Discrete Mathematics and Its Applications

Lecture 7: Graphs: Proximity

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Outline

1. Community Structures

2. Node Proximity
   - Simple Approaches
   - Graph-theoretic Approaches
   - SimRank
   - Random Walk based Approaches

Community structures

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  - Signed network: antagonistic community, quasi-antagonistic community, etc.

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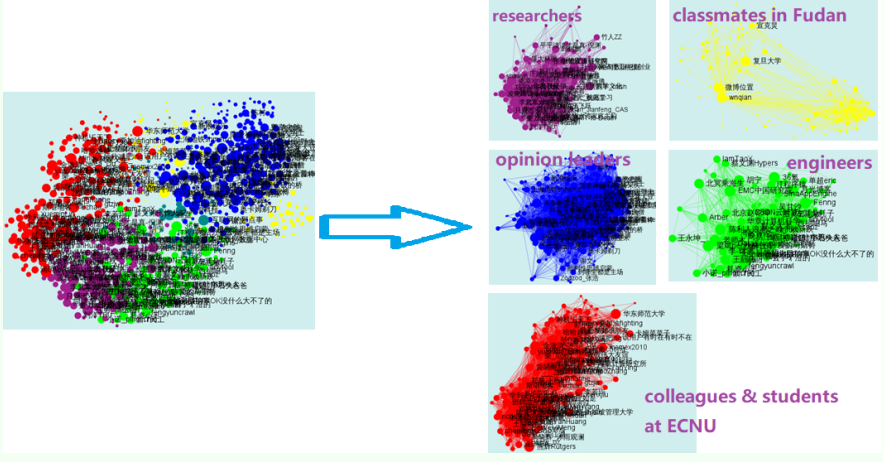

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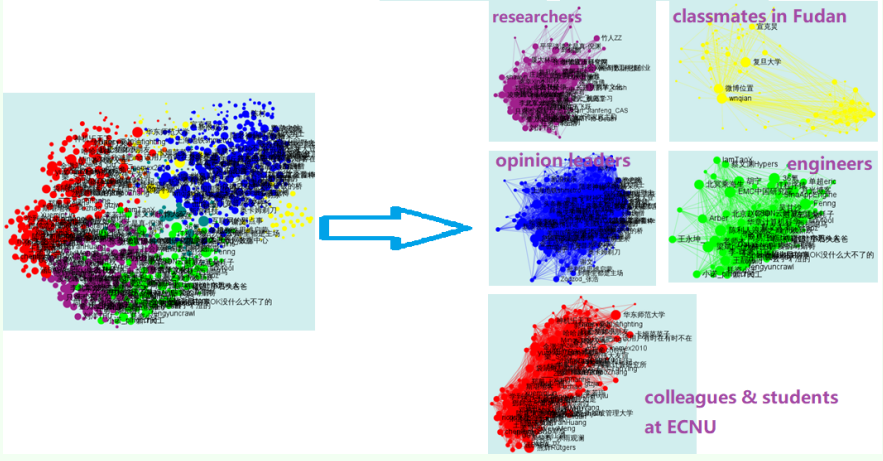

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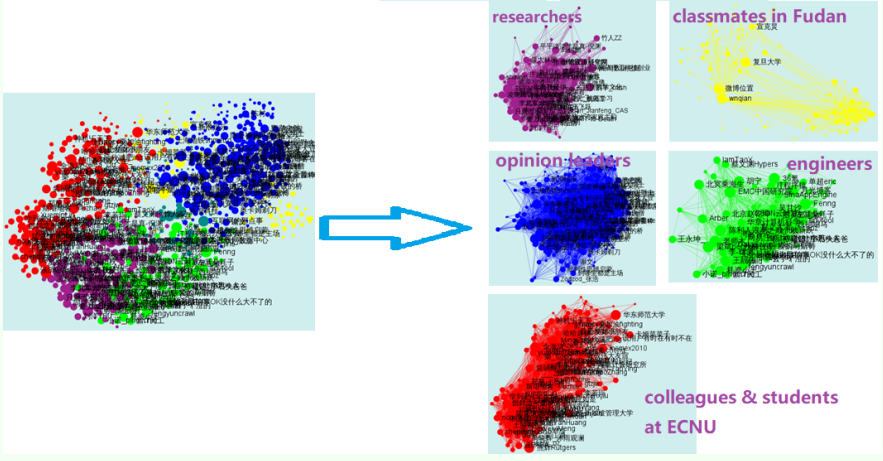

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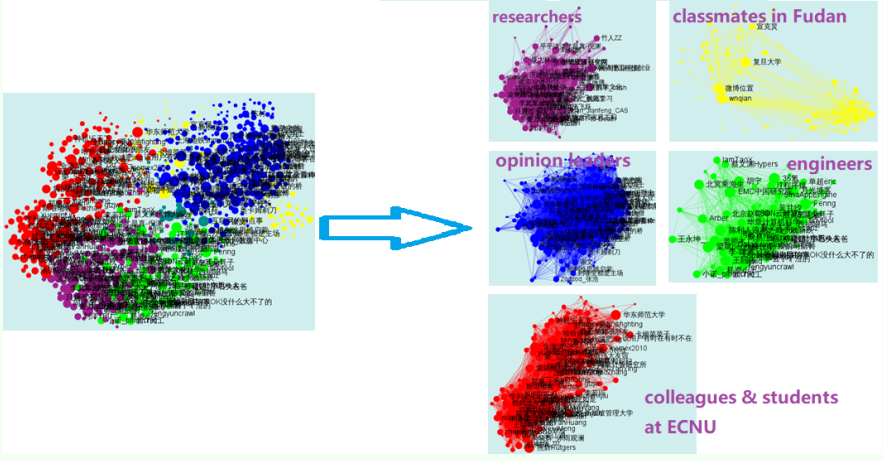

