Discrete Mathematics and Its Applications

Lecture 2: Basic Structures: Set Theory

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Outline

Set

Definition

Set A is a collection of objects (or elements).

- $a \in A$: "a is an element of A" or "a is a member of A";
- $a \notin A$: "a is not an element of A";
- $A = \{a_1, a_2, \cdots, a_n\}$: A contains a_1, $a_2, \cdots, a_n$;
- Order of elements is meaningless;
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Set equality

Sets A and B are equal if and only if they contain exactly the same elements.

- If $A = \{9, 2, 7, -3\}$ and $B = \{7, 9, 2, -3\}$, then $A = B$.
- If $A = \{9, 2, 7\}$ and $B = \{7, 9, 2, -3\}$, then $A \neq B$.;
- If $A = \{9, 2, -3, 9, 7, -3\}$ and $B = \{7, 9, 2, -3\}$, then $A = B$.

Applications

Examples

- Bag of words model: documents, reviews, tweets, news, etc;
- Transactions: shopping list, app downloading, book reading, video watching, music listening, etc;
- Records in a DB, data item in a data streaming, etc;
- Neighbors of a vertex in a graph;

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"Standard" sets

- Natural numbers: $N = \{0, 1, 2, 3, \cdots\}$
- Integers: $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
- Positive integers: $Z^+ = \{1, 2, 3, 4, \cdots\}$
- Real Numbers: $R = \{47.3, -12, -0.3, \cdots\}$
- Rational Numbers: $Q = \{1.5, 2.6, -3.8, 15, \cdots\}$

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Tabular form

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Set builder form

- $Q = \{a/b : a \in Z \wedge b \in Z \wedge b \neq 0\}$;
- $B = \{y : P(y)\}$, where $P(Y) : y \in E \wedge 0 < y \leq 50$;

Representation of sets

Remarks

- $A = \emptyset$: empty set, or null set;
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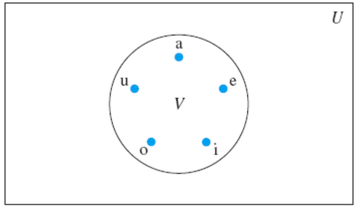
Venn diagrams



In general, a universal set is represented by a rectangle.

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- Two useful rules: (1) $A = B \Leftrightarrow (A \subseteq B) \land (B \subseteq A)$;
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- Given a set S, the power set of S is the set of all subsets of S, denoted as $\mathcal{P}(S)$. The size of $2^{|S|}$, where $|S|$ is the size of S.

Cartesian product

Definition

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Set operations

Operators

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- Union: $A \cup B = \{x : x \in A \vee x \in B\}$;
- Intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$;
- Difference: $A - B = \{x : x \in A \wedge x \notin B\}$ (sometimes denoted as $A \setminus B$);
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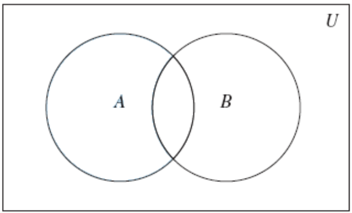

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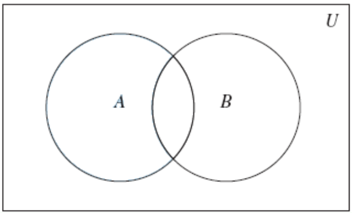

- $A - B = A \cap \overline{B}$;
- $|A \cup B| = |A| + |B| - |A \cap B|$.

Set identities

Table of logical equivalence

equivalence	name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cap A = A$ $A \cup A = A$	Idempotent laws
$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$	Associative laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cap \overline{A} = \emptyset$ $A \cup \overline{A} = U$	Complement laws

Logical equivalence Cont'd

Table of logical equivalence

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$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cap B = B \cap A$ $A \cup B = B \cup A$	Commutative laws
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$	Distributive laws
$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$	De Morgan's laws

Proof of De Morgan law

Proof

Use set builder notation and logical equivalences to establish the first De Morgan law

Steps	**Reasons**
$\overline{A \cap B}$	primise

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$= \overline{A} \cup \overline{B}$	meaning of set builder notation

Generalized unions and intersections

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- $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$;
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Intersection

- $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$;
- $A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i$;

Set covering problem

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Universal set $U = \{u_1, u_2, \cdots, u_n\}$
Subsets $S_1, S_2, \cdots, S_m \subseteq U$
Cost $c_1, c_2, \cdots, c_m$

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Applications

- Document summarization
- Natural language generation
- Information cascade

Take-aways

- Set Concepts;
- Set Operations;
- Applications.