Tutorial 6

November 24, 2019

1. Suppose that $x_1, x_2, \cdots, x_n$ is a sample with the p.d.f.

$$f(x;\theta) = e^{-(x-\theta)}, x \geq \theta$$

and $f(x;\theta) = 0$ otherwise.

   (a) Find the method of moment estimate $\hat{\theta}_1$ of θ. Is $\hat{\theta}_1$ consistent? Is $\hat{\theta}_1$ unbiased?

   (b) Find the maximum likelihood estimate $\hat{\theta}_2$ of θ. Is $\hat{\theta}_2$ consistent? Is $\hat{\theta}_2$ unbiased?

   (c) Find the sufficient statistic for θ.

2. Suppose that in the population of twins, males (M) and females (F) are equally likely to occur and that the probability that twins are identical is α. If twins are not identical, their genes are independent.

   (a) Show that

$$P(MM) = P(FF) = \frac{1+\alpha}{4}, \quad P(MF) = \frac{1-\alpha}{2}$$

   (b) Suppose that n twins are sampled. It is found that n_1 are MM, n_2 are FF, and n_3 are MF, but it is not known which twins are identical. Find the MLE of α.

3. Suppose that $x_1, x_2, \cdots, x_n$ is a sample with the p.d.f.

$$f(x;\theta) = \begin{cases} 1, & \theta - 1/2 \leq x \leq \theta + 1/2; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that both the sample mean $\bar{x}$ and $\frac{1}{2}(x_{(1)} + x_{(n)})$ are unbiased for θ. Which one is more efficient?

4. Suppose that the p.d.f. is

$$f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0.$$

Let $x_1, x_2, \cdots, x_n$ be a sample.

   (a) Find the maximum likelihood estimate of $g(\theta) = \frac{1}{\theta}$;

   (b) Find the UMVUE of $g(\theta)$;

5. Suppose that $x_1, x_2, \cdots, x_n$ is a sample from Poisson distribution $P(\lambda)$.

   (a) Find the UMVUE of λ.

   (b) Find the UMVUE of $e^{-\lambda}$. (Hint: $I\{x_1 = 0\}$ is an unbiased estimate of $e^{-\lambda}$)

6. Suppose that $x_1, x_2, \cdots, x_n$ is a sample from Geometric distribution with p.m.f.

$$f(x|\theta) = \theta(1-\theta)^{k-1}, k = 1, 2, \cdots$$

Assume that the prior of θ is a uniform distribution $U(0, 1)$.

   (a) Find the posterior of θ;

   (b) Suppose that there are four observations: $4, 3, 1, 6$. Find the Bayes estimate of θ.

7. Prove that the Gamma distribution is a conjugate prior for a Poisson distribution $P(\lambda)$.