.

Exercise: Run the algorithm on this instance.

$S_1 := R_1$



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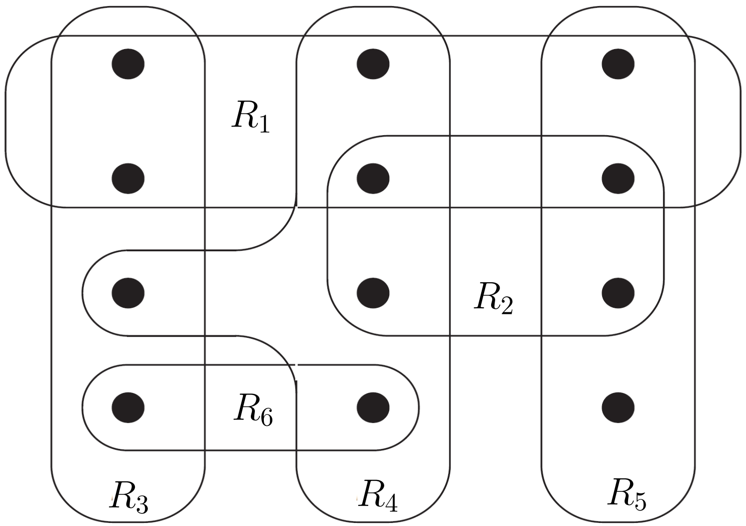
Here, $S_{<i} := \bigcup_{j=1}^{i-1} S_j$.

Exercise: Run the algorithm on this instance.

$S_1 := R_1$
$S_2 := R_4$



Greedy Algorithm

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    while $X \setminus S_{<i+1} \neq \emptyset$
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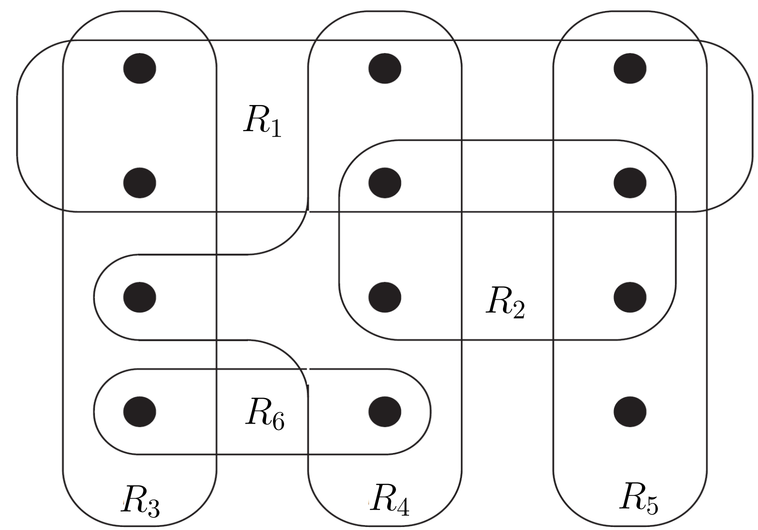
    Return $\mathcal{C} := \{S_1, \ldots, S_i\}$

Here, $S_{<i} := \bigcup_{j=1}^{i-1} S_j$.

Exercise: Run the algorithm on this instance.

$S_1 := R_1$
$S_2 := R_4$
$S_3 := R_5$



Greedy Algorithm

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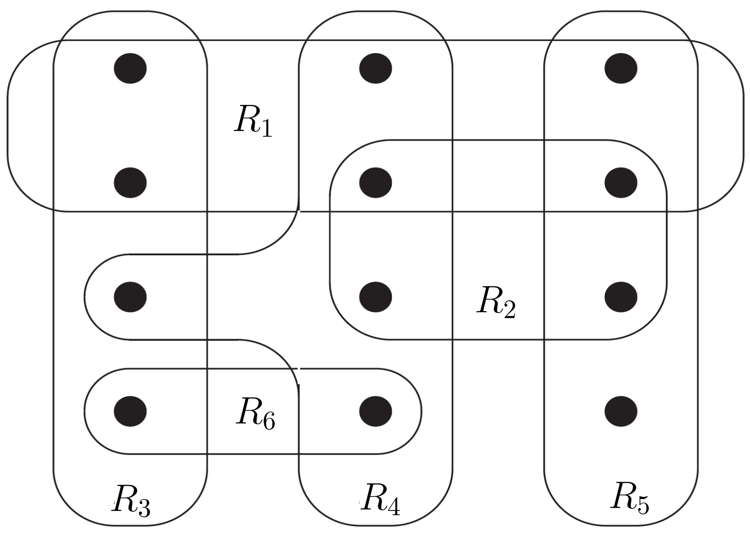
$S_1 := R_1$

