Discrete Mathematics and Its Applications

Lecture 3: Counting Principles

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Oct. 11, 2018

Outline

Let's count

How many lunches can you have?

A snack bar serves five different sandwiches and three different beverages. How many different lunches can a person order?

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Solution

1. *One way of determining the number of possible lunches is by listing or enumerating all the possibilities;*
2. *One systematic way of doing this is by means of a tree;*
3. *Counting elements in a cartesian product. A listing of possible lunches a person could have is:*

$$A \times B = \{(Beef; milk), (Beef; juice), \cdots, (Bologna; coffee)\},$$

where $A = \{beef, ham, chicken, cheese, bologna\}$, *and* $B = \{milk, juice, coffee\}$.

Counting principle

Product rule

Suppose that a procedure consists of a sequence of two tasks. If there are n_1 ways to do the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

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Extended version: A procedure is followed by tasks $T_1, T_2, \cdots, T_m$ in sequence. If each task T_i can be done in n_i ways independently, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the procedure.

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Sum rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

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Extended version: A procedure can be done by m ways, each way W_i has n_i possibilities (not intersect), then there are $n_1 + n_2 + \cdots + n_m$ ways to carry out the procedure.

How many lunches can you have?

Solution

1. *Product rule:*

Task	Number

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Task	**Number**
Task 1: Choose sandwich	*5*
Task 2: Choose beverage	*3*

Therefore, there are $5 \times 3 = 15$ *ways to order lunches.*

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Way of first order	Number

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Task 1: Choose sandwich	*5*
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Therefore, there are $5 \times 3 = 15$ *ways to order lunches.*

2. *Sum rule:*

Way of first order	Number
Way 1: beef	*3*
Way 2: ham	*3*
Way 3: chicken	*3*
Way 4: cheese	*3*
Way 5: beef	*3*

Therefore, there are $3 + 3 + 3 + 3 + 3 = 15$ *ways to order lunches.*

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In terms of the product rule:

Task	**Number**
Task 1: $a_1 \in A$	n
Task 2: $a_2 \in A$	n
	$\dots$
Task m: $a_m \in A$	n

Therefore, there are $\underbrace{n \cdot n \cdot \dots \cdot n}_{m\ times} = n^m$ functions.

Application of counting functions

1. A is a finite set, then $|P(A)| = 2^{|A|}$.
2. How many different bit strings of length seven are there? How many bit strings of length seven both begin and end with a 1?
3. A person is to complete a true-false questionnaire consisting of ten questions. How many different ways are there to answer the questionnaire?
4. A questionnaire contains four questions that have two possible answers and three questions with five possible answers. How many different answers are there?
5. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

Examples

Solution

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Task	**Number**	
Way 1:	contain 1 x	4×25^3
	Task 11: choose location of x	4
	Task 12: fill the remaining location	25^3

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Way 1:	contain 1 x	4×25^3
	Task 11: choose location of x	4
	Task 12: fill the remaining location	25^3
Way 2:	contain 2 x	6×25^2
	Task 21: choose locations of x	6
	Task 22: fill the remaining location	25^2

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Way 2:	contain 2 x	6×25^2
	Task 21: choose locations of x	6
	Task 22: fill the remaining location	25^2
Way 3:	contain 3 x	4×25
	Task 31: choose locations of x	4
	Task 32: fill the remaining location	25

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Way 1:	contain 1 x	4×25^3
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Way 2:	contain 2 x	6×25^2
	Task 21: choose locations of x	6
	Task 22: fill the remaining location	25^2
Way 3:	contain 3 x	4×25
	Task 31: choose locations of x	4
	Task 32: fill the remaining location	25
Way 4:	contain 4 x	1

Therefore, there are $4 \times 25^3 + 6 \times 25^2 + 4 \times 25 + 1$ functions.

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How many one-to-one functions are there from a set with m elements to one with n elements?

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Task	**Number**
Task 1: $a_1 \in A$	n
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	$\cdots$
Task m: $a_m \in A$	$n-m+1$

Therefore, there are $\underbrace{n \cdot (n-1) \cdot \cdots \cdot (n+m-1)}_{m\ \ times}$ functions.

