Advanced algorithms and data structures

Lecture 9: Exact exponential algorithms and parameterized complexity

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October 4th 2021

Today's Lecture

Introduction

We usually want algorithms that

1) in polynomial time,
2) for all instances,
3) find an exact solution.

Unfortunately some problems are hard, and we may have to settle for (at best) 2 out of 3. We call such algorithms

Exact exponential algorithms
: if we relax 1) to allow using exponential time.

Parameterized algorithms
: if we relax 2) to instances with small fixed values of some parameter.

Approximation algorithms
: if we relax 3) to allow approximate solutions (next 2 lectures).

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AADS Lecture 9, Part 2

Exact exponential algorithms

Exact exponential algorithms

Recall that a *decision problem* is in NP if and only if there exists:

- A function $m(x) \in \mathcal{O}(\text{poly}|x|)$; and
- a polynomial-time verifier $R(x, y)$; such that
- for every problem instance x: x is a yes-instance if and only if there exists a certificate y of size $|y| \leq m(x)$ such that $R(x, y)$ is true.

Note: A certificate is a proof that a solution exists, but does not have to *be* a solution. However, a solution is often the most natural certificate.

Note: Every optimization problem has a decision version. What is it?

Every problem in NP has a simple brute-force algorithm of the following form: Given problem instance x, try all potential certificates y with $|y| \leq m(x)$ and check if $R(x, y)$ for any of them.

Since a potential certificate is just a bit string of length at most $m(x)$ there are at most $\mathcal{O}(2^{m(x)})$ potential certificates to check, and each check takes $\mathcal{O}(\text{poly}\,|x|)$ time. Thus, if we assume $m(x)$ can be computed in $\mathcal{O}(\text{poly}|x|)$ time, the brute force running time is $\mathcal{O}(2^{m(x)}\,\text{poly}\,|x|)$.

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Notation: $\mathcal{O}^\star(\cdot)$

For any $a > 1$, $\epsilon > 0$, and any $c \in \mathbb{R}$, we have $\mathcal{O}(n^c \cdot a^n) \subset \mathcal{O}((a+\epsilon)^n)$.

So when comparing exact exponential algorithms, the polynomial factors are mostly irrelevant.

Define

$$f(n) \in \mathcal{O}^\star(g(n)) \iff \exists c \in \mathbb{R} : f(n) \in \mathcal{O}(n^c \cdot g(n))$$

In other words, $\mathcal{O}^\star(\cdot)$ is the same as $\mathcal{O}(\cdot)$ but ignores polynomial factors.

Notice that for all $a > 1$ and $\epsilon > 0$: $\mathcal{O}(a^n) \subset \mathcal{O}^\star(a^n) \subset \mathcal{O}((a+\epsilon)^n)$.

Using this notation, what is the running time for the simple brute-force algorithm?

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Using this notation, what is the running time for the simple brute-force algorithm? $\mathcal{O}^\star(2^{m(x)})$

Size of a problem

What do we mean by the "size" n of a problem? Typically:

n, or $m + n$	for graphs with n vertices and m edges.
$\lvert S \rvert$	for problems involving some set S.
#variables	for SAT-type problems.

This measure of "size" is usually sufficient to describe the running time of the natural brute-force algorithm and to show improvements in better algorithms.

Problem	**certificate size**	**brute-force time**
SAT, MIS	$m(x) = n$	$T(n) = \mathcal{O}^\star(2^n)$
TSP	$m(x) = \log_2(n!)$	$T(n) = \mathcal{O}^\star(n!)$
k-Vertex Cover	$m(x) = k \log_2(n)$	$T(n) = \mathcal{O}(n^k \cdot \text{poly}\lvert x \rvert)$
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TSP	$m(x) = \log_2(n!)$	$T(n) = \mathcal{O}^\star(n!)$	$\mathcal{O}^\star(2^n)$
k-Vertex Cover	$m(x) = k\log_2(n)$	$T(n) = \mathcal{O}(n^k \cdot \mathrm{poly}\lvert x\rvert)$	$\mathcal{O}_k(m+n)$
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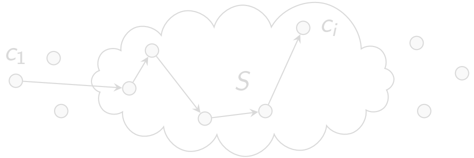
TSP via Dynamic Programming (Bellman-Held-Karp)

Problem: Given cities $c_1, \dots, c_n$, and distances $d_{ij} = d(c_i, c_j)$, find tour of minimal length, visiting all cities exactly once. Equivalently, find permutation π minimizing $d(c_{\pi(n)}, c_{\pi(1)}) + \sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)})$.

Idea: For all $S \subseteq \{c_2, \dots, c_n\}$ and $c_i \in S$ define $\mathrm{OPT}[S, c_i] :=$ minimum length of all paths in $S \cup \{c_1\}$ that starts in c_1, visits all of S once, and ends in c_i. Then $\min\{\mathrm{OPT}[\{c_2, \dots c_n\}, c_i] + d(c_i, c_1) \mid c_i \in \{c_2, \dots, c_n\}\}$ is the length of the minimal tour.

$\mathrm{OPT}[S, c_i]$:



Lemma

$$\mathrm{OPT}[S, c_i] = \begin{cases} d(c_1, c_i) & \text{if } S = \{c_i\} \\ \min\{\mathrm{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i) \mid c_k \in S \setminus \{c_i\}\} & \text{if } \{c_i\} \subset S \end{cases}$$

Proof.

Let $e = (c_k, c_i)$ be the last edge on such a path. If $k = 1$ we are done. If $k \neq 1$ the shortest length through e must be $\mathrm{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i)$. The shortest such path must use the minimum over all $c_k \in S \setminus \{c_i\}$. $\square$

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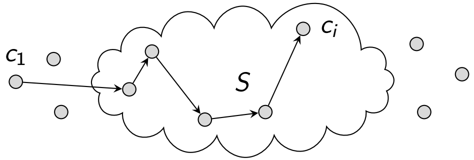

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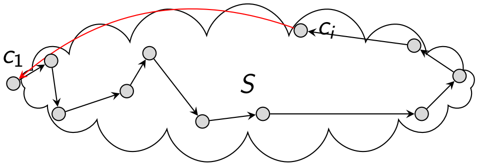

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$$\mathrm{OPT}[S, c_i] = \begin{cases} d(c_1, c_i) & \text{if } S = \{c_i\} \\ \min\{\mathrm{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i) \mid c_k \in S \setminus \{c_i\}\} & \text{if } \{c_i\} \subset S \end{cases}$$

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Let $e = (c_k, c_i)$ be the last edge on such a path. If $k = 1$ we are done. If $k \neq 1$ the shortest length through e must be $\mathrm{OPT}[S \setminus \{c_i\}, c_k] + d(c_k, c_i)$. The shortest such path must use the minimum over all $c_k \in S \setminus \{c_i\}$. $\square$

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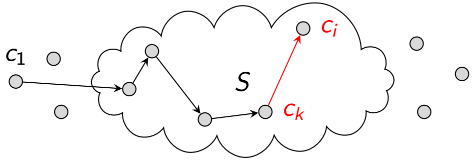

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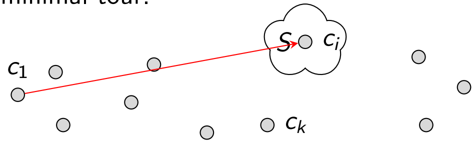

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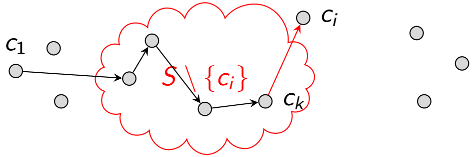

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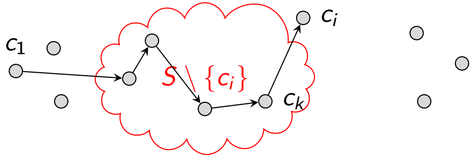

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What is the running time of the algorithm?

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What is the running time of the algorithm? $\mathcal{O}^\star(2^n)$ if we assume additions take at most polynomial time in n. Much better than $\mathcal{O}^\star(n!)$.

Dynamic Programming in general

Similar to "Divide and Conquer" in that it requires "Optimal Substructure" but subproblems may be overlapping.

Instead of recursively solving smaller disjoint subproblems, "Dynamic Programming" solves all smaller subproblems in order of increasing size.

A hybrid idea called "Memoization" (not "Memo**r**ization"!) does the same by using recursion, but caching results so each subproblem is only solved once.

In our TSP example the original problem does not have the optimal substructure property (A piece of an optimal tour does not have to be an optimal tour of some subgraph). The trick is to notice that the problem of computing $\mathrm{OPT}[\{c_2, \dots, c_n\}, c_i]$ *does* have the property, and that TSP can be solved once we know that for all c_i.

