Discrete Mathematics and Its Applications

Lecture 4: Advanced Counting Techniques: Recurrence Relation

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Nov. 2, 2018

Outline

The Fibonacci sequence

Source:

https://en.wikipedia.org/wiki/

File:Fibonacci.jpg

In 1202, Leonardo Bonacci (known as Fibonacci) asked the following question.

> "Assuming that: a newly born pair of rabbits, one male, one female, are put in an island; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on."
>
> "The puzzle that Fibonacci posed was: how many pairs will there be in one year?"

From https://en.wikipedia.org/wiki/Fibonacci_number

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Let ♠ denote a newly born rabit pair, and ♡ denote a mature rabit pair.

Month	Rabits	
1	♠	1

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Month	Rabits	
1	♠	1
2	♡	1

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2	♡	1
3	♡	

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4	♡ ♡	

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1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5

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7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

Let's try to solve Fibonacci's question.

Let $\spadesuit$ denote a newly born rabit pair, and $\heartsuit$ denote a mature rabit pair.

Month	Rabits	
1	$\spadesuit$	1
2	$\heartsuit$	1
3	$\heartsuit$ $\spadesuit$	2
4	$\heartsuit$ $\heartsuit$ $\spadesuit$	3
5	$\heartsuit$ $\heartsuit$ $\heartsuit$ $\spadesuit$ $\spadesuit$	5
6	$\heartsuit$ $\heartsuit$ $\heartsuit$ $\heartsuit$ $\heartsuit$ $\spadesuit$ $\spadesuit$ $\spadesuit$	8
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How many rabit pairs do we have at the beginning of the 8th month?

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2	♡	1
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4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
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- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.

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- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.
- The number of newly born rabbit pairs equals the number of mature rabbit pairs we have. This is also equal to the number of rabit pairs that we have in the 6th month: 8.

Thus, we will have 13+8 rabit pairs at the beginning of the 8th month.

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$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

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$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Again, what's the next number in this sequence? How can you compute it? $21+13 = 34$ is the answer. You take the last two numbers and add them up to get the next number. Why?

To be precise, let F_n be the n-th number in the Fibonacci sequence. (That is, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$ and so on.) We can define the $(n+1)$-th number as

$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$.

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$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$. Is this enough to completely specify the sequence?
No, because we do not know how to start. To get the Fibonacci sequence, we need to specify two starting values: $F_1 = 1$ and $F_2 = 1$ as well.
Now, you can see that the equation and these special values uniquely determine the sequence. It is also convenient to define $F_0 = 0$ so that the equation works for $n = 1$.

A recurrence

The equation

$$F_{n+1} = F_n + F_{n-1}$$

and the initial values $F_0 = 0$ and $F_1 = 1$ specify all values of the Fibonacci sequence. With these two initial values, you can use the equation to find the value of any number in the sequence.

This definition is called a **recurrence**. Instead of defining the value of each number in the sequence explicitly, we do so by using the values of other numbers in the sequence.

Tilings with 1x1 and 2x1 tiles

You have a walk way of length n units. The width of the walk way is 1 unit. You have unlimited supplies of 1x1 tiles and 2x1 tiles. Every tile of the same size is indistinguishable. In how many ways can you tile the walk way?

Let's consider small cases.

- When $n = 1$, there are 1 way.
- When $n = 2$, there are 2 ways.
- When $n = 3$, there are 3 ways.
- When $n = 4$, there are 5 ways.

Let's define J_n to be the number of ways you can tile a walk way of length n. From the example above, we know that $J_1 = 1$ and $J_2 = 2$.

Can you find a formula for general J_n?

Figuring out the recurrence for J_n

To figure out the general formula for J_n, we can think about the first choice we can make when tiling a walk way of length n. There are two choices:

- (1) We can start placing a 1x1 tile at the beginning, or
- (2) We can start placing a 2x1 tile at the beginning.

In each of the cases, let's think about how many ways we can tile the rest of the walk way, provided that the first step is made.

