Mathematical Statistics and Data Analysis

Lecture 7: Statistics and their distributions

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Outlines

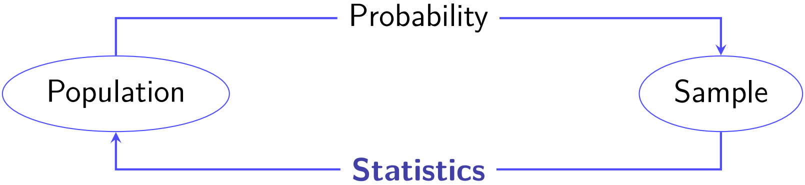
1. Sample
2. The Empirical Cumulative Distribution Function
3. Statistic
   - Sample Mean
   - Sample Variance & Sample Standard Deviation
   - Sample Moment
   - Order Statistics
   - Sample Quantiles & Sample Median
4. Distributions Derived from the Normal Distribution
   - χ^2 Distributions
   - F Distribution

Reading Material

Textbook:

- Rice: Chapter 3.7, 6, 7, 10;
- Mao: Chapter 5;

Sample



Before	After

Observed

Consider x_i's as **Random Variables**	Consider x_i's as **Observed Values**

Sample

$x_1, x_2, \cdots, x_n$

Sample

Definition

The random variables $x_1, x_2, \cdots, x_n$ are called a **simple random sample** of size n from the population $F(x)$ if $x_1, x_2, \cdots, x_n$ are mutually independent random variables and the marginal c.d.f. of each X_i is the same function $F(x)$.

Remark

- $x_1, x_2, \cdots, x_n$ are independently and identically distributed. The joint c.d.f. of $(x_1, x_2, \cdots, x_n)$ is

$$F(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} F(x_i)$$

- $F(x)$ is also called **population distribution**.

The Empirical Cumulative Distribution Function

Question:

How to find the population distribution $F(x)$?

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a simple random sample. $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ is called the **ordered sample** if the sample are sorted from the smallest to the largest, that is,

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}.$$

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The Empirical Cumulative Distribution Function

Definition

The **empirical cumulative distribution function (e.c.d.f.)** $F_n(x)$ is defined by

$$F_n(x) = \begin{cases} 0, & \text{if } x < x_{(1)}; \\ k/n, & \text{if } x_{(k)} \leq x < x_{(k+1)}, k = 1, 2, \cdots, n-1; \\ 1, & \text{if } x \geq x_{(n)}; \end{cases}$$

Property

The e.c.d.f. $F_n(x)$ is a c.d.f., that is, $F_n(x)$ satisfies that

- $F_n(x)$ is non-decreasing and right-continuous;
- $F_n(-\infty) = 0$ and $F_n(\infty) = 1$;

The Empirical Cumulative Distribution Function

Example

- Aim of study: to investigate chemical methods for detecting the presence of synthetic waxes that had been added to beeswax.
- The addition of microcrystalline wax **raises the melting point of beeswax**.
- All pure beeswax had the same melting point;
- However, the melting point and other chemical properties of beeswax vary from one beehive to another.

The Empirical Cumulative Distribution Function

Example (Con'd)

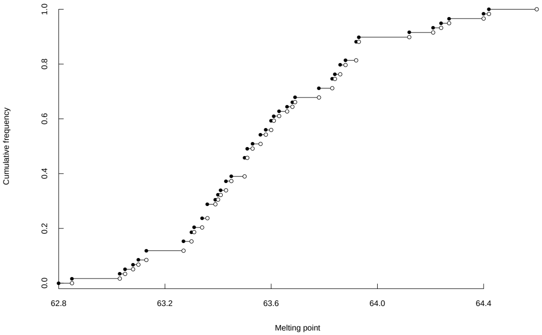
- Samples of pure beeswax are obtained from 59 sources.
- The 59 melting points (in $^\circ\mathrm{C}$) are listed as follows:

63.78	63.45	63.58	63.08	63.40	64.42	63.27	63.10
63.34	63.50	63.83	63.63	63.27	63.30	63.83	63.50
63.36	63.86	63.34	63.92	63.88	63.36	63.36	63.51
63.51	63.84	64.27	63.50	63.56	63.39	63.78	63.92
63.92	63.56	63.43	64.21	64.24	64.12	63.92	63.53
63.50	63.30	63.86	63.93	63.43	64.40	63.61	63.03
63.68	63.13	63.41	63.60	63.13	63.69	63.05	62.85
63.31	63.66	63.60					

The Empirical Cumulative Distribution Function

Example (Con'd)

