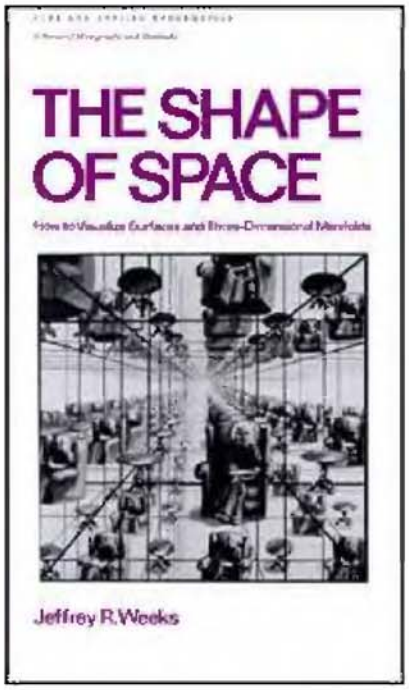

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The Shape of Space

How to Visualize Surfaces and Three-Dimensional Manifolds

Jeffrey R. Weeks

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Ithaca, New York

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PREFACE

Möbius strips and Klein bottles first caught my interest when I was in high school. I knew they were part of something called "topology" and I was eager to learn more. Sadly, neither the school library nor the public one had much on the subject. Perhaps, I thought, I will learn more about topology in college. In college I couldn't even sign up for topology until my senior year, and even then all I got was one course in extreme generalization (point-set topology) and another which developed a collection of technical tools (algebraic topology). Topology's most beautiful examples got bypassed completely.

The Shape of Space fills the gap between the simplest examples, such as the Möbius strip and the Klein bottle, and the sophisticated mathematics found in upper-level college courses. It is intended for a wide audience: I wrote it mainly for the interested nonmathematician (perhaps a high school student who has heard of

Möbius strips and wants to learn more), but it also provides the intuitive examples which are currently missing from the college and graduate school curriculum.

I still haven't told you what the book is about, or even explained the title. For that I refer you to Chapter 1.

JEFFREY R. WEEKS

ACKNOWLEDGMENTS

This book never would have gotten anywhere without the help of the people who read the various drafts and suggested improvements. Particularly useful were the comments of Beth Christie, Bill Dunbar, Charles Harris, Geoff Klineberg, Skona Libowitz, Robert Lupton, Marie McAllister, Thea Pignataro, Evan Romer, Bruce Solomon, Harry Voorhees, Eric White, and an anonymous reviewer of the first draft. I would also like to thank the Princeton University math department for providing computer resources, George Francis for giving me drawing lessons (both in person and by mail), Nadia Marano and Craig Shaw for serving as art critics, Rudy Rucker for suggesting the title, Bill Thurston for inventing the mathematics described in Chapter 18, and my geometry students at Stockton State College for being the guinea pigs.

Natalia Panczyszyn Kozlowski painted the cover picture. The painting shows the view in a three-torus. Chapter 2 introduces the three-torus, and Chapter 7 explains the strange visual effects. The three-torus is a possible shape for the universe, one in which space is finite yet has no edges.

Figures 3.3, 9.8, 10.4 and 14.6 are from ''The Mathematics of Three-dimensional Manifolds" by W. Thurston and J. Weeks (*Scientific American,* July 1984), copyright Ó 1985 by Scientific American, Inc., all rights reserved. Figures 4.9 and 5.6 are from Hilbert and Cohn-Vossen's *Anschauliche Geometrie* (English translation published by Chelsea, New York, 1952). Figure 13.8 is from Rudy Rucker's *Geometry, Relativity and the Fourth Dimension* (Dover, New York, 1977). Bill Thurston generously provided Figure 18.4. Richard Bassein did Figures 5.7, 5.8, 6.9, 6.10, 7.13, 8.2, 11.1, 11.4, 13.7, 15.1, 17.6, 18.1 and 18.2; John Heath did 2.4 through 2.8, and 4.8; Nadia Marano did 6.3; M.C. Escher did 8.1; and I did the rest.

