# Discrete Mathematics and Its Applications

## Lecture 5: Discrete Probability: Probability Basics

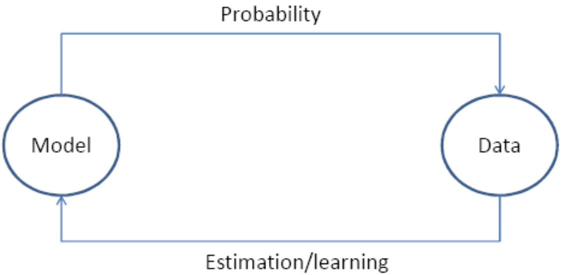
MING GAO

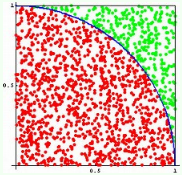
DaSE@ ECNU
(for course related communications)
mgao@dase.ecnu.edu.cn

Nov. 22, 2018

# Outline

# Introduction



Probability as a mathematical framework for:

- reasoning about uncertainty
- developing approaches to inference problems

# Experiment and sample space

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- Roll a die one time, $\Omega = \{1, 2, 3, 4, 5, 6\}$.
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The **sample space**, denoted as $\Omega$, of the experiment is the set of possible outcomes.

### Example

- Roll a die one time, $\Omega = \{1, 2, 3, 4, 5, 6\}$.
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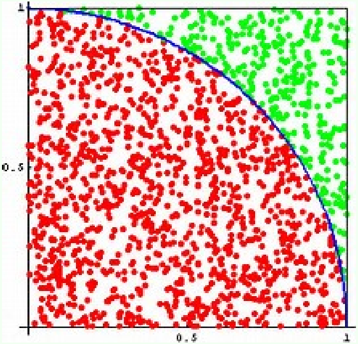
- "List" (set) of possible outcomes
- List must be:
  1. Mutually exclusive
  2. Collectively exhaustive
- Art: to be at the "right" granularity

# Continuous sample space




For this case, sample space $\Omega = \{(x, y) | 0 \leq x, y \leq 1\}$. Note that the sample space is infinite and uncountable.

In this course, we only consider the **countable sample spaces**. Thus, we call the learning content to be the discrete probability.

# Probability axioms

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- Roll an even number, $A = \{2, 4, 6\} \subset \Omega$;
- Toss at least one head $B = \{HH, HT, TH\} \subset \Omega$;
- Toss at least three head $C = \emptyset \subset \Omega$.
- There are $2^{|\Omega|}$ events for an experiments;
- Events therefore have all set operations.

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- There are $2^{|\Omega|}$ events for an experiments;
- Events therefore have all set operations.

### Axioms

- **Nonnegativity:** $P(A) \geq 0$;
- **Normalization:** $P(\Omega) = 1$ and $P(\emptyset) = 0$;
- **Additivity:** If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Furthermore, if $A_i \cap A_j = \emptyset$ for $\forall i \neq j$, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

# Finite probability

If $S$ is a finite nonempty sample space of equally likely outcomes, and $E$ is an event, that is, a subset of $S$, then the probability of $E$ is

$$p(E) = \frac{|E|}{|S|}.$$

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- Let all outcomes be equally likely;
- Computing probabilities $\equiv$ two countings;
  - Counting the successful ways of the event;
  - Counting the size of the sample space;

# Examples

## Example I

**Question:** An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

**Solution:** Let $S$ be the sample space, i.e.,

$$S = \{\bigcirc_1, \bigcirc_2, \bigcirc_3, \bigcirc_4, \bigcirc_1, \bigcirc_2, \bigcirc_3, \bigcirc_4, \bigcirc_5\}.$$

Let $E$ be the event of choosing a blue ball, i.e.,

$$E = \{\bigcirc_1, \bigcirc_2, \bigcirc_3, \bigcirc_4\}.$$

In terms of the definition, we can compute the probability as

$$P(E) = \frac{|E|}{|S|} = \frac{4}{9}.$$

# Examples Cont'd

## Example II

**Question:** What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

**Solution:**

| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
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- There are a total of 36 possible outcomes when two dice are rolled.
- There are six successful outcomes, namely, $(1,6), (2,5)$, $(3,4), (4,3)$, $(5,2)$, and $(6,1)$.
- Hence, the probability that a seven comes up when two fair dice are rolled is $6/36 = 1/6$.