- The e.c.d.f. is plotted as follows:



The Empirical Cumulative Distribution Function

$F_n(x)$ has another formula:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(-\infty,x)}(x_i)$$

where

$$I_{(-\infty,x)}(x_i) = \begin{cases} 1, & x_i \leq x; \\ 0, & x_i > x; \end{cases}$$

The random variables $I_{(-\infty,x)}(x_i)$ are independent Bernoulli random variables:

$$I_{(-\infty,x)}(x_i) = \begin{cases} 1, & \text{with probability } F(x) \\ 0, & \text{with probability } 1 - F(x); \end{cases}$$

The Empirical Cumulative Distribution Function

Thus, $nF_n(x)$ is a binomial random variable $b(n, F(x))$ and so

$$\begin{aligned} E(F_n(x)) &= F(x) \\ Var(F_n(x)) &= \frac{1}{n}F(x)(1 - F(x)) \end{aligned}$$

Theorem

Suppose that $x_1, x_2, \cdots, x_n$ are a sample from a population c.d.f $F(x)$ and $F_n(x)$ is e.c.d.f. Then,

$$P\left(\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \to 0\right) = 1$$

as $n \to \infty$

Statistic

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a sample from an unknown population. A **statistic** T is defined by a function of the sample $T = T(x_1, x_2, \cdots, x_n)$ without any unknown parameters.

Remark:

- Statistics: $\sum_{i=1}^n x_i$, $\sum_{i=1}^n x_i^2$ and $F_n(x)$;
- A statistic does not depend on unknown parameters;
- The distribution of the statistic often depend on unknown parameters;

Sample Mean

Definition

Let $x_1, x_2, \cdots, x_n$ be a sample. The **sample mean** $\bar{x}$ is defined as the arithmetic mean of a sample, i.e.

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Property

- $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$;
- $\bar{x} = \underset{c}{\operatorname{argmin}} \sum_{i=1}^{n}(x_i - c)^2$, where c is a constant;

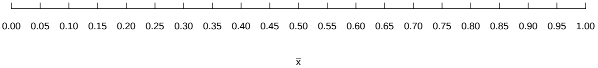
Sample Mean

Example

Suppose that $x_1, x_2, \cdots, x_{10}$ from a uniform distribution $U(0,1)$. At the i sampling, calculate the sample mean as

$$\bar{x}_i = \frac{\sum_{j=1}^{10} x_{i,j}}{10}, i = 1, 2, \cdots, 500.$$

What is the distribution of the sample mean?



Sample Mean

Theorem

Suppose that $\{x_i\}_{i=1}^n$ are a sample and $\bar{x}$ is the sample mean.

- If the population distribution is $N(\mu, \sigma^2)$, then the exact distribution of $\bar{x}$ is $N(\mu, \sigma^2/n)$;
- Suppose the population distribution is unknown. But $E(x) = \mu$ and $Var(x) = \sigma^2$. The asymptotic distribution of $\bar{x}$ is $N(\mu, \sigma^2/n)$. Denote $\bar{x} \dot{\sim} N(\mu, \sigma^2/n)$.

Proof:

- Since $\sum_{i=1}^n x_i \sim N(n\mu, n\sigma^2)$, we have
$$\bar{x} \sim N(\mu, \sigma^2/n).$$
- By CLT, $\sqrt{n}(\bar{x} - \mu)/\sigma \xrightarrow{L} N(0, 1)$. Thus, the asymptotic distribution of $\bar{x}$ is $N(\mu, \sigma^2/n)$.

Sample Variance

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a sample. The sample variance is defined by

$$s_*^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ or } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Remark:

- s^2 is also called **unbiased variance**;
- The different formula for the sample variance is

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

Sample Variance

Theorem

Suppose that the population X has first- and second- order moment, that is, $E(X) = \mu$ and $Var(X) = \sigma^2 < \infty$. Let $x_1, x_2, \cdots, x_n$ be a sample from the population. $\bar{x}$ and s^2 are, respectively, the sample mean and sample variance. Then,

$$E(\bar{x}) = \mu, \quad Var(\bar{x}) = \sigma^2/n, \quad E(s^2) = \sigma^2.$$

Proof: It is obvious that

$$\begin{aligned} E(\bar{x}) &= \frac{1}{n} E\left(\sum_{i=1}^{n} x_i\right) = \frac{n\mu}{n} = \mu, \\ Var(\bar{x}) &= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} x_i\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}. \end{aligned}$$

Theorem (Con'd)