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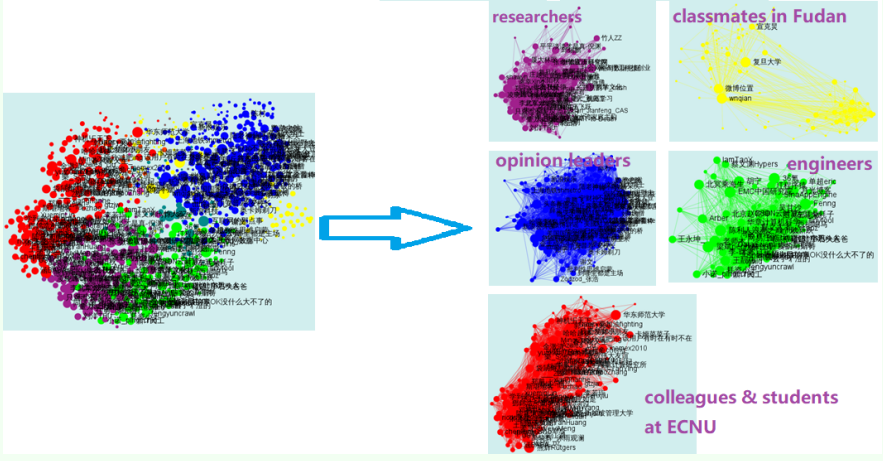

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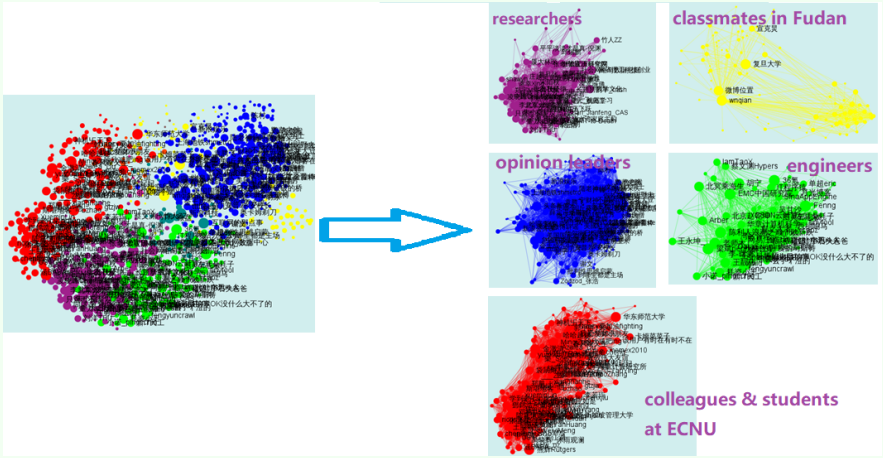

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In the study of complex networks, a network is said to have community structure if the nodes of the network can be easily grouped into (potentially overlapping) sets of nodes such that each set of nodes is densely connected internally.

