Computational Methods in Simulation hand-in week 5

KXS806

1 LINEAR ELASTIC MATERIALS

1.1 Problem description

This week we want to simulate an elastic beam with one traction field and how it behaves with outer force.

First we need to have the Cauchy equation:

$$\rho\ddot{\mathbf{x}} = \nabla \cdot \sigma + \mathbf{b}$$

Here b is body force density, ρ is spatial mass density and σ is stress tensor. We also have Cauchy's stress hypothesis fulfilled on the boundary part, so the whole model of the problem is:

$$\begin{aligned} \rho\ddot{\mathbf{x}} &= \nabla \cdot \sigma + \mathbf{b}, & \forall \mathbf{x} \in v, \\ \mathbf{t} &= \sigma\mathbf{n}, & \forall \mathbf{x} \in \partial v_t. \end{aligned}$$

Here $\partial v_t \subseteq \partial v$ is the subset of the boundary where traction is applied.

Then we need to do volume integral for the beam. First, we multiply it by a test function and do volume integral, we denote the test function as w here and get:

$$\int_{\Omega} (\rho\ddot{\mathbf{x}} - \nabla \cdot \sigma - b)^{\top} \omega d\Omega = 0$$

For this integration, we need to do an assembly for every timestamp and change it to be an integration over the material coordinate. Instead of using $\rho(x(t))$ function we denote it as $\rho(t,x)$ and add time coefficient $x(t) = \Phi(t,x)$ and then we can get $\rho(\Phi(t,x))$ as new ρ function.

Then using this we can re-write the integral as:

$$\int_{\Omega_0} (\rho\ddot{\mathbf{x}} - \nabla \cdot \sigma - b)^{\top} \omega j d\Omega_0 = 0$$

Here we have $d\Omega = jd\Omega_0$.

With the new integral in material coordinate, we can do a lot of computation easily while the domain is deforming, because in spatial coordinate the original point will change after deform so we need to re-calculate its position, but in material coordinate the position is stable.

Then we assume a Quasi-static is working, then there will not be a time derivative. Then we assume we have small displacements which means the spatial and material coordinates are almost the same. Then we can get the following equation:

$$\int_{\Omega_0} (-\nabla \cdot \sigma)^T w d\Omega_0 + \int_{\Omega_0} -b^T w d\Omega_0 = 0$$

1.2 apply integration by parts

We need to lower the requirement on the first term, first, we rewrite the first term using the product rule and change the equation as follows:

$$\int_{\Omega_0} \sigma : \nabla\omega^{\top} d\Omega_0 - \int_{\Omega_0} \nabla \cdot (\sigma w) d\Omega_0 - \int_{\Omega_0} b^{\top} w d\Omega_0 = 0$$

In this equation the first term is the elastic power term, we denote it as P_e. Since the Cauchy stress tensor is symmetric we have

$$P_e = \int_V \sigma : \frac{1}{2}\left(\nabla\mathbf{w} + \nabla\mathbf{w}^T\right) dV$$

Then we have infinitesimal strain tensor defined as $\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$, so here we can say $\varepsilon_w = \frac{1}{2}\left(\nabla\mathbf{w} + \nabla\mathbf{w}^T\right)$. We put this into the elastic term and get $P_e = \int_V \sigma : \varepsilon_w dV$.

1.3 constitutive equation

Then we want to know how the material behaves. To do that we need to have the constitutive equation, which contains a stress tensor and a strain tensor.

The stress tensor is written as $\sigma = \frac{\partial\Psi}{\partial\varepsilon}$, and strain tensor is $\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ where $u = x - X$.

For the isotropic linear elasticity example we have the strain function as

$$\Psi = \frac{\lambda}{2}\,\mathrm{tr}(\varepsilon)^2 + \mu\varepsilon : \varepsilon$$

Here λ and μ are two constants describing some nature of the material and determine what type of material we have, so we can say the stress-strain relation has the following relationship:

$$\sigma = \lambda\,\mathrm{tr}(\varepsilon)\mathbf{I} + 2\mu\varepsilon$$

In real life we have the strain-stress relationship being ploted as following:

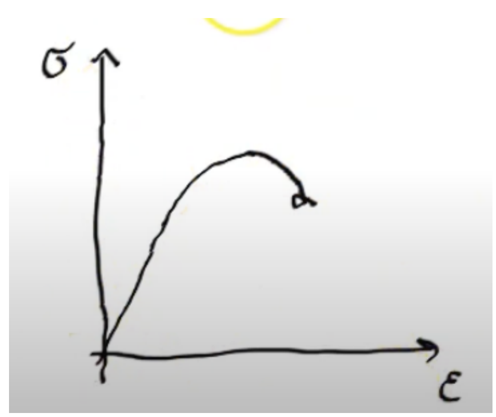

Fig. 1. strain-stress relationship

We can see in the beginning it has a linear relationship, but when it reaches a peak then the curve goes down because the inside of the material collapse.

With stress-strain relation we can write the tensor notation in component form:

$$\begin{aligned}
\sigma_{xx} &= \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)+2\mu\varepsilon_{xx},\\
\sigma_{yy} &= \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)+2\mu\varepsilon_{yy},\\
\sigma_{zz} &= \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)+2\mu\varepsilon_{zz},\\
\sigma_{xy} &= 2\mu\varepsilon_{xy},\\
\sigma_{xz} &= 2\mu\varepsilon_{xz},\\
\sigma_{yz} &= 2\mu\varepsilon_{yz}.
\end{aligned}$$

Then we can use these equations to solve the stress-strain equation and get the coefficient λ and μ. We can write these six equations into a vector and assemble a coefficient matrix.

