Discrete Mathematics and Its Applications

Lecture 7: Graphs: Euler, Hamilton and Coloring

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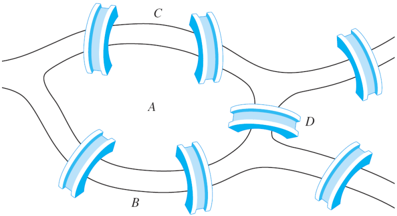
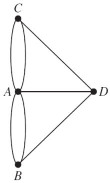
Outline

Euler circuit and euler path

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River.





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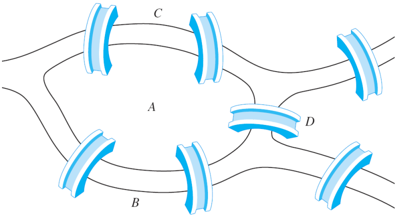

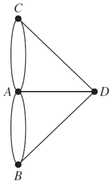

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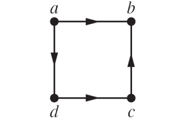
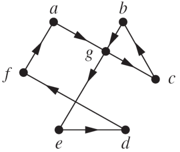
Definition

An **Euler circuit** in a graph G is a simple circuit containing every edge of G. An **Euler path** in G is a simple path containing every edge of G.

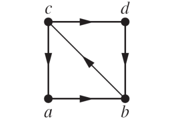
Example



H_1

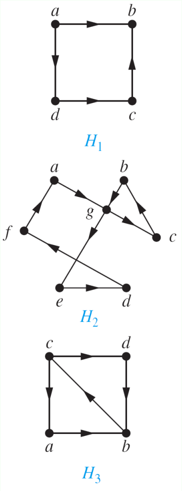


H_2



H_3

Example



Question: Which of the directed graphs in figure have an Euler circuit? Of those that do not, which have an Euler path?

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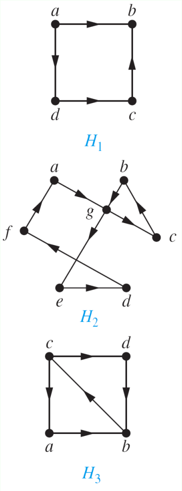

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The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$.

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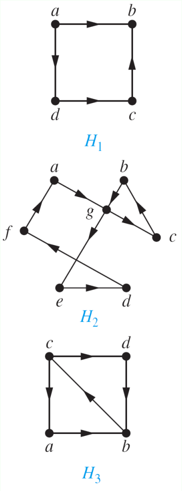

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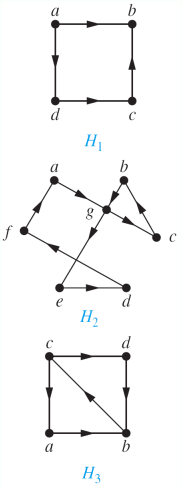

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Neither H_1 nor H_3 has an Euler circuit.
H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not.

Necessary and sufficient for Euler circuit

Theorem

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Algorithm to find Euler circuit

ALGORITHM 1 Constructing Euler Circuits.

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
H := G with the edges of this circuit removed
while H has edges
    subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of *circuit*
    H := H with edges of *subcircuit* and all isolated vertices removed
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Algorithm 1 provides an efficient algorithm for finding Euler circuits in a connected multigraph G with all vertices of even degree.

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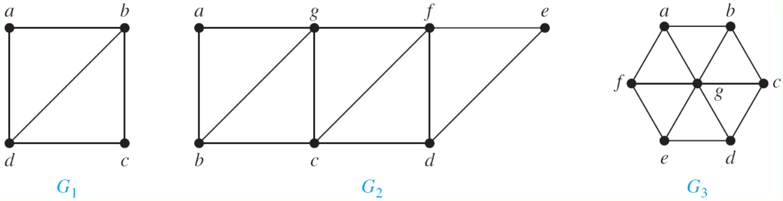

Which graphs shown in the figure have an Euler path?

Hamilton paths and circuits

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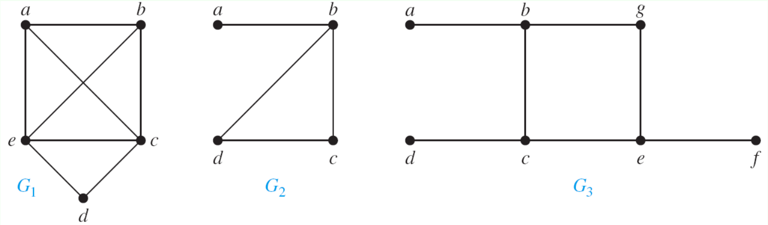

Which graphs shown in the figure have an Hamilton path?

Conditions for the existence of Hamilton circuit

Dirac's theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

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If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Planar graph

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A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint).

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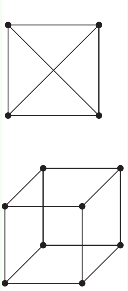

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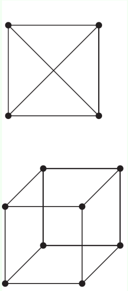

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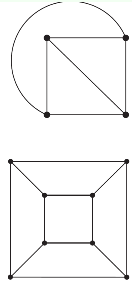

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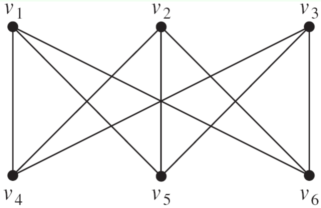

Is $K_{3,3}$ a planar graph?