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**Question:** In a lottery, players win a large prize when they pick four random digits that match, in the correct order. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

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**Small prize case:** Exactly one digit must be wrong to get three digits correct, but not all four correct. Hence, there is a total of $\binom{4}{1} \times 9 = 36$ ways to choose four digits with exactly three of the four digits correct. Thus, the probability that a player wins the smaller prize is $36/10,000 = 9/2500 = 0.0036$.

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Hence, the probabilities are

$$\frac{C(13,1)C(4,4)C(48,1)}{C(52,5)} \approx 0.00024, \frac{C(13,2)C(4,3)C(4,2)}{C(52,5)} \approx 0.0014.$$

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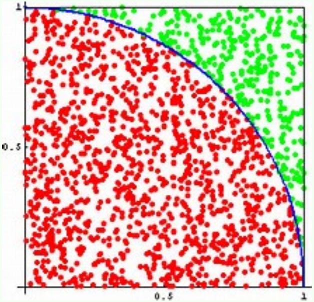
## Example V



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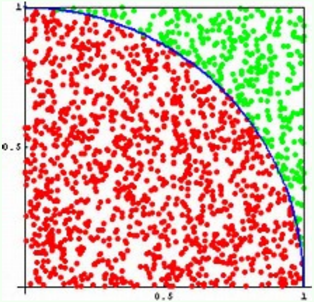


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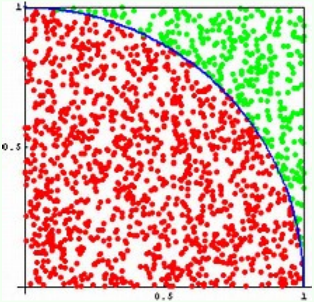


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Let $E$ be an event that the point locates in the circle area $C$, where $C = \{(x, y)|x^2 + y^2 \leq 1 \wedge x, y \geq 0\}$.

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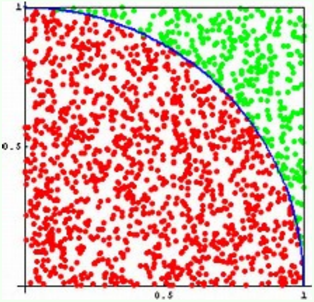


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Let $E$ be an event that the point locates in the circle area $C$, where $C = \{(x, y) | x^2 + y^2 \leq 1 \wedge x, y \geq 0\}$. Then we have

$$P(E) = \frac{S(C)}{S(\Omega)},$$

where $S(\cdot)$ is the area of a plane region.

# Assigning probabilities

Let $\Omega$ be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $P(s)$ to each outcome $s \in \Omega$. We require that two conditions be met:

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1. $0 \leq P(s) \leq 1$ for each $s \in \Omega$;
2. $\sum_{s\in\Omega} P(s) = 1$.

Function $P$ from the set of all outcomes of the sample space $\Omega$ to $[0, 1]$ is called a **probability distribution**.

We can model experiments in which outcomes are either equally likely or not equally likely by choosing the appropriate function $P(s)$.

# Example I

**Question:** What probabilities should we assign to the outcomes $H$ (heads) and $T$ (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

**Solution:**

**Fair case:** For a fair coin, the probability that heads comes up when the coin is flipped equals the probability that tails comes up, so the outcomes are equally likely, i.e., $P(H) = P(T) = \frac{1}{2}$.

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**Unfair case:** For the biased coin we have $P(H) = 2P(T)$. Since $P(H) + P(T) = 1$, it follows that $P(T) = 1/3$ and $P(H) = 2/3$.

# Uniform distribution

### Definition of uniform distribution

Suppose that $\Omega$ is a set with $n$ elements. The **uniform distribution** assigns the probability $1/n$ to each element of $\Omega$.

