Tutorial 3 Solutions

1.

(a) From the definition of marginal pdf, we have

$$f_X(x) = \begin{cases} \int_0^x 3x dy = 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_y^1 3x dx = \frac{3}{2} - \frac{3}{2}y^2, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Since $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, thus X and Y are not independent.

2. Since $Z = \max\{X_1, X_2\} - \min\{X_1, X_2\} = |X_1 - X_2|$, we have $F_Z(z) = P(|X_1 - X_2| \leq z) = P(|X_1 - X_2| \leq z) = P(X_1 - X_2 \leq z, X_1 \geq X_2) + P(X_2 - X_1 \leq z, X_2 \geq X_1)(0 \leq z \leq 1)$. Draw the event of interest and we have

$$F_Z(z) = 1 - 2\int_z^1 \int_0^{x_1 - z} 2x_2 \cdot 2x_1 dx_2 dx_1 = \frac{8}{3} - 4z + \frac{4z^3}{3}(0 \leq z \leq 1).$$

thus,

$$f_Z(z) = \begin{cases} \frac{8}{3} - 4z + \frac{4z^3}{3}, & 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

3.

(a) Since $U_1 \sim U(0,1), U_2 \sim U(0,1)$, and $Z_1 = -2\log U_1, Z_2 = 2\pi U_2$, we have

$$f_{Z_1}(z_1) = f_{U_1}(e^{-\frac{1}{2}z_1})\left|e^{-\frac{1}{2}z_1} \times (-\frac{1}{2})\right| = \frac{1}{2}e^{-\frac{1}{2}z_1}(z_1 > 0),$$

$$f_{Z_2}(z_2) = \frac{1}{|2\pi|} f_{U_2}(\frac{1}{2\pi}z_2) = \frac{1}{2\pi}(0 < z_2 < 2\pi).$$

Thus $Z_1 \sim Exp(\frac{1}{2}), Z_2 \sim U(0, 2\pi)$.

(b) Since U_1 and U_2 are independent, and $Z_1 = -2\log U_1, Z_2 = 2\pi U_2$, thus Z_1 and Z_2 are independent. We have

$$f_{Z_1 Z_2}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2) = \frac{1}{4\pi}e^{-\frac{1}{2}z_1}(z_1 > 0, 0 < z_2 < 2\pi).$$

And $X = \sqrt{Z_1} \cos Z_2, Y = \sqrt{Z_2} \sin Z_2$, thus $Z_1 = X^2 + Y^2, Z_2 = \arctan \frac{Y}{X}$, we have

$$J = \begin{bmatrix} 2x & 2y \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = 2.$$

Therefore,

$$f_{XY}(x, y) = f_{Z_1 Z_2}(x^2 + y^2, \arctan \frac{y}{x})|2| = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, -\infty < x < +\infty, -\infty < y < +\infty.$$

So $(X, Y) \sim N(0, 0, 1, 1, 0)$, is a bivariate normal distribution. X and Y are two independent normal variables.

4. Let $P = X + Y, Q = X/Y$, thus $X = \frac{PQ}{Q+1}, Y = \frac{P}{Q+1}$, we have

$$J = \begin{bmatrix} \frac{q}{q+1} & \frac{p}{(q+1)^2} \\ \frac{1}{q+1} & -\frac{p}{(q+1)^2} \end{bmatrix} = -\frac{p}{(q+1)^2}.$$

Therefore,

$$f_{PQ}(p, q) = f_{XY}(\frac{pq}{q+1}, \frac{p}{q+1})\left|-\frac{p}{(q+1)^2}\right| = \lambda^2 e^{-\lambda(\frac{pq}{q+1} + \frac{p}{q+1})} \frac{p}{(q+1)^2} = \frac{\lambda^2 p}{(q+1)^2} e^{-\lambda p}(p > 0, q > 0),$$

$$f_P(p) = \int_0^{+\infty} \frac{\lambda^2 p}{(q+1)^2} e^{-\lambda p} dq = \lambda^2 p e^{-\lambda p}(p > 0),$$

$$f_Q(q) = \int_0^{+\infty} \frac{\lambda^2 p}{(q+1)^2} e^{-\lambda p} dp = \frac{1}{(q+1)^2}(q > 0).$$

So $f_{PQ}(p, q) = f_P(p) f_Q(q)(p > 0, q > 0)$. When $p \le 0$ or $q \le 0$, $f_P(p) = 0$ or $f_Q(q) = 0$, $f_{PQ}(p, q) = f_P(p) f_Q(q) = 0$. Above all, $f_{PQ}(p, q) = f_P(p) f_Q(q)$, $P = X + Y$ and $Q = X/Y$ are independent.

