Statistics 200
Winter 2009
Homework 5 Solutions

Problem 1 (8.16)

$X_1, \ldots, X_n$ i.i.d. with density function $f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)$

(a) – (c) (See HW 4 Solutions)

(d) According to Corollary A on page 309 of the text, the maximum likelihood estimate is a function of a sufficient statistic T. In part (b), the maximum likelihood estimate was found to be

$$\hat{\sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

Therefore, a sufficient statistic $T(X_1, X_2, \ldots, X_n)$ is given by:

$$T(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n}|X_i|$$

Problem 2 (8.52)

$X_1, \ldots, X_n$ i.i.d. with density function $f(x|\theta) = (\theta+1)\,x^{\theta}, \qquad , 0 \leq x \leq 1$

(a)

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^1 x f(x|\theta)dx \\
&= \int_0^1 x^{(\theta+1)}(\theta+1)dx \\
&= \frac{\theta+1}{\theta+2}x^{(\theta+2)}\Big|_0^1 \\
&= \frac{\theta+1}{\theta+2}
\end{aligned}$$

Therefore, a method of moments estimate of θ is given by:

$$\begin{aligned} & \hat{\mu}_1 = \frac{\hat{\theta}_{MM}+1}{\hat{\theta}_{MM}+2} \\ \Rightarrow \quad & \hat{\mu}_1 \left(\hat{\theta}_{MM}+2\right) = \hat{\theta}_{MM}+1 \\ \Rightarrow \quad & (\hat{\mu}_1 - 1)\,\hat{\theta}_{MM} = 1 - 2\hat{\mu}_1 \\ \Rightarrow \quad & \hat{\theta}_{MM} = \frac{1-2\hat{\mu}_1}{\hat{\mu}_1 - 1} \end{aligned}$$

where $\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}_n$.

(b)

$$\begin{aligned} & l(\theta) = \sum_{i=1}^n \left[\log(\theta+1) + \theta \log(x_i)\right] \\ \Rightarrow \quad & \frac{d}{d\theta} l(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^n \log(x_i) \\ \Rightarrow \quad & 0 = \frac{n}{\hat{\theta}_{MLE}+1} + \sum_{i=1}^n \log(x_i) \\ \Rightarrow \quad & \hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^n \log(x_i)} - 1 \end{aligned}$$

(c)

$$\begin{aligned} I(\theta) &= -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2} \log f(X|\theta)\right] \\ &= -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2} \left(\log(\theta+1) + \theta \log X\right)\right] \\ &= -\mathrm{E}\left[\frac{\partial}{\partial\theta} \left(\frac{1}{\theta+1} + \log X\right)\right] \\ &= -\mathrm{E}\left[-\frac{1}{(\theta+1)^2}\right] \\ &= -\frac{1}{(\theta+1)^2} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\left[\hat{\theta}_{MLE}\right] &\approx \frac{1}{nI(\theta)} \\ &= \frac{(\theta+1)^2}{n} \end{aligned}$$

(d) According to Corollary A on page 309 of the text, the maximum likelihood estimate is a function of a sufficient statistic T. In part (b), the maximum likelihood estimate was found to be

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^n \log(x_i)} - 1$$

Therefore, a sufficient statistic $T(X_1, X_2, \ldots, X_n)$ is given by:

$$T(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \log(X_i)$$

Problem 3 (8.59)

Let I denote the event that a pair of twins is identical, so $P(I) = \alpha$.

(a)

$$\begin{aligned}
P(MM) &= P(MM|I)P(I) + P(MM|I^C)P(I^C) \\
&= \frac{1}{2}\alpha + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1-\alpha) \\
&= \frac{1+\alpha}{4} \\
P(FF) &= P(FF|I)P(I) + P(FF|I^C)P(I^C) \\
&= \frac{1}{2}\alpha + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1-\alpha) \\
&= \frac{1+\alpha}{4} \\
P(MF) &= 1 - (P(MM) + P(FF)) \\
&= 1 - \frac{1+\alpha}{2} \\
&= \frac{1-\alpha}{2}
\end{aligned}$$