Clustering based on vertex similarity

K-means

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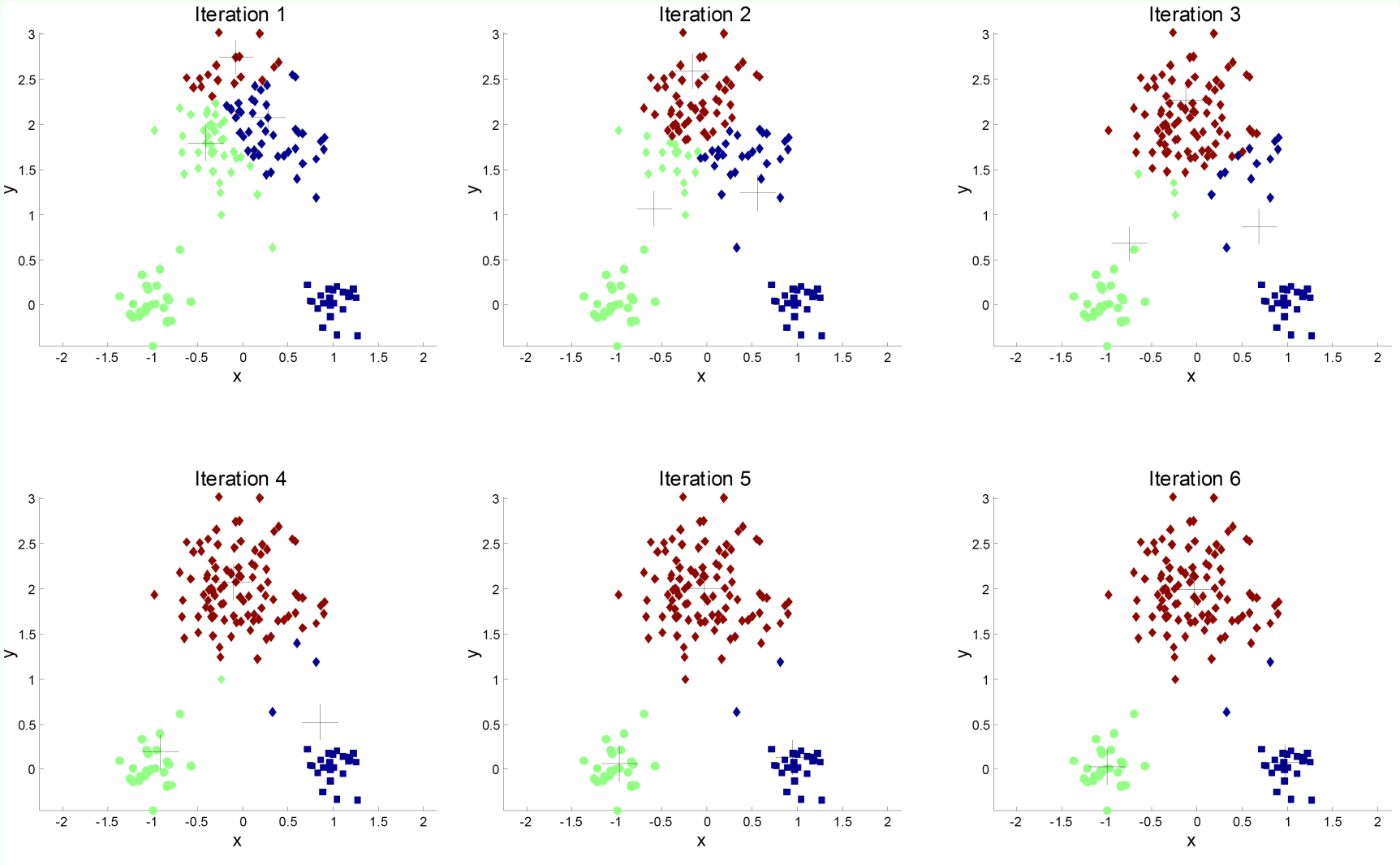
- $C(i)$ denotes cluster number for vertex v_i;
- Similarity measure or dissimilarity measure: Euclidean distance metric or Jaccard coefficient;
- K-means minimizes within-cluster point scatter:

$$W(C) = \frac{1}{2}\sum_{k=1}^{K}\sum_{C(i)=k}\sum_{C(j)=k} \| x_i - x_j \|^2$$

$$= \sum_{k=1}^{K} N_k \sum_{C(i)=k} \| x_i - m_k \|^2,$$

where N_k is the number of vertices in $k-$th cluster

K-means example



Spectral clustering

Let $\mathcal{L}$ be normalized Laplacian of graph G, the algorithm partitions nodes into two sets (B_1, B_2) based on the eigenvector $\mathbf{v}$ corresponding to the second-smallest eigenvalue of $\mathcal{L}$. Partitioning may be done in various ways:

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- Assign all nodes whose component in $\mathbf{v}$ satisfies certain condition in B_1, and B_2 otherwise, e.g., larger than median, the sign of each entry of $\mathbf{v}$.
- The algorithm can be used for hierarchical clustering by repeatedly partitioning the subsets in this fashion.

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Graph has community structure, if it is different from random graph (not expected to have community structure for random graph).

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- Modularity [Newman 2006]:

$$M = \frac{1}{2m} \sum_{i,j} (A_{ij} - \frac{d(i)d(j)}{2m}) \delta(C_i, C_j).$$

where m and C_i denote $\#$ edges and the $i-$th community in the graph.