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Similar to “Divide and Conquer” in that it requires “Optimal Substructure” but subproblems may be overlapping.

Instead of recursively solving smaller disjoint subproblems, “Dynamic Programming” solves all smaller subproblems in order of increasing size.

A hybrid idea called “Memoization” (not “Memo**r**ization”!) does the same by using recursion, but caching results so each subproblem is only solved once.

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MIS via Branching

Problem: Given undirected graph (V, E), find the maximum cardinality of $I \subseteq V$ so each edge has at most one endpoint in I.

Such a set I is called a *Maximum Independent Set (MIS)* for the graph.

Naive: Try all 2^n subsets (where $n = |V|$). This takes $\mathcal{O}^\star(2^n)$ time.

For $v \in V$ define $N[v] := \{v\} \cup \{w \in V \mid (v, w) \in E\}$. This is called the *closed neighborhood* of v.

Observation: $N[v] \cap I \neq \emptyset$ for all $v \in V$ and all MIS I.
Why?

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1: function MISsize($G = (V, E)$)
2:     if $V = \emptyset$ then return 0
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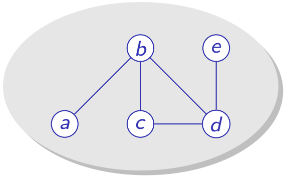

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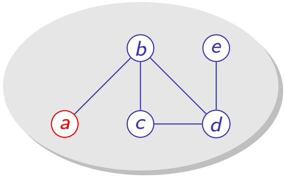

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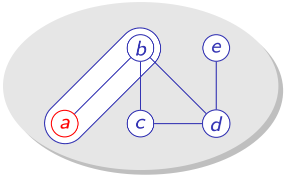

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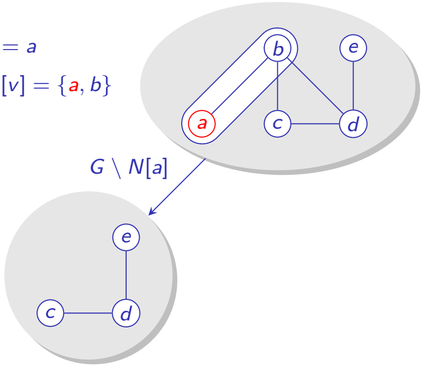

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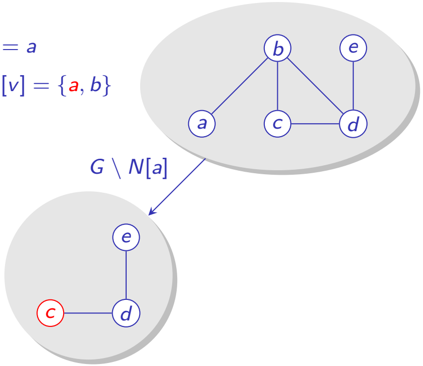
$G \setminus N[a]$

$v = c$



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$v = c$

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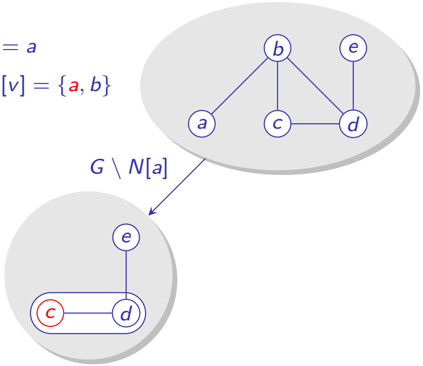

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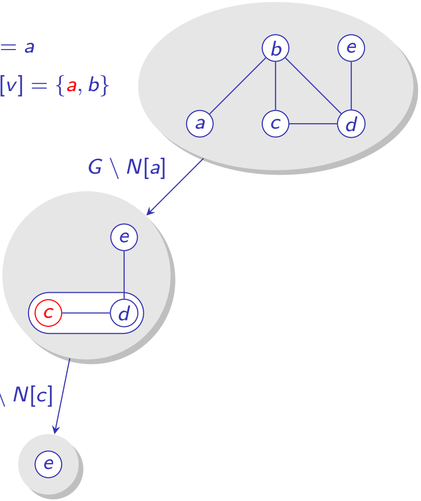

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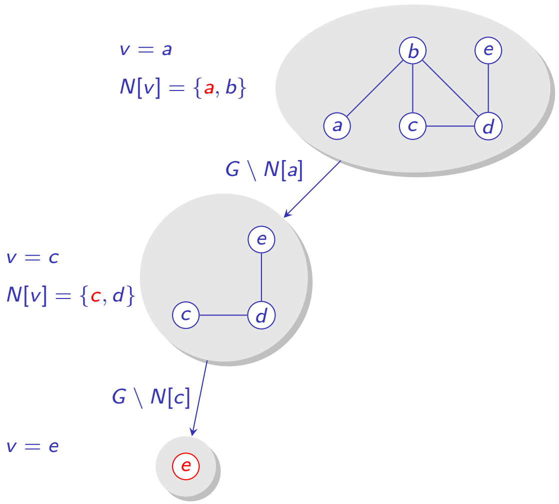

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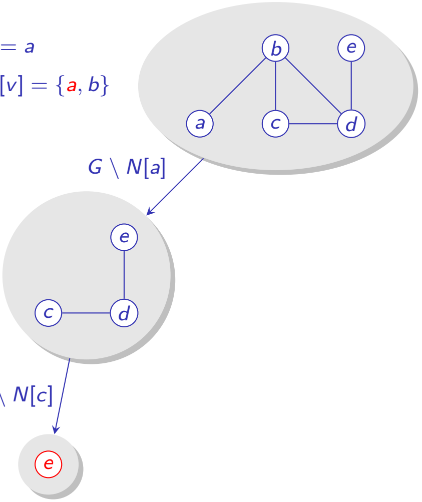

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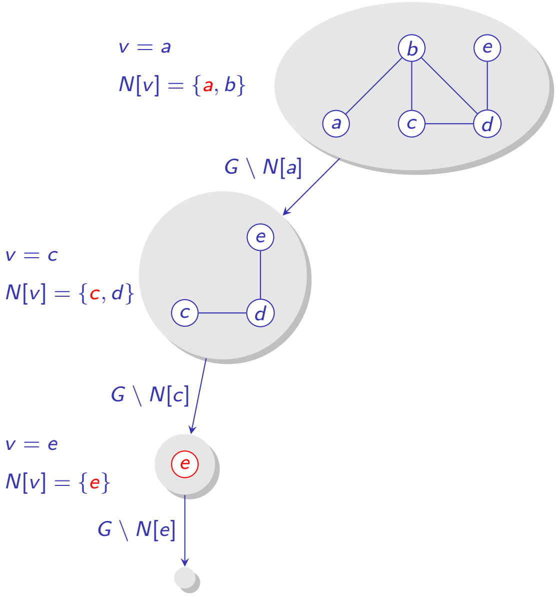

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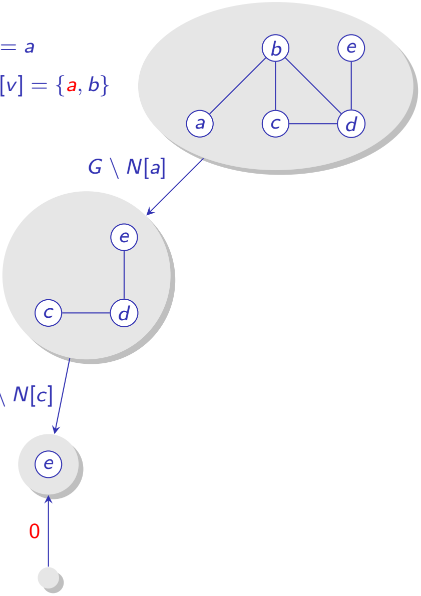

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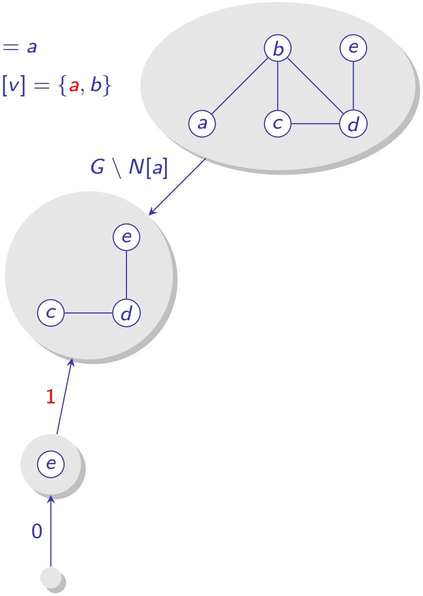