(b) We will assume that n *sets* of twins are sampled, so $n_1 + n_2 + n_3 = n$.

$$\begin{aligned}
& \text{lik}(\alpha) = \left(\frac{1+\alpha}{4}\right)^{n_1} + \left(\frac{1+\alpha}{4}\right)^{n_2} + \left(\frac{1-\alpha}{2}\right)^{n_3} \\
\Rightarrow \quad & l(\alpha) = (n_1+n_2)\log\left(\frac{1+\alpha}{4}\right) + n_3 \log\left(\frac{1-\alpha}{2}\right) \\
\Rightarrow \quad & \frac{d}{d\alpha} l(\alpha) = \frac{n_1+n_2}{1+\alpha} - \frac{n_3}{1-\alpha} \\
\Rightarrow \quad & 0 = \frac{n_1+n_2}{1+\hat{\alpha}_{MLE}} - \frac{n_3}{1-\hat{\alpha}_{MLE}} \\
\Rightarrow \quad & \hat{\alpha}_{MLE} = \frac{n_1+n_2-n_3}{n}
\end{aligned}$$

Now to compute the variance of $\hat{\alpha}_{MLE}$, we will rewrite $\hat{\alpha}_{MLE}$ as

$$\begin{aligned}
\hat{\alpha}_{MLE} &= \frac{n_1+n_2-n_3}{n} \\
&= \frac{n_1+n_2-(n-n_1-n_2)}{n} \\
&= \frac{2(n_1+n_2)-n}{n}
\end{aligned}$$

Then the variance of the MLE can be computed as

$$\begin{aligned}
\mathrm{Var}[\hat{\alpha}_{MLE}] &= \mathrm{Var}\left[\frac{2(n_1+n_2)-n}{n}\right] \\
&= \frac{4}{n^2}\mathrm{Var}[n_1+n_2] \\
&= \frac{4}{n^2}\left(\mathrm{Var}[n_1]+\mathrm{Var}[n_2]+2\mathrm{Cov}(n_1,n_2)\right)
\end{aligned}$$

We note that n_1 and n_2 are both Binomial random variables with n trials and success probability $\frac{1+\alpha}{4}$, so

$$\mathrm{Var}[n_1] = \mathrm{Var}[n_2] = n\left(\frac{1+\alpha}{4}\right)\left(\frac{3-\alpha}{4}\right)$$

Now we define $Y_i = \mathbf{1}\{\mathrm{i}^{th}\text{ set of twins is }MM\}$ and $X_i = \mathbf{1}\{\mathrm{i}^{th}\text{ set of twins is }FF\}$. Clearly $n_1 = \sum_{i=1}^n Y_i$ and $n_2 = \sum_{i=1}^n X_i$, and also $Y_iX_i = 0$ since a given set of twins cannot be both two males and two females. Using these definitions, we have

$$\begin{aligned}
\mathrm{Cov}(n_1,n_2) &= \mathrm{E}[n_1n_2]-\mathrm{E}[n_1]\mathrm{E}[n_2] \\
&= \mathrm{E}\left[\left(\sum_{i=1}^n Y_i\right)\left(\sum_{j=1}^n X_j\right)\right] - \frac{n(1+\alpha)}{4}\frac{n(1+\alpha)}{4} \\
&= \mathrm{E}\left[\sum_{i=1}^n Y_iX_i + \sum_{i\neq j} Y_iX_j\right] - n^2\left(\frac{1+\alpha}{4}\right)^2 \\
&= \mathrm{E}\left[\sum_{i=1}^n 0\right] + \mathrm{E}\left[\sum_{i\neq j} Y_iX_j\right] - n^2\left(\frac{1+\alpha}{4}\right)^2 \\
&= 0 + (n^2-n)\mathrm{E}[Y_i]\mathrm{E}[X_j] - n^2\left(\frac{1+\alpha}{4}\right)^2 \\
&= (n^2-n)\left(\frac{1+\alpha}{4}\right)^2 - n^2\left(\frac{1+\alpha}{4}\right)^2 \\
&= -n\left(\frac{1+\alpha}{4}\right)^2
\end{aligned}$$