Note that if we start by placing a 1x1 tile, we are left with a walk way of length $n-1$. From the definition of J_n, we know that there are J_{n-1} ways to tile the rest of the walk way of length $n-1$. Using similar reasoning, we know that if we start with a 2x1 tile, there are J_{n-2} ways to tile the rest of the walk way.

The recurrence for J_n

From the discussion, we have that

$$J_n = J_{n-1} + J_{n-2},$$

where $J_1 = 1$ and $J_2 = 2$.

Note that this is exactly the same recurrence as the Fibonacci sequence, but with different initial values. In fact, we have that

$$J_n = F_{n+1}.$$

The number of bit strings

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Solution: Let a_n denote the number of bit strings of length n that do not have two consecutive 0s.
Note that $a_1 = 2$, $a_2 = 3$. Assume that $n \geq 3$, we have

Way	**# bit strings**
1: ending with 1	a_{n-1}
2: ending with 10	a_{n-2}

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Way	**# bit strings**
1: ending with 1	a_{n-1}
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We conclude that

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$.

Identities on Fibonacci numbers

There are a lot of identities related to Fibonacci numbers. Let's see the first few values in the sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Now, let's add the first few numbers:

$$\begin{aligned}
0+1 &= 1 \\
0+1+1 &= 2 \\
0+1+1+2 &= 4 \\
0+1+1+2+3 &= 7 \\
0+1+1+2+3+5 &= 12 \\
0+1+1+2+3+5+8 &= 20 \\
0+1+1+2+3+5+8+13 &= 33
\end{aligned}$$

From this we can formulate the following conjecture:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Theorem: For $n \geq 0$, we have that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Proof: We shall prove by induction on n. The base case has already been demonstrated when we consider small values of n.

Inductive Step: Let's assume that the statement is true for $n = k$, for $k \geq 0$, i.e., assume that

$$F_0 + F_1 + \cdots + F_k = F_{k+2} - 1.$$

We shall prove that the statement is true when $n = k + 1$. This is not hard to show. We write

$$\begin{aligned}(F_0 + F_1 + \cdots + F_k) + F_{k+1} &= (F_{k+2} - 1) + F_{k+1} \\ &= F_{k+3} - 1,\end{aligned}$$

as required. Note that the first step follows from the induction hypothesis. ■

Another harder identity

The following identity is harder to prove:

$$F_n^2 + F_{n-1}^2 = F_{2n-1}.$$

Let's try a few values as a sanity check.

$$F_1^2 + F_2^2 = 1^2 + 1^2 = 2 = F_3$$

$$F_2^2 + F_3^2 = 1^2 + 2^2 = 5 = F_5$$

$$F_3^2 + F_4^2 = 2^2 + 3^2 = 13 = F_7$$

To see how hard it is to prove the identity, let's try to prove it by induction. (Let's jump to the inductive step.)

We use strong induction. Assume that the statement is true for $n = k, k-1, k-2, \ldots, 0$. We prove the statement for $n = k+1$.

Let's work on the left hand side.

$$\begin{aligned}
& F_{k+1}^2 + F_k^2 = (F_k + F_{k-1})^2 + (F_{k-1} + F_{k-2})^2 \\
= \quad & F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 + F_{k-1}^2 + 2F_{k-1}F_{k-2} + F_{k-2}^2 \\
= \quad & (F_k^2 + F_{k-1}^2) + 2F_kF_{k-1} + (F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} \\
= \quad & F_{2k-1} + F_{2k-3} + 2F_kF_{k-1} + 2F_{k-1}F_{k-2},
\end{aligned}$$

where the last step follows from the induction hypothesis.

Note that we end up with the terms like: $F_kF_{k-1} + F_{k-1}F_{k-2}$. We can keep expanding the terms, but we will end up with the same cross terms like this.

So, let's take a look at a few values of this expression. Maybe we can guess its values.

