Discrete Mathematics and Its Applications

Lecture 3: Counting: Pigeonhole Principle and Binomial Coefficients

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Outline

The sock problem

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Example

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.

Students with the same birthday

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- But that's the worst case scenario, as it is more common to find people with the same birthday. (In the next class, we will try to see if there is a pair of students in the class with the same birthday.)
- So, let's think about the probability that there are two students with the same birthday in a room with 40 students.

A simple case

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- Thus, the probability is $\frac{366}{366^2} = 0.0027$, very unlikely.

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  - Notice that this is the number of ordered subsets.
- Thus, the probability that they do not share birthdays is $\frac{366 \cdot 365 \cdot 364}{366^3} = 0.9918$. Thus the probability that two of them share a birthday is $1 - 0.9918 = 0.0082$.

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General case: n days k people

- Let's continue on the general case. When we have k people and a year contains n days, the probability that no two people share the same birthday is

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- If this number is very close to 0, then it is very unlikely that no two people share the same birthday, i.e., it is very likely that there exists two people with the same birthday.

A few tweaks

- Dealing with small numbers is sometimes troublesome. (The reason will be more apparent later when we start introducing the tools.) So let's consider the reciprocal instead:

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- The top term looks easy to deal with; the bottom one does not. Let's break up the product:

$$\left(\frac{n}{n}\right) \cdot \left(\frac{n}{n-1}\right) \cdot \left(\frac{n}{n-2}\right) \cdots \left(\frac{n}{n-k+1}\right).$$

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- If you look closely at this product, you can see that each term is at least one. In the beginning, the terms are very close to

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$$= \ln\left(\frac{n}{n}\right) + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n}{n-2}\right) + \cdots + \ln\left(\frac{n}{n-k+1}\right).$$

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$$=\ln\left(\frac{n}{n}\right)+\ln\left(\frac{n}{n-1}\right)+\ln\left(\frac{n}{n-2}\right)+\cdots+\ln\left(\frac{n}{n-k+1}\right).$$

- The terms do not look that much better. But there's a nice fact about the natural logarithms.

$\ln x$: the upper bound

Fact:

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So let's do that at Wolfram Alpha.

$\ln x$: the lower bound

We know that

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If we use the fact that $\ln \frac{1}{x} = -\ln x$, we can obtain the lower bound.

$$\ln x = -\ln \frac{1}{x} \geq -\left(\frac{1}{x} - 1\right) = \frac{x-1}{x}.$$

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Let's conclude by stating the lemma:

Lemma

$$\frac{x-1}{x} \leq \ln x \leq x - 1.$$

The lower bound

Let's look at each term in the sum: $\ln\left(\frac{n}{n-j}\right)$. Using the lower bound in Lemma 1, we get that

$$\ln\left(\frac{n}{n-j}\right) \geq \frac{\frac{n}{n-j} - 1}{\frac{n}{n-j}} = \frac{\frac{n-n+j}{n-j}}{\frac{n}{n-j}} = \frac{j}{n}.$$

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Thus,

$$\begin{aligned}
&\ln\left(\left(\frac{n}{n}\right)\cdot\left(\frac{n}{n-1}\right)\cdot\left(\frac{n}{n-2}\right)\cdots\left(\frac{n}{n-k+1}\right)\right)\\
&= \ln\left(\frac{n}{n}\right)+\ln\left(\frac{n}{n-1}\right)+\ln\left(\frac{n}{n-2}\right)+\cdots+\ln\left(\frac{n}{n-k+1}\right)\\
&\geq \frac{0}{n}+\frac{1}{n}+\frac{2}{n}+\cdots+\frac{k-1}{n}\\
&= \frac{1}{n}\left(1+2+\cdots+(k-1)\right) = \frac{k(k-1)}{2n}.
\end{aligned}$$

The upper bound

Again, let's look at each term in the sum: $\ln\left(\frac{n}{n-j}\right)$. Using the upper bound in Lemma 1, we get that

$$\ln\left(\frac{n}{n-j}\right) \le \frac{n}{n-j} - 1 = \frac{j}{n-j}.$$

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$$\begin{aligned}
&\le \frac{0}{n-0} + \frac{1}{n-1} + \frac{2}{n-2} + \cdots + \frac{k-1}{n-k+1} \\
&\le \frac{0}{n-k+1} + \frac{1}{n-k+1} + \frac{2}{n-k+1} + \cdots + \frac{k-1}{n-k+1} \\
&= \frac{1}{n-k+1}\left(1+2+\cdots+(k-1)\right) = \frac{k(k-1)}{2(n-k+1)}.
\end{aligned}$$

Both

Using the derived upper and lower bounds, we get

$$e^{\frac{k(k-1)}{2n}} \leq \frac{n^k}{n(n-1)(n-2)\cdots(n-k+1)} \leq e^{\frac{k(k-1)}{2(n-k+1)}}$$

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So the probability that we get no two people with the same birthday is between $1/8.42 \approx 0.118$ and $1/10.86 \approx 0.092$. So we have high chance of finding two students with the same birthday.

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Theorem: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Proof.

Suppose that none of the boxes contains more than $\lceil \frac{N}{k} \rceil - 1$ objects.

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$$k(\lceil \frac{N}{k} \rceil - 1) < k((\frac{N}{k} + 1) - 1) = N,$$

where the inequality $\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1$ has been used.

