Tutorial 1 Solutions

1.

(a) We have

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) \geq P(A_1) + P(A_2) - 1.$$

(b) We have

$$\begin{aligned}
P(A_1 \cap A_2 \cap \cdots \cap A_n) =& P(A_1) + P(A_2 \cap A_3 \cap \cdots \cap A_n) - P(A_1 \cup (A_2 \cap A_3 \cap \cdots \cap A_n)) \\
\geq& P(A_1) + P(A_2 \cap A_3 \cap \cdots \cap A_n) - 1 \\
=& P(A_1) + P(A_2) + P(A_3 \cap A_4 \cap \cdots \cap A_n) - P(A_2 \cup (A_3 \cap A_4 \cap \cdots \cap A_n)) - 1 \\
\geq& P(A_1) + P(A_2) + P(A_3 \cap A_4 \cap \cdots \cap A_n) - 2 \\
\cdots& \\
=& P(A_1) + P(A_2) + \cdots + P(A_n) - P(A_{n-1} \cup A_n) - (n-2) \\
\geq& P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1).
\end{aligned}$$

2.

(a) Since $P(A|B) > P(A|B^c)$, $0 < P(B) < 1$ and $0 < 1 - P(B) < 1$. Since $P(A) = P(B) \cdot P(A|B) + P(B^c)P(A|B^c) > (P(B) + P(B^c))P(A|B^c) = P(A|B^c) \geq 0$ and $P(A) < (P(B) + P(B^c))P(A|B) = P(A|B) \leq 1$, $0 < P(A) < 1$ and then $0 < P(A^c) < 1$. Thus,

$$\begin{aligned}
& \frac{P(AB)}{P(B)} > \frac{P(AB^c)}{P(B^c)} \\
\Rightarrow \quad & \frac{P(AB)}{P(B)} > \frac{P(A) - P(AB)}{1 - P(B)} \\
\Rightarrow \quad & P(AB) \cdot (1 - P(B)) > P(B) \cdot (P(A) - P(AB)) \\
\Rightarrow \quad & P(AB) - P(AB)P(B) > P(B)P(A) - P(B)P(AB) \\
\Rightarrow \quad & P(AB) > P(B)P(A) \\
\Rightarrow \quad & P(AB) - P(AB)P(A) > P(B)P(A) - P(AB)P(A) \\
\Rightarrow \quad & P(AB) \cdot (1 - P(A)) > P(A)(P(B) - P(AB)) \\
\Rightarrow \quad & P(B|A) > P(B|A^c)
\end{aligned}$$

(b) ($\Rightarrow$) Since A and B are independent, $P(AB) = P(A)P(B)$ and $P(AB^c) = P(A)P(B^c)$.

Then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) = \frac{P(A)P(B^c)}{P(B^c)} = \frac{P(AB^c)}{P(B^c)} = P(A|B^c).$$

($\Leftarrow$) Since $P(A|B) = P(A|B^c)$, then

$$\begin{aligned} \frac{P(AB)}{P(B)} &= \frac{P(AB^c)}{P(B^c)} \\ P(AB)(1 - P(B)) &= (P(A) - P(AB))P(B) \\ P(AB) &= P(A)P(B) \end{aligned}$$

Therefore, A and B are independent.

3.

(a)

$$P(\#\text{Undergraduate}) = 1 - \frac{A_{12}^4(\text{Permutation})}{12^4} = \frac{41}{96}.$$

(b)

$$P(\#\text{Graduate}) = 1 - \frac{A_{12}^2(\text{Permutation})}{12^2} = \frac{1}{12}.$$

4.

(a)

$$P(\#\text{Wife B and Husband O}) = 0.10 * 0.45 = 0.045.$$

(b)

$$P(\#\text{Husband B or O}) = 0.10 + 0.45 = 0.55.$$

5.

(a)

$$P(\#\text{Offspring AA}) = 1 \times 0.5 = 0.5.$$

$$P(\#\text{Offspring Aa}) = 1 \times 0.5 = 0.5.$$

(b) The first generations provide A with probability $p + 2q \times 0.5 = p + q$, and a with probability $2q \times 0.5 + r = q + r$. Thus

$$P(\#\text{SecondGenerations AA}) = (p + q)^2.$$

$$P(\#\text{SecondGenerations Aa}) = 2(p + q)(q + r).$$

$$P(\#\text{SecondGenerations aa}) = (q+r)^2.$$

Since $p+2q+r=1$, the second generations provide A with probability $(p+q)^2 + 2(p+q)(q+r)\times 0.5 = (p+q)(p+q+q+r) = p+q$, and a with probability $2(p+q)(q+r)\times 0.5 + (q+r)^2 = (q+r)(p+q+q+r) =$ q + r. Thus

$$P(\#\text{ThirdGenerations AA}) = (p+q)^2.$$

$$P(\#\text{ThirdGenerations Aa}) = 2(p+q)(q+r).$$

$$P(\#\text{ThirdGenerations aa}) = (q+r)^2.$$

Therefore, the probabilities in the second and third generations are the same.

6. Counterexample: set $C = A^c$. If A and B are independent, A^c and B are independent. But A and $C = A^c$ are opposite events and not independent.