As Lame constants usually are hard to get, so we can use Youngs moduls E and Poisson ratio v to denote it as:

$$\begin{aligned}
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)},\\
\mu &= \frac{E}{2(1+\nu)}.
\end{aligned}$$

Young Moduls denotes the stiffness of the material, and Poisson ratio denotes if the material will extend with outer force being applied to it.

Then we can rewrite the previous D matrix using E and v as:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} d_0 & d_1 & d_1 & 0 & 0 & 0\\ d_1 & d_0 & d_1 & 0 & 0 & 0\\ d_1 & d_1 & d_0 & 0 & 0 & 0\\ 0 & 0 & 0 & d_2 & 0 & 0\\ 0 & 0 & 0 & 0 & d_2 & 0\\ 0 & 0 & 0 & 0 & 0 & d_2 \end{bmatrix}$$

and we have

$$\begin{aligned}
d_0 &= (1-\nu),\\
d_1 &= \nu,\\
d_2 &= \frac{(1-2\nu)}{2}.
\end{aligned}$$

So if we dont have lame constants but we have Youngs moduls and Poisson ratio we can use this matrix to solve the constitutive equation.

1.4 vector notation of strain definition

We also need the strain definition in the equation, we already have $\varepsilon = \frac{1}{2}\left(\nabla u + \nabla v^{\top}\right)$, then we can get it in a matrix form as follows:

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ 0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial y} \end{bmatrix}\begin{bmatrix} u_x\\ u_y\\ u_z \end{bmatrix}$$

1.5 apply approximation

We already have the weak form integral equation as

$$\int_{\Omega_0} \sigma : \varepsilon_w d\Omega_0 - \int_{\sigma_t} w^{\top}\vec{t}dr_t - \int_{\Omega_0} w^{\top} b d\Omega_0 = 0$$

Then we can approximate u as the sum of shape function multiplied by the triangle node, so we have $\bar{v} = N^e\bar{u}^e = \sum_\alpha N^e_\alpha(x_0)\,\vec{u}^e_\alpha$. Also we have $\vec{\sigma} = D\vec{\varepsilon}$, $\vec{\varepsilon} = S\vec{u} = \underbrace{SN^e}_{B^e}\,\bar{u}^e$.

Then we rewrite the first term as $\delta\bar{w}^{e^T}B^{e^T}DB^e\bar{u}^e$. So we can clean the equation as :

$$\left(\int_{\Omega_0} B^{e^T}DB^e d\Omega_0\right)\vec{v}^e - \int_{\Gamma_t^e} N^{e^T} t d\Gamma_0 - \int_{\Omega_0^e}^{+} N^{e^T} b dS_0 = 0$$

In this equation, the second term is the traction term, and the third term is the body term. They are the total force added to the system, and we need to solve the first term, which contains the unknown u vector. We can denote it as

$$\left(V^eB^{e^T}DB^e\right)\bar{u}^e = f^e$$

So we only need to solve this linear system to solve the problem.

2 PRACTIAL PART

In this part, we simulated a linear elasticity problem in 2D using FEM. First, we calculated the D matrix and B matrix, then assembled the K_e array. Then using the k_e array to get the global K array. We plot the fill pattern and eigenvalues as follows:

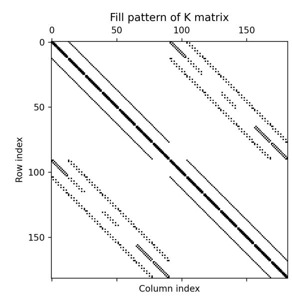

Fig. 2. fill pattern

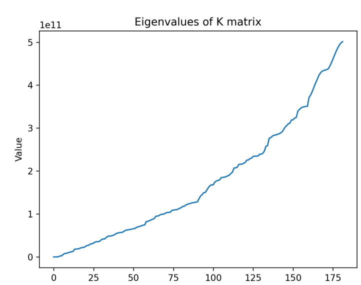

Fig. 3. eigen values

In this example we used young modulus as 69e9 and poisson ration as 0.3 and we apply a load of -10e7 to the element.

Then we added boundary conditions to the K and f matrix and solve it to get the u array, then we plot the result as follows:

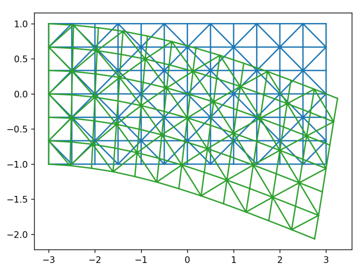

Fig. 4. result

Then we calculated the area of the undeformed beam and the area of the deformed beam and get the following result: Undeformed beam area: 12.0 Deformed beam area: 12.411255904156214 This shows there are some errors while we do the simulation, it might be because of the numerical error during the calculation or because the size of the beam is too big.

Then we also tried to see the system's performance under different load and plot the result as follows:

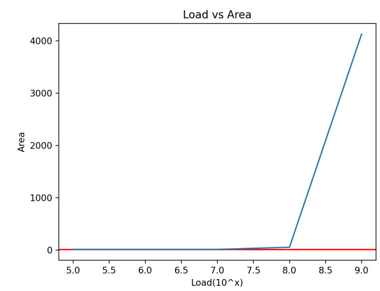

Fig. 5. load area plot

From the plot we can observe when the load is very high the system will explode and the error will become very high.