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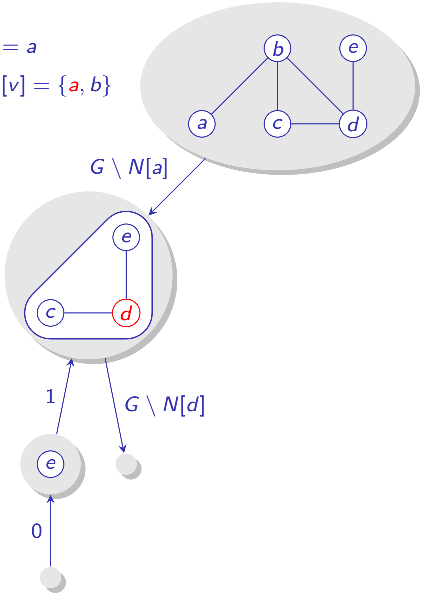

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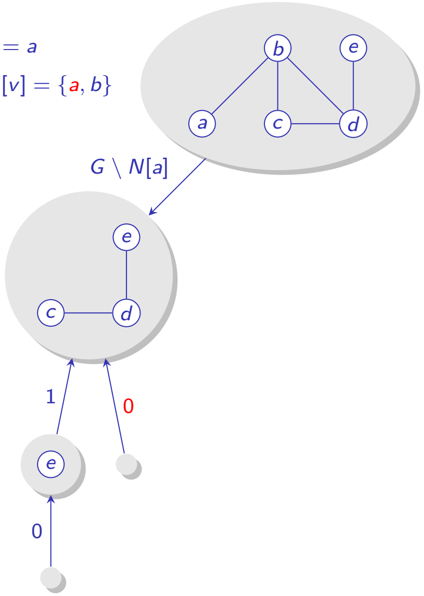

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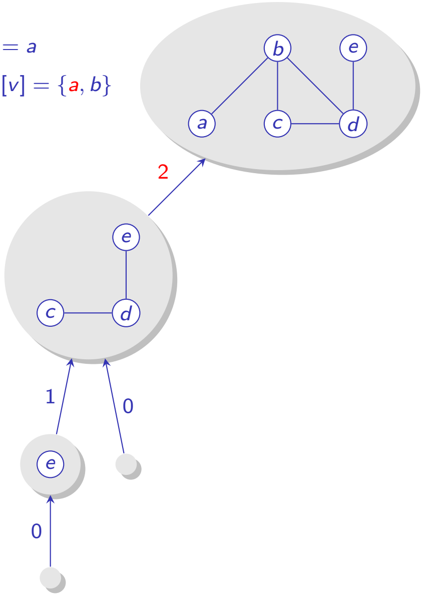

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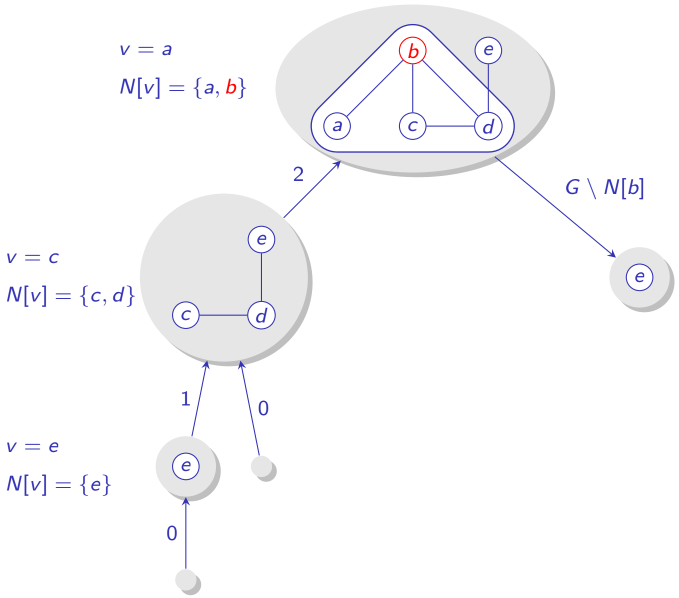

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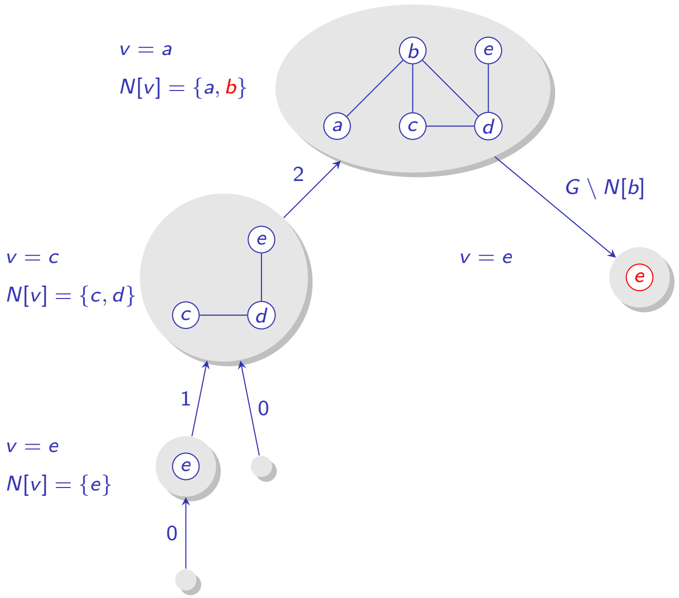

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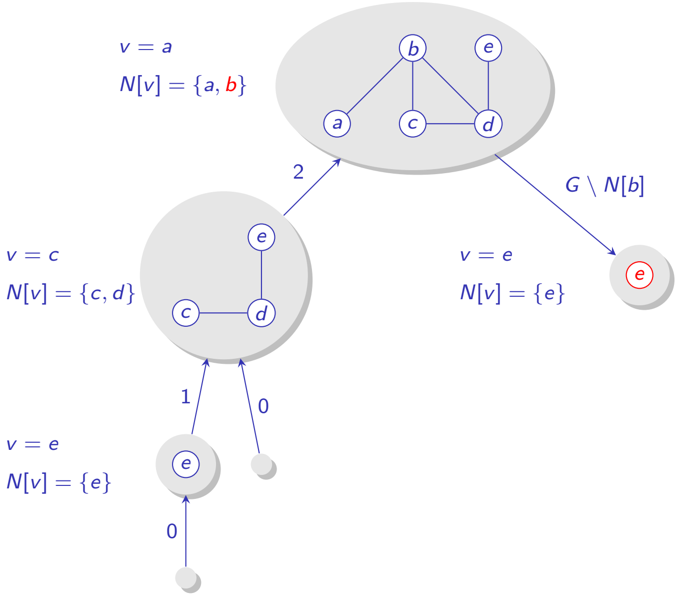

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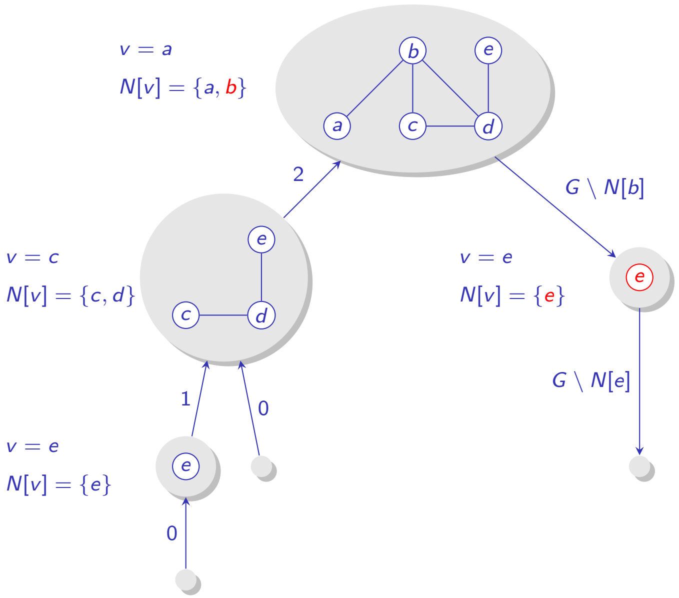