$S_2 := R_4$

$S_3 := R_5$

$S_4 := R_3$ or $S_4 := R_6$



Theorem

Thm.: For opt. sol. $\mathcal{C}^*$, we have

$$|\mathcal{C}| \leq H_{|X|} \cdot |\mathcal{C}^*|,$$

where

$$H_n := \sum_{i=1}^{n} 1/i \leq \ln n + 1.$$

```
GREEDY-SET-COVER(X, F)
   i := 0
   while X \ S_{<i+1} ≠ ∅
      i := i + 1
      Pick S_i ∈ F with max |S_i \ S_{<i}|
   Return C := {S_1, ..., S_i}
```

Hence, GREEDY-SET-COVER is a $O(\log n)$-approx. alg.

Theorem

$\boxed{\textbf{Thm.: } |\mathcal{C}| \leq H_{|X|} \cdot |\mathcal{C}^*|.}$

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Theorem

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For $x \in S_i \setminus S_{<i}$, define $c_x := \frac{1}{|S_i \setminus S_{<i}|}$.
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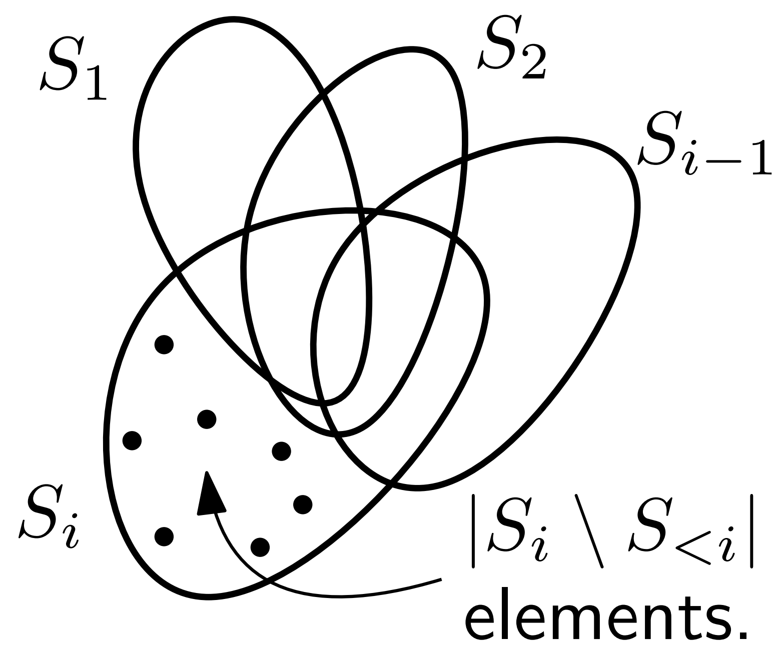

Theorem

Thm.: $|\mathcal{C}| \leq H_{|X|} \cdot |\mathcal{C}^*|$.

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Observation:

$$c(X) = \sum_{i=1}^{|\mathcal{C}|} \sum_{x \in S_i \setminus S_{<i}} c_x = \sum_{i=1}^{|\mathcal{C}|} 1 = |\mathcal{C}|.$$