We know

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\bar{x})^2 &= \sum_{i=1}^{n}(x_i^2-2\bar{x}x_i+\bar{x}^2)\\ &= \sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{x_i}+n\bar{x}^2=\sum_{i=1}^{n}x_i^2-n\bar{x}^2.\end{aligned}$$

Since $E(x_i^2)=Var(x_i)+\left(E(x_i)\right)^2=\sigma^2+\mu^2$ and $E(\bar{x}^2)=Var(\bar{x})+\left(E\bar{x}\right)^2=\sigma^2/n+\mu^2$, we have

$$E\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)=n(\mu^2+\sigma^2)-n(\mu^2+\sigma^2/n)=(n-1)\sigma^2.$$

Thus, $E(s^2)=\sigma^2$.

Sample Standard Deviation

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a sample. The **sample standard deviation** is defined by

$$s_* = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Sample Moment

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a sample.

- The k**th-order sample moment** is defined by

$$a_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$$

  Particularly, $a_1 = \bar{x}$.

- The k**th-order sample central moment** is defined by

$$b_k = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

  Particularly, $b_2 = s_*^2$.

Sample Moment

Definition

Suppose that $x_1, x_2, \cdots, x_n$ are a sample.

- The **sample coefficient of skewness** is

$$\hat{\beta}_s = \frac{b_3}{b_2^{3/2}}$$

- The **sample kurtosis** is defined by

$$\hat{\beta}_k = \frac{b_4}{b_2^2} - 3$$

Order Statistics

Definition

Suppose that $x_1, \cdots, x_n$ are a sample. The **ith order statistic** is defined by $x_{(i)}$. Particularly,

- the **minimum statistic** is defined by $x_{(1)} = \min\{x_1, \cdots, x_n\}$;
- the **maximum statistic** is defined by $x_{(n)} = \max\{x_1, \cdots, x_n\}$.

Theorem

Suppose the p.d.f. is $f(x)$ and the c.d.f. is $F(x)$. Let $x_1, x_2, \cdots, x_n$ be a sample. Then the p.d.f. of the kth order statistic $x_{(k)}$ is

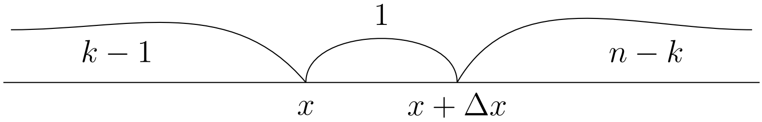
$$f_k(x) = \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1}(1-F(x))^{n-k}f(x).$$

Proof: For any x, the event $x \leq x_{(k)} \leq x + \Delta x$ occurs.

Order Statistics

Theorem (Con'd)

This is equivalent to that $k-1$ observations are less than x, one observation is in the interval $[x, x+\Delta x]$, and $n-k$ observations are greater than $x+\Delta x$.



Then, for each $x_{(i)}$, we have

$$\begin{aligned} P(x_{(i)} \le x) &= F(x) \\ P(x < x_{(i)} \le x+\Delta x) &= F(x+\Delta x) - F(x) \\ P(x_{(i)} > x+\Delta x) &= 1 - F(x+\Delta x) \end{aligned}$$

Order Statistics

Theorem (Con'd)

There are $\frac{n!}{(k-1)!1!(n-k)!}$ such arrangements. Let $F_k(x)$ be the c.d.f. of $x_{(k)}$. Thus, by the multinomial distribution,

$$F_k(x+\Delta x) - F_k(x) \approx \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1} \cdot (F(x+\Delta x) - F(x))(1 - F(x+\Delta x))^{n-k}$$

Both sides are divided by Δx, and let $\Delta x \to 0$, that is,

$$\begin{aligned} f_k(x) &= \lim_{\Delta x \to 0} \frac{F_k(x+\Delta x) - F_k(x)}{\Delta x} \\ &= \frac{n!}{(k-1)!(n-k)!}(F(x))^{k-1} f(x)(1-F(x))^{n-k}, \end{aligned}$$

where the non-zero intervals of $f_k(x)$ and $f(x)$ are the same.