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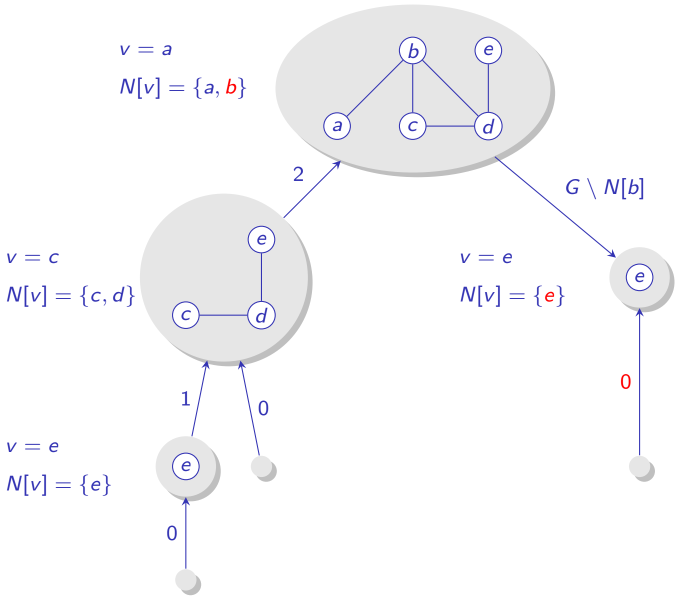

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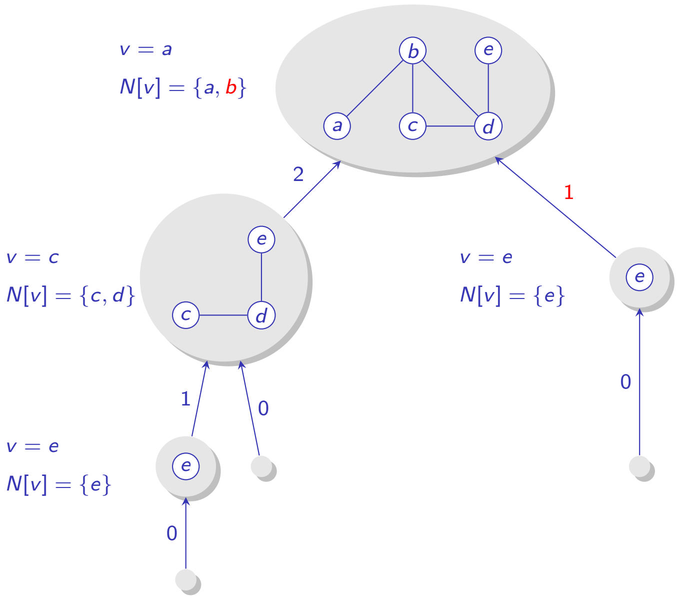

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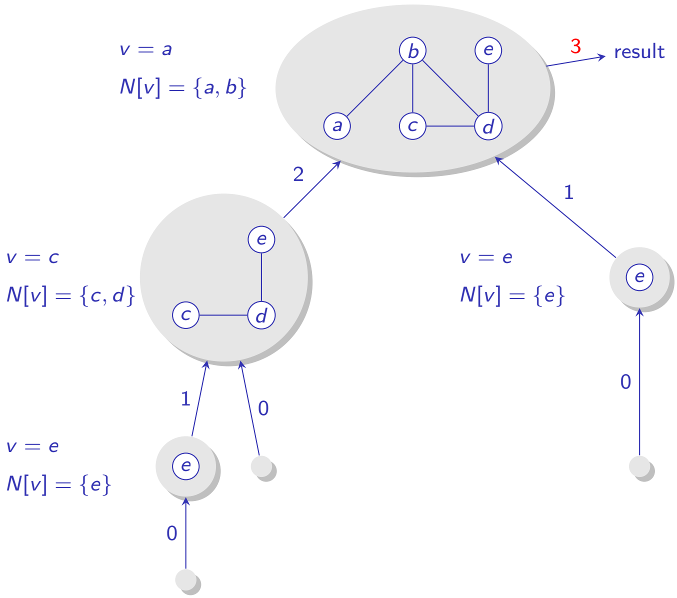

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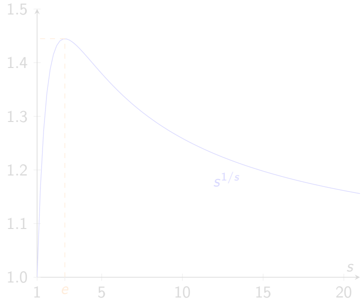

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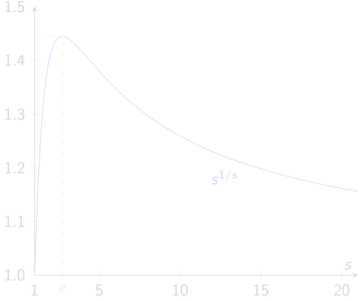

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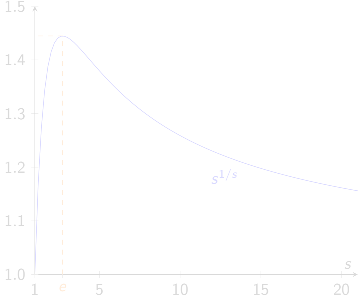

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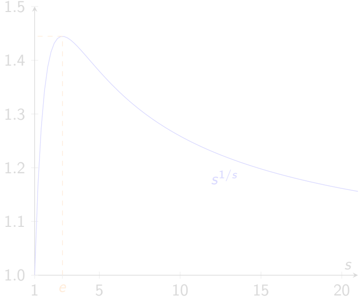

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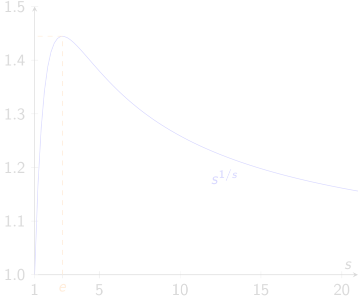

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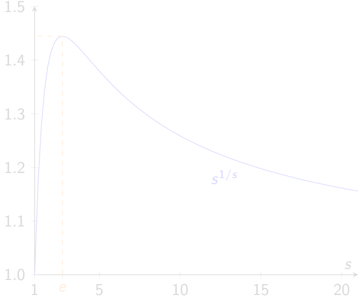

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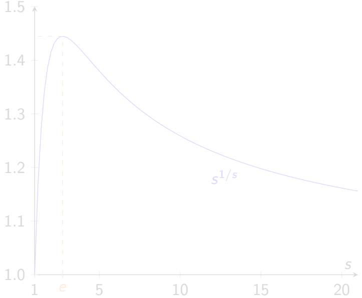

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&\le 1 + s + s^2 + \cdots + s^{n/s} \\
&= \frac{s^{1+n/s}-1}{s-1} < 2s^{n/s} \qquad (\text{for } s \ge 2) \\
&\in \mathcal{O}(s^{n/s}) \subseteq \mathcal{O}(3^{n/3}) \qquad \square
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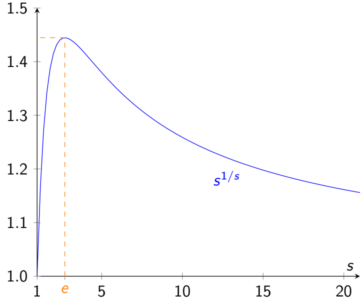
$$



MIS via Branching

Let $T(n)$ be the maximum number of subproblems considered by the branching algorithm on a graph with n vertices, then (very loosely):

$$\begin{aligned}
T(0) &= 1 \\
T(n) &\leq 1 + \sum_{w \in N[v]} T(n - (d(w)+1)) && \text{for some } v \in V \text{ of minimum degree} \\
&\leq 1 + (d(v)+1) \cdot T(n - (d(v)+1)) && \text{since } T(\cdot) \text{ is nondecreasing} \\
&= 1 + s \cdot T(n-s) && \text{where } s = d(v)+1 \text{ and thus } s \in \{1, \ldots, n-2\}
\end{aligned}$$

Lemma

$T(n) \in \mathcal{O}(3^{n/3}) \subset \mathcal{O}(1.44225^n)$

"Proof" (spot the error).

$$\begin{aligned}
T(n) &\leq 1 + s \cdot T(n-s) \\
&\leq 1 + s + s^2 + \cdots + s^{n/s} \\
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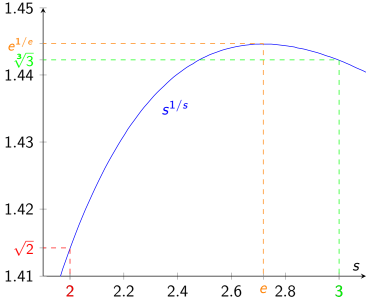

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Lemma
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"Proof" (Error: Depends on v).

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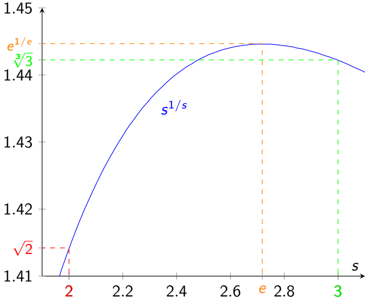

AADS Lecture 9, Part 3

Parameterized problems

"Bar fight prevention" aka k-Vertex Cover

Problem: Bouncer in a small city wants to block people at the door to prevent fights. Assume he knows everyone and knows which pairs of people would fight if they were both let in. Management only allows him to block k of the n people who wants in. Is that enough to prevent fights, and if so, who should be blocked?