Let's plug in a few values:

$$F_3F_2 + F_2F_1 = 2 \cdot 1 + 1 \cdot 1 = 3 = F_4$$

$$F_4F_3 + F_3F_2 = 3 \cdot 2 + 2 \cdot 1 = 8 = F_6$$

$$F_5F_4 + F_4F_3 = 5 \cdot 3 + 3 \cdot 2 = 21 = F_8$$

$$F_6F_5 + F_5F_4 = 8 \cdot 5 + 5 \cdot 3 = 55 = F_{10}$$

From this, we can make another conjecture:

Conjecture 2:

$$F_{n+1}F_n + F_nF_{n-1} = F_{2n}.$$

Let's assume that Conjecture 2 is true and see if we can prove the identity that we want.
Recall that we have

$$\begin{aligned}
F_{k+1}^2 + F_k^2 &= F_{2k-1} + F_{2k-3} + 2F_kF_{k-1} + 2F_{k-1}F_{k-2} \\
&= F_{2k-1} + F_{2k-3} + 2(F_kF_{k-1} + F_{k-1}F_{k-2}) \\
&= F_{2k-1} + F_{2k-3} + 2F_{2k-2} \qquad \text{(from Conj 2)} \\
&= (F_{2k-1} + F_{2k-2}) + (F_{2k-2} + F_{2k-3}) \\
&= F_{2k} + F_{2k-1} \\
&= F_{2k+1},
\end{aligned}$$

as required. We use Conjecture 2 to show the second step.

This means that assuming the Conjecture 2, we can show the identity $F_n^2 + F_{n-1}^2 = F_{2n-1}$.

Let's prove Conjecture 2

Conjecture 2: $F_{n+1}F_n + F_nF_{n-1} = F_{2n}$.

Proof: Let's do so by induction. Since we have plugged in many small values, we can only consider the inductive step now. Assume that the statement is true for $n = k, k-1, k-2, \ldots, 0$. We prove the statement for $n = k+1$. We write

$$\begin{aligned} & F_{k+2}F_{k+1} + F_{k+1}F_k = (F_{k+1} + F_k)F_{k+1} + (F_k + F_{k-1})F_k \\ = \ & F_{k+1}^2 + F_kF_{k+1} + F_k^2 + F_{k-1}F_k \\ = \ & (F_kF_{k+1} + F_{k-1}F_k) + F_{k+1}^2 + F_k^2 = F_{2k} + F_{k+1}^2 + F_k^2. \end{aligned}$$

(Note that the 4th step uses the induction hypothesis.) Do you see any familiar terms?

Yes, the terms $F_{k+1}^2 + F_k^2$ is the left hand side of the identity we have just proven. Actually, we cannot use it directly here, because we use Conjecture 2 to prove it and now we are trying to prove the conjecture itself. **Using it results in a circular reasoning.**

We can actually prove the conjecture using that identity, but we first have to break our circular reasoning by proving both statements together. Formally, let's define predicates P and Q:

$$P(n) : F_n^2 + F_{n-1}^2 = F_{2n-1}$$

$$Q(n) : F_{n+1}F_n + F_nF_{n-1} = F_{2n}$$

We will prove that for all integer $n \geq 0$, $P(n) \wedge Q(n)$.

Base Case: We have shown that $P(1)$ and $Q(1)$ are true.

Inductive Step: Assume that the statements are true for $n = k, k-1, \ldots, 1$ for $k \geq 1$. We will prove $P(k+1) \wedge Q(k+1)$.

- $P(k+1)$ can be proved as in the proof of the identity previously.
- To prove $Q(k+1)$, we can use the induction hypotheses and also $P(k+1)$.

Simultaneous induction

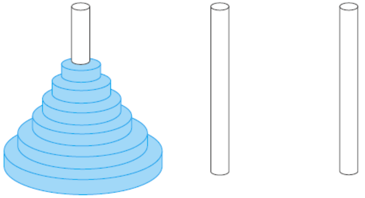
Let's prove $Q(k+1)$. We can continue from our "broken" proof. We have that

$$\begin{aligned} F_{k+2}F_{k+1} + F_{k+1}F_k &= F_{2k} + (F_{k+1}^2 + F_k^2) \\ &= F_{2k} + F_{2k+1} = F_{2k+2}, \end{aligned}$$

as required. Note that the second step uses $P(k+1)$. ■

The technique we use to prove P and Q together is called **simultaneous induction**.