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This is a contradiction because there are a total of N objects. □

Some elegant applications of the Pigeonhole principle I

Question: During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution

Let a_j be the number of games played on or before the $j-$th day of the month. Then $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{30} \leq 45$.

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Moreover, $15 \leq a_i + 14 \leq 59$ is also an increasing sequence of distinct positive integers. The 60 positive integers have
$1 \leq a_1, a_2, \cdots, a_{30}, a_1 + 14, a_2 + 14, \cdots, a_{30} + 14 \leq 59$.

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Moreover, $15 \le a_i + 14 \le 59$ is also an increasing sequence of distinct positive integers. The 60 positive integers have
$1 \le a_1, a_2, \cdots, a_{30}, a_1 + 14, a_2 + 14, \cdots, a_{30} + 14 \le 59$.
Hence, by the pigeonhole principle two of these integers are equal. Because a_j are all distinct and $a_j + 14$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i.

Some elegant applications of the Pigeonhole principle II

Question: Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution

Write each of the $n+1$ integers $a_1, a_2, \cdots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \cdots, n+1$, where k_j is a nonnegative integer and q_j is odd. Integers $q_1, q_2, \cdots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of integers $q_1, q_2, \cdots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i.

Ramsey theory

Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is increasing or decreasing strictly.
Proof: Let a_i be a sequence of n^2+1 distinct real numbers. Associate (i_k, d_k) to term a_k, where i_k and d_k are the lengths of the longest increasing and longest decreasing subsequences starting at a_k, respectively.

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The binomial coefficients

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem, and
- advanced counting with binomial coefficients.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x+y$
- $(x+y)^2 = x^2 + 2\cdot xy + y^2$
- $(x+y)^3 = x^3 + 3\cdot x^2y + 3\cdot xy^2 + y^3$
- $(x+y)^4 = x^4 + 4\cdot x^3y + 6\cdot x^2y^2 + 4\cdot xy^3 + y^4$.

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- $(x+y)^4 = x^4 + 4\cdot x^3y + 6\cdot x^2y^2 + 4\cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?*
Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x+y)^4$ again. It is

$$(x+y)(x+y)(x+y)(x+y).$$

- How do we get x^4 in the expansion?

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- How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors).

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- How do we get x^4 in the expansion? For every factory, you have to pick x.
- How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3}=\binom{4}{1}$ ways to do so.

The binomial theorem

Theorem

If you expand $(x+y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \cdots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients.

Corollary 1

Let n be a nonnegative integer, then

$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n[\text{Hint: } (1+1)^n=2^n].$$

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Corollary 1

Let n be a nonnegative integer, then

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n [\text{Hint: } (1+1)^n = 2^n].$$

Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.

Corollary

Corollary 2

Let n be a nonnegative integer. Then

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.[\text{Hint: } (1-1)^n = 0]$$

Corollary

Corollary 2

Let n be a nonnegative integer. Then

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.[\text{Hint: } (1-1)^n = 0]$$

Corollary 3

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n [\text{Hint: } (1+2)^n = 3^n].$$

Triangle binomial coefficient

$$
\begin{array}{cccccc}
\binom{0}{0} & & & & & \\
\binom{1}{0} & \binom{1}{1} & & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
$$

$$
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
$$

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.

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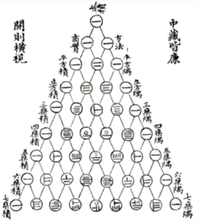
$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & \vdots & & \vdots & & & & & & & & \vdots &
\end{array}
$$

The table and the binomial coefficients have many other interesting properties.

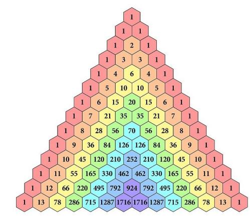
Pascal's triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

Yanghui's triangle



Pascal's triangle



Pascal's triangle was known in the early 11th century through the work of Chinese mathematicians Jia Xian (1010-1070) and Yang Hui (1238-1298).

Next observation I

Pascal's identity

Theorem: Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

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Pascal's identity

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$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Vandenmonde's identity

Theorem: Let m, n and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Next observation II

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$1^2 = 1$$

Next observation II

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2
\end{aligned}
$$

Next observation II

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2 \\
1^2 + 2^2 + 1^2 &= 6
\end{aligned}
$$

Next observation II

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2 \\
1^2 + 2^2 + 1^2 &= 6 \\
1^2 + 3^2 + 3^2 + 1^2 &= 20
\end{aligned}
$$

Next observation II

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2 \\
1^2 + 2^2 + 1^2 &= 6 \\
1^2 + 3^2 + 3^2 + 1^2 &= 20 \\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70
\end{aligned}
$$

Theorem: Let n be a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

□

Corollary 4: Let n and r be a nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}.$$

Proof.

Let $\binom{n+1}{r+1}$ counts the bit strings of length $n+1$ containing $r+1$ ones.

Ways	**Location of last 1**	**Counting**

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$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}.$$

Proof.

Let $\binom{n+1}{r+1}$ counts the bit strings of length $n+1$ containing $r+1$ ones.

Ways	**Location of last 1**	**Counting**
Way 1:	$r+1$	$\binom{r}{r}$
Way 2:	$r+2$	$\binom{r+1}{r}$
Way 3:	$r+3$	$\binom{r+2}{r}$
$\cdots$		$\cdots$
Way $n-r+1$:	$n+1$	$\binom{n}{r}$

□

Take-aways

Conclusions

- Pigeonhole Principle
- Binomial Coefficient
- Pascal's Triangle