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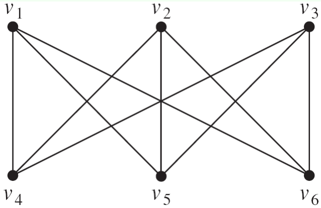
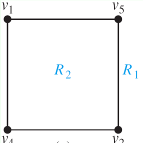
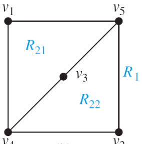


Is $K_{3,3}$ a planar graph?

Solution: In any planar representation of $K_{3,3}$, v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2, as shown in Figure (a). v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}, as shown in Figure (b).
Note that there is no way to place v_6.



(a)



(b)

Outline

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Basic step: The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1, because $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$.

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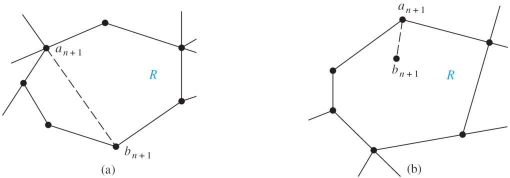

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We have completed the induction argument. Hence, $r_n = e_n - v_n + 2$ for all n. □

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$$2e = \sum_{\text{all regions } R} deg(R) \geq 3r.$$

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$$2e = \sum_{\text{all regions } R} deg(R) \geq 3r.$$

Using Eulers formula, we obtain $e - v + 2 = r \leq (2/3)e$. This shows that $e \leq 3v - 6$. □

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If G has one or two vertices, the result is true. If G has at least three vertices, by Corollary 1 we know that $e \leq 3v - 6$, so $2e \leq 6v - 12$.

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If the degree of every vertex were at least six, then because $2e = \sum_{v \in V} deg(v)$ (by the handshaking theorem), we would have $2e \geq 6v$. But this contradicts the inequality $2e \leq 6v - 12$.

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It follows that there must be a vertex with degree no greater than five. $\square$

Corollary III

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Proof.

The proof of this corollary is similar to that of Corollary 1, except that in this case the fact that there are no circuits of length three implies that the degree of a region must be at least four. □

Corollary III

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Question: Determine whether K_5 and $K_{3,3}$ are planar graphs or not.

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Solution:
For K_5, we have $e = 10$ and $3v - 6 = 9$. However, the inequality $e \leq 3v - 6$ is not satisfied. Therefore, K_5 is not planar.

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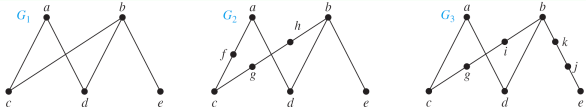
Because $K_{3,3}$ has no circuits of length three, we have $e = 9$ and $2v - 4 = 8$. Since Corollary 3 is not satisfied, $K_{3,3}$ is nonplanar.

Outline

Homeomorphic

Definition

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an **elementary subdivision**. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

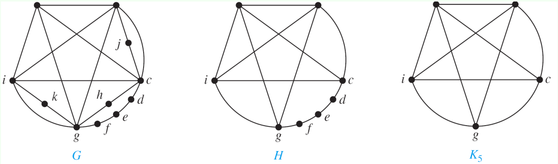


Question: Determine whether the graphs G_1, G_2, and G_3 are all homeomorphic or not.

Homeomorphic application

Kuratowski's theorem

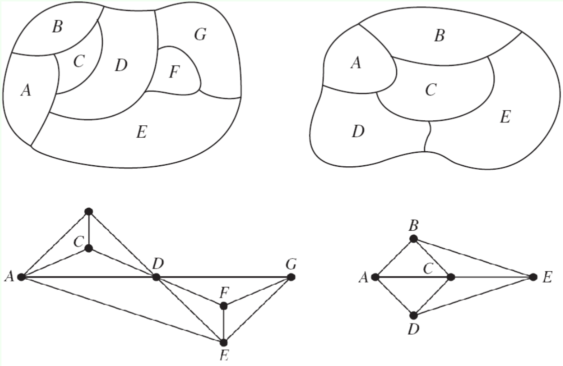
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.



Question: Determine whether the graph G shown in the figure is planar.

Problem formulation

Motivation



Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.

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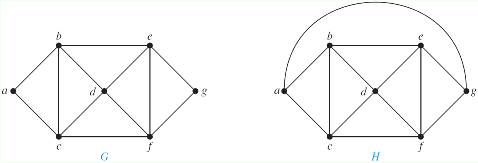
Chromatic number

The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$.



Question: What are the chromatic numbers of graphs G and H?

The four color theorem

Theorem

The chromatic number of a planar graph is no greater than four.

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Remarks:

- Note that the four color theorem applies only to planar graphs.
- Nonplanar graphs can have arbitrarily large chromatic numbers.

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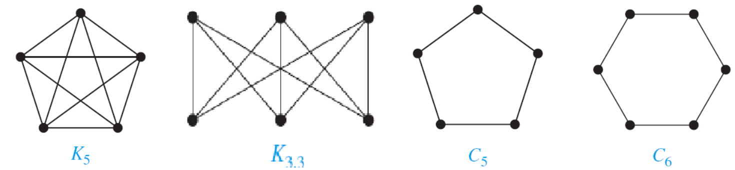

Question: What are the chromatic numbers of graphs K_5, $K_{3,3}$, C_5 and C_6?

Take-aways

Conclusions

- Euler Paths and Circuits
- Hamilton Paths and Circuits
- Planar Graphs
  - Euler's Formula
  - Homeomorphic
- Graph Coloring