The Tower of Hanoi problem

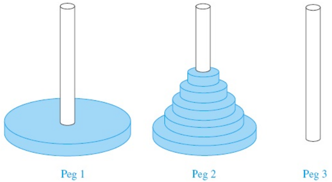


Source: https://en.wikipedia.org/wiki/File:Fibonacci.jpg

In 1883, French mathematician Édouard Lucas asked the following question (called the Tower of Hanoi).

The puzzle consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in Figure). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for sequence $\{H_n\}$.



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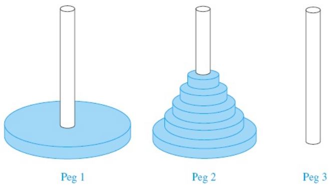

Tower of Hanoi

Step	**# moves**
1: top $n-1$ disk to peg 2	H_{n-1}
2: largest disk to peg 3	1
3: $n-1$ disks from peg 2 to 3	H_{n-1}

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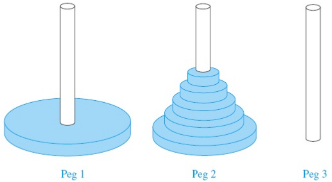

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Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that $H_1 = 1$, and when $n \geq 2$

$$\begin{aligned} H_n &= 2H_{n-1} + 1 = 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &= \cdots = 2^{n-1} H_1 + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1 \end{aligned}$$

Particularly, $2^{64} - 1 = 18,446,744,073,709,551,615$.

Codeword Enumeration

Question: A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. How many valid n-digit codewords?

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Note that $a_1 = 9$ since only 0 is invalid. Assume that $n \geq 2$, we have

Way		**# digits**
Way 1:	ending with non-zero	$9a_{n-1}$
	Task 11: choose the ending digit	9
	Task 12: choose the remaining digits	a_{n-1}
Way 2:	ending with zero	$10^{n-1} - a_{n-1}$

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We conclude that

$$a_n = 8a_{n-1} + 10^{n-1}$$

for $n \geq 2$.

Specifying the order of multiplication

Question: Find the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdots x_n$, to specify the order of multiplication.

Specifying the order of multiplication

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Solution: Note that $C_0 = C_1 = 1$. Assume that $n \geq 2$, we have

	Location of last operation ·	**# orders of multip.**
Way 1:	between x_0 and x_1	$C_0 C_{n-1}$
Way 2:	between x_1 and x_2	$C_1 C_{n-2}$
...	...	...
Way k:	between x_{k-1} and x_k	$C_{k-1} C_{n-k}$
...	...	...
Way n:	between x_{n-1} and x_n	$C_{n-1} C_0$

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We conclude for $n \geq 2$ that

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

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Steps:

1. Define subproblems;
2. Write down the recurrence that relates subproblems;
3. Recognize and solve the base cases.

Example (1-dimensional DP)

Given n, find the number of different ways to write n as the sum of $1, 3, 4$. For example: when $n = 5$, the answer is 6.

$$\begin{aligned} 5 &= 1+1+1+1+1 = 1+1+3 = 1+3+1 \\ &= 3+1+1 = 1+4 = 4+1 \end{aligned}$$

How to solve the 1-dimensional DP?

Solution

Step 1: Define subproblems *Let D_n be the number of ways to write n as the sum of $1, 3, 4$.*

Step 2: Find the recurrence

- *Consider one possible solution $n = x_1 + x_2 + \cdots + x_m$;*
- *If $x_m = 1$, the rest of the terms must sum to $n - 1$. Thus, the number of sums that end with $x_m = 1$ is equal to D_{n-1};*
- *Similarly, recurrence is then $D_n = D_{n-1} + D_{n-3} + D_{n-4}$.*

Step 3: Solve the base cases *We can set $D_0 = D_1 = D_2 = 1$, and $D_3 = 2$.*

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Implement:

```
D[0] = D[1] = D[2] = 1; D[3] = 2;
for(i = 4; i <= n; i++)
    D[i] = D[i-1] + D[i-3] + D[i-4];
```

2-dimensional DP Example

Question: Given two strings x and y, find the longest common subsequence (LCS). For example, x : $A\mathbf{BC}BD\mathbf{AB}$ and y : $\mathbf{B}D\mathbf{CAB}C$, then $BCAB$ is the longest subsequence found in both sequences.