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Part I

Surfaces and Three-Manifolds

1

Flatland

In 1884 an extraordinary individual named A Square succeeded in publishing his memoirs. Actually an intermediary by the name of Abbott published them for him--A Square himself was in prison for heresy at the time. A Square was extraordinary not because he had such an odd name, but rather because he had such a descriptive and accurate name. For you see, A Square was a square.

Now you might be wondering just where A Square lived. After all, you wouldn't expect to find a two-dimensional square living in a three-dimensional universe such as ours. You might allow for a slightly thickened square, say a creature with the dimensions of a sheet of paper, but certainly not a completely flat individual like A Square. Anyhow, A Square didn't live in our three-dimensional universe. He lived in Flatland, a two-dimensional universe resembling a giant plane.

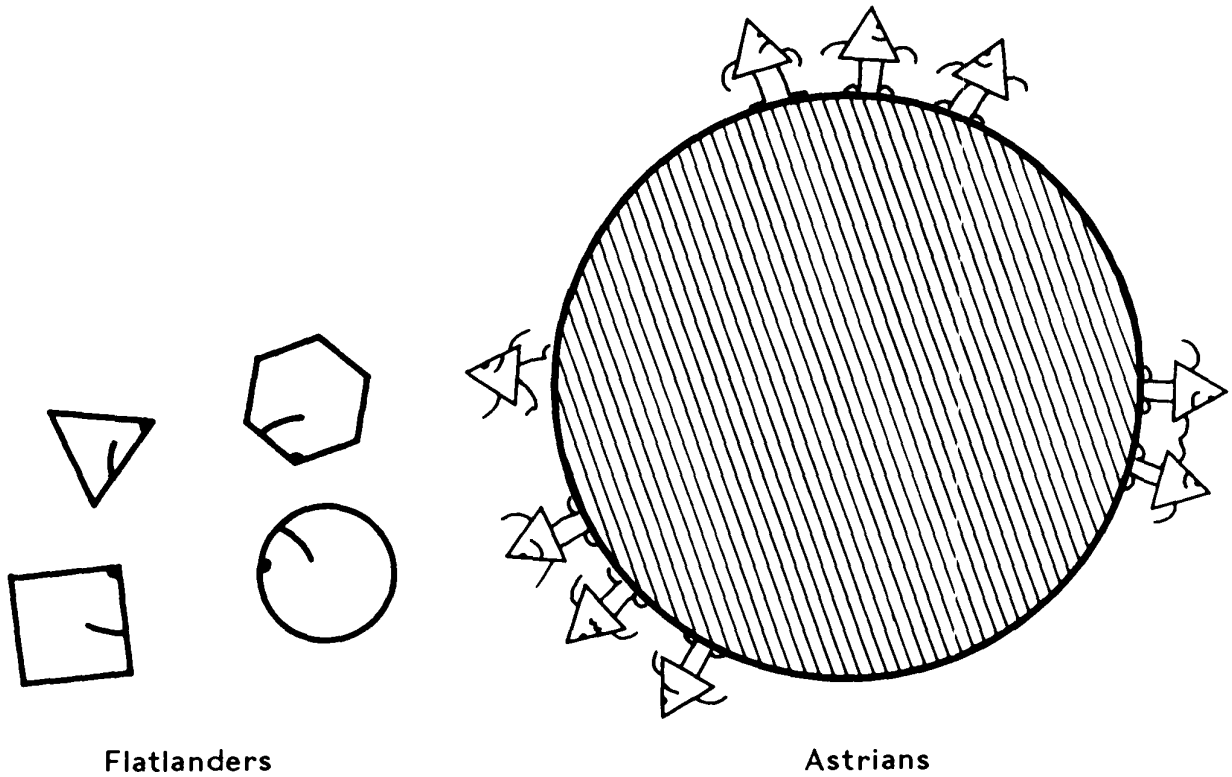

Figure 1.1: Flatlanders move freely in a "plane", while Astrians are confined to the edge of their disk-shaped planet Astria.

