Discrete Mathematics and Its Applications

Lecture 1: The Foundations: Logic and Proofs (1.1-1.2)

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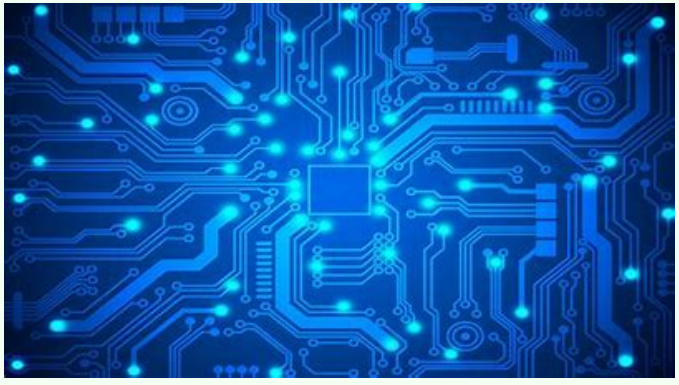
Outline

Why do we need to learn logic



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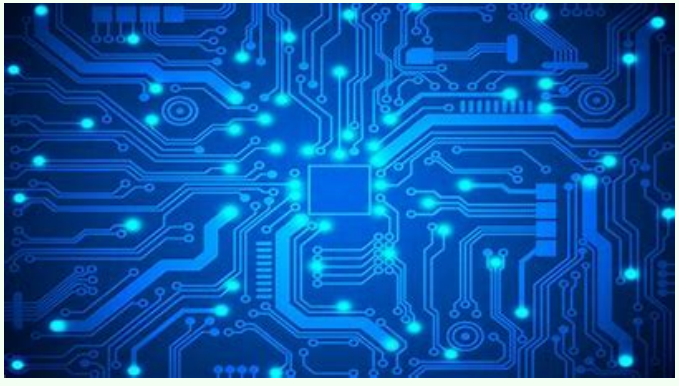
Why do we need to learn logic



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Logic-based approach

i.e. (Peter FatherOf Tom) -> (Tom SonOf Peter)

(Peter ColleagueOf Tom), (Sarah ColleagueOf Peter)
-> (Peter ColleaugeOf Sarah)

Automatic Theorem Proving

- Resolution: if we have
  older(joanne, jake) ← mother(joanne, jake)
  wiser(joanne, jake) ← older(joanne, jake)
- Using resolution, we can construct
  wiser(joanne, jake) ← mother(joanne, jake)

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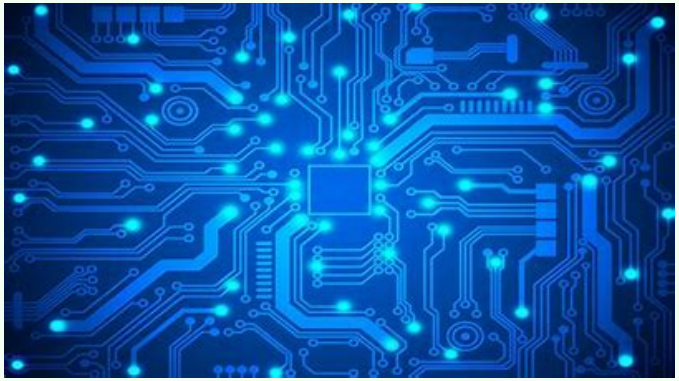

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It will help computer to be more intelligent and smart.

Proposition

Definition

A *proposition* is a **statement** which is either **true** or **false**.

- It is our basic unit of mathematical "facts".
- Examples:
  - Algorithm `A1` is correct.
  - $10^2 = 90$.
  - $\sqrt{2}$ is irrational.

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- Examples:
  - Algorithm `A1` is correct.
  - $10^2 = 90$.
  - $\sqrt{2}$ is irrational.
- Examples of statements which are not propositions (why?):
  - Is Paris the capital of France?
  - $x > 10$.
  - This algorithm is fast.
  - Run, run quickly.
- We usually use the lowercases to represent the propositions, e.g., p, q, r, s.

Logical connectives

Connectives

- **Negation (not)**: $\neg$ (unary connective)
- **Conjunction (and)**: $\wedge$ (binary connective)
- **Disjunction (or)**: $\vee$ (binary connective)
- **Exclusive or**: $\oplus$ (binary connective)
- **Implication (if $\cdots$, then)**: $\rightarrow$ (binary connective)
- **Biconditional (if and only if)**: $\leftrightarrow$ (binary connective)

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Combining proposition

Given two atomic propositions p and q,

Truth tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	F	T	T	F		
T	F	F	F	T	T		
F	T	T	F	T	T		
F	F	T	F	F	F		

- $\neg p$ ("not p"): it is not the case that p.
- $p \wedge q$ ("p and q"): it is true when both p and q are true.
- $p \vee q$ ("p or q"): it is false when both p and q are false.
- $p \oplus q$ ("p exclusive or q"): it is true when exactly one of them is true.
- $p \rightarrow q$ ("if p, then q"): it is false when q is true and p is false.
- $p \leftrightarrow q$ ("p if and only if q"): it is true when p and q have the same truth values.