Substituting these results back into the expression for $\mathrm{Var}[\hat{\alpha}_{MLE}]$, we have

$$\begin{aligned}
\mathrm{Var}[\hat{\alpha}_{MLE}] &= \frac{4}{n^2}\left[n\left(\frac{1+\alpha}{4}\right)\left(\frac{3-\alpha}{4}\right)+n\left(\frac{1+\alpha}{4}\right)\left(\frac{3-\alpha}{4}\right)+2\left(-n\left(\frac{1+\alpha}{4}\right)^2\right)\right] \\
&= \frac{1}{n}\left[\frac{(1+\alpha)(3-\alpha)}{2}+\frac{(1+\alpha)^2}{2}\right] \\
&= \frac{(1+\alpha)4}{2n} \\
&= \frac{2(1+\alpha)}{n}
\end{aligned}$$

Problem 4 (8.68)

$X_1, \ldots, X_n$ i.i.d. with probability mass function function $p(x|\lambda) = \frac{1}{x!}\lambda^x e^{-\lambda}$

(a) To show that $T = \sum_{i=1}^n X_i$ is sufficient for λ, we first note that T has a Poisson distribution with parameter $n\lambda$, so we have:

$$\begin{aligned}
&P\left(X_1 = x_1,\, X_2 = \; x_2, \ldots, X_n = x_n \middle|\, T = t\right) \\
&\quad = \frac{P\left(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, T = t\right)}{P(T = t)} \\
&\quad = \frac{P\left(X_1 = x_1, X_2 = x_2, \ldots, X_n = t - \sum_{i=1}^{n-1} x_i\right)}{P(T = t)} \\
&\quad = \frac{\left[\prod_{i=1}^{n-1} \lambda^{x_i} e^{-\lambda}/x_i!\right] \left[\lambda^{(t-\sum_{i=1}^{n-1} x_i)} e^{-\lambda} / \left(t - \sum_{i=1}^{n-1} x_i\right)!\right]}{(n\lambda)^t e^{-n\lambda}/t!} \\
&\quad = \frac{e^{-(n-1)\lambda} \lambda^{\sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n-1} 1/x_i!\right) \lambda^{(t-\sum_{i=1}^{n-1} x_i)} e^{-\lambda} / \left(t - \sum_{i=1}^{n-1} x_i\right)!}{(n\lambda)^t e^{-n\lambda}/t!} \\
&\quad = \frac{e^{-n\lambda}\lambda^t \left(\prod_{i=1}^{n-1} 1/x_i!\right) \left[1/\left(t - \sum_{i=1}^{n-1} x_i\right)!\right]}{(n\lambda)^t e^{-n\lambda}/t!} \\
&\quad = \frac{\left(\prod_{i=1}^{n-1} 1/x_i!\right) \left[1/\left(t - \sum_{i=1}^{n-1} x_i\right)!\right]}{n^t/t!}
\end{aligned}$$

Since the distribution of $X_1, \ldots, X_n$ given T does not depend on λ, $T = \sum_{i=1}^n X_i$ is sufficient.

(b) To show that X_1 is not sufficient, we again compute the distribution of $X_1, \ldots, X_n$ given X_1:

$$\begin{aligned}
&P\left(X_1 = x_1,\, X_2 = \; x_2, \ldots, X_n = x_n \middle|\, X_1 = x_1\right) \\
&\quad = \frac{P\left(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, X_1 = x_1\right)}{P(X_1 = x_1)} \\
&\quad = \frac{P\left(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right)}{P(X_1 = x_1)} \\
&\quad = \frac{\prod_{i=1}^{n} \lambda^{x_i} e^{-\lambda}/x_i!}{\lambda^t e^{-\lambda}/x_1!} \\
&\quad = \prod_{i=2}^{n} \lambda^{x_i} e^{-\lambda}/x_i!
\end{aligned}$$

Since this distribution still depends on λ, X_1 is not sufficient.