Permutations

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Examples

1. In how many ways can we select three students from a group of five students to stand in line for a picture?
2. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
3. A saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

The number of permutations

Theorem

If n is a positive integer and m is an integer with $1 \leq m \leq n$, then there are

$$P(n, m) = n(n-1)(n-2) \cdot \cdots \cdot (n-m+1),$$

m-permutations of a set with n distinct elements.

Proof.

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Therefore, there are $\underbrace{n\cdot(n-1)\cdot\dots\cdot(n+m-1)}_{m\ times}$ functions. $\square$

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We have proved this corollary.

Corollary: The number of permutations of a set with n elements is $n!$.

Remark: $0! = 1$, and $1! = 1$.

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Abbreviations: We shall call a set with n elements as an n-**set**. We shall call a subset with k elements as a k-**subset**. In general, elements in a given set is unordered. I.e., sets $\{1,2,3\}$ and $\{3,1,2\}$ are the same set.

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However, sometimes, it is useful to treat sets as ordered.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

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- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.

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- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- Therefore, we conclude that the number of ways is $10 \cdot 9 \cdot 8$.

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We can arrive at the same answer by a different way of counting.

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  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are 7! of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.

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$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

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A group of committees

For student 1, 2, and 3. Ordered sets

$$\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 1, 2\}$$

are the same group of committees. Remember that the number of the ordered sets is .

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The number of different groups of committees should be

$$\frac{P(10,3)}{P(3,3)}.$$

Combinations

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Theorem: The number of m-combinations of a set with n elements, where n is a nonnegative integer and m is an integer with $0 \leq m \leq n$, equals

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where $C(n, m)$ is also denoted as $\binom{n}{m}$.

Corollary: The number of m-subsets of an n-set is

$$C(n, m) = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-m+1)}{m!} = \frac{n!}{(n-m)!m!}.$$

Corollary proof

Proof.

Consider the following process for choosing an ordered subsets with k elements of an n-set.

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Consider the following process for choosing an ordered subsets with k elements of an n-set. First, we choose a k-subset, then we permute it. Let B be the number of k-subsets. For each subset that we choose in the first step, the second step has $k!$ choices.

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$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of k-subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required. □

Examples of combinations

Applications

1. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
2. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
3. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?
4. How many bit strings of length n contain exactly r 1s?

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

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- $\binom{n}{n} = 1$ (why?),
- $\binom{n}{0} = 1$ (why?), and,
- when $k > n$, $\binom{n}{k} = 0$.

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How big is 100! ?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some conditions).
- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.
  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.) How about 2^n? Is it a lower bound? How about 3^n or 5^n? Are they lower bounds of $n!$?

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

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Are they any good?

n	2^{n-1}	$n!$	n^{n-1}
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	$3,628,800$	$1,000,000,000$

A better bound?

Let's consider $n!$ again, but for simplicity, let's consider only the case when n is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2-1) \cdot (n/2) \cdot (n/2+1) \cdots n$$

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To get a better lower bound, we may move our cutting point from 2 to, say, $n/2$. Note that at least $n/2$ factors are at least $n/2$. Thus,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots n \\ &\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2} \\ &= (n/2)^{n/2} = \sqrt{(n/2)^n}. \end{aligned}$$

Better?

n	2^{n-1}	$\sqrt{(n/2)^n}$	$n!$	n^{n-1}
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	7,776
10	512	3,125	3,628,800	1,000,000,000
12	2,048	46,656	479,001,600	743,008,370,688

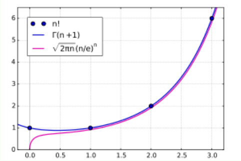
OK. A bit better.

Stirling's formula

Theorem

Theorem (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$



When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$.

