STAT 3202: Practice 03

Spring 2019, OSU

Exercise 1

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

(a) Obtain a method of moments **estimator** for λ, $\tilde{\lambda}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Solution:

Recall that for a Poisson distribution we have $\text{E}[X] = \lambda$.

Now to obtain the method of moments estimator we simply equate the first population mean to the first sample mean. (And then we need to "solve" this equation for λ...)

$$\text{E}[X] = \bar{X}\lambda = \bar{X}$$

Thus, after "solving" we obtain the method of moments *estimator*.

$$\boxed{\tilde{\lambda} = \bar{X}}$$

Thus for the given data we can use this estimator to calculate the *estimate*.

$$\tilde{\lambda} = \bar{x} = \frac{1}{4}(1 + 2 + 4 + 2) = \boxed{2.25}$$

(b) Find the maximum likelihood **estimator** for λ, $\hat{\lambda}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Solution:

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} (x_i!)}$$

$$\log L(\lambda) = \left(\sum_{i=1}^{n} x_i\right) \log \lambda - n\lambda - \sum_{i=1}^{n} \log (x_i!)$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\frac{d^2}{d\lambda^2}\log L(\lambda) = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0$$

We then have the *estimator*, and for the given data, the *estimate*.

$$\boxed{\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{1}{4}(1+2+4+2) = \boxed{2.25}$$

(c) Find the maximum likelihood **estimator** of $P[X=4]$, call it $\hat{P}[X=4]$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Solution:

Here we use the invariance property of the MLE. Since $\hat{\lambda}$ is the MLE for λ then

$$\boxed{\hat{P}[X=4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!}}$$

is the maximum *likelihood estimator* for $P[X=4]$.

For the given data we can calculate an *estimate* using this estimator.

$$\hat{P}[X=4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!} = \frac{2.25^4 e^{-2.25}}{4!} = \boxed{0.1126}$$

Exercise 2

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$.

Find a method of moments **estimator** for the *parameter vector* (θ, σ^2).

Solution:

Since we are estimating two parameters, we will need two population and sample moments.

$$\mathrm{E}[X] = \theta$$

$$\mathrm{E}\left[X^2\right] = \mathrm{Var}\left[X\right] + \left(\mathrm{E}[X]\right)^2 = \sigma^2 + \theta^2$$

We equate the first population moment to the first sample moment, $\bar{x}$ and we equate the second population moment to the second sample moment, $\overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

$$\mathrm{E}\left[X\right] = \bar{X}$$

$$\mathrm{E}\left[X^2\right] = \overline{X^2}$$

For this example, that is,

$$\theta = \bar{X}$$

$$\sigma^2 + \theta^2 = \overline{X^2}$$

Solving this system of equations for θ and σ^2 we find the method of moments estimators.

$$\boxed{\begin{aligned} &\tilde{\theta} = \bar{X} \\ &\tilde{\sigma}^2 = \overline{X^2} - (\bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{aligned}}$$

Exercise 3

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(1, \sigma^2)$.

Find a method of moments **estimator** of σ^2, call it $\tilde{\sigma}^2$.

Solution:

The first moment is not useful because it is not a function of the parameter of interest σ^2.

$$\mathrm{E}[X] = 1$$

As a results, we instead use the second moment

$$\mathrm{E}\left[X^2\right] = \mathrm{Var}\left[X\right] + \left(\mathrm{E}[X]\right)^2 = \sigma^2 + 1^2 = \sigma^2 + 1$$

We equate this second population moment to the second population moment, $\overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$$\mathrm{E}\left[X^2\right] = \overline{X^2}$$

$$\sigma^2 + 1 = \overline{X^2}$$

Now solving for σ^2 we obtain the method of moments estimator.

$$\boxed{\tilde{\sigma}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - 1}$$

Exercise 4

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pdf

$$f(x \mid \theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}, \quad 0 < x < 1,\ 0 < \theta < \infty$$

(a) Find the maximum likelihood **estimator** of θ, call it $\hat{\theta}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 0.10,\ x_2 = 0.22,\ x_3 = 0.54,\ x_4 = 0.36.$$

Solution:

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{(1-\theta)/\theta} = \theta^{-n} \left(\prod_{i=1}^{n} x_i \right)^{\frac{1-\theta}{\theta}}$$

$$\log L(\theta) = -n \log \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log x_i = -n \log \theta + \frac{1}{\theta} \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} \log L(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i = 0$$

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i$$

Note that $\hat{\theta} > 0$, since each $\log x_i < 0$ since $0 < x_i < 1$.