Flatland also happens to be the title under which A Square's memoirs were published. It's now available in paperback, and I recommend it highly. In 1907 C. H. Hinton published a similar book, *An Episode of Flatland*. The chief difference between these books is that the residents of Flatland proper can move freely about their two-dimensional universe, whereas the inhabitants of Hinton's world are constrained by gravity to living on the circular edge of their disk-shaped planet Astria (Figure 1.1). For the full story on the lore of Astria, see A. K. Dewdney's *The Planiverse*.

Getting back to the subject at hand, the Flatlanders all thought that Flatland was a giant plane, what we Spacelanders would call a

Euclidean plane. To be accurate, I should say that they *assumed* that Flatland was a plane, since nobody ever gave the issue any thought. Well, almost nobody. Once a physicist by the name of A Stone had proposed an alternative theory, something about Flatland having a finite area, yet having no boundary. He compared Flatland to a circle. For the most part people didn't understand him. It was obvious that a circle had a finite circumference and no endpoints, but what did that have to do with Flatland, which obviously had an infinite area? At least part of the problem was linguistic: The only word for "plane" was the word for "Flatland" itself, so to express the idea that Flatland was *not* a plane, one was trapped into stating that "Flatland is not Flatland". Needless to say, this theory attracted few disciples.

A Square, though, was among the few. He was particularly intrigued by the idea that a person could set out in one direction and come back from the opposite direction, without ever having turned around. He was so intrigued that he wanted to try it out. The Flatlanders were for the most part a timid lot, and few had ever travelled more than a day or two's journey beyond the outlying farms of Flatsburgh. A Square reasoned that if he were willing to spend a month tromping eastward through the woods, he might just have a shot at coming back from the west.

He was delighted when two friends volunteered to go with him. The friends, A Pentagon and A Hexagon, didn't believe any of A Square's theories--they just wanted to keep him out of trouble. To this end they insisted that A Square buy up all the red thread he

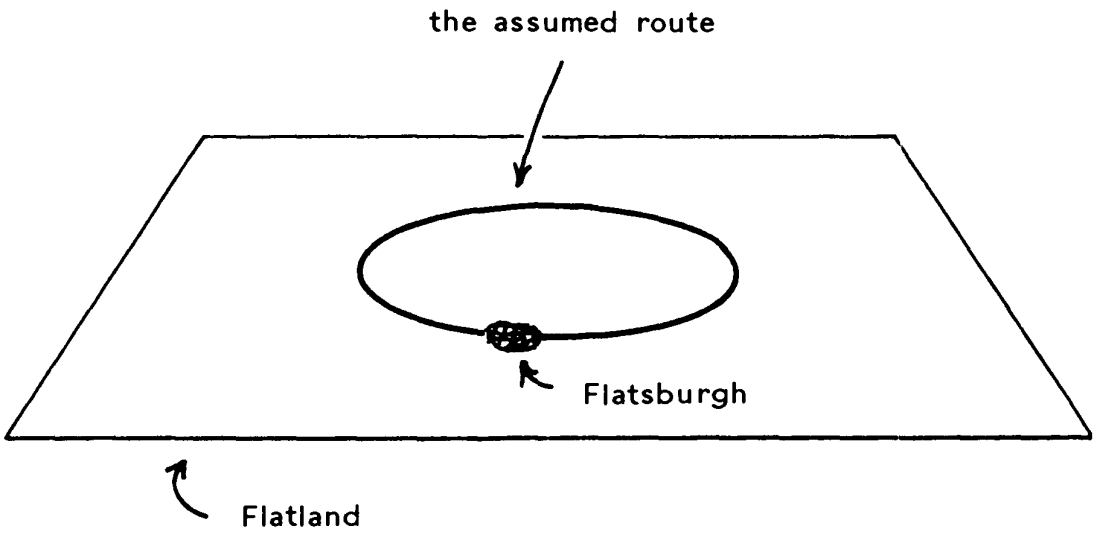

Figure 1.2: Even A Square's companions thought they had veered in a circle.

could find in Flatsburgh. The idea was that they would lay out a trail of red thread behind them, so that after they had travelled for a month and given up, they could then find their way back to Flatsburgh.