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Question: Given two strings x and y, find the longest common subsequence (LCS). For example, x : A**BC**BD**AB** and y : **B**D**CAB**C, then $BCAB$ is the longest subsequence found in both sequences.

Solution

Step 1: Define subproblems *Let D_{ij} be the length of the LCS of $x_{1\ldots i}$ and $y_{1\ldots j}$;*

Step 2: Find the recurrence

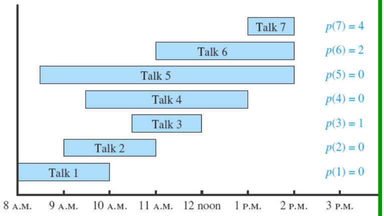
- *If $x_i = y_j$, they both contribute to the LCS, then $D_{ij} = D_{i-1,j-1} + 1$;*
- *Otherwise, either x_i or y_j does not contribute to the LCS, so one can be dropped $D_{ij} = \max\{D_{i-1,j}, D_{i,j-1}\}$.*

Step 3: Solve the base cases *We can set $D_{i0} = D_{0j} = 0$.*

Implementation

```
for(i = 0; i <= n; i++) D[i][0] = 0;
for(j = 0; j <= m; j++) D[0][j] = 0;
for(i = 1; i <= n; i++) {
    for(j = 1; j <= m; j++) {
        if(x[i] == y[j])
            D[i][j] = D[i-1][j-1] + 1;
        else
            D[i][j] = max(D[i-1][j], D[i][j-1]);
    }
}
```

Talk scheduling



We need to schedule as many talks as possible in a single lecture hall. These talks have preset start and end times; once a talk starts, it continues until it ends; no two talks can proceed at the same time; and a talk can begin at the same time another one ends. Our goal is to have the largest possible combined attendance of the scheduled talks.

We formalize this problem by supposing that we have n talks, where talk j begins at time s_j, ends at time e_j , and will be attended by w_j students. We define

$$p(j) = \begin{cases} \max\{i\}, & \text{if } i < j \wedge e_i \leq s_j; \\ 0, & \text{otherwise.} \end{cases}$$

DP for talk scheduling

Solution

Step 1: Define subproblems $T(j)$ *is the maximal number of total attendees for an optimal schedule for the first* j *talks.*

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- *When talk j belongs to the optimal schedule,* $T(j) = w_j + T(p(j))$;
- *When talk j does not belong to the optimal schedule,* $T(j) = T(j-1)$;
- *Thus,* $T(j) = \max\{w_j + T(p(j)), T(j-1)\}$;

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- *Thus,* $T(j) = \max\{w_j + T(p(j)), T(j-1)\}$;

Step 3: Solve the base cases *We can set* $T(0) = 0$.

Implementation

procedure *Maximum Attendees* ($s_1, s_2, \ldots, s_n$: start times of talks;
$e_1, e_2, \ldots, e_n$: end times of talks; $w_1, w_2, \ldots, w_n$: number of attendees to talks)
sort talks by end time and relabel so that $e_1 \leq e_2 \leq \cdots \leq e_n$

Initialize p(j)

for $j := 1$ **to** n
  if no job i with $i < j$ is compatible with job j
    $p(j) = 0$
  else $p(j) := \max\{i \mid i < j$ and job i is compatible with job $j\}$
$T(0) := 0$

Find the optimal solution

for $j := 1$ **to** n
  $T(j) := \max(w_j + T(p(j)), T(j-1))$
return $T(n)\{T(n)$ is the maximum number of attendees$\}$

An explicit form of the Fibonacci sequence

While the recurrence for F_n completely specifies the sequence, it is hard to find the value of, say, F_{20} quickly. We really have to enumerate the sequence from $F_0, F_1, \ldots,$ to get to F_{20}. Also, with the definition based on the recurrence, other properties of the sequence is unclear (e.g., how fast the sequence grows).
Therefore, it might be useful to find the explicit definition of the Fibonacci sequence.

