Discrete Mathematics and Its Applications

Lecture 6: Discrete Probability: Relations and Their Properties

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Outline

Binary relation

Definition

Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$, written $R : A \leftrightarrow B$, is a subset of $A \times B$.

- The notation aRb means $(a, b) \in R$;
- If aRb, we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".

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Example

$<: N \leftrightarrow N :\equiv \{(n, m) | n < m\}$. $a < b$ means $(a, b) \in <$.
A binary relation R corresponds to a predicate function $P_R : A \times B \rightarrow \{T, F\}$ defined over the two sets A and B.

Examples of binary relations

- Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. For instance, we have $0Ra$, $0Rb$, etc..
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  Can we have visualized expressions of relations?
- Let A be the set of all cities, and let B be the set of the 50 states in the USA. Define the relation R by specifying that (a, b) belongs to R if city a is in state b. For instance, $(Boulder, Colorado), (Bangor, Maine)$, $(AnnArbor, Michigan)$, $(Middletown, NewJersey), (Middletown, NewYork)$, $(Cupertino, California)$, and $(RedBank, NewJersey)$ are in R.

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- "eats" $:\equiv \{(a, b)|$ organism a eats food $b\}$.

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- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph. This can be done by assigning to an element a of A the unique element $b \in B$ such that $(a, b) \in R$.

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- Relations are a generalization of graphs of functions; they can be used to express a much wider class of relationships between sets. (Recall that the graph of the function f from A to B is the set of ordered pairs $(a, f(a))$ for $a \in A$.)

Relations on a set

Definition

A relation on a set A is a relation from A to A. That is, a relation on a set A is a subset of $A \times A$.

- Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b)|a \text{ divides } b\}$?
- Consider these relations on the set of integers:

$$R_1 = \{(a, b)|a \leq b\},$$
$$R_2 = \{(a, b)|a > b\},$$
$$R_3 = \{(a, b)|a = \pm b\},$$
$$R_4 = \{(a, b)|a + b \leq 3\}.$$

  Which of these relations contain each of the pairs $(1, 1), (1, 2)$, $(2, 1), (1, -1)$, and $(2, 2)$?

Relation representation I

Matrix representing

Suppose that R is a relation from $A = \{a_1, a_2, \cdots, a_m\}$ to $B = \{b_1, b_2, \cdots, b_n\}$. (Here the elements of A and B have been listed in a particular, but arbitrary, order.) The relation R can be represented by $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R; \\ 0, & \text{otherwise.} \end{cases}$$

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- Let $A = \{0, 1, 2\}$, $B = \{a, b\}$, and $R = \{(0, a), (0, b), (1, a), (2, b)\}$. Then

$$M_R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Relation representation II

Digraph definition

A directed graph, or digraph, consists of (V, E), where V and E denote the sets of vertices (nodes) and edges (or arcs). In the edge (a, b), a and b are called the initial vertex and the terminal vertex.

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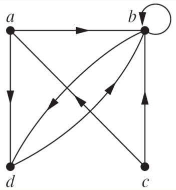
- Edge (a, a) is represented using an arc from the vertex a back to itself, called a **loop**.
- The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b), where $(a, b) \in R$, as edges.



The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b)$, $(b, d), (c, a)$, (c, b), and (d, b) is displayed in the figure.

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A set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$.
Thus, there are 2^{n^2} relations on a set with n elements.
For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$.

Reflexive

Definition

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

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Example

Consider the following relations on $\{1, 2, 3, 4\}$:

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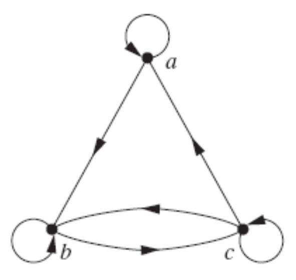
Which of these relations are reflexive?

Property of reflexive



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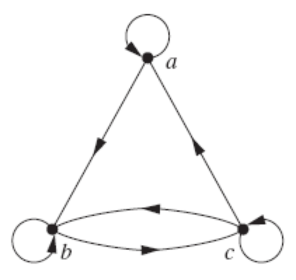

- $$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
- A relation is reflexive if and only if there is a loop at every vertex of the directed graph.

Symmetric and antisymmetric

Definition

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.

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- Similarly, the relation R on the set A is antisymmetric if

$$\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)).$$

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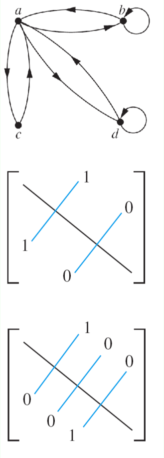

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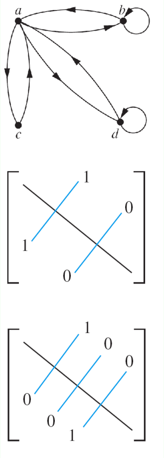

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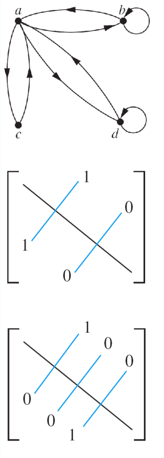

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- Similarly, a relation is antisymmetric if and only if: (1) there are never two edges in opposite directions between distinct vertices; (2) the matrix is a antisymmetric one.