As it turned out, the thread was unnecessary. Much to A Square's delight--and A Pentagon's and A Hexagon's relief--they returned from the west after three weeks of travel. Not that this convinced anyone of anything. Even A Pentagon and A Hexagon thought that they must have veered slightly to one side or the other, bending their route into a giant circle in the plane of Flatland (Figure 1.2). A Square had no reply to their theory, but this did little to dampen his enthusiasm. He was ready to try it again!

By now red thread was in short supply in Flatland, so this time A Square laid out a trail of blue thread to mark his route. He set

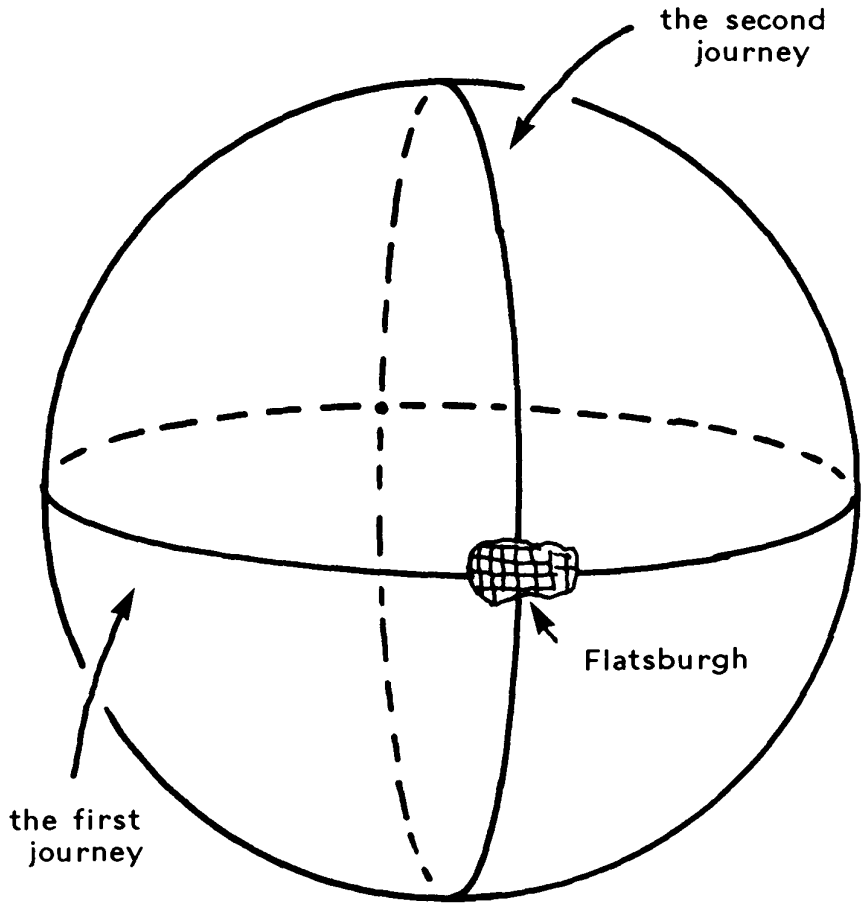

Figure 1.3: The two threads ought to cross, even if Flatland were a "hypercircle" (i.e. a sphere).

out to the north, and, sure enough, returned two weeks later from the south. Again everyone assumed that he had simply veered in a circle, and counted him lucky for getting back at all.