(c) According to Theorem A of Section 8.8.1, the statistic T is sufficient if and only if the density $f(x_1,\ldots,x_n|\lambda)$ can be factored as

$$f(x_1,\ldots,x_n|\lambda) = g(T(x_1,\ldots,x_n),\lambda)h(x_1,\ldots,x_n)$$

For the Poisson density and the statistic $T = \sum_{i=1}^n X_i$, we can write

$$\begin{aligned}
f(x_1,\ldots,x_n|\lambda) &= \prod_{i=1}^n \frac{\lambda^{x_i}e^{-\lambda}}{x_i!} \\
&= \left[\lambda^{\sum_{i=1}^n x_i}e^{-n\lambda}\right]\left[\prod_{i=1}^n \frac{1}{x_i!}\right] \\
&= \left[\lambda^T e^{-n\lambda}\right]\left[\prod_{i=1}^n \frac{1}{x_i!}\right] \\
&= g(T(x_1,\ldots,x_n),\lambda)h(x_1,\ldots,x_n)
\end{aligned}$$

where

$$g(T,\lambda) = \lambda^T e^{-n\lambda}$$

$$h(x_1,\ldots,x_n) = \prod_{i=1}^n \frac{1}{x_i!}$$

Problem 5

$X_1,\ldots,X_n$ i.i.d. with density function $f(x|\mu,\tau^2,p) = pf_1(x|\mu) + (1-p)f_2(x|\mu,\tau^2)$, where

$$f_1(x|\mu) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2}\right\}$$

is the $\mathcal{N}(\mu,1)$ density, and

$$f_2(x|\mu,\tau) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left\{-\frac{(x-\mu)^2}{2\tau^2}\right\}$$

is the $\mathcal{N}(\mu,\tau^2)$ density. Then the expectation of a random variable with this mixture density is given by:

$$\begin{aligned}
\mathrm{E}[X_i] &= \int_{-\infty}^{\infty} xf(x|\mu,\tau^2,p)dx \\
&= \int_{-\infty}^{\infty} x\left(pf_1(x|\mu) + (1-p)f_2(x|\mu,\tau^2)\right)dx \\
&= p\int_{-\infty}^{\infty} xf_1(x|\mu)dx + (1-p)\int_{-\infty}^{\infty} xf_2(x|\mu,\tau^2)dx \\
&= p\mu + (1-p)\mu \\
&= \mu
\end{aligned}$$

To calculate the variance of a random variable with this mixture density, we use the fact that $\mathrm{Var}[X_i] = \mathrm{E}[X_i^2] - (\mathrm{E}[X_i])^2$, where $\mathrm{E}[X_i^2]$ is given by:

$$
\begin{aligned}
\mathrm{E}[X_i^2] &= \int_{-\infty}^{\infty} x^2 f(x|\mu, \tau^2, p) dx \\
&= \int_{-\infty}^{\infty} x^2 \left(p f_1(x|\mu) + (1-p) f_2(x|\mu, \tau^2) \right) dx \\
&= p \int_{-\infty}^{\infty} x^2 f_1(x|\mu) dx + (1-p) \int_{-\infty}^{\infty} x^2 f_2(x|\mu, \tau^2) dx \\
&= p(1+\mu^2) + (1-p)(\tau^2 + \mu^2) \\
&= \mu^2 + p + (1-p)\tau^2
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathrm{Var}[X_i] &= \mathrm{E}[X_i^2] - (\mathrm{E}[X_i])^2 \\
&= \mu^2 + p + (1-p)\tau^2 - \mu^2 \\
&= p + (1-p)\tau^2
\end{aligned}
$$

Problem 6

For the mixture density of problem 5, the variance of the sample mean is given by

$$
\begin{aligned}
\mathrm{Var}[\bar{X}_n] &= \frac{\mathrm{Var}[X_i]}{n} \\
&= \frac{p + (1-p)\tau^2}{n}
\end{aligned}
$$