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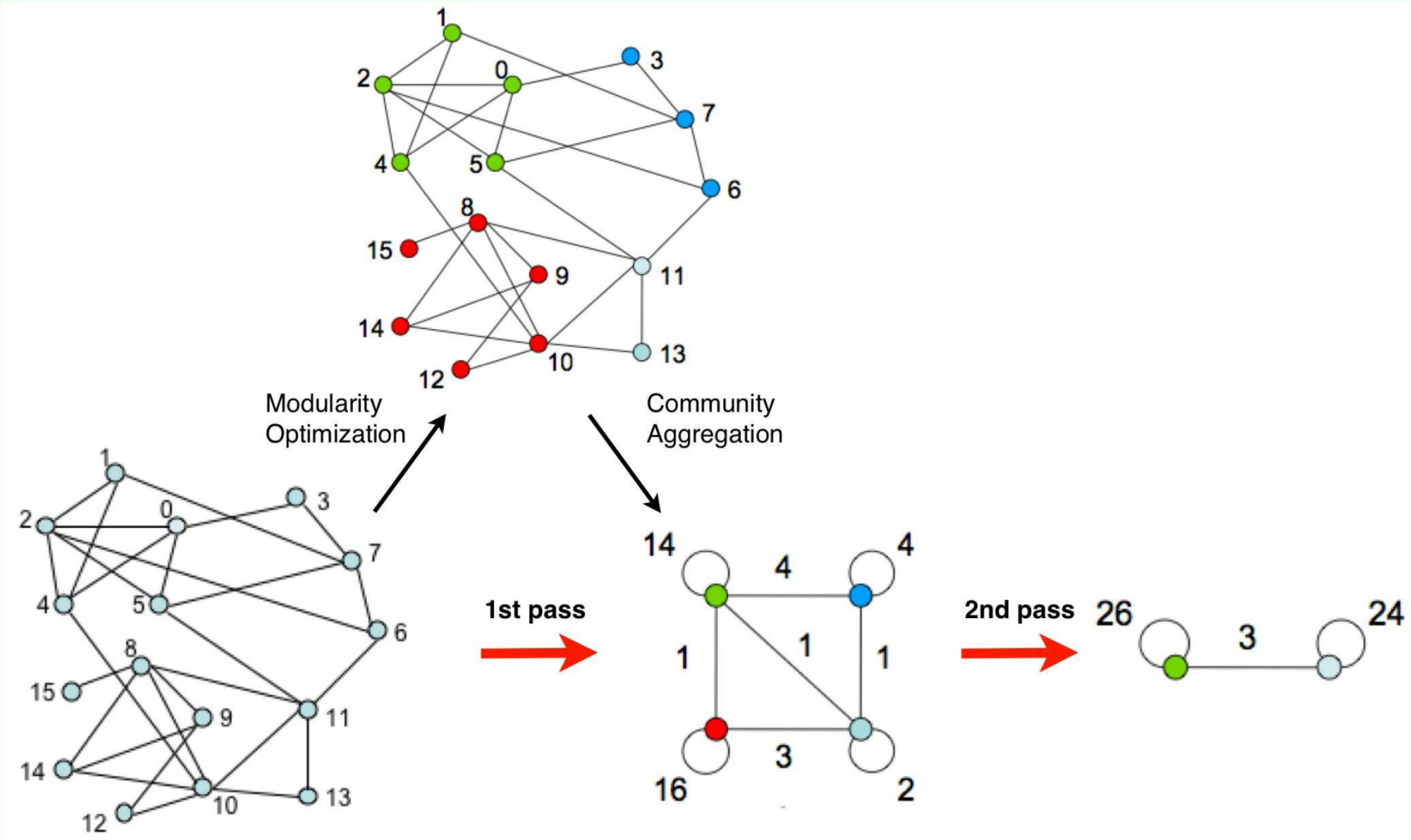
- It can be used as a measure to evaluate the communities quality.

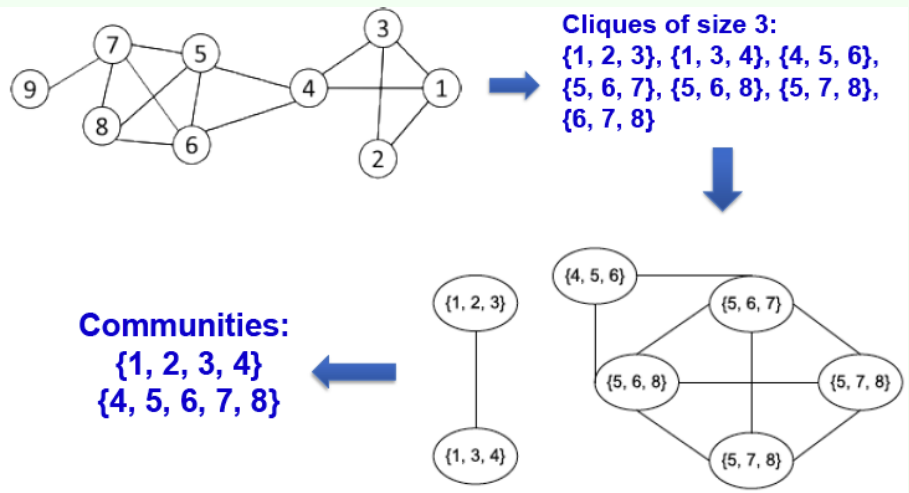
Louvain method



Clique

Definition

A clique is a subset of vertices of an undirected graph such that its induced subgraph is complete. A **maximal** clique is a clique that cannot be extended by including one more adjacent vertex.
Normally use cliques as a core or a seed to find larger communities.



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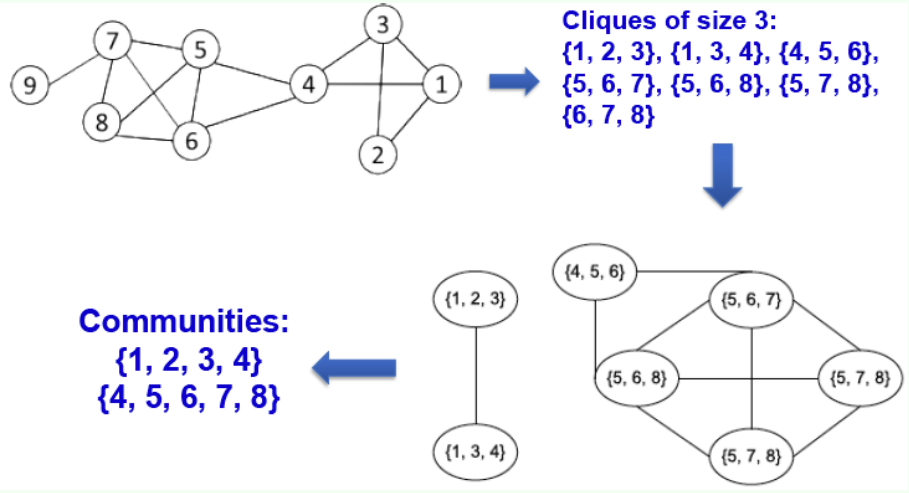