A Square was mystified that his journey was so much shorter this time, but something else bothered him even more: he had never come across the red thread they laid out on the first journey. The physicists of Flatland were equally intrigued. They confirmed that even if Flatland were a so-called "hypercircle" as A Stone had suggested, the two threads would still cross (Figure

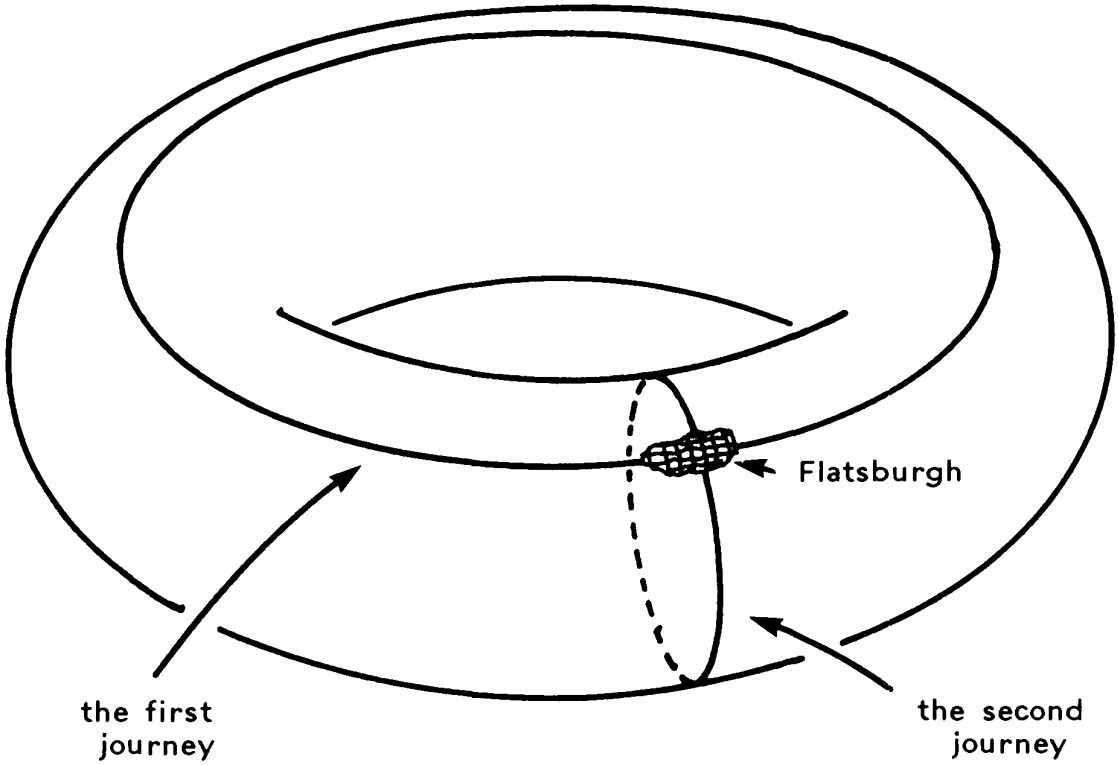

Figure 1.4: Spacelanders sometimes visualize a torus as the surface of a doughnut.

1.3). There was, of course, the possibility that the red thread had broken for one reason or another. To investigate this possibility, the scientists formed two expeditions: one party retraced the red thread, the other retraced the blue. Both threads were found to be intact.

The Mystery of the Nonintersecting Threads remained a mystery for quite a few years. Some of the bolder Flatlanders even took to retracing the threads periodically as a sort of pilgrimage. The first hint of a resolution came when a physicist proposed that Flatland should be regarded neither as a "Flatland" (i.e. a plane) nor as a hypercircle, but as something he called a "torus". At