Order Statistic

Remark:

- The p.d.f. of $x_{(1)}$ is

$$f_1(x) = n(1 - F(x))^{n-1} f(x);$$

- The p.d.f. of $x_{(n)}$ is

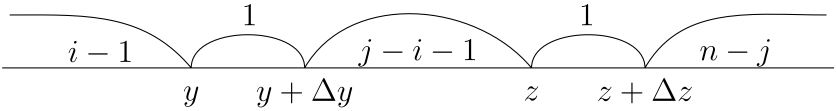
$$f_n(x) = n(F(x))^{n-1} f(x).$$

Order Statistic

Theorem

The p.d.f. of the order statistics $(x_{(i)}, x_{(j)})$ is

$$\begin{aligned} f_{i,j}(y,z) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}(F(y))^{i-1} \\ &\cdot(F(z)-F(y))^{j-i-1}(1-F(z))^{n-j}f(y)f(z), y \le z \end{aligned}$$



Order Statistic

Example

Suppose that $x_1, x_2, \cdots, x_n$ are a sample from a uniform distribution $U(0,1)$. Then the p.d.f. of the kth order statistic is

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}, 0 < x < 1.$$

Thus, $x_{(k)} \sim Be(k, n-k+1)$ and $E(x_{(k)}) = \frac{k}{n+1}$.
The joint p.d.f. of $(Y, Z) = (x_{(1)}, x_{(n)})$ is

$$f(y,z) = n(n-1)(z-y)^{n-2}, 0 < y < z < 1.$$

Let $R = Z - Y$. Since $R > 0$ and $0 < Y < Z < 1$,

$$0 < Y = Z - R \leq 1 - R.$$

Order Statistic

Example

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The joint p.d.f. of $(Y, Z) = (x_{(1)}, x_{(n)})$ is

$$f(y,z) = n(n-1)(z-y)^{n-2}, 0 < y < z < 1.$$

Let $R = Z - Y$. Since $R > 0$ and $0 < Y < Z < 1$,

$$0 < Y = Z - R \leq 1 - R.$$

Example (Con'd)

The joint p.d.f. of R is

$$f(y, r) = n(n-1)r^{n-2}, y > 0, r > 0, y + r < 1,$$

Then the marginal p.d.f. of R is

$$\begin{aligned} f(r) &= \int_0^{1-r} n(n-1)r^{n-2}\mathrm{d}y \\ &= n(n-1)r^{n-2}(1-r), 0 < r < 1 \end{aligned}$$

Thus, $R \sim Be(n-1, 2)$.

Sample Quantiles & Sample Median

Definition

Suppose that $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ are a ordered sample. The **pth sample quantile** is defined by

$$m_p = \begin{cases} x_{([np+1])}, & \text{if } np \text{ is not an integer;} \\ \frac{1}{2}(x_{(np)} + x_{(np+1)}), & \text{if } np \text{ is an integer;} \end{cases}$$

Particularly, the **sample median** is defined by

$$m_{0.5} = \begin{cases} x_{\left(\frac{n+1}{2}\right)}, & \text{if } n \text{ is odd;} \\ \frac{1}{2}\left(x_{\left(\frac{1}{2}\right)} + x_{\left(\frac{1}{2}+1\right)}\right), & \text{if } n \text{ is even;} \end{cases}$$

Sample Quantiles & Sample Median

Theorem

Suppose that the p.d.f. of a population is $f(x)$ and x_p is the pth sample quantile. $f(x)$ is continuous at the point $x = x_p$ and $f(x_p) > 0$. The asymptotic distribution of the pth sample quantile m_p is

$$m_p \dot{\sim} N\left(x_p, \frac{p(1-p)}{n \cdot f^2(x_p)}\right).$$

Particularly, the asymptotic distribution of the sample median is

$$m_{0.5} \dot{\sim} N\left(x_{0.5}, \frac{1}{4n \cdot f^2(x_{0.5})}\right)$$

Sample Quantiles & Sample Median

Example

The population distribution is Cauchy distribution. The p.d.f. is

$$f(x) = \frac{1}{\pi(1+(x-\theta))^2}, -\infty < x < \infty$$

Then the c.d.f. is

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x-\theta)$$

It is obvious that θ is the median of the Cauchy distribution, that is, $x_{0.5} = \theta$. Let $x_1, x_2, \cdots, x_n$ be a sample. Then, the asymptotic distribution of the sample median is

$$m_{0.5} \dot{\sim} N(\theta, \frac{\pi^2}{4n}).$$

χ^2 Distributions

Review

The p.d.f. of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Since $U = Z^2 \geq 0$, $F_U(u) = 0$ if $u \leq 0$. Thus, $f_U(u) = 0$ if $u \leq 0$. If $u > 0$, we have

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u}) \\ &= 2\Phi(\sqrt{y}) - 1 \end{aligned}$$

Then, the c.d.f. of U is

$$F_U(u) = \begin{cases} 2\Phi(\sqrt{y}) - 1, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

χ^2 Distributions

Review (Con'd)

The p.d.f. of Y is

$$
\begin{aligned}
f_U(u) &= \begin{cases} \phi(\sqrt{y})y^{-1/2}, & y > 0, \\ 0, & y \leq 0, \end{cases} \\
&= \begin{cases} \dfrac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}, & y > 0, \\ 0, & y \leq 0. \end{cases}
\end{aligned}
$$

Thus, $U \sim Ga(1/2, 1/2)$.