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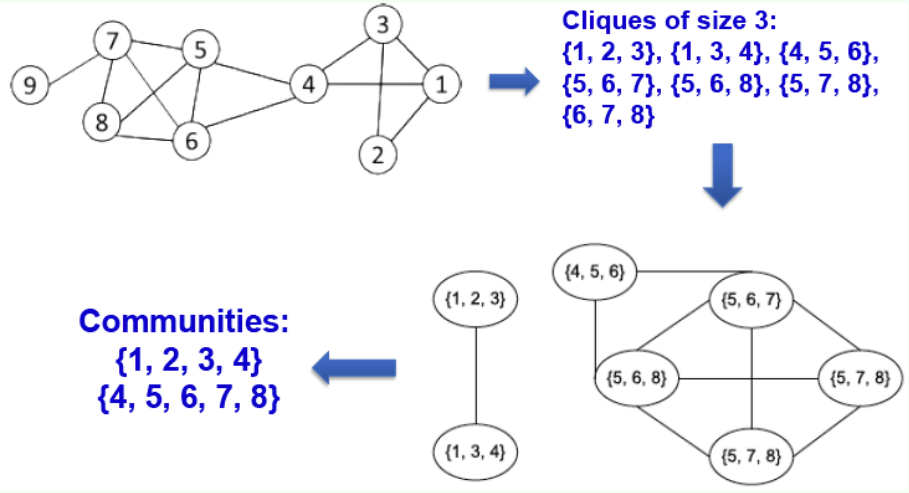

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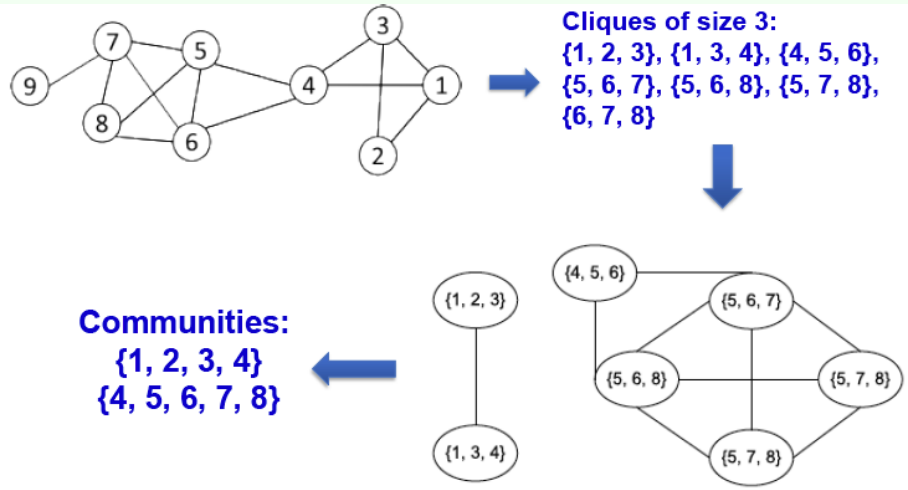

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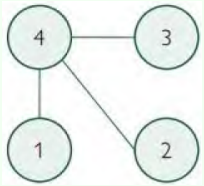
- Each connected component in the clique graph forms a community

Extensions of clique

$k-$clique



Maximal subgroup, where the largest geodesic distance between any pair of nodes is not greater than k. It is a clique if $k = 1$.

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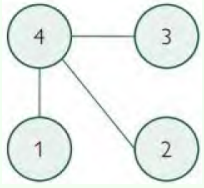

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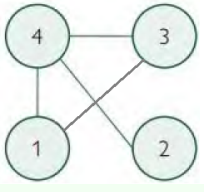

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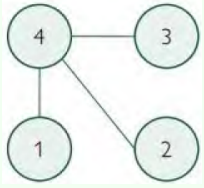

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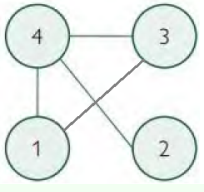

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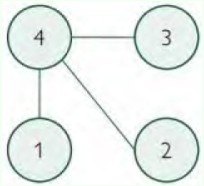

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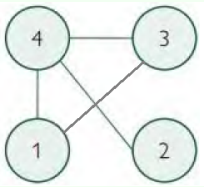
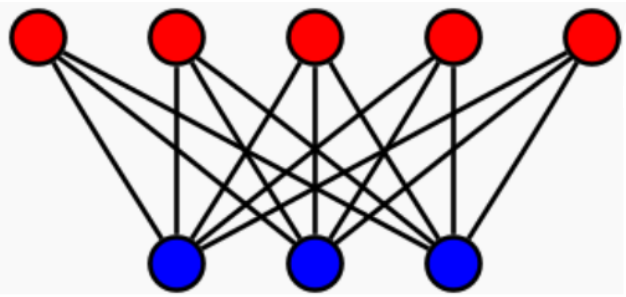


- Node degree: every node in induced subgraph is adjacent to at least $\gamma(n-1)$ other nodes.
- Edge density: Number of edges in subgraph is at least $\gamma n(n-1)/2$, where n denotes # nodes in subgraph.