Examples

Let two atomic propositions p and q be "It is raining" and "I am at classroom".

- $\neg p$ ("not p"): It is not the case that it is raining, or it is not raining.
- $p \wedge q$ ("p and q"): It is raining and I am at classroom.
- $p \vee q$ ("p or q"): It is raining or I am at classroom.
- $p \oplus q$ ("p exclusive or q"): It is raining or I am at classroom, but not both.
- $p \rightarrow q$ ("if p, then q"): If it is raining, then I am at classroom.
- $p \leftrightarrow q$ ("p if and only if q"): It is raining if and only if I am at classroom.

Tautology and contradiction

Tautology

A tautology is a statement that is always true, i.e., a propositional form which is always true regardless of the truth values of its variables is called a *tautology*. Examples:

- $r \vee \neg r$;
- $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$;
- If $p \rightarrow q$ is a tautology, we write $p \Rightarrow q$;
- If $p \leftrightarrow q$ is a tautology, we write $p \Leftrightarrow q$.

Tautology and contradiction

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Contradiction

A contradiction is a statement that is always false, i.e., a propositional form which is always false regardless of the truth values of its variables is called a *contradiction.* Examples:

- $r \wedge \neg r$;
- $\neg\big(\neg(p \wedge q)\big) \leftrightarrow \neg p \vee \neg q$.
- The negation of any tautology is a contradiction, and the negation of contradiction is a tautology.

Implications

Given p and q, an implication

$$p \to q$$

stands for "if p, then q". This is a very important propositional form. It states that "when q is true, p must be true". Let's try to fill in its truth table:

Implications

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

What?

- The propositional form "If you have smart phone, then $2 + 3 = 5$" is always true.
- Yes, when p is false, $p \to q$ is **always true** no matter what truth value of q is.
- We say that in this case, the statement $p \to q$ is *vacuously true.*
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if p, then q we sometimes mean something more than that in the logical expression "$p \to q$."

One explanation

- But let's look closely at what it means when we say that:

  if p is true, q must be true.

- Note that this statement does not say anything about the case when p is false, i.e., it only considers the case when p is true.
- Therefore, having that $p \rightarrow q$ is true is OK with the case that (1) q is false when p is false, and (2) q is true when p is false.
- This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if p, then q" for $p \to q$, but there are other ways to say this. E.g., we can write (1) q if p, (2) p only if q, or (3) when p, then q.

Noticing if-then

We can write "if p, then q" for $p \rightarrow q$, but there are other ways to say this. E.g., we can write (1) q if p, (2) p only if q, or (3) when p, then q.

Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- You will feel dizzy during class if you do not have enough sleep.
- You can get A from this course, only if you work fairly hard.
- When you eat a lot and you do not have enough exercise, then you will get fat.

Only-if

Let p be "you work fairly hard."
Let q be "you get A from this course."
Let r be "You can get A from this course, only if you work fairly hard."
Let's think about the truth values of r.

Only if you work fairly hard.

p	q	$p \rightarrow q$	r
T	T	T	
T	F	F	
F	T	T	
F	F	T	

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Let q be "you get A from this course."
Let r be "You can get A from this course, only if you work fairly hard."
Let's think about the truth values of r.

Only if you work fairly hard.

p	q	$p \to q$	r
T	T	T	
T	F	F	
F	T	T	
F	F	T	

Thus, r should be logically equivalent to $p \to q$. (We write $r \equiv p \to q$ in this case.)

If-Then in programming languages

If-then

If-then in many programming languages is different from that used in logic.

if p **then** S;
where S is a program segment (one or more statements to be executed). For example:
if $2 + 2 = 4$ then $x : x + 1$;
If $x = 0$ before this statement is encountered, what is the value of the variable x after the statement?

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If $x = 0$ before this statement is encountered, what is the value of the variable x after the statement?

If-then-else

How about If-then-else?

if p **then** S_1**;**
else S_2**;**
where S_1 and S_2 are two program segments. For example:
if $2+2=5$ then $x : x+1$;
else $x : x+2$;
If $x = 0$ before this statement is encountered, what is the value of the variable x after the statement?

If and only if: $(\leftrightarrow)$

Given p and q, we denote by

$$p \leftrightarrow q$$

the statement "p if and only if q."