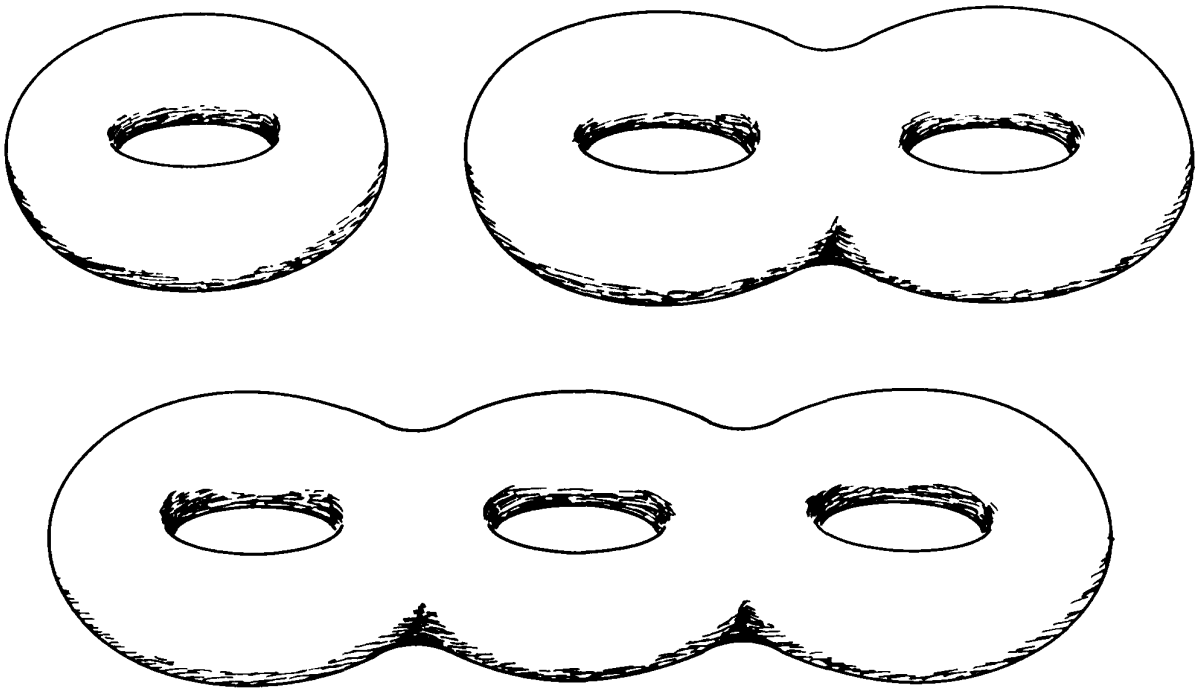

Figure 1.5: Some possible shapes for Flatland.

first no one had any idea what he was talking about. Gradually though, people agreed that this theory resolved the Mystery of the Nonintersecting Threads, and everyone was happy about that. So for many years Flatland was thought to be a torus (Figure 1.4).

Until one day somebody came up with yet another theory on the "shape" of Flatland. This theory explained the Mystery of the Nonintersecting Threads just as well as the torus theory did, but it gave a different overall view of Flatland.

And this new theory was just the first of many. For the next few months people were constantly coming up with new possibilities for the shape of Flatland (Figure 1.5). Soon a vast Universal Survey was undertaken to map all of Flatland and thereby determine

its true shape once and for all . . .

(STORY TO BE CONTINUED IN CHAPTER 4)

As Spacelanders we have three dimensions available for drawing pictures like Figure 1.5, so it's easy for us to understand how a two-dimensional universe can close back on itself. The Flatlanders inhabiting such a universe would have a much tougher time. To sympathize with their feelings, try imagining yourself in each of the following situations.[1]

1. You are on an expedition to a distant galaxy in search of intelligent life. When you reach the galaxy you head for the most hospitable-looking planet you can find, only to discover that you're back on Earth.
2. You are an astronomer. You seem to be observing the exact same object in two different locations in the sky.
3. You are a radio astronomer searching for signals from extraterrestrials. You have detected a faint signal coming from a distant galaxy. Once you tune it in you recognize it as a broadcast of the old TV show "Father Knows Best".

Each of the above situations leads you to suspect that space is built differently than you thought it was. That is, space seems to have a d -ent shape than the obvious one you assumed it had.

---

[1] Warning: These situations are designed to stimulate the imagination. Don't worry about technical complications!

Not that you have any idea what this actual shape is!

In fact, no one knows what the shape of the real universe is. But people do know a fair amount about what the *possible* shapes are. These possible shapes are the topic of this book. Such a possible shape is called a **three-dimensional manifold**, or **three-manifold** for short. (Similarly, a two-dimensional shape for Flatland is called a **two-dimensional manifold**, or, more commonly, a **surface**.) At this point your conception of a three-manifold is probably pretty vague. Don't worry: we'll start seeing some examples in Chapter 2. The main thing now is to realize that our universe might conceivably close back on itself, just as the various surfaces representing Flatland close back on themselves.