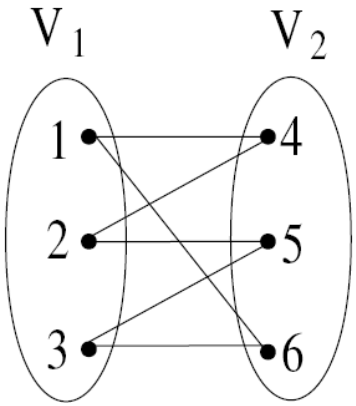
Local community structure in bipartite graph



Biclique

A biclique is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.

- A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$.



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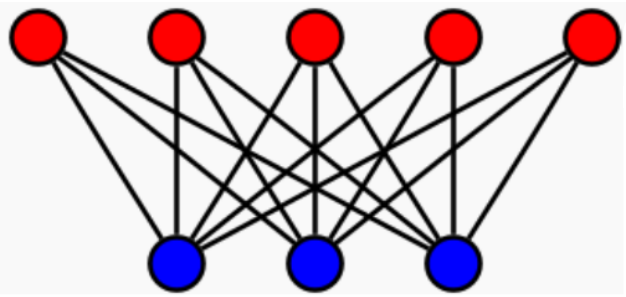

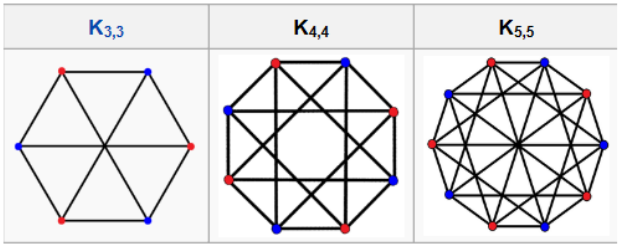

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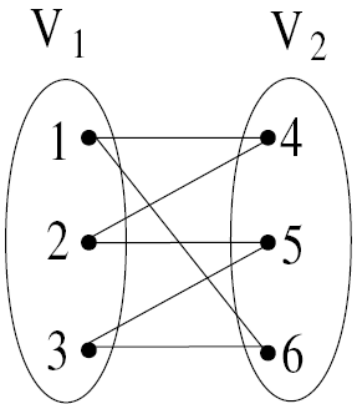

Quasi-bi-clique

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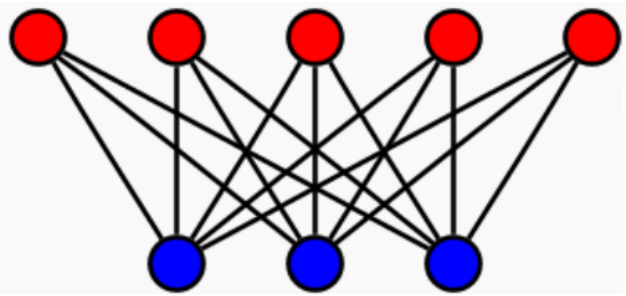

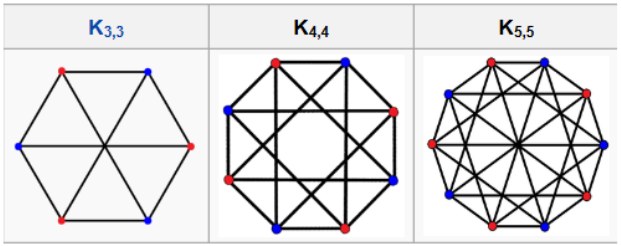

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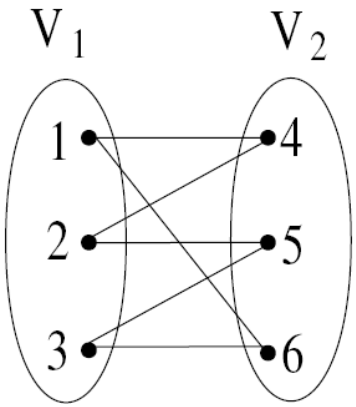

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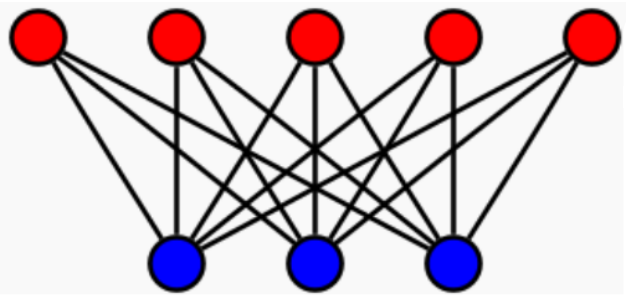