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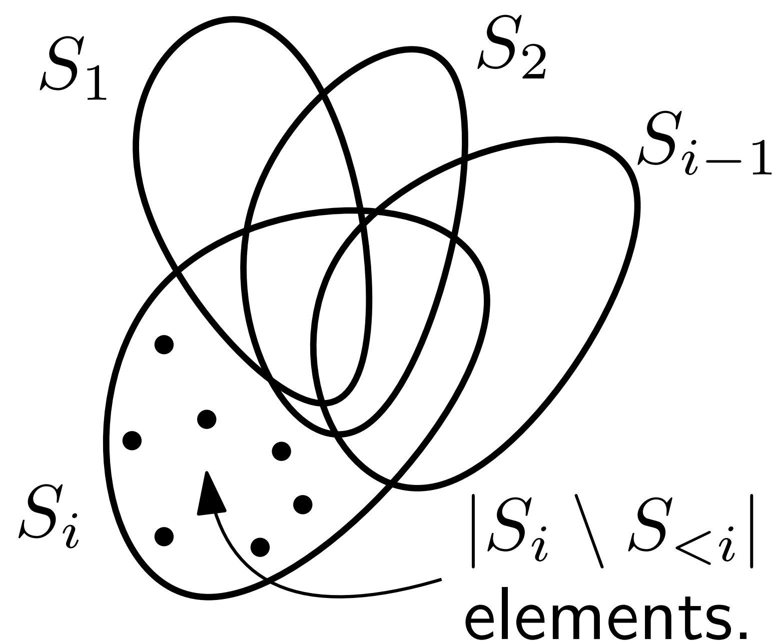

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Lemma: For all $S \in \mathcal{F}$:

$$c(S) \leq \sum_{i=1}^{|S|} \frac{1}{i} = H_{|S|}.$$

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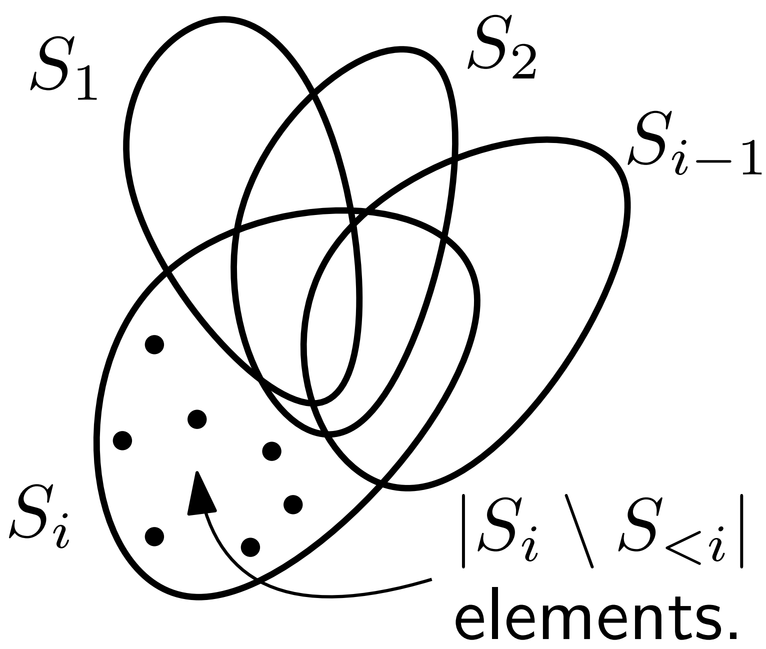

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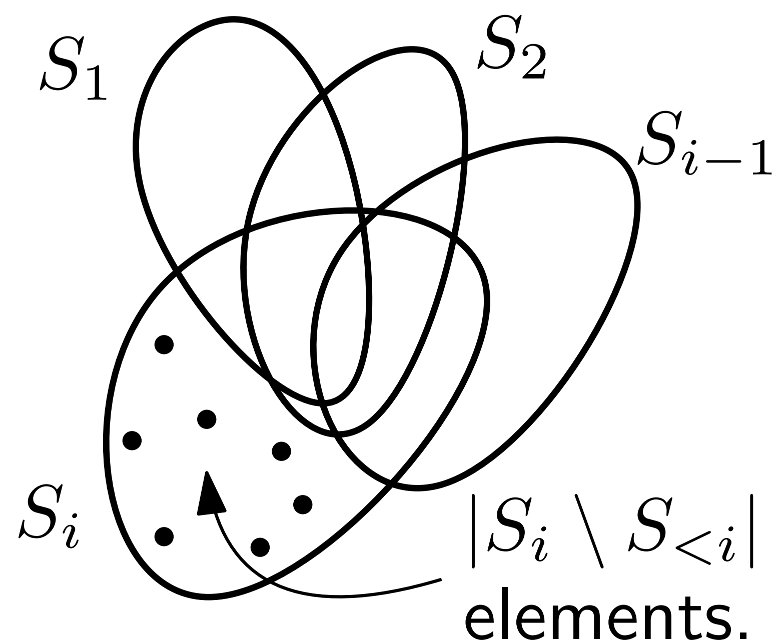

