Computational Methods in Simulation hand-in week 6

KXS806

1 INTRODUCTION OF FVM

FVM is a method to discretize PDEs into a system of equations, but it's different from the other two in how it deals with the original PDE. FVM get the PDE and calculate its volume integral and use the Gauss divergence theorem to rewrite it into surface integrals, then cut these integrals into geometric sub-pieces.

Also FVM uses a collection of control volumes which can encompass the space and then approximate the solution at their centers, this differs from elements in FEM and grid in FDM.

FVM possess both local and global conservation properties.

Local conservation refers to FVM can ensure the preservation of conservation laws locally within each individual control volume. This can make the integral form of the conservation equations discretized over each control volume.

Global conservation refers to the FVM can ensure the overall conservation of the conserved quantities over the entire computational domain, it can guarantee the sum of the conserved variables integrated over all control volumes to be a constant regardless of time.

For example, for the following domain:

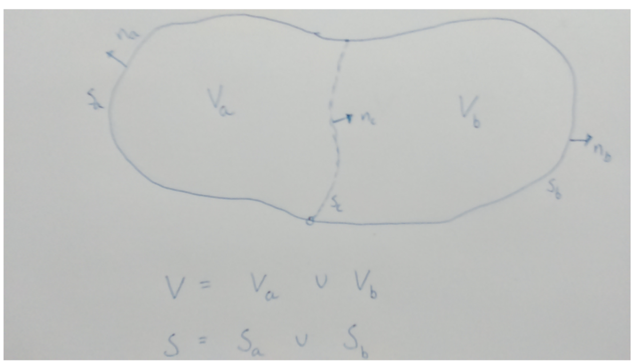

Fig. 1. splited domain

We have V_a and V_b share the common boundary S_c and boundary of V_a is $S_a \cup S_c$, boundary of V_b is $S_b \cup S_c$.

Then for local conservation we can write integral form over the sub-volumes and add up as:

$$\int_{V_a} \frac{\partial g(\mathbf{u})}{\partial t} dV = -\left(\int_{s_a} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_a dS + \int_{s_c} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_c dS \right)$$
$$\int_{V_b} \frac{\partial g(\mathbf{u})}{\partial t} dV = -\left(\int_{s_b} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_b dS + \int_{s_c} \mathbf{f}(\mathbf{u}) \cdot (-\mathbf{n}_c)\, dS \right)$$

For the global conservation we add both on left and right hand side and get

$$\int_{V_a \cup V_b} \frac{\partial g(\mathbf{u})}{\partial t} dV = -\int_{s_a} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_a dS - \int_{s_b} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}_b dS$$
$$= -\int_{s_a \cup s_b} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS,$$

where $S_a \cup S_b$ is the boundary of $V_a \cup V_b$.

2 INTRODUCTION OF CONTROL VOLUME

A control volume is a small cell within the computational domain and it can be thought of a net over a computational mesh. It is a discrete region and can vary in shape and size depending on the problem.

The control volume is the fundamental units when solving the PDE, the equations are discretized over each control volume, and we can get the result by solving equations within each control volume. Control volume's size and shape depends on the mesh, and the computationsl mesh can influence the accuracy of the simulation we get, also a mesh with smaller grid can cost more time to compute, but it will result in a better solution with less error.

3 APPLY FVM ON A TOY EXAMPLE

3.1 PDE of toy exapmle

We have a known vector function $f(u)$ and a scalar function $g(u)$ and we have an equation:

$$\frac{\partial g(\mathbf{u})}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = \mathbf{0}$$

We want to solve it and get a solution for u.

3.2 get volume integral

Then we need to integrate the PDE over a volume V and get:

$$\int_V \frac{\partial g(\mathbf{u})}{\partial t} dV = -\int_V \nabla \cdot \mathbf{f}(\mathbf{u}) dV$$

3.3 apply Gauss-Divergence theorem

We have Gauss Divergence Theorem as follows:

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot \mathbf{n} dS$$

It is used to rewrite volume integrals into surface integrals and it can show the flux of a vector field across a closed surface. It can simplify calculations by reducing the derivative level.

So here we apply this theorem to the right hand side and get:

$$\int_V \frac{\partial g(\mathbf{u})}{\partial t} dV = -\int_S \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

3.4 Piecewise continuous integral

As the whole boundary is not continuous we need to do a rewriting to right hand side.

Piecewise continuous integral can be used to divide the entire domain into small continuous intervals and sum the integral inside these intervals up. As the boundary is piece-wise continuous, so we replace the right hand side with the summation over continuous pieces and we have:

$$\int_V \frac{\partial}{\partial t} g(\mathbf{u}) dV = -\sum_e \int_{S_e} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

3.5 numerical integration

As the volume V is independent of time, so we can take the derivative out from the integral and get:

$$\frac{\partial}{\partial t} \int_V g(\mathbf{u}) dV = -\sum_e \int_{S_e} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

We have Midpoint Approximation rule, which can be used to approximate the value of a definite integral by dividing the integral into subintevals and then calculate the volume using the hight at midpoint of each subinterval.

The approximation assumes that the function remains constant over each subinterval and the height of functions is the midpoint's height, so here we assume the outward unit n normal is constant and use the midpoint approximation rule and get the following equation:

$$\frac{\partial}{\partial t} [g(\mathbf{u})]_c V = -\sum_e [\mathbf{f}(\mathbf{u})]_e \cdot \mathbf{n}_e l_e$$

Here I_e is the area of the S_e surface.

3.6 Apply Finite Difference

Then we can apply finite difference on left-hand side and get the final discrete version of the original integral form:

$$\int_V \frac{\partial g(\mathbf{u})}{\partial t} dV = -\int_S \mathbf{f}(\mathbf{u}) \cdot \mathbf{n} dS$$

And we can denote it as

$$Au = b$$

Then we can solve the system to get u.

4 PRACTICAL PART

In this part we first defined M as $[0,0]^T$ outside the unit circle and $[0,-1]^T$ inside, and then we fill the matrix A and we get the results shown as follows:

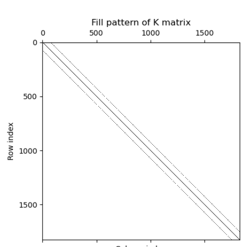

Fig. 2. fill pattern

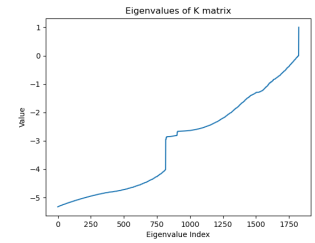

Fig. 3. eigen values

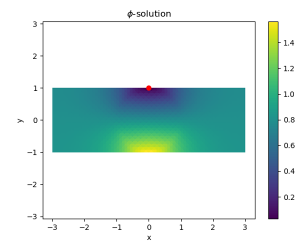

Fig. 4. Φ solution

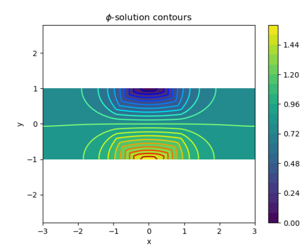

Fig. 5. contours

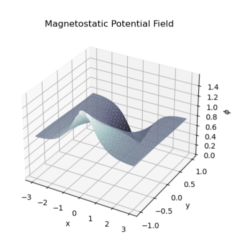

Fig. 6. magneto static Potential field