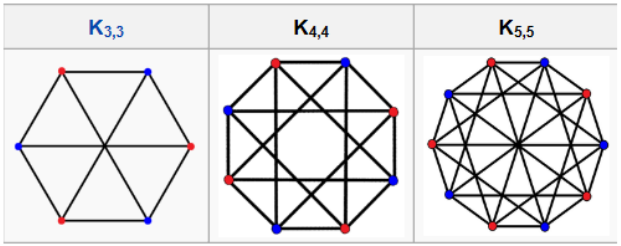

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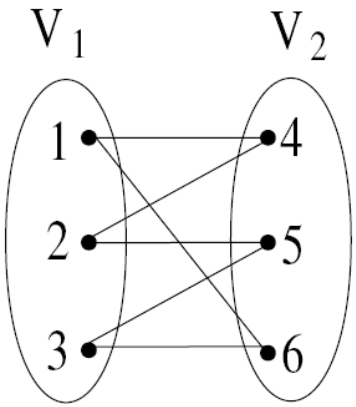

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- Relative version of quasi-bi-clique.
- Absolute version of quasi-bi-clique.

Node proximity

Node proximity (= similarity or closeness, but $\neq$ distance) measures:

- Information exchange
- Latency/speed of information exchange
- Likelihood of future link
- Propagation of a product/idea/service/ disease
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Approaches

- Simple approaches
- Graph-theoretic approaches
- SimRank
- Random walk with restarts

Outline

Simple approaches

Similarity metrics

Given a graph $G = (V, E)$, $N(v_i)$ denotes the neighbors of node v_i.

- Common neighbors: $|N(v_i) \cap N(v_j)|$.
- Jaccard coefficient: $\frac{|N(v_i) \cap N(v_j)|}{|N(v_i) \cup N(v_j)|}$.
- Adamic/Adar: $\sum_{v \in N(v_i) \cap N(v_j)} \frac{1}{\log |N(v)|}$.
- Preferential attachment: $|N(v_i)| \times |N(v_j)|$.

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Drawbacks

- Jaccard coefficient treats a graph as a set of transactions which are independent.
- Thus, it loss the topological information of a graph.

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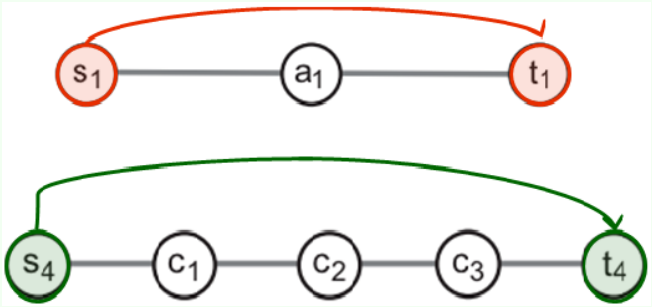
Graph-theoretic approaches

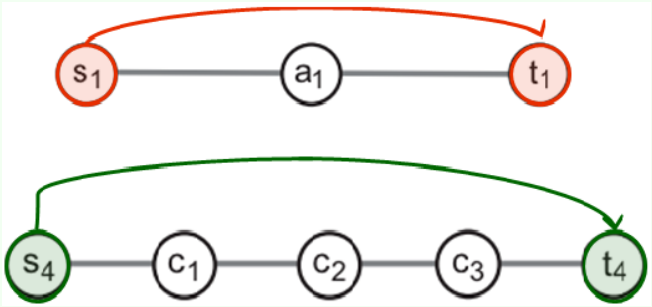
Idea



- (s_1, t_1) is more similar than (s_4, t_4).
- Simple metrics
  - Number of hops
  - Sum of weights of hops

Graph-theoretic approaches

Idea



- (s_1, t_1) is more similar than (s_4, t_4).
- Simple metrics
  - Number of hops
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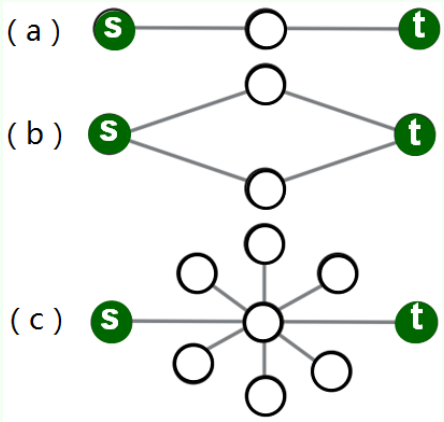
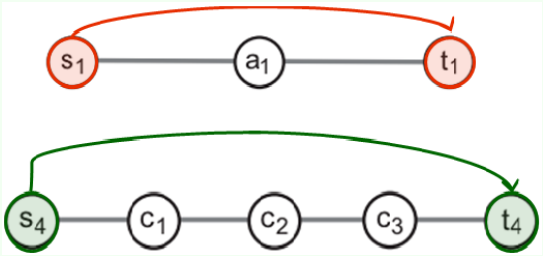
Drawbacks



- (s, t) in (b) more similar than in (a) and (c) because of linked via more paths.
- In (c), s and t are probably unrelated since (s, t) linked via high degree node.

Graph-theoretic approaches cont.