Ratios

To get started, we might want to look for a common form of the function. We can start by looking at the numbers in the sequence.

n	F_n	ratio F_n/F_{n-1}
1	1	
2	1	1.0000000000
3	2	2.0000000000
4	3	1.5000000000
5	5	1.6666666667
6	8	1.6000000000
7	13	1.6250000000
8	21	1.6153846154
9	34	1.6190476190
10	55	1.6176470588
11	89	1.6181818182
12	144	1.6179775281

n	F_n	ratio F_n/F_{n-1}
13	233	1.6180555556
14	377	1.6180257511
15	610	1.6180371353
16	987	1.6180327869
17	1597	1.6180344478
18	2584	1.6180338134
19	4181	1.6180340557
20	6765	1.6180339632
21	10946	1.6180339985
22	17711	1.6180339850
23	28657	1.6180339902
24	46368	1.6180339882

The 1st guess: a^n

We can see that the ratio between two consecutive Fibonacci numbers is close to 1.61803. We may guess that the explicit form for F_n is an exponential function a^n. (While we know that this is not true, it may give us hints on the correct function.)

Let's try to figure out the exact value for a. The value a must satisfy the recurrence $F(n+1) = F(n) + F(n-1)$, i.e.,

$$a^{n+1} = a^n + a^{n-1}.$$

We can try to solve for a. Dividing the equation by a^{n-1}, we get

$$a^2 = a + 1,$$

i.e., $a^2 - a - 1 = 0$.

Solutions (1)

We can use a standard formula to get the values of a, i.e., a can be $\frac{1+\sqrt{1^2+4\cdot 1\cdot 1}}{2\cdot 1}$, $\frac{1-\sqrt{1^2+4\cdot 1\cdot 1}}{2\cdot 1}$, or

$$\tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}.$$

These look nice because $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

These solutions give us two candidates for F_n:

$$g(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n, \text{ and } h(n) = \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

But we can see that while both $g(n)$ and $h(n)$ satisfy $g(n+1) = g(n) + g(n-1)$ and $h(n+1) = h(n) + h(n-1)$, they are not the correct function for F_n. (We can just plug in various values of n to check.)

Linear recurrence relations

Definition

A linear homogeneous recurrence relation (shorted in LHR^2) of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \cdots, c_k$ are real numbers, and $c_k \neq 0$.

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Examples

- $P_n = (1.11)P_{n-1}$, $f_n = f_{n-1} + f_{n-2}$ and $a_n = 5a_{n-5}$ are linear homogeneous recurrence relations associated with of degrees one, two and five;

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- $P_n = (1.11)P_{n-1}$, $f_n = f_{n-1} + f_{n-2}$ and $a_n = 5a_{n-5}$ are linear homogeneous recurrence relations associated with of degrees one, two and five;
- $a_n = a_{n-1} + a_{n-2}^2$ is not linear; $H_n = 2H_{n-1} + 1$ is not homogeneous; $B_n = nB_{n-1}$ does not have constant coefficients.

Solving LHR^2 with constant coefficients

Guess: The basic approach for solving LHR^2 is to look for solutions of the form $a_n = r^n$, where r is a constant.
If $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

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Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation.
We call this the **characteristic equation** of the recurrence relation.
The solutions of this equation are called the **characteristic roots** of the recurrence relation.

LHR^2 with degree two I

Theorem

Let c_1 and c_2 be real numbers $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \cdots$, where α_1 and α_2 are constants.

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Proof

Note that r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$. Thus, we have $r_i^2 = c_1 r_i + c_2$ for $i = 1, 2$.

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If $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, we have

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n = a_n \end{aligned}$$

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If $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, we have

$$\alpha_1 = \frac{C_1 - r_2 C_0}{r_1 - r_2}$$

$$\alpha_2 = C_0 - \alpha_1 = \frac{r_1 C_0 - C_1}{r_1 - r_2}$$

Solutions for Fibonacci numbers

Since $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$ are two roots of $r^2 - r - 1 = 0$.
We define $g(n)$ and $h(n)$ as follows:

$$g(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n, \text{ and } h(n) = \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

Furthermore, we have $\ell(n) = \alpha g(n) + \beta h(n)$.