And for large n, we have that

$$
\sqrt{n}\,(M_n - M) \to_d \mathcal{N}\left(0, \frac{1}{4 f^2(M)}\right)
$$

where M is the true median of the distribution. Since this distribution is symmetric about μ, we have that $M = \mu$, and therefore

$$
\begin{aligned}
f^2(M) &= f^2(\mu) \\
&= \left(p \frac{1}{\sqrt{2\pi}} + (1-p) \frac{1}{\sqrt{2\pi\tau^2}} \right)^2 \\
&= p^2 \frac{1}{2\pi} + p(1-p) 2 \frac{1}{2\pi\tau} + (1-p)^2 \frac{1}{2\pi\tau^2}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \mathrm{Var}[\sqrt{(n)}\,(M_n - M)] \approx \frac{1}{4f^2(M)} \\
\Rightarrow \quad & \mathrm{Var}[\sqrt{(n)}M_n] \approx \frac{1}{4f^2(M)} \\
\Rightarrow \quad & n\mathrm{Var}[M_n] \approx \frac{1}{4f^2(M)} \\
\Rightarrow \quad & \mathrm{Var}[M_n] \approx \frac{1}{4nf^2(M)} \\
\Rightarrow \quad & \mathrm{Var}[M_n] \approx \frac{1}{4n\left(p^2\frac{1}{2\pi} + p(1-p)2\frac{1}{2\pi\tau} + (1-p)^2\frac{1}{2\pi\tau^2}\right)}
\end{aligned}
$$

When $p = 0.9$ and $\tau = 5$,

$$
\begin{aligned}
\mathrm{Var}[\bar{X}_n] &= \frac{0.9 + 0.1(25)}{n} \\
&= \frac{1}{n}3.4 \\
\mathrm{Var}[M_n] &\approx \frac{1}{4n\left(0.9^2\frac{1}{2\pi} + 0.9(0.1)2\frac{1}{10\pi} + 0.1^2\frac{1}{50\pi}\right)} \\
&= \frac{1}{n}1.8559
\end{aligned}
$$

So the ratio of the asymptotic variances for $p = 0.9$ and $\tau = 5$ is

$$
\begin{aligned}
\frac{\mathrm{Var}[\bar{X}_n]}{\mathrm{Var}[M_n]} &= \frac{3.4/n}{1.8559/n} \\
&= 1.8320
\end{aligned}
$$

A confidence interval for μ based on $\bar{X}_n$ is given by

$$\bar{X}_n \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}[\bar{X}_n]}$$

In order for a 95% confidence interval to have length 0.1, we must have

$$
\begin{aligned}
& z_{1-0.05/2}\sqrt{\mathrm{Var}[\bar{X}_n]} = 0.05 \\
\Rightarrow \quad & 1.96\sqrt{\frac{3.4}{n}} = 0.05 \\
\Rightarrow \quad & 39.2 = \sqrt{\frac{n}{3.4}} \\
\Rightarrow \quad & 1536.64 = \frac{n}{3.4} \\
\Rightarrow \quad & n = 5224.576
\end{aligned}
$$

Thus a sample of size 5225 is needed to give a 95% confidence interval based on $\bar{X}_n$ with length ≤ 0.1.

Similarly, a confidence interval for μ based on M_n is given by

$$M_n \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}[M_n]}$$

In order for this 95% confidence interval to have length 0.1, we must have

$$
\begin{aligned}
& z_{1-0.05/2}\sqrt{\mathrm{Var}[M_n]} = 0.05 \\
\Rightarrow \quad & 1.96\sqrt{\frac{1.8559}{n}} = 0.05 \\
\Rightarrow \quad & 39.2 = \sqrt{\frac{n}{1.8559}} \\
\Rightarrow \quad & 1536.64 = \frac{n}{1.8559} \\
\Rightarrow \quad & n = 2851.85
\end{aligned}
$$

So a sample of size 2852 is needed to give a 95% confidence interval based on M_n with length ≤ 0.1.