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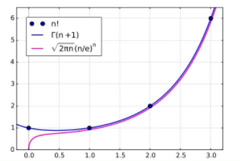

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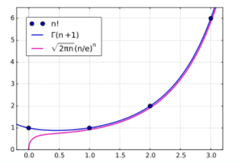

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$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100\log(100/e) + \log(200\pi) \approx 157.9696.$$

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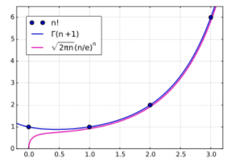

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Note that the correct answer is 158 digits.

Subtraction rule

Subtraction rule

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways. The rule is also called the principle of inclusion-exclusion, i.e.,

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Application

Question: How many bit strings of length eight either start with a 1 bit or end with two bits 00?

Solution: Set A is bit strings of length eight start with a 1 bit;
Set B is bit strings of length eight end with two bits 00;
Therefore, $|A \cup B| = |A| + |B| - |A \cap B| = 180$, where $|A| = 2^7$, $|B| = 2^6$, and $|A \cap B| = 2^5$.

Quick questions (1)

There are 40 students in the classroom. There are 35 students who like Naruto, 10 students who like Bleach, and 7 students who like both of them. How many students in this classroom who do not like either Bleach or Naruto?

Quick questions (2)

There are 35 students in the classroom. There are 25 students who like Naruto, 15 students who like Bleach, 12 students who like One Piece. There are 10 students who like both Naruto and Bleach, 7 students who like both Bleach and One Piece, and 9 students who like both Naruto and One Piece. There are 5 students who like all of them.
How many students in this classroom who do not like any of Bleach, Naruto, or One Piece?

Is this correct?

The answer from the previous quick question is

$$35 - (25 + 15 + 12 - 10 - 7 - 9 + 5) = 4.$$

Is this correct? Why?

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Is this correct? Why?

Let's try to argue that this answer is, in fact, correct and try to find general answers to this kind of counting questions.

Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O									

Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
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Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-									

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Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-	*								
⋮	⋮									

Let's look at an individual student (2)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O	1	-1		-1			$+1$		0
Bobby	B	1		-1						0
Cathy	B,O	1		-1	-1		$+1$			0
Dave	N,B,O	1	-1	-1	-1	$+1$	$+1$	$+1$	-1	0
Eddy	-	1								1
$\vdots$	$\vdots$									

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Do you see any patterns here?

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Do you see any patterns here? How about

$$1-\binom{5}{1}+\binom{5}{2}-\binom{5}{3}+\binom{5}{4}-\binom{5}{5}\quad ?$$

Underlying structures

Let's write 1 as $\binom{5}{0}$. Also, let's separate plus terms and minus terms:

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} \quad \heartsuit \quad \binom{5}{1} + \binom{5}{3} + \binom{5}{5}$$

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Note that the left terms are the number of even subsets and the right terms are the number of odd subsets.

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Theorem: Let A_i be one of n sets, then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} |A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|.$$

Example (1)

Question: How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A and B be the sets of positive integers not exceeding 1000 that are divisible by 7 and 11, respectively.

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| && (1)\\
&= \lfloor \frac{1000}{7} \rfloor + \lfloor \frac{1000}{11} \rfloor - \lfloor \frac{1000}{7 \times 11} \rfloor && (2)\\
&= 142 + 90 - 12 = 220. && (3)
\end{aligned}
$$

Onto functions

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$$\begin{aligned}
|\overline{P_1 \cup P_2 \cup P_3}| &= N - [|P_1| + |P_2| + |P_3|] \\
&+ [|P_1 \cap P_2| + |P_2 \cap P_3| + |P_1 \cap P_3|] - |P_1 \cap P_2 \cap P_3| \\
&= 3^6 - C(3,1)2^6 + C(3,2)1^6 = 729 - 192 + 3 = 540.
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Theorem: Let $m, n \in \mathcal{Z}^+$ $m \geq n$. Then, there are

$$\sum_{k=0}^{n-1} (-1)^k C(n,k)(n-k)^m = n^m - C(n,1)(n-1)^m + \cdots + (-1)^{n-1} C(n, n-1) 1^m$$

onto functions from a set with m elements to a set with n elements.

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Question: A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. How many cases does no one receive the correct hat?