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} \log x_i$$

$$\frac{d^2}{d\theta^2} \log L(\hat{\theta}) = \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3} \left(-n\hat{\theta} \right) = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

We then have the *estimator*, and for the given data, the *estimate*.

$$\boxed{\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i} = -\frac{1}{4} \log(0.10 \cdot 0.22 \cdot 0.54 \cdot 0.36) = \boxed{1.3636}$$

(b) Obtain a method of moments **estimator** for θ, $\tilde{\theta}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 0.10,\ x_2 = 0.22,\ x_3 = 0.54,\ x_4 = 0.36.$$

Solution:

$$\mathrm{E}[X] = \int_0^1 x \cdot \frac{1}{\theta} x^{(1-\theta)/\theta} dx = \ldots \text{ some calculus happens} \ldots = \frac{1}{\theta + 1}$$

$$\mathrm{E}[X] = \bar{X}$$

$$\frac{1}{\theta + 1} = \bar{X}$$

Solving for θ results in the method of moments *estimator*.

$$\boxed{\tilde{\theta} = \frac{1 - \bar{X}}{\bar{X}}}$$

$$\bar{x} = \frac{1}{4}(0.10 + 0.22 + 0.54 + 0.36) = 0.305$$

Thus for the given data we can calculate the *estimate*.

$$\tilde{\theta} = \frac{1 - \bar{x}}{\bar{x}} = \frac{1 - 0.305}{0.305} = \boxed{2.2787}$$

Exercise 5

Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x$$

Obtain the maximum likelihood **estimator** for θ, $\hat{\theta}$.

Solution:

First, be aware that the values of x for this pdf are restricted by the value of θ.

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} \quad 0 < \theta \leq x_i \text{ for all } x_i$$

$$= \frac{\theta^n}{\prod_{i=1}^{n} x_i^2} \quad 0 < \theta \leq \min\{x_i\}$$

$$\log L(\theta) = n \log \theta - 2 \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} \log L(\theta) = \frac{n}{\theta} > 0$$

So, here we have a log-likelihood that is increasing in regions where it is not zero, that is, when $\theta \min\{x_i\}$. Thus, the likelihood is the largest allowable value of θ in this region, thus the maximum likelihood estimator is given by

$$\boxed{\hat{\theta} = \min\{X_i\}}$$

Exercise 6

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \alpha) = \alpha^{-2} x e^{-x/\alpha}, \quad x > 0, \ \alpha > 0$$

(a) Obtain the maximum likelihood **estimator** of α, $\hat{\alpha}$. Calculate the **estimate** when

$$x_1 = 0.25, \ x_2 = 0.75, \ x_3 = 1.50, \ x_4 = 2.5, \ x_5 = 2.0.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\alpha) = \prod_{i=1}^{n} f(x_i; \alpha) = \prod_{i=1}^{n} \alpha^{-2} x_i e^{-x_i/\alpha} = \alpha^{-2n} \left(\prod_{i=1}^{n} x_i \right) \exp \left(\frac{-\sum_{i=i}^{n} x_i}{\alpha} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\alpha) = -2n \log \alpha + \sum_{i=i}^{n} \log x_i - \frac{\sum_{i=i}^{n} x_i}{\alpha}$$

To maximize this function, we take a **derivative** with respect to α.

$$\frac{d}{d\alpha} \log L(\alpha) = \frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2}$$

We set this derivative equal to **zero**, then **solve** for α.

$$\frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\boxed{\hat{\alpha} = \frac{\sum_{i=i}^{n} x_i}{2n} = \frac{\bar{x}}{2}}$$

Using the given data, we obtain an *estimate*.

$$\hat{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

(We should also verify that this point is a maxmimum, which is omitted here.)