If and only if: $(\leftrightarrow)$

Given p and q, we denote by

$$p \leftrightarrow q$$

the statement "p if and only if q." It is logically equivalent to

$$(p \leftarrow q) \wedge (p \rightarrow q),$$

i.e., $p \leftrightarrow q \equiv (p \leftarrow q) \wedge (p \rightarrow q)$.
Let's fill in its truth table.

p	q	$p \rightarrow q$	$p \leftarrow q$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

An implication and its friends

When you have two propositions

- $p =$ "I own a cell phone", and
- $q =$ "I bring a cell phone to class".

We have

- an implication $p \to q \equiv$
  "If I own a cell phone, I'll bring it to class",
- its **converse** $q \to p \equiv$
  "If I bring a cell phone to class, I own it", and
- its **contrapositive** $\neg q \to \neg p \equiv$
  "If I do not bring a cell phone to class, I do not own one".

Quick check 3

Let's consider the following truth table:

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Quick check 3

Let's consider the following truth table:

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Quick check 3

Let's consider the following truth table:

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Do you notice any equivalence? Right, $p \to q \equiv \neg q \to \neg p$.

Precedence of logical operators

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Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

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Operator	Precedence
$\neg$	1
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$\rightarrow$	4
$\leftrightarrow$	5

- $(p \vee q) \wedge \neg r = (p \vee q) \wedge (\neg r)$;
- $p \vee q \rightarrow \neg r = (p \vee q) \rightarrow (\neg r)$;
- $p \wedge q \vee r = (p \wedge q) \vee r$.

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$\neg$	1
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- $p \vee q \rightarrow \neg r = (p \vee q) \rightarrow (\neg r)$;
- $p \wedge q \vee r = (p \wedge q) \vee r$.

Because some rules may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Bit and bit operations

A **bit** is a symbol with two possible values, namely, 0 (false) and 1 (true).
A variable is called a **Boolean variable** if its value is either true or false.

Operations

x	y	$x \vee y(OR)$	$x \wedge y(AND)$	$x \oplus y(XOR)$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

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1	0	1	0	1
1	1	1	1	0

Example

Let 01 1011 0110 and 11 0001 1101 be two bit strings. Please find the bitwise OR, AND, and XOR of them.

01 1011 0110	01 1011 0110	01 1011 0110
11 0001 1101	11 0001 1101	11 0001 1101
11 1011 1111 (OR)	01 0001 0100 (AND)	10 1010 1011 (XOR)

Translating sentences

How can the following sentence be translated into a logical expression?

Example

You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.

Solution:

- $p =$ "You can ride the roller coaster;"
- $q =$ "you are under 4 feet tall;"
- $r =$ "you are older than 16 years old."

Then the sentence can be translated to $(q \wedge \neg r) \rightarrow \neg p$, or $p \rightarrow (\neg q \vee r)$.

System specifications

Example

Determine whether these system specifications are consistent?

- "The diagnostic message is stored in the butter or it is retransmitted;"
- "The diagnostic message is not stored in the butter;"
- "If the diagnostic message is stored in the butter, then it is retransmitted."

Let p denote *The diagnostic message is stored in the butter* and q denote *The diagnostic message is retransmitted*.

- $p \vee q$
- $\neg p$
- $p \rightarrow q$

Logic puzzles

Example

There are two kinds of inhabitants in an island, knights, who always tell the truth, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?
Solution:

- p : *"A is a knight;"*
- q : *"B is a knight;"*

- We first consider the possibility that A is a knight. The statement given by B can be represented as $(\neg p \wedge q) \vee (p \wedge \neg q)$. Thus, we have $p \wedge q \wedge ((\neg p \wedge q) \vee (p \wedge \neg q))$. According to the truth table, it is a contradiction. Consequently, we can conclude that A is not a knight.

Logic puzzles Cont'd

Example

There are two kinds of inhabitants in an island, knights, who always tell the truth, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?

