**Tutorial 2 Solutions**

**1.**

**(a)** The pmf of binomial distribution can be expressed as:

$$bin(n,\theta) = C_n^x \theta^x (1-\theta)^{n-x} = C_n^x e^{\log \theta^x (1-\theta)^{n-x}} = C_n^x e^{x \log \frac{\theta}{1-\theta} + n \log(1-\theta)},$$

we have

$$h(x) = C_n^x, \eta(\theta)^T = \log \frac{\theta}{1-\theta}, T(x) = x, \zeta(\theta) = -n \log(1-\theta).$$

So binomial distribution is an exponential family.

**(b)** The pdf of possion distribution can be expressed as:

$$P(\theta) = \frac{\theta^x e^{-\theta}}{x!} = \frac{1}{x!} e^{xln\theta - \theta},$$

we have

$$h(x) = \frac{1}{x!}, \eta(\theta)^T = ln\theta, T(x) = x, \zeta(\theta) = \theta.$$

So possion distribution is an exponential family.

**(c)** The pmf of negative binomial distribution can be expressed as:

$$NB(r,\theta) = C_{x-1}^{r-1} \theta^r (1-\theta)^{x-r} = C_{x-1}^{r-1} e^{\log(1-\theta)x + r \log \frac{\theta}{1-\theta}},$$

we have

$$h(x) = C_{x-1}^{r-1}, \eta(\theta)^T = \log(1-\theta), T(x) = x, \zeta(\theta) = -r \log \frac{\theta}{1-\theta}.$$

So negative binomial distribution is an exponential family.

**(d)** The pdf of exponential distribution can be expressed as:

$$Exp(\theta) = \theta e^{-\theta x} = e^{-\theta x + \log \theta},$$

we have

$$h(x) = 1, \eta(\theta)^T = -\theta, T(x) = x, \zeta(\theta) = -\log \theta.$$

So exponential distribution is an exponential family.

**(e)** The pdf of gamma distribution can be expressed as:

$$Ga(\alpha,\gamma) = \frac{\gamma^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\gamma x} = \frac{1}{x} e^{\alpha \log x - \gamma x + \log \frac{\gamma^\alpha}{\Gamma(\alpha)}},$$

we have

$$h(x) = \frac{1}{x}, \eta(\alpha, \gamma)^T = (\alpha, -\gamma)^T, T(x) = (\log x, x), \zeta(\alpha, \gamma) = -\log \frac{\gamma^\alpha}{\Gamma(\alpha)}.$$

So gamma distribution is an exponential family.

**(f)** The pdf of beta distribution can be expressed as:

$$Be(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{x(1-x)} e^{\alpha \log x + \beta \log(1-x) + \log \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}},$$

we have

$$h(x) = \frac{1}{x(1-x)}, \eta(\alpha, \beta)^T = (\alpha, \beta)^T, T(x) = (\log x, \log(1-x)), \zeta(\alpha, \gamma) = -\log \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

So beta distribution is an exponential family.

**2.**

**(a)** Since

$$\int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi} (\int_0^\infty \frac{x}{1+x^2} dx + \int_{-\infty}^0 \frac{-x}{1+x^2} dx) = \frac{1}{2\pi} (\log(1+x^2)|_0^\infty - \log(1+x^2)|_{-\infty}^0) = \infty,$$

therefore the expectation does not exist.

**(b)** The pdf of standard Cauchy distribution is an even function, so the median is 0.

**3.** We have

$$\begin{aligned} F_Y(y) =& P(Y \leq y) \\ =& P(x^{1/\gamma} \leq y)(y > 0) \\ =& P(x \leq y^\gamma)(\gamma > 0) \\ =& \int_0^{y^\gamma} \lambda e^{-\lambda x} dx \\ =& 1 - e^{-\lambda y^\gamma}, \end{aligned}$$

thus $f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda\gamma y^{\gamma-1}e^{-\lambda y^\gamma}(y > 0)$. Therefore,

