Discrete Mathematics and Its Applications

Lecture 2: Basic Structures: Function

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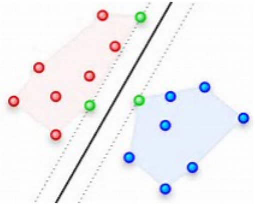
Outline

Motivations

Classifier



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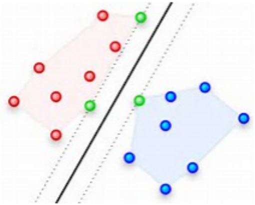
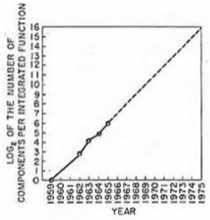


Moore's law



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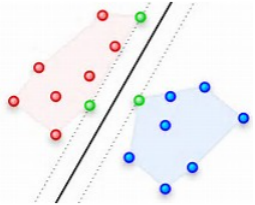

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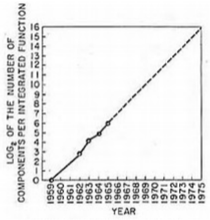
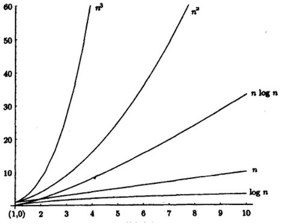


Complexity analysis



Motivations

- Functions connect the relationships between different objects;
- A function may tell us a rule;
- Models in data mining, machine learning, etc., are usually presented in functions.

Function

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to *each element* of A. We write $f(a) = b$ if b is the unique element of B assigned by function f to element a of A. If f is a function from A to B, we write $f : A \to B$.

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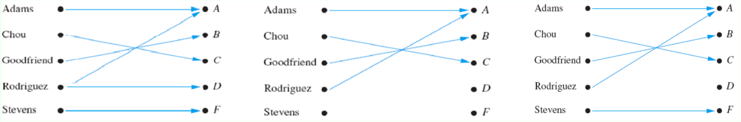

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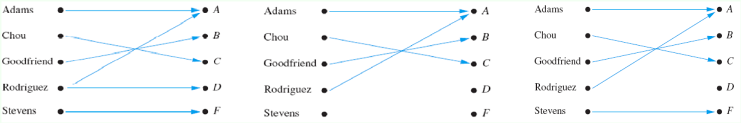

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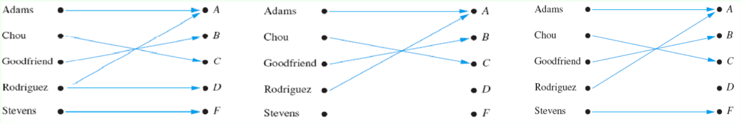

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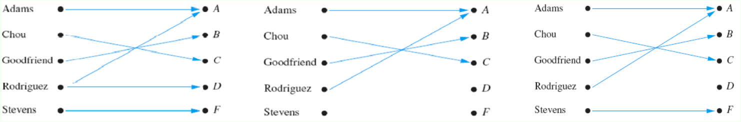

- Functions are sometimes also called mappings or transformations;
- Functions are specified in many different ways, such as assignment rules, formula, relations, etc;
- This function is defined by assignment $f(a) = b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

Domain and codomain

Definition

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

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- The domain of g is set $\{Adams, Chou, Goodfriend, Rodriguez, Stevens\}$, and the codomain is set $\{A, B, C, D, F\}$.
- The range of g is set $\{A, B, C, F\}$, because each grade except D is assigned to some students.

Add and product

Definition

Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to R defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

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Example

Let f_1 and f_2 be functions from R to R such that $f_1(x) = (x+1)^2$ and $f_2(x) = -(x-1)^2$. Thus, we have

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = (x+1)^2 - (x-1)^2 = 4x,$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = -(x+1)^2 (x-1)^2 = -(x^2-1)^2.$$

Projection

Definition

Let f be a function from A to B and let S be a subset of A. The image of S under f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t | \forall s \in S, t = f(s)\}.$$

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Example

Let $f(x)$ be function from R to R such that $f(x) = x + 1$, and Z be the set of integers. Then

$$f(Z) = Z.$$

One-to-one function

Definition

A function f is said to be one-to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

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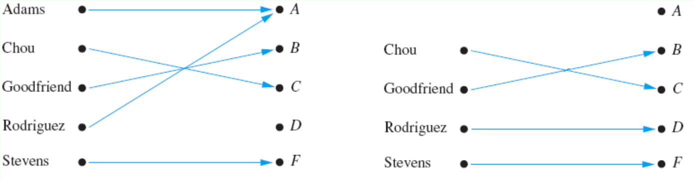
Injunctive function



Remark: f is one-to-one $\Leftrightarrow \forall a \forall b(f(a) = f(b) \rightarrow a = b)$ or $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$.