Equivalent Problem: Given a graph (V, E) with $n = |V|$ vertices, is there a subset $C \subseteq V$ of size $|C| \leq k$ such that every edge has at least one endpoint in C? Such a set C is called a *k-Vertex Cover* in the graph, and its complement $V \setminus C$ is an *Independent Set* of size $n - k$.

For concreteness in the following, suppose $n = 1000$ and $k = 10$.
Naive 1: Try all 2^n subsets of people. ($2^{1000} \approx 1.07 \cdot 10^{301}$ cases).
Naive 2: Use MIS algorithm. ($\mathcal{O}^\star(3^{1000/3}) \approx 2.195 \cdot 10^{159}$ cases).
Better 1: Try all $\binom{n}{k}$ subsets of k people. ($\binom{1000}{10} \approx 2.63 \cdot 10^{23}$ cases).

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Consider the conflict graph $G = (V, E)$.

Idea: If $d(v) = 0$: let v in and drop v from G.
Why?

Idea: If $d(v) \geq k + 1$: reject v, drop v from G, and decrease k.
Why?

Note: If $d(v) \leq k$ for all v and $|E| > k^2$, there is no solution.
Why?

Better 2: The above ideas reduce to a graph H with $|V| \leq 2k^2$ vertices.
Why?

Now try all $\binom{2k^2}{k}$ subsets of k people. $(\binom{2\cdot 10^2}{10} \approx 2.24 \cdot 10^{16})$.

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Better 2: The above ideas reduce to a graph H with $|V| \leq 2k^2$ vertices.
Why? $|V| = \sum_{v \in V} 1 \leq \sum_{v \in V} d(v) = 2|E| \leq 2k^2$
Now try all $\binom{2k^2}{k}$ subsets of k people. $\left(\binom{2 \cdot 10^2}{10} \approx 2.24 \cdot 10^{16}\right)$.

Idea: If $N[v] = \{v, w\}$: let v in, reject w, drop $N[v]$ from G, and decrease k.
Why? In any solution that lets w in, we can let v in instead. Never worse.

Better 3: The above ideas reduce to a graph H with $|V| \leq k^2$ vertices.
Why? $|V| = \sum_{v \in V} 1 = \frac{1}{2} \sum_{v \in V} 2 \leq \frac{1}{2} \sum_{v \in V} d(v) = |E| \leq k^2$
Now try all $\binom{k^2}{k}$ subsets of k people. $\left(\binom{10^2}{10} \approx 1.73 \cdot 10^{13}\right)$.

“Bar fight prevention” via Kernelization

```
function BarFightPrevention($(k, G)$)
    $k', H, C \leftarrow$ BFP-Kernel($(k, G)$)
    if $H$ has $\leq (k')^2$ edges and BFP-Brute-Force($(k', H)$) returns a solution $C'$ then
        return $C \cup C'$
    return “No solution”

function BFP-Kernel($(k, G)$)
    $k' \leftarrow k$, $H \leftarrow G$, $C \leftarrow \emptyset$
    loop
        if Some $v$ has $d(v) = 0$ then
            $H \leftarrow H \setminus \{v\}$
        elseif Some $v$ has $d(v) \geq k' + 1$ then
            $H \leftarrow H \setminus \{v\}$, $C \leftarrow C \cup \{v\}$, $k' \leftarrow k' - 1$
        elseif Some $v$ has $N[v] = \{v, w\}$ for some $w$ then
            $H \leftarrow H \setminus N[v]$, $C \leftarrow C \cup \{w\}$, $k' \leftarrow k' - 1$
        else
            return $k', H, C$

function BFP-Brute-Force($(k, G = (V, E))$)
    for every subset $C \subseteq V$ of size $k$ do
        if $C$ is a vertex cover of $G$ then
            return $C$
    return “No solution”
```

Kernelization

The subgraph H we reduced to before brute-forcing is called a *Kernel* for the Bar Fight Prevention problem, and the process of finding such a kernel is called *Kernelization*.

The general idea is to use the parameter k to quickly reduce to a smaller subproblem of the same type, whose size ideally depends only on k and not on n. For the bar fight prevention problem we have just shown that:

- If there is a solution for a given k and a given graph G with n vertices and m edges, then we can find a kernel H with at most k^2 vertices.
- Furthermore, such a kernel can be found in $\mathcal{O}(m+n)$ time, and checking if a given subset of size at most k is a solution can be done in $\mathcal{O}(k^2)$ time.
- Thus, for any fixed k, the total running time of this algorithm is $\mathcal{O}(m+n+\binom{k^2}{k}k^2) \subseteq \mathcal{O}(m+n+(ke)^{2k+2}) = \mathcal{O}_k(m+n)$.

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“Bar fight prevention” via Bounded Search Tree

Note: For each edge $(u, v) \in E$, at least one of u, v must be rejected.

Idea: Pick arbitrary edge (u, v), and recursively try with u rejected and with v rejected.

```
function BFP-Bounded-Search($k$, $G$)
    if $G$ has an no edges then
        return $\emptyset$
    if $k > 0$ then
        Let $(u, v)$ be an arbitrary edge of $G$
        for $w \in \{u, v\}$ do
            if BFP-Bounded-Search($k - 1$, $G \setminus \{w\}$) returns a solution $C$ then
                return $C \cup \{w\}$
    return “No solution”
```

This recursive procedure has depth at most k.
Thus the total number of subproblems considered at most 2^k.
If we start by rejecting all vertices of degree $d(v) \geq k + 1$ (like in the kernelization approach), the resulting graph has at most $|E| = \frac{1}{2} \sum_{v \in V} d(v) \leq \frac{1}{2} nk$ edges, so constructing each subproblem can be done in $\mathcal{O}(nk)$ time.
The total running time is then $\mathcal{O}(m + nk \cdot 2^k)$. $(1000 \cdot 10 \cdot 2^{10} \approx 10^7)$
Part of Assignment 5 asks you to improve this.

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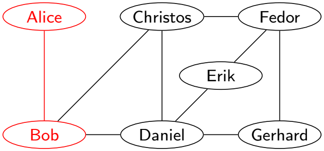
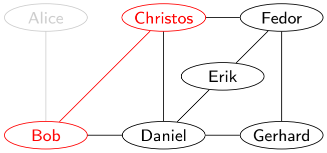


$k = 3$

Alice vs Bob

“Bar fight prevention” via Bounded Search Tree



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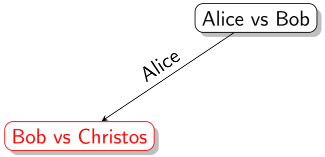

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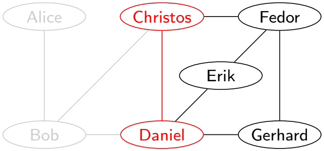

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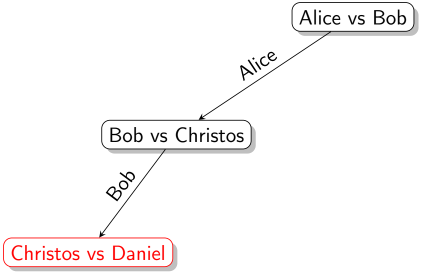

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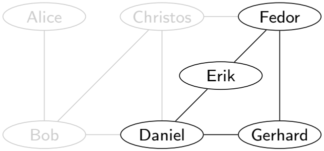
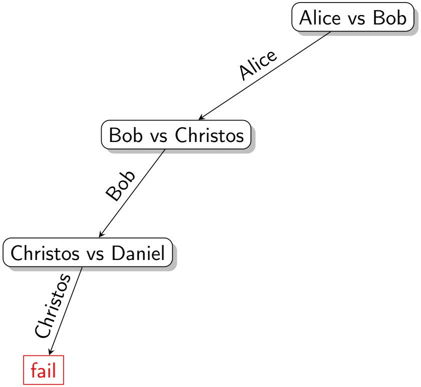


$k = 0$



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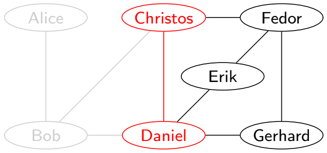

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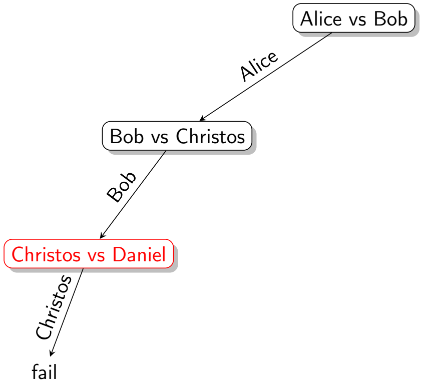