$$
\begin{aligned}
\#\text{Expectation} =& E(Y) \\
=& \int_0^\infty \lambda\gamma y^{\gamma-1}e^{-\lambda y^\gamma} y dy \\
=& \int_0^\infty \lambda\gamma y^{\gamma}e^{-\lambda y^\gamma} dy \\
=& \lambda^{-\frac{1}{\gamma}} \int_0^\infty (\lambda y^\gamma)^{\frac{1}{\gamma}} e^{-\lambda y^\gamma} d\lambda y^\gamma \\
=& \lambda^{-\frac{1}{\gamma}} \Gamma(\frac{1}{\gamma} + 1) \\
E(Y^2) =& \int_0^\infty \lambda\gamma y^{\gamma-1}e^{-\lambda y^\gamma} y^2 dy \\
=& \int_0^\infty \lambda\gamma y^{\gamma+1}e^{-\lambda y^\gamma} dy \\
=& \lambda^{-\frac{2}{\gamma}} \int_0^\infty (\lambda y^\gamma)^{\frac{2}{\gamma}} e^{-\lambda y^\gamma} d\lambda y^\gamma \\
=& \lambda^{-\frac{2}{\gamma}} \Gamma(\frac{2}{\gamma} + 1) \\
\#\text{Variance} =& E(Y^2) - E^2(Y) \\
=& \lambda^{-\frac{2}{\gamma}} \Gamma(\frac{2}{\gamma} + 1) - (\lambda^{-\frac{1}{\gamma}} \Gamma(\frac{1}{\gamma} + 1))^2 \\
F(y_\alpha) =& 1 - e^{-\lambda y_\alpha^\gamma} \\
=& \alpha \\
\#\alpha\text{th quantiles} =& y_\alpha \\
=& (-\frac{\log(1-\alpha)}{\lambda})^{\frac{1}{\gamma}}
\end{aligned}
$$

**4.**

**(a)** We have

$$
\begin{aligned}
\#\text{Left} =& E(X) \\
=& \sum_{k=0}^{\infty} kP(X=k) \\
=& \sum_{k=1}^{\infty} kP(X=k) \\
=& \sum_{k=1}^{\infty}\sum_{i=1}^{k} P(X=k) \\
\#\text{Right} =& \sum_{k=1}^{\infty} P(X \geq k) \\
=& \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} P(X=i)
\end{aligned}
$$

By using matrices to represent the two double summation,

$$
\begin{bmatrix}
\clubsuit & \clubsuit & \cdots & \clubsuit \\
\vdots & \clubsuit & \cdots & \clubsuit \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \cdots & \clubsuit
\end{bmatrix}
\begin{bmatrix}
\spadesuit & \cdots & \cdots & \cdots \\
\spadesuit & \spadesuit & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\spadesuit & \spadesuit & \cdots & \spadesuit
\end{bmatrix}
$$

We have $\#\text{Left} = \sum \clubsuit, \#\text{Right} = \sum \spadesuit$. Thus $\#\text{Left} = \#\text{Right}, E(X) = \sum_{k=1}^{\infty} P(X \geq k)$.

**(b)** We have

$$
\begin{aligned}
\#\text{Left} =& \sum_{k=0}^{\infty} kP(X > k) \\
=& \sum_{k=1}^{\infty} kP(X > k) \\
=& \sum_{k=1}^{\infty} k(P(X \geq k) - P(X = k)) \\
=& \sum_{k=1}^{\infty} kP(X \geq k) - \sum_{k=1}^{\infty} P(X = k) \\
=& \sum_{k=1}^{\infty} k \sum_{i=k}^{\infty} P(X = i) - \sum_{k=1}^{\infty} P(X = k) \\
=& \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} kP(X = i) - \sum_{k=1}^{\infty} P(X = k) \\
\#\text{Right} =& \frac{1}{2}(E(X^2) - E(X)) \\
=& \frac{1}{2}(\sum_{k=0}^{\infty} k^2 P(X = k) - \sum_{k=0}^{\infty} kP(X = k)) \\
=& \frac{1}{2}(\sum_{k=1}^{\infty} k^2 P(X = k) - \sum_{k=1}^{\infty} kP(X = k)) \\
=& \frac{1}{2}(\sum_{k=1}^{\infty} k^2 P(X = k) + \sum_{k=1}^{\infty} kP(X = k)) - \sum_{k=1}^{\infty} P(X = k) \\
=& \sum_{k=1}^{\infty} \frac{k(k+1)}{2} P(X = k) - \sum_{k=1}^{\infty} P(X = k) \\
=& \sum_{k=1}^{\infty} \sum_{i=0}^{k} iP(X = k) - \sum_{k=1}^{\infty} P(X = k) \\
=& \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} kP(X = i) - \sum_{k=1}^{\infty} P(X = k) \text{ (Similar to (a))}
\end{aligned}
$$

Thus #Left = #Right, $\sum_{k=0}^{\infty} kP(X > k) = \frac{1}{2}(E(X^2) - E(X))$.