This book centers on a series of examples of three-manifolds. Rather than developing an extensive theory of these manifolds, you'll come to know each of them in a visual and intuitive way. Obviously this is not an easy task. Imagine the difficulties A Square would have in communicating to A Hexagon the true nature of a torus. A Square cannot draw a definitive picture of a torus, being confined to two dimensions as he is. Similarly, we cannot draw a definitive picture of any three-manifold.

There is some hope, though. You can use tricks to define various three-manifolds, and as you work with them over a period of time you'll find your intuition for them growing steadily. The human mind is remarkably flexible in this regard. *Just be sure to read slowly and give things plenty of time to digest.* At most a

chapter, and often as little as a single exercise, will be plenty for one sitting.

This book provides not a series of answers, but rather a series of questions designed to lead you to your own intuitive understanding of three-manifolds. Prepare your imagination for a workout!

2

Gluing

A popular video game pits two players in biplanes in aerial combat on a TV screen. An interesting feature of the game is that when a biplane flies off one edge of the screen it doesn't crash, but rather it comes back from the opposite edge of the screen. Mathematically speaking, the screen's edges have been "glued" together. (The gluing is purely abstract: there is no need to *physically* connect the edges.) A square or rectangle whose opposite edges are abstractly glued in this fashion is called a **torus** or, more precisely, a **flat two-dimensional torus.** There is a connection between this flat two-dimensional torus and the doughnut-surface torus of Chapter 1, but for the time being *you should forget the doughnut surface entirely.*