Definition

If Z is a standard normal r.v., the distribution of $U = Z^2$ is called **Chi-squared (χ^2)** distribution with 1 degree of freedom.

χ^2 Distributions

Review (Con'd)

The p.d.f. of Y is

$$\begin{aligned} f_U(u) &= \begin{cases} \phi(\sqrt{y})y^{-1/2}, & y > 0, \\ 0, & y \leq 0, \end{cases} \\ &= \begin{cases} \dfrac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}, & y > 0, \\ 0, & y \leq 0. \end{cases} \end{aligned}$$

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Definition

If Z is a standard normal r.v., the distribution of $U = Z^2$ is called **Chi-squared (χ^2)** distribution with 1 degree of freedom.

χ^2 Distributions

Review

If $U_1 \sim Ga(\alpha_1, \lambda)$, $U_2 \sim Ga(\alpha_2, \lambda)$ and U_1 and U_2 are independent, then $V = U_1 + U_2 \sim Ga(\alpha_1 + \alpha_2, \lambda)$.
Since $V = U_1 + U_2 \geq 0$, the p.d.f. of V is $f_V(v) = 0$ if $v \leq 0$.
If $v > 0$, the p.d.f. of

$$
\begin{aligned}
f_V(v) &= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z (z-y)^{\alpha_1-1} e^{-\lambda(z-y)} y^{\alpha_2-1} e^{-\lambda y} \mathrm{d}y \\
&= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z (z-y)^{\alpha_1-1} y^{\alpha_2-1} \mathrm{d}y \\
&= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1+\alpha_2-2} \int_0^z \left(1 - \frac{y}{z}\right)^{\alpha_1-1} \left(\frac{y}{z}\right)^{\alpha_2-1} \mathrm{d}y \\
&= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1+\alpha_2-1} \int_0^1 (1-t)^{\alpha_1-1} (t)^{\alpha_2-1} \mathrm{d}t
\end{aligned}
$$

χ^2 Distributions

Review (Con'd)

$$
\begin{aligned}
f_V(v) &= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1+\alpha_2)} z^{\alpha_1+\alpha_2-1} \\
&\quad \cdot \int_0^1 \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (1-t)^{\alpha_1-1} (t)^{\alpha_2-1}\, \mathrm{d}t \\
&= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda z}}{\Gamma(\alpha_1+\alpha_2)} z^{\alpha_1+\alpha_2-1}
\end{aligned}
$$

Thus, $V \sim Ga(\alpha_1+\alpha_2, \lambda)$.

- Z_i's are independently and identically distributed Gamma random variables $Ga(\alpha_i, \lambda)$. Then, $\sum_{i=1}^n Z_i \sim Ga(\sum_{i=1}^n \alpha_i, \lambda)$.

χ^2 Distributions

Definition

If $Z_1, Z_2, \cdots, Z_n$ are independently and identically distributed standard normal r.v.s, then $Z_1^2 + Z_2^2 + \cdots + Z_n^2$ is distributed as **Chi-squared (χ^2)** distribution with n degrees of freedom.

Remarks

- In fact, $Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim Ga(n/2, 1/2)$.
- The χ^2 distribution is a special case of the Gamma distribution.
- Properties:

$$E(Z_1^2 + Z_2^2 + \cdots + Z_n^2) = n$$

  and

$$Var(Z_1^2 + Z_2^2 + \cdots + Z_n^2) = 2n.$$

χ^2 Distributions

Example

Suppose that $x_1, x_2, \cdots, x_n$ is a sample from a normal population $N(\mu, \sigma^2)$, where the expectation μ is known. What is the distribution of

$$T = \sum_{i=1}^{n} (x_i - \mu)^2.$$

Solution: Let $y_i = (x_i - \mu)/\sigma, i = 1, 2, \cdots, n$. Then $y_1, y_2, \cdots, y_n$ are independently and identically distributed random variables. The distribution of y_1 is $N(0, 1)$. From the definition,

$$\frac{T}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} y_i^2 \sim \chi^2(n).$$

