Tutorial 3

October 8, 2019

1. The joint p.d.f. of a random bivariate vector $(X, Y)^\tau$ is

$$f(x, y) = \begin{cases} 3x, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

Answer the following questions:

(a) What are the marginal p.d.f.s of X and Y, $f_X(x)$ and $f_Y(y)$?

(b) Are X and Y independent?

2. Suppose that X_1 and X_2 are independent and identically distributed with a density

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the p.d.f. of $Z = \max\{X_1, X_2\} - \min\{X_1, X_2\}$.

3. Suppose that U_1 and U_2 are two independent uniform variables on the interval $(0, 1)$. Show that

(a) $Z_1 = -2 \ln U_1 \sim Exp(1/2)$ and $Z_2 = 2\pi U_2 \sim U(0, 2\pi)$;

(b) $X = \sqrt{Z_1} \cos Z_2$ and $Y = \sqrt{Z_1} \sin Z_2$ are two independent standard normal variables.

4. Find the joint the density of $X + Y$ and X/Y, where X and Y are independent exponential random variables with parameter λ. Show that $X + Y$ and X/Y are independent.

5. Let X_1 and X_2 be independent standard normal random variables. Show that the joint distribution of

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 + b_1 \\ Y_2 &= a_{21}X_1 + a_{22}X_2 + b_2 \end{aligned}$$

is bivariate normal. Write down $\mathrm{E}Y_1$, $\mathrm{E}Y_2$, $\mathrm{Var}(Y_1)$, $\mathrm{Var}(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

6. Suppose random variables $X_1, X_2, \cdots, X_n$ are independent and identically distributed as a uniform distribution $U(0, \theta)$. Let

$$Y = \max\{X_1, X_2, \cdots, X_n\} \text{ and } Z = \min\{X_1, X_2, \cdots, X_n\}.$$

Find $\mathrm{E}(Y)$ and $\mathrm{E}(Z)$.

7. Suppose that X and Y are independently and identically distributed. Find $P(X = k|X+Y=m)$

   (a) when X and Y are distributed as a geometric distribution $Ge(p)$;

   (b) when X and Y are distributed as a binomial distribution $b(n,p)$;

8. Suppose that $X \sim N(\mu, 1)$, $Y \sim N(0,1)$ and X and Y are independent. Let

$$I = \begin{cases} 1, & Y < X \\ 0, & X \leq Y \end{cases}$$

   Prove that

   (a) $\mathrm{E}(I|X=x) = \Phi(x)$;

   (b) $\mathrm{E}(\Phi(X)) = P(Y < X)$;

   (c) $\mathrm{E}(\Phi(X)) = \Phi(\mu/\sqrt{2})$;