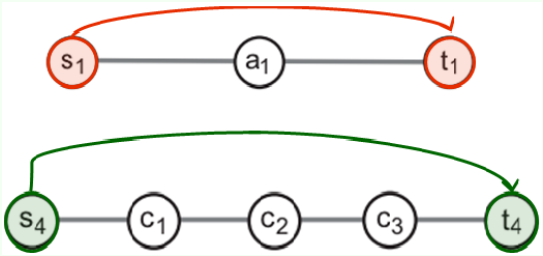
Max-flow approach



- Heavy weighted links and number of paths matter, but path length doesn't.
- The max flow of (s_1, t_1) is the same to that of (s_4, t_4).

Graph-theoretic approaches cont.

Max-flow approach



- Heavy weighted links and number of paths matter, but path length doesn't.
- The max flow of (s_1, t_1) is the same to that of (s_4, t_4).

Katz

$$K = \sum_{l \geq 1} \beta^l |\text{paths}_{v_i, v_j}^{<l>}|,$$

where $\beta \in (0, 1)$ is a pre-defined parameter.

- $\text{paths}_{v_i, v_j}^{<l>}$ is the set of exact l-length paths from v_i to v_j.
- $|\text{paths}_{v_i, v_j}^{<l>}| = 1$ if and only if v_i and v_i are connected by a link.
- $K = \beta A + \beta^2 A^2 + \cdots = (I - \beta A)^{-1} - I$.

Outline

SimRank [G. Jeh and J. Widom KDD 2002]

Idea

Two objects are similar if they are connected to similar objects.

- Iterative computation with initial value

$$s^{(0)}(a,b) = \begin{cases} 1, & a = b; \\ 0, & a \neq b. \end{cases}$$

$$s^{(k+1)}(a,b) = \begin{cases} 1, & a = b; \\ \frac{\alpha}{|N(a)N(b)|} \sum_{c \in N(a)} \sum_{d \in N(b)} s^{(k)}(c,d), & a \neq b. \end{cases}$$

  where $\alpha \in [0,1]$ is a constant.

- However, it needs $O(n^3)$ runtime. Many works improve the runtime.
  - Simrank++ [VLDB 2008]
  - $S = C(A^T S A) + I$ with Kronecker product [EDBT 2010]
  - Parallel SimRank [VLDB 2015]

Outline

Random walk

Hitting time

Hitting time H_{v_i,v_j} is the expected number of steps required for a random walk starting at v_i to reach v_j.

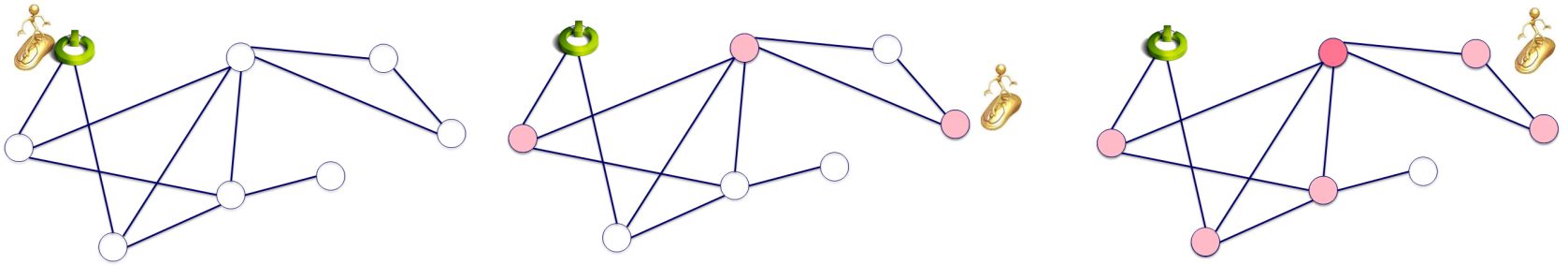
- Similar to Katz, all paths need to enumerate when hitting time is computed, but longer paths have smaller probabilities.
- $H = W + 2W^2 + \cdots = ((I - W)^{-1} - I)(I - W)^{-1}$.

Commute time

Commute time $H_{v_i,v_j} + H_{v_j,v_i}$ is the expected number of steps required for a random walk starting at v_i to reach v_j, then return to v_i.

- For an undirected graph, $H_{v_i,v_j} = H_{v_j,v_i}$.
- Similar, we can compute commute time for every pair of nodes.

Personalized random walk with restarts



Idea

$$\mathbf{r}_i = cW\mathbf{r}_i + (1-c)\mathbf{e}_i,$$

where $\mathbf{r}_i \in R^{n\times 1}$ is ranking score w.r.t. v_i, W is the transition probability matrix, and $\mathbf{e}_i \in R^{n\times 1}$ is start score w.r.t. v_i with $r_{ii} = 1$, 0 otherwise.

- $(I - cD^{-1}A)\mathbf{r}_i = (1-c)\mathbf{e}_i$.
- $\mathbf{r}_i = (1-c)(I - cD^{-1}A)^{-1}\mathbf{e}_i$.

Take-home messages

- Community detection
  - Global structure
  - Local structure
- Node proximity
  - Simple approaches
  - Graph-theoretic approaches
  - SimRank
  - Random walk based approaches