Onto function

Definition

A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

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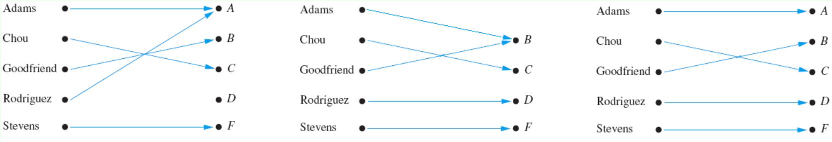
Surjective function



Remark: A function f is onto if $\forall y \exists x (f(x) = y)$.

Monotonicity

Definition

A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called decreasing if $f(x) \geq f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

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Remarks

- A function f is increasing if $\forall x \forall y(x < y \to f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \to f(x) < f(y))$;
- A function f is decreasing if $\forall x \forall y(x < y \to f(x) \geq f(y))$, strictly increasing if $\forall x \forall y(x < y \to f(x) > f(y))$.

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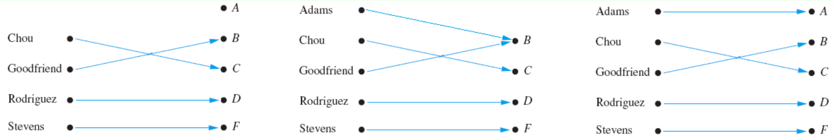

Remarks: f is a one-to-one correspondence if and only if $\Leftrightarrow$ $\forall a \forall b(a = b \leftrightarrow f(a) = f(b))$.

Inverse function

Definition

Let f be a one-to-one correspondence from set A to set B. The inverse function of f is the function that assigns an element $b \in B$ to a unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

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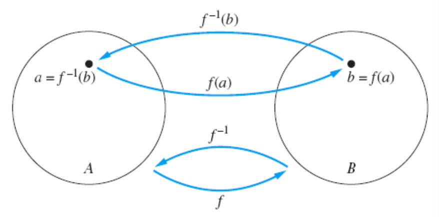

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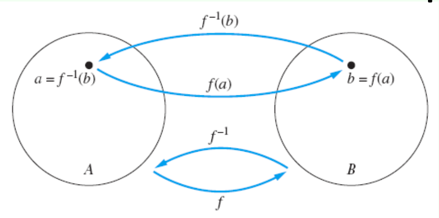

Remarks:

- $f^{-1} \neq \frac{1}{f}$;
- A one-to-one correspondence is called invertible because we can define an inverse of this function.

Composition of functions

Definition

Let g be a function from set A to set B and let f be a function from set B to set C. The composition of functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

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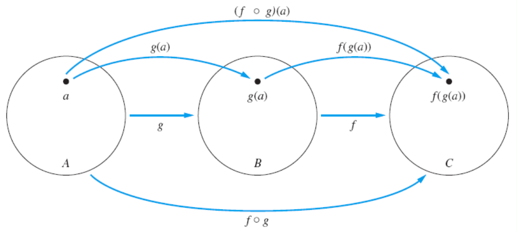

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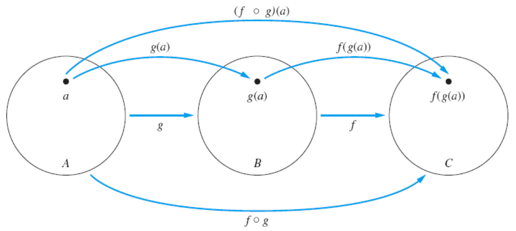

Remarks: The commutative law does not hold for the composition of functions, i.e.,

$$f \circ g \neq g \circ f.$$

Examples

Example I

Let $f(x) = 2x + 3$ and $g(x) = 3x + 2$ from Z to Z. What is the composition of f and g? What is the composition of g and f ?

- $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$;
- $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.

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Example II

If $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}x - \frac{2}{3}$, how about the answers?

- $(f \circ g)(x) = f(g(x)) = f(\frac{1}{3}x - \frac{2}{3}) = 3(\frac{1}{3}x - \frac{2}{3}) + 2 = x$;
- $(g \circ f)(x) = g(f(x)) = g(3x + 2) = \frac{1}{3}(3x + 2) - \frac{2}{3} = x$.

Remarks:

- $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$, i.e., $(f \circ f^{-1}) = \mathcal{I}_B$;
- $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$, i.e., $(f^{-1} \circ f) = \mathcal{I}_A$.

Inverse of composition of functions

Theorem

Let f and g be invertible functions such that their composition $f \circ g$ is well defined. Then $f \circ g$ is invertible and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proof.

Let A, B, and C be sets such that $g : A \to B$ and $f : B \to C$.
Then the following two equations must be shown to hold:
$(g^{-1} \circ f^{-1}) \circ (f \circ g) = \mathcal{I}_A$, and $(f \circ g) \circ (g^{-1} \circ f^{-1}) = \mathcal{I}_B$.