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i.e., a relation is antisymmetric if and only if there are no pairs of distinct elements a and b with aRb and bRa.

Example of transitive

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

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Which of these relations are transitive?

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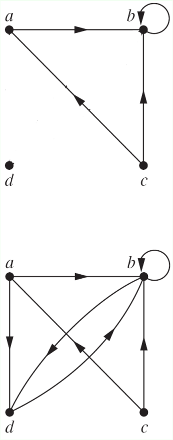

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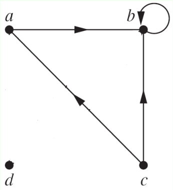

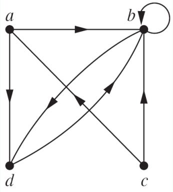

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- A relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z (completing a triangle where each side is a directed edge with the correct direction).

n-ary relations

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Let $A_1, A_2, \cdots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \cdots, A_n$ are called the domains of the relation, and n is called its degree.

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- Let R be the relation on $N \times N \times N$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3.

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- Let R be the relation on $Z \times Z \times Z^+$ consisting of triples (a, b, m), where a, b, and m are integers with $m \geq 1$ and $a \equiv b (\text{mod } m)$.

Databases and relations

Database

TABLE 1 **Students.**

Student_name	*ID_number*	*Major*	*GPA*
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Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
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Field

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- Relations used to represent databases are also called tables.
- A domain of an n-ary relation is called a primary key when the value of the n-tuple from this domain determines the n-tuple. Which domains are primary keys for the n-ary relation displayed in the table?

Relation operators

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

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- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$;
- $R_1 \cap R_2 = \{(1, 1)\}$;
- $R_1 - R_2 = \{(2, 2), (3, 3)\}$;
- $R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

Relation operators

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.
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- $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$.

Composite

Definition

Let R be a relation from A to B and S a relation from B to C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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Example

Let R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$. What is the composite of the relations R and S?

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

Composing the parent relation with itself

Power

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \cdots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

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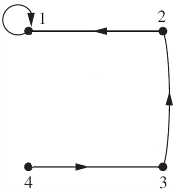

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers $R^n, n = 2, 3, 4, \cdots$

- $R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$;
- $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$;
- $R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$;
- $R^n = R^3$ when $n = 5, 6, \cdots$

Property of transitive

Theorem

The relation R on a set A is transitive if and only if $R^n \subset R$ for $n = 1, 2, 3, \cdots$

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Proof.

$\Rightarrow$:
We suppose that $R^n \subset R$ for $n = 1, 2, 3, \cdots$. Note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subset R$, this means that $(a, c) \in R$. Hence, R is transitive.

$\Leftarrow$: (Mathematical induction)
Assume that $R^n \subset R$ for $n \geq 1$ (inductive hypothesis).

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$\Leftarrow$: (Mathematical induction)

Assume that $R^n \subset R$ for $n \geq 1$ (inductive hypothesis). Let $(a, b) \in R^{n+1}$. Because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$.

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Matrix representation

Question: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

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Solution:

Because all the diagonal elements of this matrix are equal to 1, R is reflexive.

Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

Matrix operations

Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2}, respectively.

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Thus, the matrices representing the union, intersection, composite and power of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2},$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2},$$

$$M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2},$$

$$M_{R^n} = M_R^{[n]}.$$

Examples

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_{R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

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Thus, the matrices representing the union and intersection of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Examples Cont'd

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Thus, the matrices representing the composite and power of these relations are

$$M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{R_1^2} = M_{R_1}^{[2]} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Selection

Definition

Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the **selection operator** s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

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Example

TABLE 1 **Students.**

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Field

Record

- C_1 : Major = "Computer Science";
- C_2 : GPA > 3.5;
- C_3 : $C_1 \wedge C_2$;

Projection

Definition

The projection $P_{i_1 i_2, \cdots, i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, \cdots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \cdots, a_{i_m})$, where $m \leq n$.

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Example

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Record

TABLE 2 GPAs.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Join

Definition

Let R be a relation of degree m and S be a relation of degree n. The join $J_p(R,S)$ is a relation of degree $m+n-p$ that consists of all $(m+n-p)$-tuples $(a_1,\cdots,a_{m-p},c_1,\cdots,c_p,b_1,\cdots,b_{n-p})$, where the m-tuple $(a_1,\cdots,a_{m-p},c_1,\cdots,c_p)\in R$ and the n-tuple $(c_1,\cdots,c_p,b_1,\cdots,b_{n-p})\in S$.

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Let R be a relation of degree m and S be a relation of degree n. The join $J_p(R, S)$ is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, \cdots, a_{m-p}, c_1, \cdots, c_p, b_1, \cdots, b_{n-p})$, where the m-tuple $(a_1, \cdots, a_{m-p}, c_1, \cdots, c_p) \in R$ and the n-tuple $(c_1, \cdots, c_p, b_1, \cdots, b_{n-p}) \in S$.

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

Join Cont'd

What relation results when the join operator J_2 is used to combine the relation displayed in above two tables?

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

SQL

Example

TABLE 8 Flights.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

SQL

Example

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Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
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Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'

Take-aways

Conclusions

- Relation and Its Representation
- Properties of Relations
- n-ary Relations and Their Applications
- Relation Operators
  - Operations on Binary Relations
  - Operations on n-ary Relations
  - Structure Query Language