5. Since X_1 and X_2 are independent standard normal random variables, we have

$$\begin{aligned}
E(Y_1) =& E(a_{11}X_1 + a_{12}X_2 + b_1) = a_{11}E(X_1) + a_{12}E(X_2) + b_1 = b_1, \\
E(Y_2) =& E(a_{21}X_1 + a_{22}X_2 + b_2) = a_{21}E(X_1) + a_{22}E(X_2) + b_2 = b_2, \\
Var(Y_1) =& Var(a_{11}X_1 + a_{12}X_2 + b_1) = a_{11}^2 Var(X_1) + a_{12}^2 Var(X_2) = a_{11}^2 + a_{12}^2, \\
Var(Y_2) =& Var(a_{21}X_1 + a_{22}X_2 + b_2) = a_{21}^2 Var(X_1) + a_{22}^2 Var(X_2) = a_{21}^2 + a_{22}^2, \\
Corr(Y_1, Y_2) =& \frac{Cov(Y_1, Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}} = \frac{E(Y_1Y_2) - E(Y_1)E(Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}} = \frac{a_{11}a_{21} + a_{12}a_{22}}{\sqrt{a_{11}^2 + a_{12}^2}\sqrt{a_{21}^2 + a_{22}^2}}.
\end{aligned}$$

And suppose $a_{11}a_{22} - a_{21}a_{12} \neq 0$, thus,

$$X_1 = \frac{a_{22}y_1 - a_{12}y_2 - a_{22}b_1 + a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}},$$

$$X_2 = \frac{a_{11}y_2 - a_{21}y_1 + a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}.$$

We have

$$J = \begin{bmatrix} \frac{a_{22}}{a_{11}a_{22} - a_{21}a_{12}} & -\frac{a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ -\frac{a_{11}}{a_{21}a_{22} - a_{21}a_{12}} & \frac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}.$$

Therefore,

$$\begin{aligned}
f_{Y_1Y_2}(y_1, y_2) =& f_{X_1X_2}\left(\frac{a_{22}y_1 - a_{12}y_2 - a_{22}b_1 + a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}, \frac{a_{11}y_2 - a_{21}y_1 + a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}\right)|J| \\
=& f_{X_1}\left(\frac{a_{11}y_2 - a_{21}y_1 + a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}\right) f_{X_2}\left(\frac{a_{11}y_2 - a_{21}y_1 + a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}\right)|J| \\
=& \frac{1}{2\pi} e^{-\frac{\left(\frac{a_{22}y_1 - a_{12}y_2 - a_{22}b_1 + a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}\right)^2 + \left(\frac{a_{11}y_2 - a_{21}y_1 + a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}\right)^2}{2}} |J| \\
=& \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y_1 - \mu_{Y_1}}{\sigma_{Y_1}}\right)^2 + \left(\frac{y_2 - \mu_{Y_2}}{\sigma_{Y_2}}\right)^2 - \frac{2\rho(y_1 - \mu_{Y_1})(y_2 - \mu_{Y_2})}{\sigma_X \sigma_Y}\right\}\right.,
\end{aligned}$$

where $\mu_{Y_1} = E(Y_1), \mu_{Y_2} = E(Y_2), \sigma_{Y_1} = \sqrt{Var(Y_1)}, \sigma_{Y_2} = \sqrt{Var(Y_2)}, \rho = Corr(Y_1, Y_2)$. So joint distribution of Y_1, Y_2 is bivariate normal.