χ^2 Distributions

Example (Con'd)

Then, the p.d.f. of T is

$$f_T(t) = \frac{1}{(2\sigma^2)^{n/2}\Gamma(n/2)} \exp\left\{-\frac{t}{2\sigma^2}\right\} t^{\frac{n}{2}-1}$$

So,

$$T \sim Ga\left(\frac{n}{2}, \frac{1}{2\sigma^2}\right).$$

χ^2 Distributions

Theorem

Suppose that $x_1, x_2, \cdots, x_n$ is a sample from a normal distribution $N(\mu, \sigma^2)$. The sample mean and sample variance is respectively

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Then,

- $\bar{x}$ and s^2 are independent;
- $\bar{x} \sim N(\mu, \sigma^2/n)$;
- $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$.

χ^2 Distributions

Theorem (Con'd)

Proof: The joint p.d.f. of

$$
\begin{aligned}
f(x_1, x_2, \cdots, x_n) &= (2\pi\sigma^2)^{-n/2} \exp\left\{ -\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2} \right\} \\
&= (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{\sum_{i=1}^{n} x_i^2 - 2\bar{x}n\mu + n\mu^2}{2\sigma^2} \right\}
\end{aligned}
$$

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)'$.

χ^2 Distributions

Theorem (Con'd)

Proof:

$$A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2\cdot 1}} & \frac{1}{\sqrt{2\cdot 1}} & 0 & \cdots & 0; \\ \frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & \cdots & 0; \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{n\cdot(n-1)}} & \frac{1}{\sqrt{n\cdot(n-1)}} & \frac{1}{\sqrt{n\cdot(n-1)}} & \cdots & \frac{1}{\sqrt{n\cdot(n-1)}} \end{pmatrix}$$

As we know, the matrix A is orthogonal. Let $\boldsymbol{y} = A\boldsymbol{x}$. The Jacobian determinant is 1. Then,

$$\bar{x} = \frac{1}{\sqrt{n}} y \text{ and } \sum_{i=1}^{n} y_i^2 = \boldsymbol{y}' y == \boldsymbol{x}' A' A x$$

χ^2 Distributions

Theorem (Con'd)

The joint p.d.f. of $y_1, y_2, \cdots, y_n$ is

$$\begin{aligned} f(y_1, y_2, \cdots, y_n) &= (2\pi\sigma)^{-n/2} \exp\left\{ -\frac{\sum_{i=1}^n y_i - 2\sqrt{n} y_1 \mu + n\mu^2}{2\sigma^2} \right\} \\ &= (2\pi\sigma)^{-n/2} \exp\left\{ -\frac{\sum_{i=2}^n y_i + (y_1 - \sqrt{n}\mu)^2}{2\sigma^2} \right\} \end{aligned}$$

Then, $y_1, y_2, \cdots, y_n$ are independent and are distributed as a normal distribution with the variance σ^2. Thus, the mean of $y_2, y_3, \cdots, y_n$ is 0 and the mean of y_1 is $\sqrt{n}\mu$.

χ^2 distribution

Theorem (Con'd)

Since

$$\begin{aligned}(n-1)s^2 &= \sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n} x_i^2 - (\sqrt{n}\bar{x})^2 \\ &= \sum_{i=1}^{n} y_1^2 - y_1^2 = \sum_{i=2}^{n} y_i^2.\end{aligned}$$

Then, $y_2, \cdots, y_n$ are independent and identically distributed. And X_i's are distribution $N(0,1)$. Therefore,

$$\frac{(n-1)s^2}{\sigma^2} = \sum_{i=2}^{n}\left(\frac{y_i}{\sigma}\right)^2 \sim \chi^2(n-1)/$$

F distribution

Definition

Let U and V be independent Chi-square random variables with m and n degrees of freedom, respectively. The distribution of

$$F = \frac{U/m}{V/n}$$

is called the **F distribution** with m and n degrees of freedom and is denoted by $F_{m,n}$ or $F(m,n)$.

Proposition

The p.d.f. of F is given by

$$f(y) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{\frac{m}{2}} y^{\frac{m}{2}-1} \left(1 + \frac{m}{n}y\right)^{-\frac{m+n}{2}}, w > 0$$

F distribution

How to derive the p.d.f. of the F distribution?