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To get the actual function, we observe that if both $g(n)$ and $h(n)$ are solutions to our recurrence, then for any α and β, we have

$$\alpha\left(\tfrac{1+\sqrt{5}}{2}\right)^0 + \beta\left(\tfrac{1-\sqrt{5}}{2}\right)^0 = \alpha + \beta = 0,$$

and

$$\alpha\left(\tfrac{1+\sqrt{5}}{2}\right)^1 + \beta\left(\tfrac{1-\sqrt{5}}{2}\right)^1 = \alpha\left(\tfrac{1+\sqrt{5}}{2}\right) + \beta\left(\tfrac{1-\sqrt{5}}{2}\right) = 1.$$

Final solution

The first equation gives $\beta = -\alpha$. Put that in the second equation to get

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) - \alpha\left(\frac{1-\sqrt{5}}{2}\right) = \frac{2\alpha\sqrt{5}}{2} = \alpha\sqrt{5} = 1,$$

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implying that $\alpha = 1/\sqrt{5}$ and $\beta = -1/\sqrt{5}$.
Using the obtained α and β, our solution to F_n becomes

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Note that $\frac{1+\sqrt{5}}{2} \approx 1.61803$ is the golden ratio. Also observe that $|\frac{1-\sqrt{5}}{2}| \approx |-0.61803| < 1$; therefore, the term $\left(\frac{1-\sqrt{5}}{2}\right)^n$ goes to zero as n goes to infinity. This explains why we only observe only the ratio $\frac{1+\sqrt{5}}{2}$ in F_n as n gets large.

LHR^2 with degree two II

Theorem

Let c_1 and c_2 be real numbers $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \cdots$, where α_1 and α_2 are constants. [The proof is left as an exercise]

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What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

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Solution: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

Using the initial conditions, it follows that $a_0 = 1 = \alpha_1$, $a_1 = 6 = 3\alpha_1 + 3\alpha_2$. That is $a_n = 3^n + n3^n$.

LHR^2 with degree k I

Theorem

Let $c_1, c_2, \cdots, c_k$ be real numbers $c_k \neq 0$. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots r_i for $i = 1, 2, \cdots, k$. Then sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \cdots$, where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.
[The proof is left as an exercise]

Example

Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5$, and $a_2 = 15$.

Solution:

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Solution: The characteristic equation of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

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Solution: The characteristic equation of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

The characteristic roots are $r = 1, r = 2$, and $r = 3$. Hence, the solutions to this recurrence relation are of form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

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Using the initial conditions, it gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3, a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3,$$
$$a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3.$$

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Hence, we have $a_n = 1 - 2^n + 2 \cdot 3^n$. ($\alpha_1 = 1, \alpha_2 = -1$, and $\alpha_3 = 2$)

LHR^2 with degree k II

Theorem

Let $c_1, c_2, \cdots, c_k$ be real numbers $c_k \neq 0$. Suppose that characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots r_i with multiplicities m_i (for $i = 1, 2, \cdots, t$), respectively (that is $\sum_{i=1}^{t} m_i = k$). Then sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \cdots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$. [The proof is left as an exercise]

Example

Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with the initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$.
Solution:

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Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with the initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

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Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

The characteristic root is $r = -1$ of multiplicity three of the equation. Hence, the solutions to this recurrence relation are of form

$$a_n = \alpha_0 \cdot (-1)^n + \alpha_1 \cdot n \cdot (-1)^n + \alpha_2 \cdot n^2 \cdot (-1)^n.$$

Using the initial conditions, it gives

$$a_0 = 1 = \alpha_0, a_1 = -2 = -\alpha_0 - \alpha_1 - \alpha_2,$$
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$$a_0 = 1 = \alpha_0, a_1 = -2 = -\alpha_0 - \alpha_1 - \alpha_2,$$
$$a_2 = -1 = \alpha_0 + 2\alpha_1 + 4\alpha_2.$$

Hence, $a_n = (1 + 3n + -2n^2)(-1)^n$. ($\alpha_0 = 1, \alpha_1 = 3$, and $\alpha_2 = -2$)

LNR^2 with constant coefficients

Definition

A **linear nonhomogeneous recurrence relation** (shorted in LNR^2) with constant coefficients of degree k is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \cdots, c_k$ are real numbers ($c_k \neq 0$), and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the **associated homogeneous recurrence relation**.