(b) Obtain the method of moments **estimator** of α, $\tilde{\alpha}$. Calculate the **estimate** when

$$x_1 = 0.25, \ x_2 = 0.75, \ x_3 = 1.50, \ x_4 = 2.5, \ x_5 = 2.0.$$

Hint: Recall the probability density function of an exponential random variable.

$$f(x \mid \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0,\ \theta > 0$$

Note that, the moments of this distribution are given by

$$E[X^k] = \int_0^\infty \frac{x^k}{\theta} e^{-x/\theta} = k! \cdot \theta^k.$$

This hint will also be useful in the next exercise.

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the second moment of an exponential distribution.

$$\mathrm{E}[X] = \int_0^\infty x \cdot \alpha^{-2} x e^{-x/\alpha} dx = \frac{1}{\alpha} \int_0^\infty \frac{x^2}{\alpha} e^{-x/\alpha} dx = \frac{1}{\alpha}(2\alpha^2) = 2\alpha$$

We then set the first population moment, which is a function of α, equal to the first **sample moment**.

$$2\alpha = \frac{\sum_{i=i}^n x_i}{n}$$

Solving for α, we obtain the method of moments *estimator*.

$$\boxed{\tilde{\alpha} = \frac{\sum_{i=i}^n x_i}{2n} = \frac{\bar{x}}{2}}$$

Using the given data, we obtain an *estimate*.

$$\tilde{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

Note that, in this case, the MLE and MoM estimators are the same.

Exercise 7

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x \mid \beta) = \frac{1}{2\beta^3} x^2 e^{-x/\beta}, \quad x > 0,\ \beta > 0$$

(a) Obtain the maximum likelihood **estimator** of β, $\hat{\beta}$. Calculate the **estimate** when

$$x_1 = 2.00,\ x_2 = 4.00,\ x_3 = 7.50,\ x_4 = 3.00.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\beta) = \prod_{i=1}^{n} f(x_i; \beta) = \prod_{i=1}^{n} \frac{1}{2\beta^3} x^2 e^{-x/\beta} = 2^{-n}\beta^{-3n}\left(\prod_{i=1}^{n} x_i\right)\exp\left(\frac{-\sum_{i=i}^{n} x_i}{\beta}\right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\beta) = -n\log 2 - 3n\log\beta + \sum_{i=i}^{n}\log x_i - \frac{\sum_{i=i}^{n} x_i}{\beta}$$

To maximize this function, we take a **derivative** with respect to β.

$$\frac{d}{d\beta}\log L(\beta) = \frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2}$$

We set this derivative equal to **zero**, then **solve** for β.

$$\frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\boxed{\hat{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}}$$

Using the given data, we obtain an *estimate*.

$$\hat{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

(We should also verify that this point is a maxmimum, which is omitted here.)

(b) Obtain the method of moments **estimator** of β, $\tilde{\beta}$. Calculate the **estimate** when

$$x_1 = 2.00,\ x_2 = 4.00,\ x_3 = 7.50,\ x_4 = 3.00.$$

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the third moment of an exponential distribution.

$$\mathrm{E}[X] = \int_0^\infty x \cdot \frac{1}{2\beta^3} x^2 e^{-x/\beta} dx = \frac{1}{2\beta^2}\int_0^\infty \frac{x^3}{\beta} e^{-x/\beta} dx = \frac{1}{2\beta^2}(6\beta^3) = 3\beta$$

We then set the first population moment, which is a function of β, equal to the first **sample moment**.

$$\mathrm{E}[X] = \bar{X}$$

$$3\beta = \frac{\sum_{i=i}^{n} x_i}{n}$$

Solving for β, we obtain the method of moments *estimator*.

$$\boxed{\tilde{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}}$$

Using the given data, we obtain an *estimate*.

$$\tilde{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

Note again, the MLE and MoM estimators are the same.

Exercise 8

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a distribution with pdf

$$f(y \mid \alpha) = \frac{2}{\alpha} \cdot y \cdot \exp\left\{-\frac{y^2}{\alpha}\right\}, \; y > 0, \; \alpha > 0.$$

(a) Find the maximum likelihood **estimator** of α.