**5.**

**(a)** We have

$$a = \int_a^b af(x)dx \leq E(X) = \int_a^b xf(x)dx \leq \int_a^b bf(x)dx = b,$$

thus $a \leq E(X) \leq b$.

**(b)** Let $Y = \frac{X-a}{b-a}$, thus $Y \in [0,1]$. And

$$\begin{aligned}
Var(X) =& (b-a)^2 Var(Y) \\
=& (b-a)^2 (E(Y^2) - E^2(Y)) \\
\leq& (b-a)^2 (E(Y) - E^2(Y)) (Y \in [0,1], Y^2 \leq Y) \\
=& (b-a)^2 (\frac{1}{4} - (E(Y) - \frac{1}{2})^2) \\
\leq& \frac{(b-a)^2}{4},
\end{aligned}$$

therefore $Var(X) \leq (\frac{b-a}{2})^2$.

**6.**

**(a)** Let $t > c$, thus $2c - t > c$. We have

$$\begin{aligned}
P(X > c) &= \int_c^{\infty} f(t)dt, \\
P(X < c) &= \int_{-\infty}^{c} f(2c-t)d(2c-t) \\
&= \int_{-\infty}^{c} f(t)d(2c-t) \\
&= \int_{\infty}^{c} f(t) - dt \\
&= \int_c^{\infty} f(t)dt.
\end{aligned}$$

Therefore $P(X > c) = P(X < c)$, and the median of $X$ is the number $c$.

**(b)** Let $t > c$, thus $2c - t > c$. We have

$$\begin{aligned}
E(X) &= \int_c^{\infty} tf(t)dt + \int_{-\infty}^{c} (2c-t)f(2c-t)d(2c-t) \\
&= \int_c^{\infty} tf(t)dt + 2c\int_{-\infty}^{c} f(2c-t)d(2c-t) - \int_{-\infty}^{c} tf(2c-t)d(2c-t) \\
&= \int_c^{\infty} tf(t)dt + 2c \times \frac{1}{2} - \int_c^{\infty} tf(t)dt \\
&= c
\end{aligned}$$

**(c)** If $c = 0$, $f(x)$ is an even function. Thus,

$$\begin{aligned}
&\int_{-\infty}^{x_\alpha} f(x)dx = \alpha \\
\Leftrightarrow &\int_{x_\alpha}^{\infty} f(x)dx = 1 - \alpha \\
\Leftrightarrow &\int_{-\infty}^{-x_\alpha} f(x)dx = 1 - \alpha
\end{aligned}$$

And $\int_{-\infty}^{x_{1-\alpha}} f(x)dx = 1 - \alpha$, so $x_\alpha = -x_{1-\alpha}$.

**7.** Since $Y = a + bX$, we have

$$E(Y) = bE(X) + a, Var(Y) = b^2 Var(X).$$

From the definition of coefficient of skewness and coefficient of kurtosis, we have

$$\begin{aligned}
\beta_s(Y) =& \frac{E(Y - E(Y))^3}{[Var(Y)]^{3/2}} \\
=& \frac{E(a + bX - bE(X) - a)^3}{[b^2 Var(X)]^{3/2}} \\
=& \frac{E(bX - bE(X))^3}{[b^2 Var(X)]^{3/2}} \\
=& \frac{E(X - E(X))^3}{[Var(X)]^{3/2}} \\
=& \beta_s(X) \\
\beta_k(Y) =& \frac{E(Y - E(Y))^4}{[Var(Y)]^2} - 3 \\
=& \frac{E(a + bX - bE(X) - a)^4}{[b^2 Var(x)]^2} - 3 \\
=& \frac{E(X - E(X))^4}{[Var(X)]^2} - 3 \\
=& \beta_k(X)
\end{aligned}$$