Proof of Thm.:

$$|\mathcal{C}| = c(X) \leq \sum_{S \in \mathcal{C}^*} c(S) \leq \sum_{S \in \mathcal{C}^*} H_{|S|} \leq \sum_{S \in \mathcal{C}^*} H_{|X|} = |\mathcal{C}^*| \cdot H_{|X|}.$$

Lemma: Idea and Example

Lemma: For all $S \in \mathcal{F}$:

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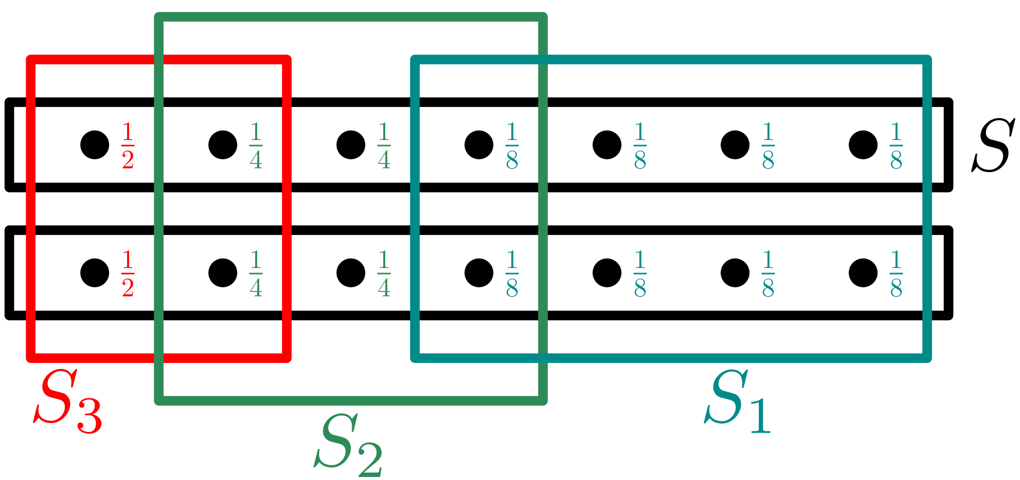
Idea: 1st element in S to be covered has $c_x \le \frac{1}{|S|}$, 2nd has $c_x \le \frac{1}{|S|-1}$, ...

Example:



$$\begin{aligned} c(S) &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &\le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \\ &= H_{|S|}. \end{aligned}$$

Proof of Lemma

Lemma: For all $S \in \mathcal{F}$:

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Proof: Let $S = \{x_k, x_{k-1}, \ldots, x_1\}$, where x_k covered first, then x_{k-1}, etc. (break ties arbitrarily).

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x_j covered first by $S_i \implies |S \setminus S_{<i}| \ge j$
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by greedy choice of S_i

Proof of Lemma

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by greedy choice of S_i

$$c(S) = c_{x_1} + c_{x_2} + \ldots + c_{x_k} \leq 1 + \frac{1}{2} + \ldots + \frac{1}{k} = H_{|S|}$$

Using greedy algorithm for vertex cover

GREEDY-VERTEX-COVER(G)
    $C := \emptyset$
    while $E \neq \emptyset$
        Choose $v \in V$ of maximum degree
        $C := C \cup \{u\}$
        Remove edges incident to u from E
    return C

Using greedy algorithm for vertex cover

```
GREEDY-VERTEX-COVER(G)
  C := ∅
  while E ≠ ∅
    Choose v ∈ V of maximum degree
    C := C ∪ {u}
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Exercise: Find graph G where GREEDY-VERTEX-COVER does not produce optimal solution.

Using greedy algorithm for vertex cover

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    return C
```

Exercise: Find graph G where GREEDY-VERTEX-COVER does not produce optimal solution.

The algorithm only gives a $\Theta(\log |E|)$-approximation.