Theorem of derangement

Theorem: The number of derangements of a set with n elements is

$$D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}].$$

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$$\begin{aligned} D_n &= |\overline{P_1 \cup P_2 \cup P_3 \cup \cdots \cup P_n}| = N - \sum_i |P_i| \\ &+ \sum_{i<j} |P_i \cap P_j| - \sum_{i<j<k} |P_i \cap P_j \cap P_k| + \cdots + (-1)^n |P_1 \cap \cdots \cap P_n| \\ &= n! + \sum_{k=1}^{n} (-1)^k C(n,k)(n-k)! = n! + \sum_{k=1}^{n} (-1)^k \frac{n!}{k!(n-k)!}(n-k)! \end{aligned}$$

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Question: A new employee checks the hats of 5 people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. How many cases does at least one receive the correct hat?

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Task	**Number**		
Way 1:	derangement of 4 hats	$C(5,1)D_4$	45
Way 2:	derangement of 3 hats	$C(5,2)D_3$	20
Way 3:	derangement of 2 hats	$C(5,3)D_2$	10
Way 4:	derangement of 1 hats	$C(5,4)D_1$	0
Way 5:	derangement of 0 hats	$C(5,5)D_0$	1

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Therefore, the total number of cases is 76. (Note that the case of D_1 is impossible, i.e,. $D_1 = 0$.)

Division rule

Subtraction rule

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Recap

Question: How many different committees of three students can be formed from a group of ten students?

Solution: We obtain $P(10,3)$ ordered 3$-$subsets. However, for a given ordered 3$-$subsets (e.g., $\{1,2,3\}$), there are $P(3,3)$ replicates for the group of committee members $\{1,2,3\}$.

Therefore, total number of groups of committees is $\frac{P(10,3)}{P(3,3)} = \binom{10}{3}$.

Example

The problem of seating around a circular table

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However, each of the four choices for seat 1 leads to the same arrangement in a circle, as we distinguish two arrangements only when a person has a different immediate left or immediate right neighbor.

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However, each of the four choices for seat 1 leads to the same arrangement in a circle, as we distinguish two arrangements only when a person has a different immediate left or immediate right neighbor.

By the division rule, there are $24/4 = 6$ different seating arrangements of four people around the circular table.

Example

The coefficient of a polynomial

Question: What is the coefficient of x^2 in polynomial $(x^2+3x+1)^5$?

Example

The coefficient of a polynomial

Question: What is the coefficient of x^2 in polynomial $(x^2+3x+1)^5$?

Solution:

Way	To do	Coefficient

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Solution:

Way	**To do**	**Coefficient**
Way 1:	a x^2 and four 1	$C(5,1)C(4,4)x^2 \cdot 1^4$
	Task 11: select a x^2	$C(5,1)x^2$
	Task 12: select four 1	$C(4,4)1^4$
Way 2:	two x and three 1	$C(5,2)C(3,3)(3x)^2 \cdot 1^3$
	Task 21: select two x^2	$C(5,2)(3x)^2$
	Task 22: select three 1	$C(3,3)1^3$

Therefore, the coefficient of x^2 in the polynomial is 95.

Combinations with repetition

The coefficient of a polynomial

Question: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

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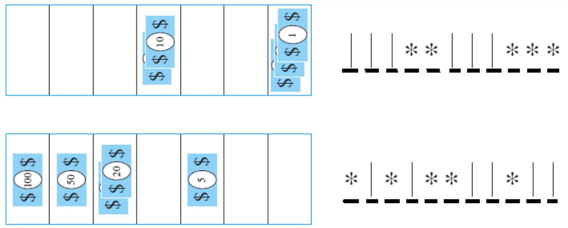
Solution:



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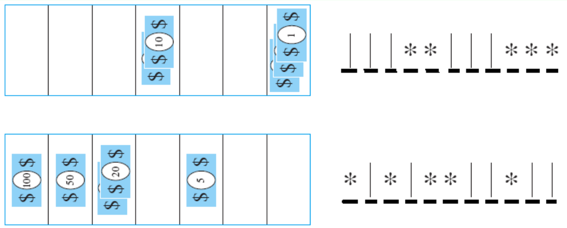

Therefore, it corresponds to the number of unordered selections of 5 objects from a set of 11 objects, which can be done in $C(11,5)$ ways.