Problem 7

$X_1, \ldots, X_n$ i.i.d. according to the Cauchy distribution with density function

$$
f(x|\theta) = \frac{1}{\pi} \cdot \frac{1}{1 + (x-\theta)^2}
$$

If $\tilde{\theta}_n$ is the sample median, we have for large n that

$$
\sqrt{n}\left(\tilde{\theta}_n - M\right) \to_d \mathcal{N}\left(0, \frac{1}{4f^2(M)}\right)
$$

where M is the true median of the distribution. Since this distribution is symmetric about θ, we have that $M = \theta$, and therefore $f^2(M) = f^2(\theta) = \left(\frac{1}{\pi}\right)^2$. This gives

$$
\begin{aligned}
& \sqrt{n}\left(\tilde{\theta}_n - \theta\right) \to_d \mathcal{N}\left(0, \frac{\pi^2}{4}\right) \\
\Rightarrow \quad & \frac{\sqrt{n}\left(\tilde{\theta}_n - \theta\right)}{\pi/2} \to_d \mathcal{N}(0,1)
\end{aligned}
$$

Using this limiting distribution, we have

$$
\begin{aligned}
P\left(\left|\tilde{\theta}_n - \theta\right| \leq \frac{1}{5}\right) &= P\left(-\frac{1}{5} \leq \tilde{\theta}_n - \theta \leq \frac{1}{5}\right) \\
&= P\left(-\frac{\sqrt{n}}{5} \leq \sqrt{n}\left(\tilde{\theta}_n - \theta\right) \leq \frac{\sqrt{n}}{5}\right) \\
&= P\left(-\frac{\sqrt{n}}{5}\frac{2}{\pi} \leq \frac{\sqrt{n}\left(\tilde{\theta}_n - \theta\right)}{\pi/2} \leq \frac{\sqrt{n}}{5}\frac{2}{\pi}\right) \\
&= \Phi\left(\frac{2\sqrt{n}}{5\pi}\right) - \Phi\left(-\frac{2\sqrt{n}}{5\pi}\right) \\
&= \Phi\left(\frac{2\sqrt{101}}{5\pi}\right) - \Phi\left(-\frac{2\sqrt{101}}{5\pi}\right) \\
&= 0.7993
\end{aligned}
$$

If instead we use an efficient estimator $\hat{\theta}_n$ that satisfies

$$
\begin{aligned}
& \sqrt{n}\left(\hat{\theta}_n - \theta\right) \rightarrow_d \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) \\
\Rightarrow \qquad & \sqrt{n}\left(\hat{\theta}_n - \theta\right) \rightarrow_d \mathcal{N}\left(0, 2\right) \\
\Rightarrow \qquad & \frac{\sqrt{n}\left(\hat{\theta}_n - \theta\right)}{\sqrt{2}} \rightarrow_d \mathcal{N}\left(0, 1\right)
\end{aligned}
$$

then we have

$$
\begin{aligned}
P\left(\left|\hat{\theta}_n - \theta\right| \leq \frac{1}{5}\right) &= P\left(-\frac{1}{5} \leq \hat{\theta}_n - \theta \leq \frac{1}{5}\right) \\
&= P\left(-\frac{\sqrt{n}}{5} \leq \sqrt{n}\left(\hat{\theta}_n - \theta\right) \leq \frac{\sqrt{n}}{5}\right) \\
&= P\left(-\frac{\sqrt{n}}{5}\frac{1}{\sqrt{2}} \leq \frac{\sqrt{n}\left(\tilde{\theta}_n - \theta\right)}{\sqrt{2}} \leq \frac{\sqrt{n}}{5}\frac{1}{\sqrt{2}}\right) \\
&= \Phi\left(\frac{\sqrt{n}}{5\sqrt{2}}\right) - \Phi\left(-\frac{\sqrt{n}}{5\sqrt{2}}\right) \\
&= \Phi\left(\frac{\sqrt{101}}{5\sqrt{2}}\right) - \Phi\left(-\frac{\sqrt{101}}{5\sqrt{2}}\right) \\
&= 0.8448
\end{aligned}
$$

Thus the efficient estimator $\hat{\theta}_n$ has a higher probability of being within 0.2 of the true value θ, as expected.