Solution:

The likelihood function of the data is the joint distribution viewed as a function of the parameter, so we have:

$$L(\alpha) = \frac{2^n}{\alpha^n} \left\{\prod_{i=1}^{n} y_i\right\} \exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n} y_i^2\right\}$$

We want to maximize this function. First, we can take the logarithm:

$$\log L(\alpha) = n \log 2 - n \log \alpha + \sum_{i=1}^{n} \log y_i - \frac{1}{\alpha}\sum_{i=1}^{n} y_i^2$$

And then take the derivative:

$$\frac{d}{d\alpha} \log L(\alpha) = -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2$$

Setting this equal to 0 and solving for α:

$$\begin{aligned}
& -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2 = 0 \\
\iff & \frac{n}{\alpha} = \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2 \\
\iff & \alpha = \frac{1}{n}\sum_{i=1}^{n} y_i^2
\end{aligned}$$

So, our candidate for the MLE is

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} y_i^2.$$

Taking the second derivative,

$$\frac{d^2}{d\alpha^2}\log L(\alpha) = \frac{n}{\alpha^2} - \frac{2}{\alpha^3}\sum_{i=1}^{n} y_i^2 = \frac{n}{\alpha^2} - \frac{2n}{\alpha^3}\hat{\alpha}$$

so that:

$$\frac{d^2}{d\alpha^2}\log L(\hat{\alpha}) = \frac{n}{\hat{\alpha}^2} - \frac{2n}{\hat{\alpha}^3}\hat{\alpha} = -\frac{n}{\hat{\alpha}^2} < 0$$

Thus, the (log-)likelihood is concave down at $\hat{\alpha}$, which confirms that the value of α that maximizes the likelihood is:

$$\boxed{\hat{\alpha}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2}$$

(b) Let $Z_1 = Y_1^2$. Find the distribution of Z_1. Is the MLE for α an unbiased estimator of α?

Solution:

If $Z_i = Y_i^2$, then $Y_i = \sqrt{Z_i}$, and $\frac{dy_i}{dz_i} = \frac{1}{2}\frac{1}{\sqrt{z_i}}$, so that:

$$f_Z(z) = \frac{2}{\alpha}\sqrt{z}\cdot\exp\left\{-\frac{z}{\alpha}\right\}\frac{1}{2}\frac{1}{\sqrt{z}} = \boxed{\frac{1}{\alpha}\exp\left\{-\frac{z}{\alpha}\right\}}$$

which is the pdf of an exponential distribution with parameter α. Thus,

$$\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right] = \mathrm{E}\left[\bar{Z}\right] = \mathrm{E}[Z_1] = \alpha,$$

so that $\hat{\alpha}_{\text{MLE}}$ is unbiased for α.

Note: I typically do not remember the "formula" for the pdf of a transformed variable, so I typically start from:

$$\text{for positive } z, \quad F_Z(z) = P(Z \le z) = P(Y^2 \le z) = P(Y \le \sqrt{z}) = F_Y(\sqrt{z})$$

and then take a derivative:

$$f_Z(z) = \frac{d}{dz}P(Z \le z) = \frac{d}{dz}F_Y(\sqrt{z}) = f_Y(\sqrt{z})\frac{d}{dz}\left\{\sqrt{z}\right\}$$

Exercise 9

Let X be a single observation from a $\text{Binom}(n, p)$, where p is an unknown parameter. (In this case, we will consider n known.)

(a) Find the maximum likelihood **estimator** (MLE) of p.

Solution:

We just have *one observation*, so the likelihood is just the pmf:

$$L(p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 < p < 1, \quad x = 0, 1, \ldots n$$

The log-likelihood is:

$$\log L(p) = \log \left\{ \binom{n}{x} \right\} + x \log(p) + (n - x) \log(1 - p).$$

The derivative of the log-likelihood is:

$$\frac{d}{dp} \log L(p) = \frac{x}{p} - \frac{n-x}{1-p}.$$

Setting this to be 0, we solve:

$$\frac{x}{p} - \frac{n-x}{1-p} = 0 \iff x - px = np - px \iff p = \frac{x}{n}.$$

Thus, $\hat{p} = \frac{x}{n}$ is our candidate.

We take the second derivative:

$$\frac{d^2}{dp^2} \log L(p) = -\frac{x}{p^2} - \frac{n-x}{(1-p)^2}$$

which is always less than 0; thus

$$\boxed{\hat{p} = \frac{X}{n}}$$

is the maximum likelihood **estimator** for p.