Generalization

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

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Examples

- Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
- How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers? $(C(11+3-1, 11) = C(11+3-1, 3-1))$

Easy anagrams

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  - Since each anagram is counted in 4! twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

General anagrams

Let's try to use the same approach to count the anagram of *HELLOWORLD*. (It has 3 L's, 2 O's, H, E, W, R, and D.)

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The number of permutation of alphabets in *HELLOWORLD*, treating each character differently is 10!. However, each anagram is counted for 3!2! times because of the 3 copies of L and the 2 copies of O. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

Generalization

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,$\cdots$, and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing presents

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- To see how many times each distribution is counted in the 9! ways, we can let children form a line and let each child permute his or her presents. Each child has 3! choices. Thus, one distribution appears 3!3!3! times.

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- To see how many times each distribution is counted in the 9! ways, we can let children form a line and let each child permute his or her presents. Each child has 3! choices. Thus, one distribution appears 3!3!3! times.
- Thus, the number of ways we can distribute presents is

$$\frac{9!}{3!3!3!}$$

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.

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- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

Presents	1	2	3	4	5	6	7	8	9
Children	A	B	A	B	C	B	C	A	C

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- This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, we reach the same solution of

$$\frac{9!}{3!3!3!}.$$

Generalization

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \cdots, k$, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Indistinguishable objects and distinguishable boxes

How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. Consequently, there are

$$C(8+10-1,10)=C(17,10).$$

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Remark: This means that there are $C(n+r-1,n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.

Distributing coins

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it, given that some student may not get any coins?

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins.

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- Since each coin is identical, we can let the first student picks the coin from the beginning of the line. Then the second one pick the next set of coins, and so on.

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This is a fairly surprising use of binomial coefficients.

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Let's consider a general problem where we have n identical coins to give out to k students so that each student gets at least one coin. In how many ways can we do that?

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> There are $\binom{n-1}{k-1}$ ways to distribute n identical coins to k children so that each child get at least one coin.

Distinguishable objects and indistinguishable boxes

How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

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Solution:

Way	**Counting**
Way 1: $oooo$	$C(4,4)$
Way 2: $ooo\|o$	$C(4,3)$
Way 3: $oo\|oo$	$C(4,2)/P(2,2)$
Way 4: $oo\|o\|o$	$C(4,2)/P(2,2)$

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Therefore, we find that there are 14 ways to put four different employees into three indistinguishable offices.

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- Considering boxes are distinguishable;
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  - There are $\sum_{k=0}^{j-1}(-1)^k C(j,k)(j-k)^n$ onto functions from a set with m elements to a set with n elements.

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- Hence, $S(n,j) = \frac{1}{j!}\sum_{k=0}^{j-1}(-1)^k C(j,k)(j-k)^n$.

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- Hence, $S(n,j) = \frac{1}{j!}\sum_{k=0}^{j-1}(-1)^k C(j,k)(j-k)^n$.

Therefore, there are $\sum_{j=1}^{m} S(n,j)$ ways to distribute n distinguishable objects into m indistinguishable boxes.

Indistinguishable objects and indistinguishable boxes

How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

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Solution:

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Way 3: $oooo\|oo$	1
Way 4: $oooo\|o\|o$	1
Way 5: $ooo\|ooo$	1
Way 6: $ooo\|oo\|o$	1
Way 7: $ooo\|o\|o\|o$	1
Way 8: $oo\|oo\|oo$	1
Way 9: $oo\|oo\|o\|o$	1

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Way 8: $oo|oo|oo$	1
Way 9: $oo|oo|o|o$	1

We conclude that there are nine allowable ways to pack the books, because we have listed them all.