First, we derive the p.d.f. of $Z = \frac{U}{V}$. Let the $f_U(u)$ and $f_V(v)$ be respectively the p.d.f. of U and V. Then, the p.d.f. of Z is

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty v f_U(zv) f_V(v) \mathrm{d}v \\
&= \frac{z^{\frac{m}{2}-1}}{\Gamma(m/2)\Gamma(n/2)\cdot 2^{\frac{m+n}{2}}} \int_0^\infty v^{\frac{m+n}{2}-1} e^{-\frac{v}{2}(1+z)} \mathrm{d}v \\
&= \frac{z^{\frac{m}{2}-1}}{\Gamma(m/2)\Gamma(n/2)\cdot 2^{\frac{m+n}{2}}} \frac{\Gamma((m+n)/2)}{((1+z)/2)^{\frac{m+n}{2}}} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} z^{\frac{m}{2}-1}(1+z)^{-\frac{m+n}{2}}, z > 0
\end{aligned}
$$

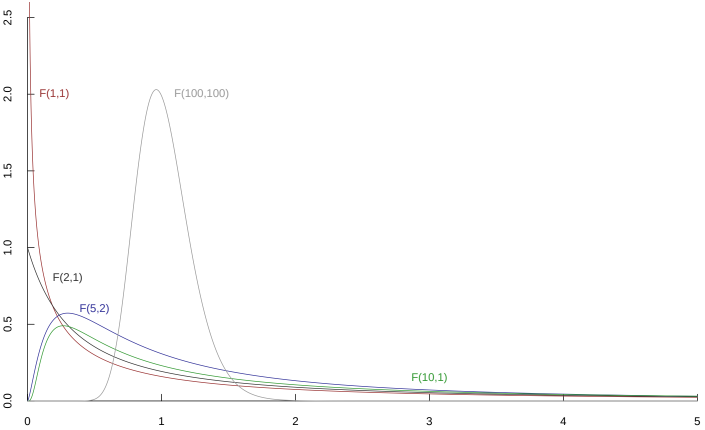
How to derive the p.d.f. of the F distribution? (Con'd)

Second, let $F = \frac{n}{m} Z$. For any $w > 0$, we have

$$
\begin{aligned}
f_F(y) &= p_Z\left(\frac{m}{n} y\right) \cdot \frac{m}{n} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n} y\right)^{\frac{m}{2}-1} \left(1 + \left(\frac{m}{n} y\right)\right)^{-\frac{m+n}{2}} \cdot \frac{m}{n} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} y^{\frac{m}{2}-1} \left(1 + \frac{m}{n} y\right)^{-\frac{m+n}{2}}
\end{aligned}
$$

F distribution

The p.d.f.s of F distribution are shown as follows:



F distribution

Proposition

Suppose that $x_1, x_2, \cdots, x_m$ is a sample from $N(\mu_1, \sigma_1^2)$ and $y_1, y_2, \cdots, y_n$ is a sample from $N(\mu_2, \sigma_2^2)$. Two samples are independent. Let

$$s_x^2 = \frac{1}{m-1}\sum_{i=1}^{m}(x_i - \bar{x})^2, \quad s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

where $\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$. Then

$$F = \frac{s_x^2/\sigma_1^2}{s_y^2/\sigma_2^2} \sim F(m-1, n-1).$$

Particularly, if $\sigma_1^2 = \sigma_2^2$, then $F = s_x^2/s_y^2 \sim F(m-1, n-1)$.

t distribution

Definition

If $Z \sim N(0,1)$ and $U \sim \chi^2_n$ and Z and U are independent, then the distribution of

$$t = \frac{Z}{\sqrt{U/n}}$$

is called the t **distribution** with n degrees of freedom.

How to derive the t distribution?

t distribution

How to derive the p.d.f. of the t distribution?

Z and $-Z$ are identically distributed for the p.d.f. of a standard normal distribution is symmetric. Then, t and $-t$ are also identically distributed. For any y,

$$P(0 < t < y) = P(0 < -t < y) = P(-y < -t < 0)$$

Thus,

$$P(0 < t < y) = \frac{1}{2}P(t^2 < y^2)$$

where

$$t^2 = \frac{Z^2}{U/n} \sim F(1, n).$$

t distribution

How to derive the p.d.f. of the t distribution? (Con'd)