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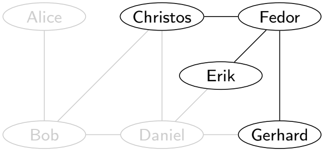

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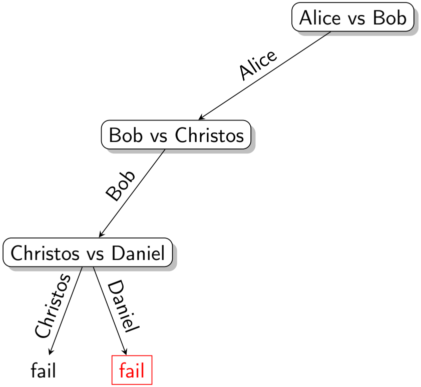

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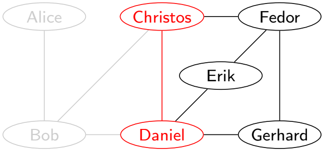

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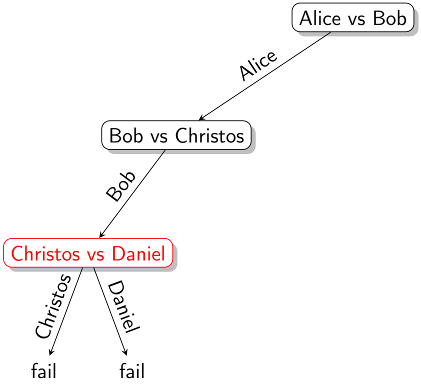

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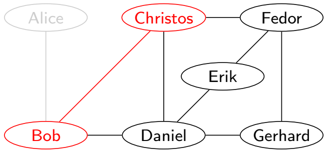
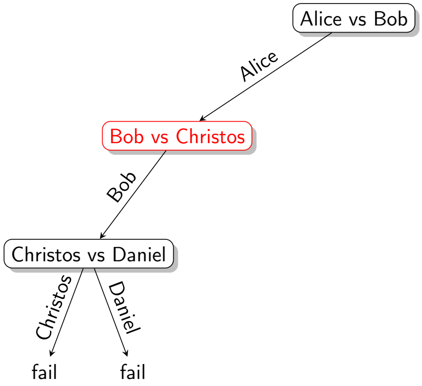


$k = 2$



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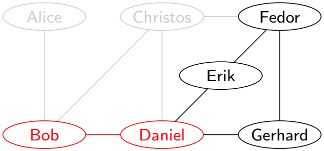

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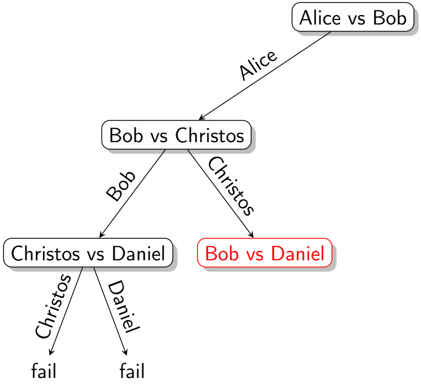

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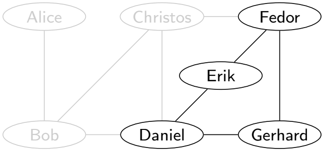

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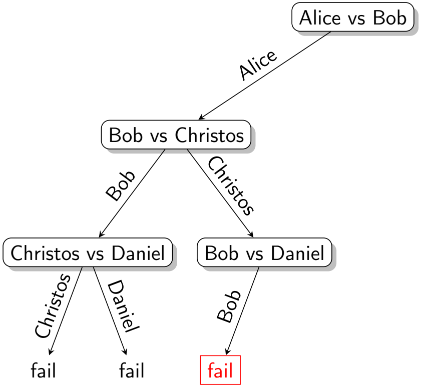

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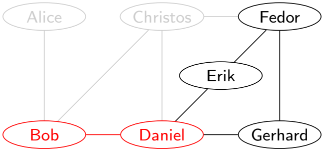

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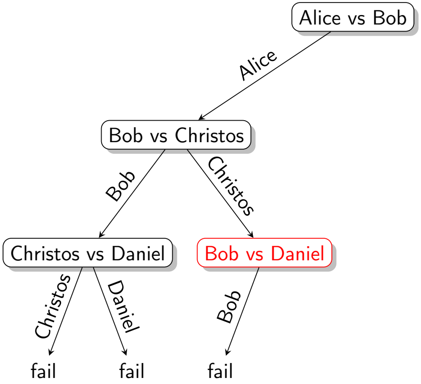

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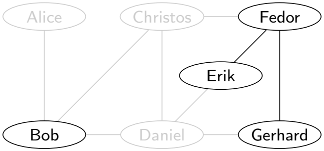

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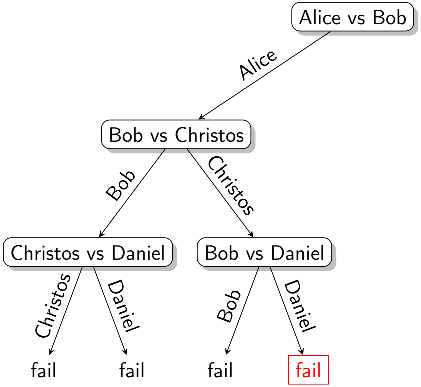

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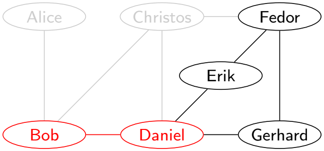

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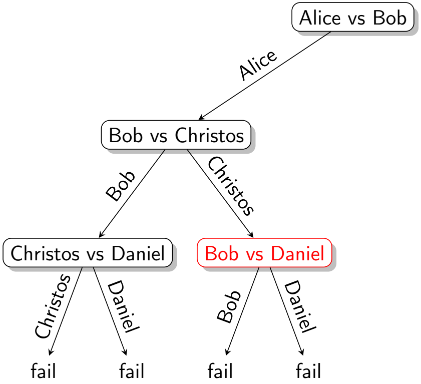

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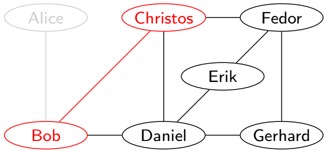

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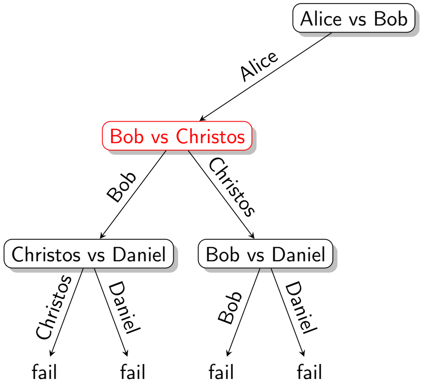

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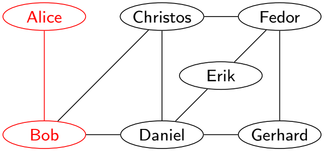

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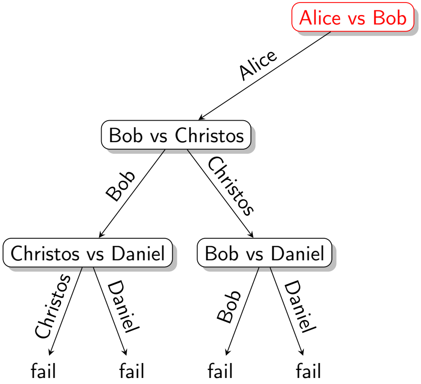

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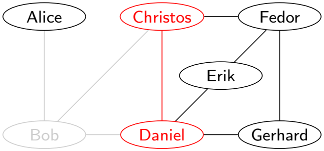

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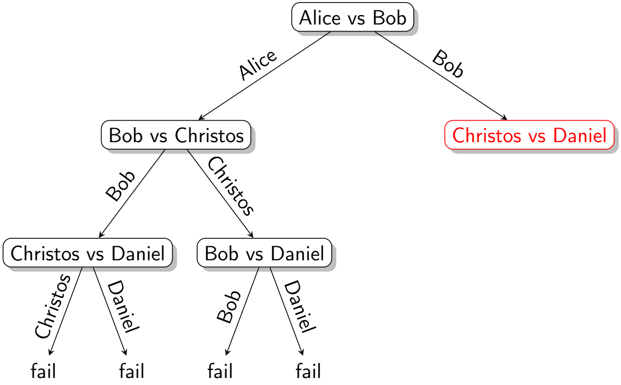