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## Definition of event probability

The probability of event $E$ is the sum of the probabilities of the outcomes in $E$ ($E$ is a countable set). That is,

$$P(E) = \sum_{s \in E} P(s).$$

(Note that when $E$ is an infinite set, $\sum_{s \in E} P(s)$ is a convergent infinite series.)

# Probability operators

## Operators

Let $\Omega$ be the sample space, $A$ and $B$ be two events:

1. If $A \cap B = \emptyset$, then

$$P(A \cup B) = P(A) + P(B);$$

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3. If $A$ and $B$ are independent, then
$$P(A \cap B) = P(A) \cdot P(B);$$

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2. $$P(\overline{A}) = P(\Omega) - P(A) = 1 - P(A);$$
3. If $A$ and $B$ are independent, then
$$P(A \cap B) = P(A) \cdot P(B);$$
4. The conditional probability of $A$ given $B$, denoted by $P(A|B)$, is computed as
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

# Probability of union of two events

**Theorem**

Let $E_1$ and $E_2$ be events in the sample space $\Omega$. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

*Proof.*

Since we have $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$,

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$$\begin{aligned} P(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|\Omega|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|\Omega|} \\ &= \frac{|E_1|}{|\Omega|} + \frac{|E_2|}{|\Omega|} - \frac{|E_1 \cap E_2|}{|\Omega|} \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2). \end{aligned}$$

□

# Probability of union Cont'd

- If $E_1 \cap E_2 = \emptyset$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2);$$

- If $E_i \cap E_j = \emptyset$ for $\forall i, j$, then

$$P(\bigcup_{i=0}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i);$$

-

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = & P(E_1) + P(E_2) + P(E_3) \\ & - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) \\ & + P(E_1 \cap E_2 \cap E_3) \end{aligned}$$

# Example II

**Question:** What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

**Solution:** Let $E_1$ be the event that the integer selected at random is divisible by 2, and let $E_2$ be the event that it is divisible by 5.

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$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} \\ &= 0.6 \end{aligned}$$

# Probability of complement of an event

**Theorem**

Let $E$ be an event in a sample space $\Omega$. The probability of event $\overline{E} = \Omega - E$, the complementary event of $E$, is given by

$$P(\overline{E}) = 1 - P(E).$$

*Proof.*

Since we have $|\Omega| = |E| + |\overline{E}|$,

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$$P(\overline{E}) = \frac{|\overline{E}|}{|\Omega|} = \frac{|\Omega| - |E|}{|\Omega|} = \frac{|\Omega|}{|\Omega|} - \frac{|E|}{|\Omega|} = 1 - P(E).$$

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$$P(E) = 1 - P(\overline{E}) = 1 - \frac{1}{2^{10}} = \frac{1023}{1024}.$$

# Running example

## Tossing coins

We toss a coin twice (Head = H, Tail = T), then $\Omega = \{HH, HT, TH, TT\}$.

We define three events:

1. $A$ : the first toss is $H$;
2. $B$ : the second toss is $H$;
3. $C$ : the first and second toss give the same results.

Hence, we have

- $P(A) = P(B) = P(C) = \frac{1}{2}$;
- $P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{|\{HH\}|}{|\Omega|} = \frac{1}{4}$;
- $P(A \cap B \cap C) = \frac{|\{HH\}|}{|\Omega|} = \frac{1}{4}$;

# Independence

## Definition

- Events $E_1$ and $E_2$ are **pair-wise independent** if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2);$$

- Events $E_1, E_2, \cdots, E_n$ are **mutually independent** if

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = P(E_{i_1})P(E_{i_2}) \cdots P(E_{i_m}),$$

  where $i_j, j = 1, 2, \cdots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $m \geq 2$.

Note that mutually independent must be pair-wise independent, but pair-wise independent may not imply mutually independent (shown in previous example).

