# Tutorial 5

## October 20, 2019

1. Suppose that $x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}$ is a sample from an unknown population and the sample size is $m+n$. Let
$$\bar{x} = \frac{1}{m+n}\sum_{i=1}^{m+n} x_i \text{ and } s^2 = \frac{1}{(m+n)-1}\sum_{i=1}^{m+n}(x_i - \bar{x})^2.$$
Suppose the $\bar{x_1}$ and $s_1^2$ are the sample mean and the sample variance of the first $m$ individuals, and $\bar{x_2}$ and $s_2^2$ are the sample mean and the sample variance of the last $n$ individuals. Prove that
$$\bar{x} = \frac{m\bar{x}_1 + n\bar{x}_2}{m+n},$$
and
$$s^2 = \frac{(m-1)s_1^2 + (n-1)s_2^2}{m+n-1} + \frac{mn(\bar{x}_1 - \bar{x}_2)^2}{(m+n)(m+n-1)}.$$

2. There exists the third moment of a population. Suppose that $x_1, x_2, \cdots, x_n$ is a sample from the population and $\bar{x}$ and $s^2$ are, respectively, the sample mean and the sample variance. Prove that $Cov(\bar{x}, s^2) = \frac{v_3}{n}$ where $v_3 = E(x - E(x))^3$.

3. Suppose that the population is a Weibull distribution where the p.d.f. is
$$f(x) = \frac{mx^{m-1}}{\eta^m}\exp\left\{-\left(\frac{x}{\eta}\right)^m\right\}, x > 0, m > 0, \eta > 0$$
If $x_1, x_2, \cdots, x_n$ is a sample from the Weibull distribution, then prove that the distribution of $x_{(1)}$ is also a Weibull distribution and write down the parameters.

4. Suppose $x_1, x_2, \cdots, x_{10}$ is a sample from a uniform distribution $U(0,1)$.
   (a) Find the asymptotic distribution of $\bar{x}$.
   (b) Find the asymptotic distribution of $m_{0.5}$
   (c) Follow the instructions:
      i. Generate random sample from a uniform $U(0,1)$: $x_1, x_2, \cdots, x_{10}$;
      ii. Calculate the sample median $m_{0.5}$;
      iii. Repeat ii. 1000 times.

   Based on these 1000 samples, make a plot of e.c.d.f., a boxplot and a histogram.

5. Suppose that $x_1, x_2, \cdots, x_{2n} (n \geq 1)$ is a sample from the normal distribution $N(\mu, \sigma^2)$ and $\bar{x} = \frac{1}{2n}\sum_{i=1}^{2n} x_i$. Find the expectation of a statistic $y = \sum_{i=1}^{n}(x_i + x_{n+i} - 2\bar{x})^2$.

6. Suppose that $x_1, x_2, \cdots, x_n$ is a sample from a population. Prove that $T = \sum_{i=1}^{n} x_i$ is a sufficient statistic for $\theta$.
   (a) If the population is $P(\theta)$.
   (b) If the population is $N(\theta, 1)$.