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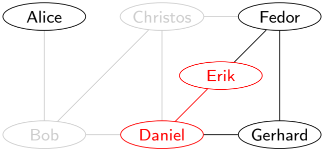

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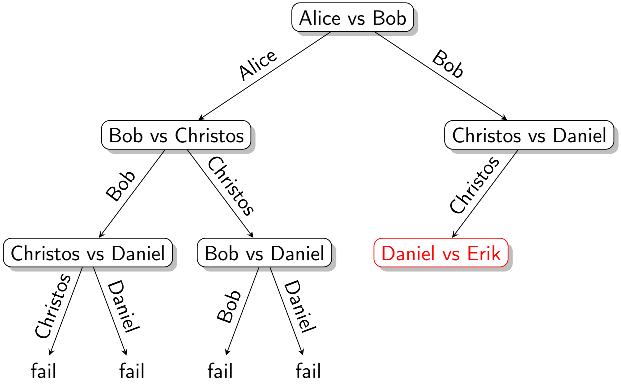

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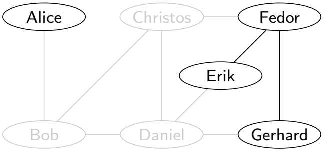

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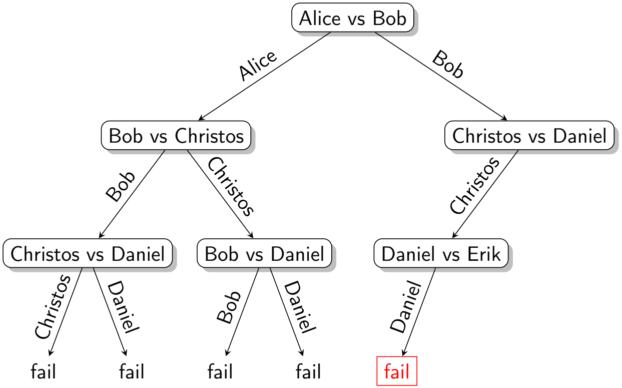

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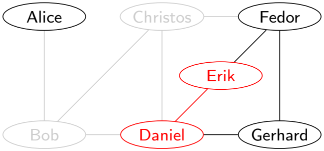

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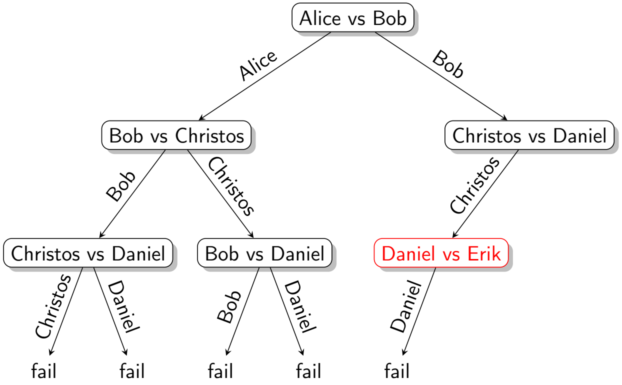

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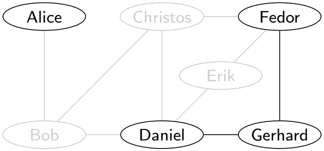
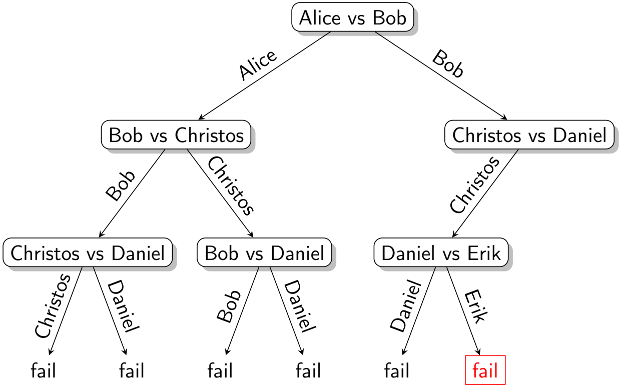


$k = 0$



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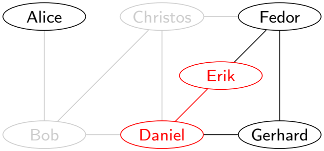
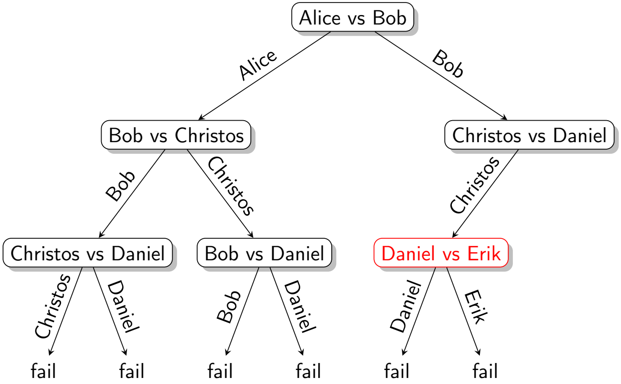


$k = 1$



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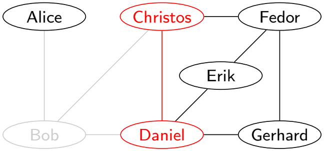
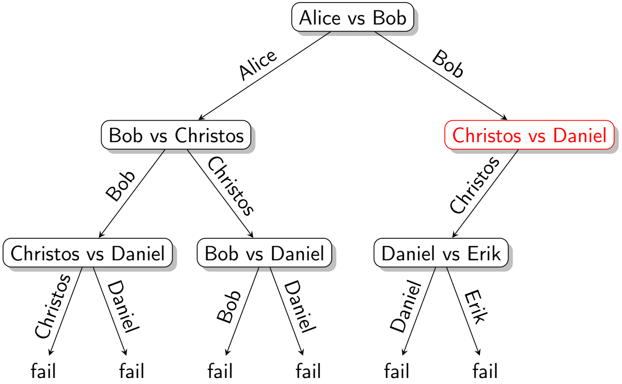


$k = 2$



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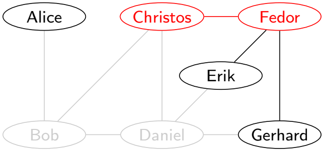
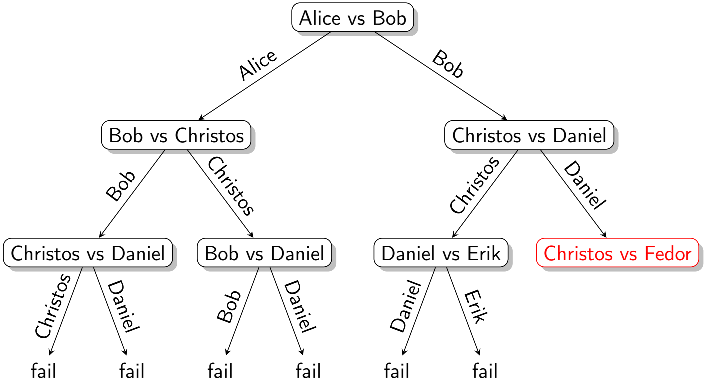


$k = 1$



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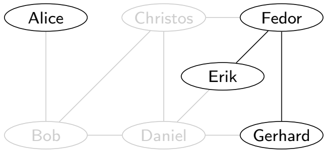
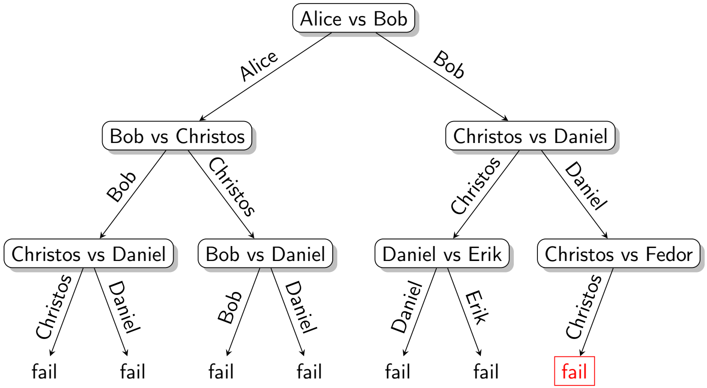


$k = 0$



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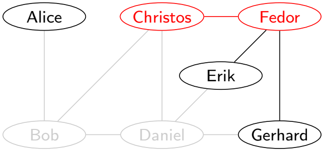