(b) Suppose you roll a 6-sided die 40 times and observe eight rolls of a 6. What is the maximum likelihood **estimate** of the probability of observing a 6?

Solution:

Here, we can let X be the number of sixes in 40 (independent) rolls of the die: $X \sim \text{Binom}(40, p)$, where p is the probability of rolling a 6 on this die.

Then

$$\boxed{\hat{p} = \frac{8}{40} = 0.2}$$

is the maximum likelihood **estimate** for p.

(c) Using the same observed data, suppose you now plan to perform a second experiment with the same die, and will roll the die 5 more times. What is the maximum likelihood **estimate** of the probability that you will observe no 6's in this next experiment?

Solution:

Let $Y \sim \text{Binom}(5, p)$ represent the number of sixes you will obtain in this second experiment. Based on the pmf of the binomial, we know that:

$$P(Y = 0) = \binom{5}{0} p^0 (1-p)^{5-0} = (1-p)^5$$

Let us call this new parameter of interest θ. Then we have

$$\theta = (1-p)^5$$

We are asked to find the MLE $\hat{\theta}$.

Based on the invariance property of the MLE,

$$\hat{\theta} = (1-\hat{p})^5$$

With the observed data, the maximum likelihood estimate is thus

$$\boxed{(1-0.2)^5 = 0.33}$$

Thus, our best guess (using the maximum likelihood framework) at the chance that we will observe no sixes in the next 5 rolls is 33%.

Exercise 10

Suppose that a random variable X follows a discrete distribution, which is determined by a parameter θ which can take *only two values*, $\theta = 1$ or $\theta = 2$. The parameter θ is unknown.

- If $\theta = 1$, then X follows a Poisson distribution with parameter $\lambda = 2$.
- If $\theta = 2$, then X follows a Geometric distribution with parameter $p = \frac{1}{4}$.

Now suppose we observe $X = 3$. Based on this data, what is the maximum likelihood **estimate** of θ?

Solution:

Because there are only two possible values of θ (1 and 2) rather than a whole range of possible values (like examples with $0 < \theta < \infty$) the approach of taking the derivative of something with respect to θ will not work. Instead, we need to think about the definition of the MLE. Instead, we just want to determine *which value of* θ makes our observed data, $X = 3$, most likely.

If $\theta = 1$, then X follows a Poisson distribution with parameter $\lambda = 2$. Thus, if $\theta = 1$,

$$P(X = 3) = \frac{e^{-2} \cdot 2^3}{3!} = 0.180447$$

If $\theta = 2$, then X follows a Geometric distribution with parameter $p = \frac{1}{4}$. Thus, if $\theta = 2$,

$$P(X = 3) = \frac{1}{4}\left(1 - \frac{1}{4}\right)^{3-1} = 0.140625$$

Thus, observing $X = 3$ is more likely when $\theta = 1$ (0.18) than when $\theta = 2$ (0.14), so $\boxed{1}$ is the maximum likelihood **estimate** of θ.

Exercise 11

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with pdf

$$f(y \mid \theta) = \frac{2\theta^2}{y^3}, \quad \theta \le y < \infty$$

Find the maximum likelihood **estimator** of θ..

Solution:

The likelihood is:

$$L(\theta) = \prod_{i=1}^{n} \frac{2\theta^2}{y_i^3} = \frac{2^n \theta^{2n}}{\prod_{i=1}^{n} y_i^3}, \; 0 < \theta \le y_i < \infty, \text{ for every } i.$$

Note that

$$0 < \theta \le y_i < \infty \text{ for every } i \iff 0 < \theta \le \min\{y_i\}.$$

To understand the behavior of $L(\theta)$, we can take the log and take the derivative:

$$\log L(\theta) = n \log 2 + (2n)\log\theta - \log\left(\prod_{i=1}^{n} y_i^3\right)$$

$$\frac{d}{d\theta}\log L(\theta) = \frac{2n}{\theta} > 0 \text{ on } \theta \in (0, \min\{y_i\})$$

Thus, the MLE is the largest possible value of θ:

$$\boxed{\hat{\theta} = \min\{Y_i\}}$$