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Example

Recurrence relations $a_n = 3a_{n-1} + 2n$, $a_n = a_{n-2} + n^2 + 1$, $a_n = 3a_{n_1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + n!$ are examples of LNR^2 with constant coefficients

Solution for LNR^2

Theorem

If $\{a_n^{(p)}\}$ is a particular solution of LNR^2 with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of LHR^2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

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$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof

Because $\{a_n^{(p)}\}$ is a particular solution of the LNR^2, we therefore have

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Proof Cont'd

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Since $\{a_n^{(p)}\}$ is a particular solution of the LNR^2, we therefore have

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the LNR^2, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Thus, we have

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.

Consequently, we have $b_n = \{a_n^{(p)} + a_n^{(h)}\}$ for all n.

Example I

Find the solution to recurrence relation $a_n = 3a_{n-1} + 2n$ with initial conditions $a_1 = 3$.

Solution:

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Find the solution to recurrence relation $a_n = 3a_{n-1} + 2n$ with initial conditions $a_1 = 3$.

Solution:

Step 1: find the solution for LHR^2. The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solution is $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Example I

Find the solution to recurrence relation $a_n = 3a_{n-1} + 2n$ with initial conditions $a_1 = 3$.

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Step 1: find the solution for LHR^2. The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solution is $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Step 2: find a particular solution. Let $p_n = cn + d$ be a particular solution of the recurrence relation, where c and d are constants. Then, we have $cn + d = 3(c(n-1) + d) + 2n$, that is $(2+2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$, i.e., $c = -1$ and $d = -3/2$.

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Consequently, $a_n = \{a_n^{(p)} + a_n^{(h)}\} = -n - \frac{3}{2} + \alpha \cdot 3^n$. Using the initial conditions, it gives $\alpha = \frac{11}{6}$. Thus,

$$a_n = (\frac{11}{6})3^n + n - \frac{3}{2}.$$

Forms of the particular solutions

Theorem: Suppose that $\{a_n\}$ satisfies LNR^2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \cdots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \cdots, b_t$ and s are real numbers.

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where $b_0, b_1, \cdots, b_t$ and s are real numbers.

- If s is not a root of the characteristic equation of the associated LHR^2, there is a particular solution of form

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- If s is not a root of the characteristic equation of the associated LHR^2, there is a particular solution of form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n;$$

- Else, s is a root of multiplicity m, the particular solution is of form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Example II

Let a_n be the sum of the first n positive integers, i.e., $a_n = \sum_{k=1}^{n} a_k$.
Solution: Note that a_n satisfies LNR^2 $a_n = a_{n-1} + n$.

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Let a_n be the sum of the first n positive integers, i.e., $a_n = \sum_{k=1}^{n} a_k$.
Solution: Note that a_n satisfies LNR^2 $a_n = a_{n-1} + n$.
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Step 2: find a particular solution. Since $F(n) = n = n \cdot (1)^n$, and $s = 1$ is a root of the characteristic equation of the associated LHR^2. Thus, there is a particular solution $n(cn + d) = cn^2 + dn$.
Then, we have $cn^2 + dn = c(n-1)^2 + d(n-1) + n$, that is $(2c - 1)n + (c - d) = 0$. It follows that $cn + d$ is a solution if and only if $2c - 1 = 0$ and $c - d = 0$, i.e., $c = d = \frac{1}{2}$.

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Thus, there is a particular solution $n(cn + d) = cn^2 + dn$.
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Consequently, $a_n = \{a_n^{(p)} + a_n^{(h)}\} = \frac{n(n+1)}{2} + \alpha$. Using the initial conditions, it gives $\alpha = 0$. Thus,

$$a_n = \frac{n(n+1)}{2}.$$

Take-aways

Conclusions

- Applications of recurrence relations
- Algorithms and recurrence relations
- Solving linear recurrence relations
  - Solving LHR^2
  - Solving LNR^2