# Independence

## Theorem

Events $E$ and $F$ are pair-wise independent, then

- $E$ and $\overline{F}$ are pair-wise independent;
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Proof.

$$\begin{aligned} P(E) &= P(E \cap (F \cup \overline{F})) = P((E \cap F) \cup (E \cap \overline{F})) \\ &= P(E \cap F) + P(E \cap \overline{F}) \\ P(E \cap \overline{F}) &= P(E) - P(E \cap F) = P(E) - P(E)P(F) \\ &= P(E)(1 - P(F)) = P(E)P(\overline{F}) \end{aligned}$$

Hence, we have $E$ and $\overline{F}$ are independent. □

# Example IV

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Because $P(E)P(F) = 3/8$, it follows that $P(E \cap F) = P(E)P(F)$, so $E$ and $F$ are independent.

# Conditional probability

## Definition

Let $E$ and $F$ be events with $P(F) > 0$. The **conditional probability** of $E$ given $F$, denoted by $P(E|F)$, is defined as

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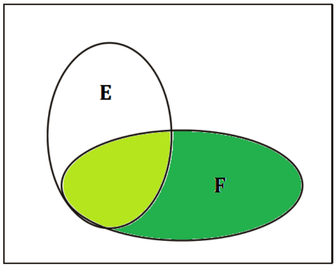

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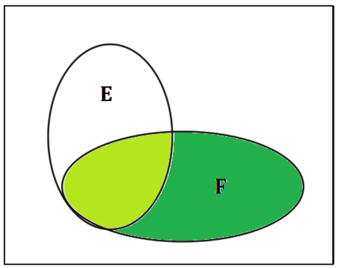


- $P(E|F)$ is the probability of $E$, given that $F$ occurred
- $F$ is our new sample space;
- $P(E|F)$ is undefined if $P(F) = 0$.

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We conclude that

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

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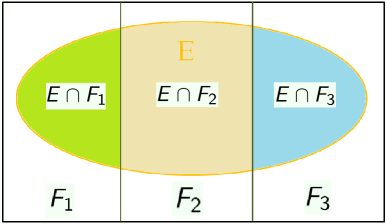

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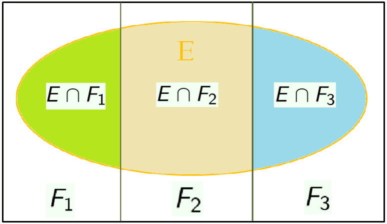


- $P(E) = P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + P(F_3) \cdot P(E|F_3)$ if $F_1 \cup F_2 \cup F_3 = \Omega$ and $F_i \cap F_j = \emptyset$ for $i \neq j$ (**Total probability theorem**);

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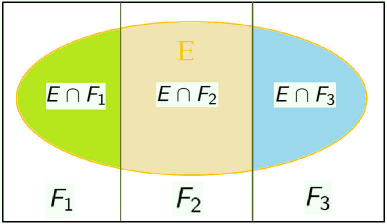


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- $P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(E)} = \frac{P(F_i) \cdot P(E|F_i)}{\sum_i P(F_i) \cdot P(E|F_i)}$ (**Bayes rule**).

# Proof of the total probability theorem

**Theorem**

Let $F_i$ (for $i = 1, 2, \cdots, n$) be a partition of sample space $\Omega$, for any event $E$, then

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**Proof:**

$$\begin{aligned}
P(E) &= P(E \cap \Omega) = P(E \cap (F_1 \cup F_2 \cup \cdots \cup F_n)) \\
&= P((E \cap F_1) \cup (E \cap F_2) \cup \cdots \cup (E \cap F_n)) \\
&= P(E \cap F_1) + P(E \cap F_2) + \cdots + P(E \cap F_n) \\
&= P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n) \\
&= \sum_{i=1}^{n} P(F_i) \cdot P(E|F_i).
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In terms of the total probability theorem, condition on whether the first card is an ace or not:

$$\begin{aligned} P(F) &= P(E) \cdot P(F|E) + P(\overline{E}) \cdot P(F|\overline{E}) \\ &= (\frac{4}{52})(\frac{3}{51}) + (\frac{48}{52})(\frac{4}{51}) \\ &= \frac{3 \cdot 4 + 48 \cdot 4}{51 \cdot 52} = \frac{4}{52} \end{aligned}$$

# Take-aways

## Conclusions

- Introduction
- Sample Space and Events
- Probability and Set Operations
  - Probability of Union
  - Probability of Complement
  - Independence
  - Conditional Probability