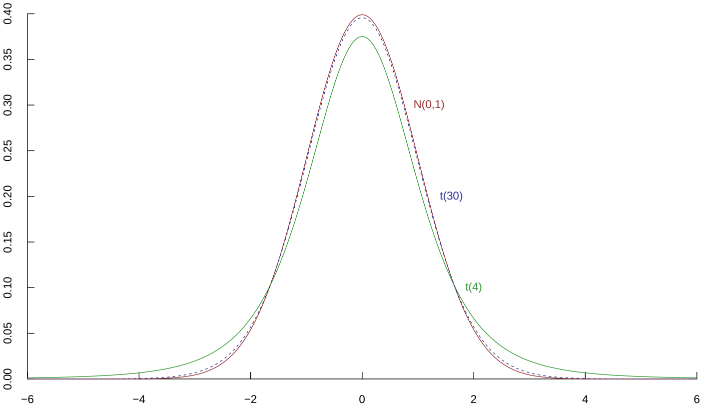
$$
\begin{aligned}
f_t(y) &= y f_F(y^2) = \frac{\Gamma\left(\frac{1+n}{2}\right)\left(\frac{1}{n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}(y^2)^{\frac{1}{2}-1}\left(1+\frac{1}{n}y^2\right)^{-\frac{1+n}{2}} \cdot y \\
&= \frac{\Gamma\left(\frac{1+n}{2}\right)\left(\frac{1}{n}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{1}{n}y^2\right)^{-\frac{1+n}{2}}, -\infty < y < \infty
\end{aligned}
$$

Remark

- If $n = 1$, then it is a standard Cauchy distribution;
- If $n > 1$, then the expectation exists and equals 0;
- If $n > 2$, then the variance exists and equals $n/(n-2)$;
- If $n \geq 30$, then $N(0,1)$ can be used as an approximate distribution.

t distribution

The p.d.f.s of t distribution are shown as follows:



t distribution

Proposition

Suppose that $x_1, x_2, \cdots, x_n$ is a sample from a normal population $N(\mu, \sigma^2)$, and $\bar{x}$ and s^2 are respectively the sample mean and sample variance. Then

$$t = \frac{\sqrt{n}(\bar{x} - \mu)}{s} \sim t(n-1)$$

Proof: Since

$$\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sim N(0, 1)$$

then

$$\frac{\sqrt{n}(\bar{x} - \mu)}{s} = \frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)s^2/\sigma^2}{n-1}}} \sim t(n-1)$$

t distribution

Proposition

Suppose that $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Let

$$s_w^2 = \frac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2} = \frac{\sum_{i=1}^{m}(x_i - \bar{x})^2 + \sum_{i=1}^{n}(y_i - \bar{y})^2}{m+n-2}.$$

Then

$$\frac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{s_w\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t(m+n-2)$$

Proof: Since $\bar{x} \sim N(\mu_1, \sigma^2/m)$, $\bar{y} \sim N(\mu_2, \sigma^2/n)$ and $\bar{x}$ and $\bar{y}$ are independent. Then,

$$\bar{x} - \bar{y} \sim N\left(\mu_1 - \mu_2, \left(\frac{1}{m} + \frac{1}{n}\right)\sigma^2\right).$$

t distribution

Proposition (Con'd)

Thus,

$$\frac{(\bar{x} - \bar{(}y)) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim N(0,1).$$

As we know, $\frac{(m-1)s_x^2}{\sigma^2} \sim \chi^2(m-1)$, $\frac{(n-1)s_y^2}{\sigma^2} \sim \chi^2(n-1)$ and they are independent. Then,

$$\frac{(m+n-2)s_w^2}{\sigma^2} = \frac{(m-1)s_x^2 + (n-1)s_y^2}{\sigma^2} \sim \chi^2(m+n-2)$$

Because $\bar{x} - \bar{y}$ and s^2 are independent,

$$\frac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{s_w\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t(m+n-2)$$

χ^2 distribution, F distribution & t distribution

Remark

- If $F \sim F(m, n)$, then $\frac{1}{F} \sim F(n, m)$.
- If $t \sim t(n)$, then $t^2 \sim F(1, n)$.
- If $X \sim F_{m,n}$, then $\frac{(m/n)X}{1+(m/n)X} \sim Be(m/2, n/2)$.
- Suppose that $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ are two independent samples from the standard normal population.

Distribution	Structure	Expectation	Variance
$\chi^2(n)$	$x_1^2 + x_2^2 + \cdots + x_n^2$	n	$2n$
$F(m, n)$	$\frac{y_1^2+y_2^2+\cdots+y_m^2}{x_1^2+x_2^2+\cdots+x_n^2}$	$\frac{n}{n-2}$ $(n > 2)$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$ $(n > 4)$
$t(n)$	$\frac{y_1}{\sqrt{(x_1^2+x_2^2+\cdots+x_n^2)/n}}$	0 $(n > 1)$	$\frac{n}{n-2}$ $(n > 2)$