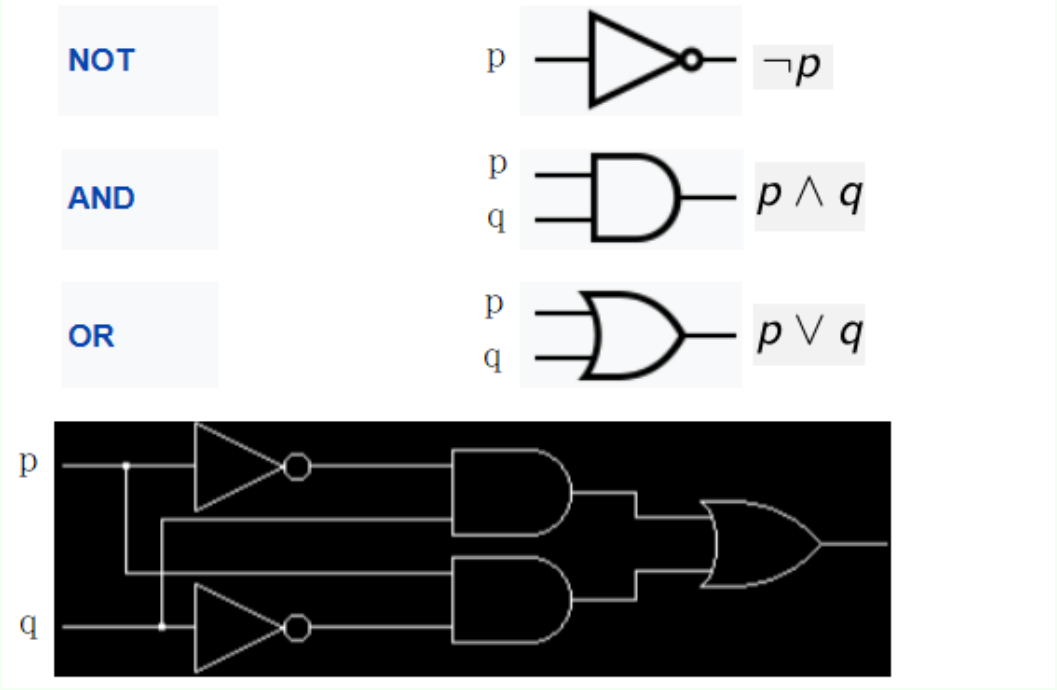
Solution:

- p : *"A is a knight;"*
- q : *"B is a knight;"*
- If A is a knave, then B is also a knave. The statement given by B can be represented as $(p \wedge q) \vee (\neg p \wedge \neg q)$. Thus, we have $\neg p \wedge \neg q \wedge ((p \wedge q) \vee (\neg p \wedge \neg q))$. According to the truth table, it is true when both $\neg p$ and $\neg q$ are true. Consequently, we can conclude that both A and B are knaves.

Logic circuits

Example

A logic circuit (or digital circuit) receives input signals $p_1, \cdots, p_n$, and produces output signals in bits $s_1, \cdots, s_m$.
Determine the output for the combinatorial circuit in the Figure.



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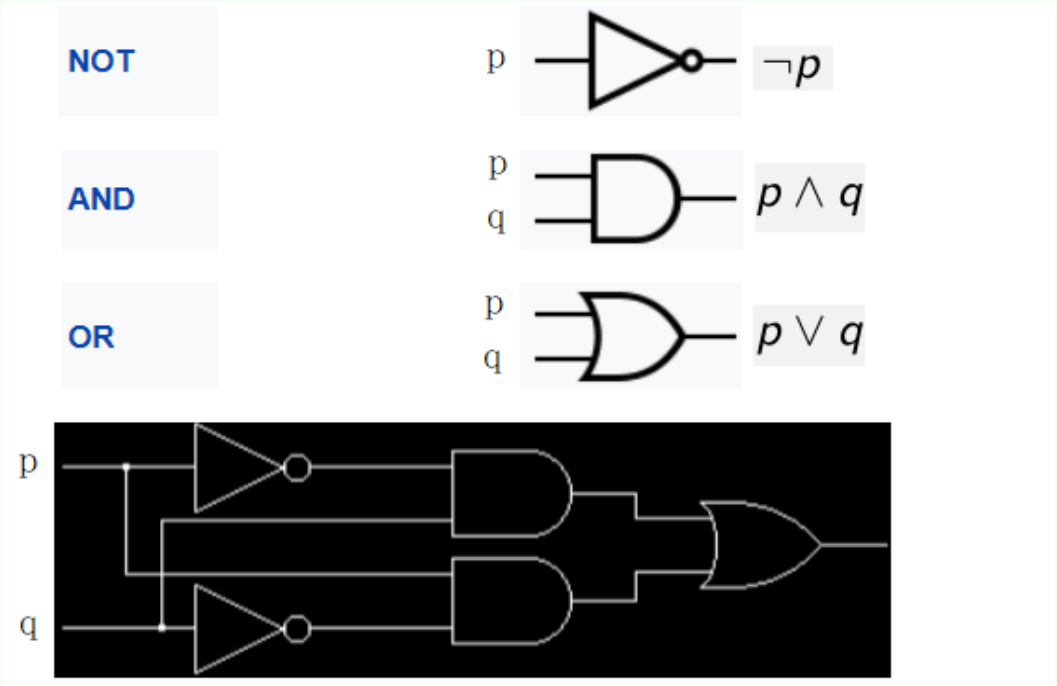

The output signal is $(\neg p \wedge q) \vee (p \wedge \neg q)$.

Take-aways

Conclusion

- Logic is important to
  1. Mathematical reasoning
  2. Computer circuits
  3. Construction of computer programs
  4. Automatical proof
  5. etc
- Proposition
- Operations of propositions
- Applications of propositional logic