Example

The problem of seating around a circular table

Question: How many permutations of number 1, 2, 3, 4, 5, 6, 7, and 8 contain

- number string 234?
- number strings 23 and 45?
- number strings 234 and 456?

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Solution:

- $P(6,6)$;
- $P(6,6)$;
- $P(4,4)$.

Example

The problem of grouping

Question: How many cases for grouping eight books into groups?

- each group has four books?
- each group has two books?
- the numbers of books in the four groups are 1, 2, 2, 3?
- three persons take 2, 2, and 4 books?

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Solution:

- $\frac{C(8,4)C(4,4)}{P(2,2)}$;
- $\frac{C(8,2)C(6,2)C(4,2)C(2,2)}{P(4,4)}$;
- $\frac{C(8,1)C(7,2)C(5,2)C(3,3)}{P(2,2)}$;
- $\frac{C(8,2)C(6,2)C(4,4)}{P(2,2)} \cdot P(3,3)$;

Generating permutations

Example: Permutation 23415 of set $\{1, 2, 3, 4, 5\}$ precedes the permutation 23514. Similarly, permutation 41532 precedes 52143.

Question: What is the next permutation in lexicographic order after 32541?

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Demonstration

Step	Result

procedure *next permutation*($a_1 a_2 \dots a_n$: permutation of $\{1, 2, \dots, n\}$ not equal to $n\ n-1\ \dots\ 2\ 1$)

$j := n - 1$

while $a_j > a_{j+1}$

  $j := j - 1$

$\{j$ is the largest subscript with $a_j < a_{j+1}\}$

$k := n$

while $a_j > a_k$

  $k := k - 1$

$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$

interchange a_j and a_k

$r := n$

$s := j + 1$

while $r > s$

  interchange a_r and a_s

  $r := r - 1$

  $s := s + 1$

{this puts the tail end of the permutation after the jth position in increasing order}

$\{a_1 a_2 \dots a_n$ is now the next permutation$\}$

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32541	$3\underline{25}41$
$3\underline{25}41$	$3\underline{2}5\underline{4}1$
$3\underline{2}5\underline{4}1$	$3\underline{4}5\underline{2}1$
$34\underline{5}2\underline{1}$	$34\underline{1}2\underline{5}$

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Generating the next larger bit string

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Algorithm:

procedure *next bit string*($b_{n-1}\ b_{n-2} \dots b_1 b_0$: bit string not equal to $11\dots11$)
$i := 0$
while $b_i = 1$
  $b_i := 0$
  $i := i + 1$
$b_i := 1$
$\{b_{n-1}\ b_{n-2} \dots b_1 b_0$ is now the next bit string$\}$

Demonstration:

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Demonstration:

Step	Result
100010011$\underline{1}$	1000100110
10001001$\underline{1}$0	1000100100
1000100$\underline{1}$00	1000100000
100010$\underline{0}$000	100010$\underline{1}$000

Generating the next r-combination in lexicographic order.

Question: Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

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procedure *next r-combination*($\{a_1, a_2, \ldots, a_r\}$: proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{n-r+1, \ldots, n\}$ with $a_1 < a_2 < \cdots < a_r$)

$i := r$

while $a_i = n - r + i$

  $i := i - 1$

$a_i := a_i + 1$

for $j := i + 1$ **to** r

  $a_j := a_i + j - i$

$\{\{a_1, a_2, \ldots, a_r\}$ is now the next combination$\}$

Demonstration: (where $r = 4$ and $n = 6$)

Step	Result

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{$\{a_1, a_2, \ldots, a_r\}$ is now the next combination}

Demonstration: (where $r = 4$ and $n = 6$)

Step	**Result**
$\{1, \underline{2}, 5, 6\}$	$\{1, \underline{3}, 5, 6\}$
$\{1, 3, \underline{5}, 6\}$	$\{1, 3, \underline{4}, 6\}$
$\{1, 3, 4, \underline{6}\}$	$\{1, 3, 4, \underline{5}\}$

Take-aways

Conclusions

- Counting principles
  - Product rule
  - Sum rule
  - Subtraction rule
  - Division rule
- Permutations
- Combinations