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In terms of associative rule for function composition (homework), we have

$$\begin{aligned}(g^{-1} \circ f^{-1}) \circ (f \circ g) &= g^{-1} \circ ((f^{-1} \circ f) \circ g) \\ &= g^{-1} \circ (\mathcal{I}_B \circ g) \\ &= g^{-1} \circ g = \mathcal{I}_A\end{aligned}$$

Similarly, we have $(f \circ g) \circ (g^{-1} \circ f^{-1}) = \mathcal{I}_B$. □

Graph of functions

Definition

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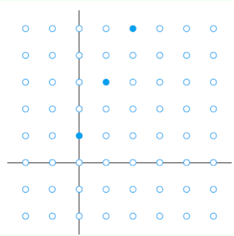
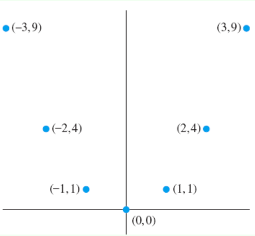


$f(n) = 2n + 1$ from Z to Z



$f(x) = x^2 + 1$ from Z to Z

Some important functions——Floor and ceiling functions

Definition

- The floor function assigns to a real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$;
- The ceiling function assigns to a real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

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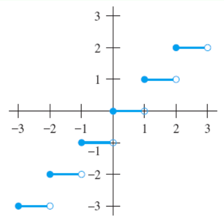
Remarks

- $\lfloor x \rfloor = \max\{m \in Z : m \leq x\}$;
- $\lceil x \rceil = \min\{m \in Z : m \geq x\}$.
- There are many applications, including data transmission, and data storage, etc;

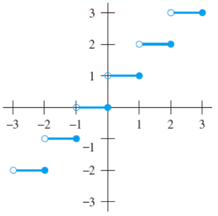
Graphs of floor and ceiling functions

Graphs

Floor function



Ceiling function



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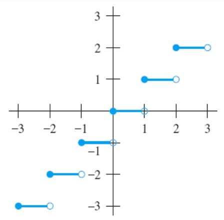
Floor function



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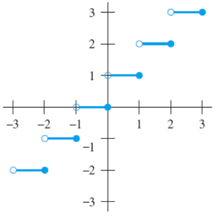

Examples

- $\lfloor \frac{1}{2} \rfloor = 0$, $\lceil \frac{1}{2} \rceil = 1$;
- $\lfloor -\frac{1}{2} \rfloor = -1$, $\lceil -\frac{1}{2} \rceil = 0$;
- $\lfloor -2.1 \rfloor = -3$, $\lceil -2 \rceil = -2$;

Useful properties of floor and ceiling functions

Properties

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- $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$;

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- $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$;
- $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$;
- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$;

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$\square$

Some important functions——Indicator function

Definition

The indicator function of a subset A of set S is a function

$$I_A : X \to \{0, 1\},$$

defined as

$$I_A(x) := \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{else.} \end{cases}$$

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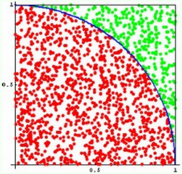

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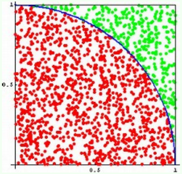

- $A = \{(x, y) : x^2 + y^2 \leq 1 \wedge x \geq 0 \wedge y \geq 0\}$, and $S = \{(x, y) : 1 \geq x \geq 0 \wedge 1 \geq y \geq 0\}$
- $\forall P_i \in S$, we define $I_A(P_i)$ and $I_{S-A}(P_i)$;
- $\pi \approx 4 \frac{\sum_{i=1}^{n} I_A(P_i)}{\sum_{i=1}^{n} I_A(P_i) + \sum_{i=1}^{n} I_{S-A}(P_i)}$

Set covering problem

Input

Universal set $U = \{u_1, u_2, \cdots, u_n\}$
Subsets $S_1, S_2, \cdots, S_m \subseteq U$
Cost $c_1, c_2, \cdots, c_m$

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The problem can be solved by linear programming or submodular.

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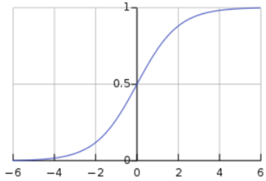
Logistic function

Graphs



General form

$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}},$$

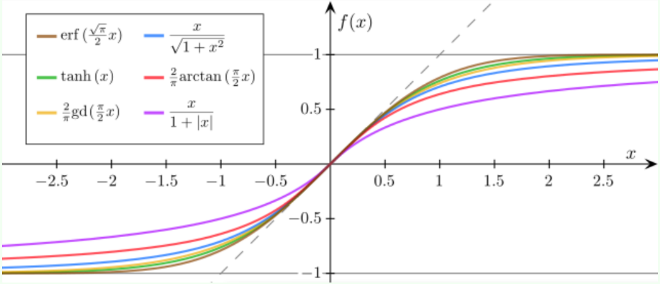
where x_0 is the x-value of the sigmoid's midpoint, L is the maximum value, and k is the steepness of the curve.

Some important functions——Sigmoid functions

Applications

- Binary classification in logistic regression model;
- Activation function in artificial neurons;
- Cumulative distribution function in statistics.

Other forms



Take-aways

Conclusions

- Function
- One-to-one and onto functions
- Inverse functions and compositions of functions
- Some important functions