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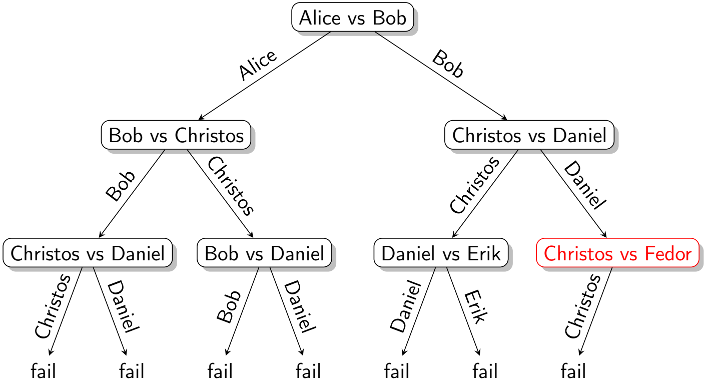

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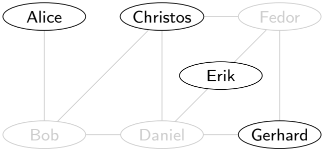
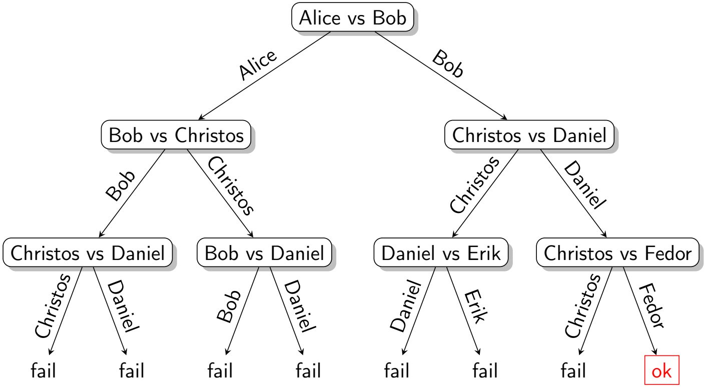


$k = 0$ $C = \emptyset$



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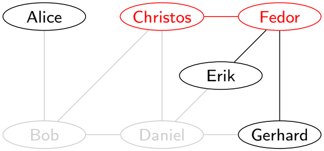
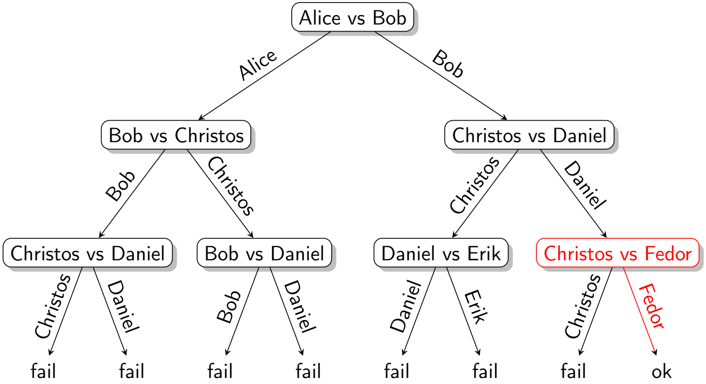


$k = 1$ $C = \{\text{Fedor}\}$



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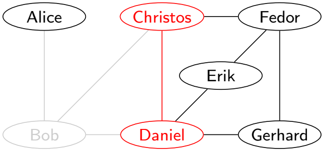
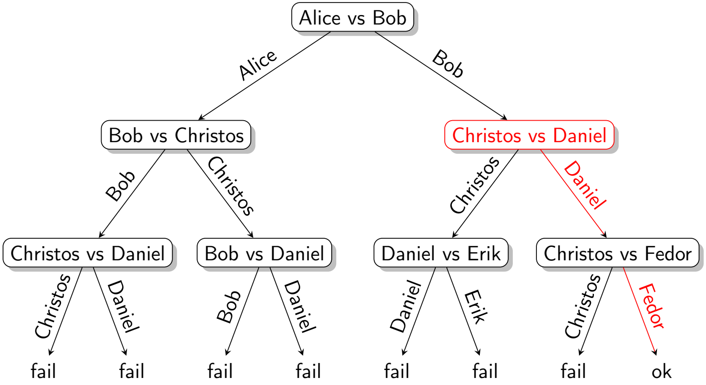


$k = 2$ $\qquad C = \{\text{Fedor}, \text{Daniel}\}$



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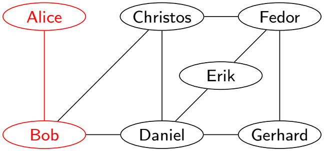
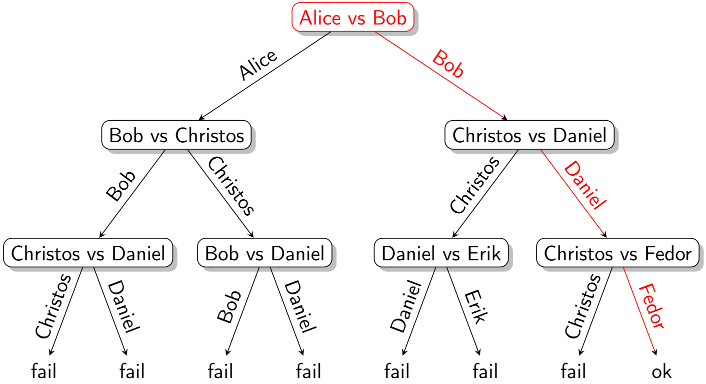


$k = 3$ $C = \{\text{Fedor}, \text{Daniel}, \text{Bob}\}$



AADS Lecture 9, Part 4

FPT vs XP

FPT vs XP

An important feature of the Bar Fight Prevention problem is the existence of the *parameter* k. The problem of finding the minimum k that works is NP-complete, but for any fixed constant k we have just seen two linear-time algorithms!

We say the problem is *parameterized* by the *parameter* k. In this case k is the maximum solution size, but other problems may have different parameters (and may have more than one).

Definition: A parameterized problem is *Fixed Parameter Tractable (FPT)* if it is has an algorithm with running time $f(k) \cdot n^c$ for some function f and some constant $c \in \mathbb{R}$.

Definition: A parameterized problem is *Slice-wise Polynomial (XP)* if it is has an algorithm with running time $f(k) \cdot n^{g(k)}$ for some functions f, g.

Note: FPT $\subset$ XP, why?

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Note: FPT $\subset$ XP, why? Simply set $g(k) = c$.

Example: Vertex k-Coloring

Problem: Given graph G and an integer k, does G have a proper vertex coloring with k colors?

Lemma

Unless $P = NP$, this problem is not XP and therefore not FPT.

Proof.

The problem is NP-hard even for $k = 5$, so unless $P = NP$ there can be no algorithm for general k with running time $f(k) \cdot n^{g(k)}$. $\square$

Example: k-Clique

Problem: Given graph G and an integer k, does G have a clique of size k?

Lemma

k-clique is XP.

Proof.

A simple brute-force algorithm is to check every k-subset of the vertices. There are $\binom{n}{k} \leq n^k$ such subsets, and we can check in $\mathcal{O}(k^2)$ time whether a given subset forms a clique. Thus the running time of this algorithm is $\mathcal{O}(k^2 \cdot n^k)$ which proves the problem is in XP. □

It is unknown whether k-clique is FPT, but it is widely believed that $\mathcal{O}(n^k)$ is optimal which would prove it is not.

Example: k-Clique parameterized by Δ

Problem: Given graph G with maximum degree Δ, does G have a clique of size k?

Lemma

k-clique is FPT when parameterized by the maximum degree Δ.

Proof.

A naive algorithm is for each vertex to try all subsets of its neighbors. There are at most $n \cdot 2^{\Delta}$ such subsets and each can be checked in $\mathcal{O}(\Delta^2)$ time. The total time is thus $\mathcal{O}((2^{\Delta} \cdot \Delta^2) \cdot n)$, which proves the problem is FPT. □

In fact, we can easily improve this algorithm to run in $\mathcal{O}(\binom{\Delta}{k-1} \cdot k^2 \cdot n) \subseteq \mathcal{O}(\Delta^{k-1} \cdot k^2 \cdot n)$ time.

There are often many possible choices of parameter. Choosing the right one for a specific problem is an art.

Summary

Todays topics were “Exact exponential algorithms” and “Parameterized Complexity”. We have covered

- The natural brute force algorithm for problems in NP.
- An exact $\mathcal{O}^\star(2^n)$-time dynamic programming algorithm for TSP.
- An exact $\mathcal{O}^\star(3^{n/3})$-time branching algorithm for MIS.
- A kernelization for the “Bar Fight Prevention” problem, a.k.a. k-vertex cover.
- A bounded search tree algorithm for k-vertex cover.
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- Examples of problems in FPT, XP but not FPT, and not XP.
- Next time: Approximation algorithms

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