6. Since $X_1, X_2, \cdots, X_n$ are i.i.d $U(0, \theta)$, $Y = \max\{X_1, X_2, \cdots, X_n\}, Z = \min\{X_1, X_2, \cdots, X_n\}$, we have

$$f_Y(y) = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n},$$

$$f_Z(z) = n\left(1 - \frac{z}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{n(\theta - z)^{n-1}}{\theta^n}.$$

Thus,

$$E(Y) = \int_0^\theta \frac{ny^{n-1}}{\theta^n} y dy = \frac{n}{n+1}\theta,$$

$$E(Z) = \int_0^\theta \frac{n(\theta - z)^{n-1}}{\theta^n} z dz = \int_0^\theta \frac{np^{n-1}(\theta - p)}{\theta^n} dp (p = \theta - z) = \frac{\theta}{n+1}.$$

7.

(a) By using convolution theorem, we have

$$\begin{aligned} P(X+Y=m) =& \sum_{i=1}^{m-1} P(X=i)P(Y=m-i) \\ =& \sum_{i=1}^{m-1} (1-p)^{i-1} p (1-p)^{m-i-1} p \\ =& \sum_{i=1}^{m-1} (1-p)^{m-2} p^2 \\ =& (m-1)(1-p)^{m-2} p^2 \end{aligned}$$

Thus

$$\begin{aligned} P(X=k|X+Y=m) =& \frac{P(X=k, X+Y=m)}{P(X+Y=m)} \\ =& \frac{P(X=k)P(Y=m-k)}{P(X+Y=m)} \\ =& \frac{(1-p)^{k-1} p (1-p)^{m-k-1} p}{(m-1)(1-p)^{m-2} p^2} \\ =& \frac{1}{m-1} \end{aligned}$$

(b) By using convolution theorem, we have

$$\begin{aligned} P(X+Y=m) =& \sum_{i=0}^{m} P(X=i)P(Y=m-i) \\ =& \sum_{i=0}^{m} C_n^i p^i (1-p)^{n-i} C_n^{m-i} p^{m-i} (1-p)^{n-m+i} \\ =& \sum_{i=0}^{m} C_n^i C_n^{m-i} p^m (1-p)^{2n-m} \\ =& C_{2n}^m p^m (1-p)^{2n-m} \end{aligned}$$

Thus

$$\begin{aligned}
P(X=k|X+Y=m) =& \frac{P(X=k, X+Y=m)}{P(X+Y=m)} \\
=& \frac{P(X=k)P(Y=m-k)}{P(X+Y=m)} \\
=& \frac{C_n^k p^k (1-p)^{n-k} C_n^{m-k} p^{m-k} (1-p)^{n-m+k}}{C_{2n}^m p^m (1-p)^{2n-m}} \\
=& \frac{C_n^k C_n^{m-k}}{C_{2n}^m}
\end{aligned}$$

8. Proof:

(a) We have

$$\begin{aligned}
E(I|X=x) =& P(I=1|X=x) \times 1 + P(I=0|X=x) \times 0 \\
=& P(Y<X|X=x) \\
=& \int_{-\infty}^{x} f(y|x)dy \\
=& \int_{-\infty}^{x} f(y)dy \text{ (X and Y are independent)} \\
=& \Phi(x)
\end{aligned}$$

(b) We have

$$\begin{aligned}
E(\Phi(X)) =& \int_{-\infty}^{+\infty} \Phi(x) f(x) dx \\
=& \int_{-\infty}^{+\infty} \int_{-\infty}^{x} f(y) f(x) dy dx \\
=& P(Y<X)
\end{aligned}$$

(c) Since $X \sim N(\mu, 1), Y \sim N(0, 1)$ and X and Y are independent, $X - Y \sim N(\mu, 2)$, thus,

$$
\begin{aligned}
E(\Phi(X)) =& P(Y < X) \\
=& P(X - Y > 0) \\
=& P(\frac{X - Y - \mu}{\sqrt{2}} > \frac{-\mu}{\sqrt{2}}) \\
=& P(\frac{X - Y - \mu}{\sqrt{2}} < \frac{\mu}{\sqrt{2}}) \\
=& \Phi(\frac{\mu}{\sqrt{2}})
\end{aligned}
$$