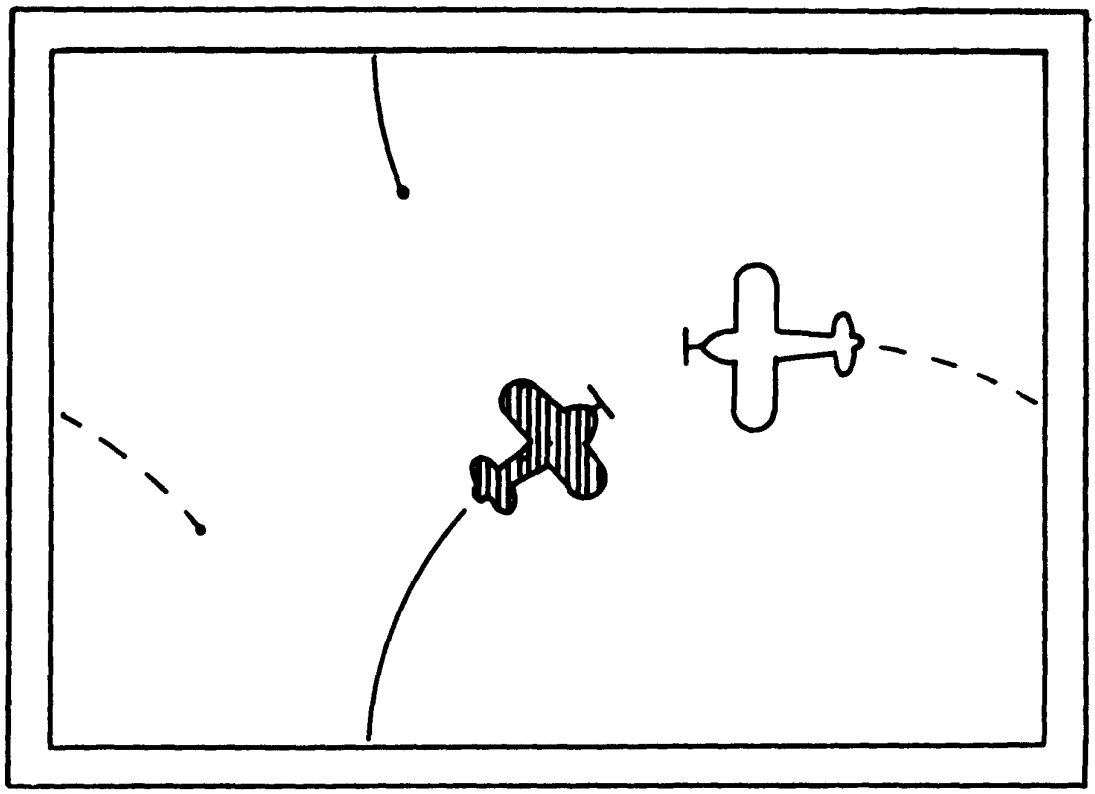

Figure 2.1: The biplanes can fly across the edges of the TV screen.

Exercise 2.1. Play a few games of torus tic-tac-toe with a willing friend. The rules are the same as in traditional tic-tac-toe, except here the opposite sides of the board are glued to form a torus, just as the opposite sides of the TV screen are glued in the biplane game. So, for example, the three X's in Figure 2.2 constitute a winning three-in-a-row. □

Exercise 2.2. Figure 2.3 shows a torus tic-tac-toe game in progress. It's X's turn. Where should he move? If it were O's turn instead, where would his best move be? *(Answers to exercises*

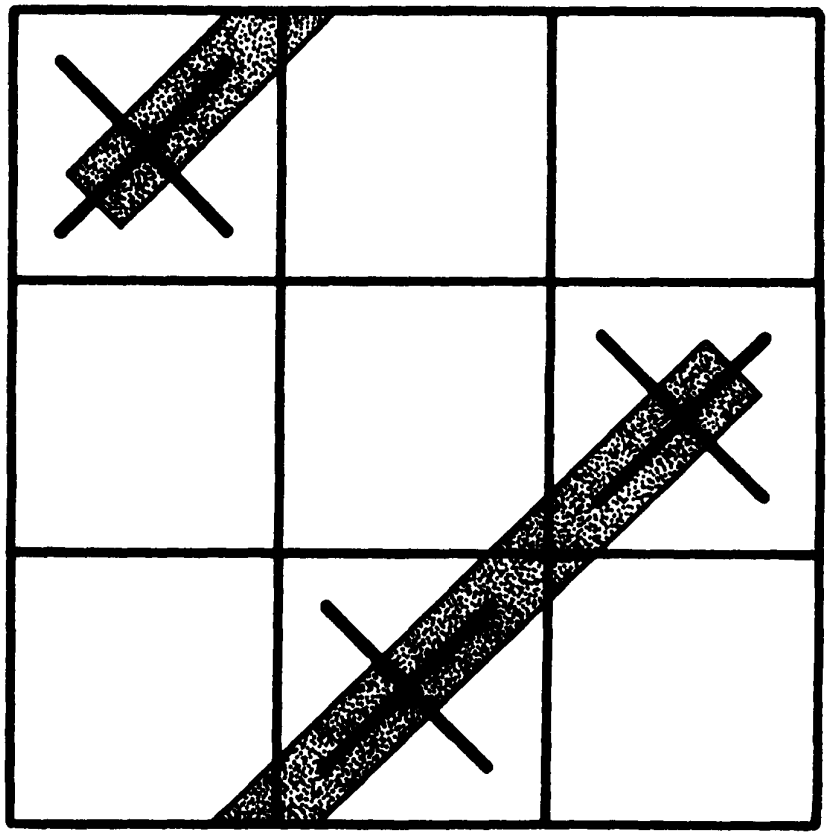

Figure 2.2: These X's are three-in-a-row if the board is imagined to represent a torus.

are found at the back of the book.) □

The positions shown in Figure 2.4 are all equivalent in torus tic-tac-toe. The second position is obtained from the first by moving everything "up" one notch (when the top row moves "up" it naturally reappears at the bottom). Similarly, the third position is the result of moving everything in the second position one notch to the right. The fourth position is obtained a little differently: it results from rotating the third position one quarter turn clockwise.

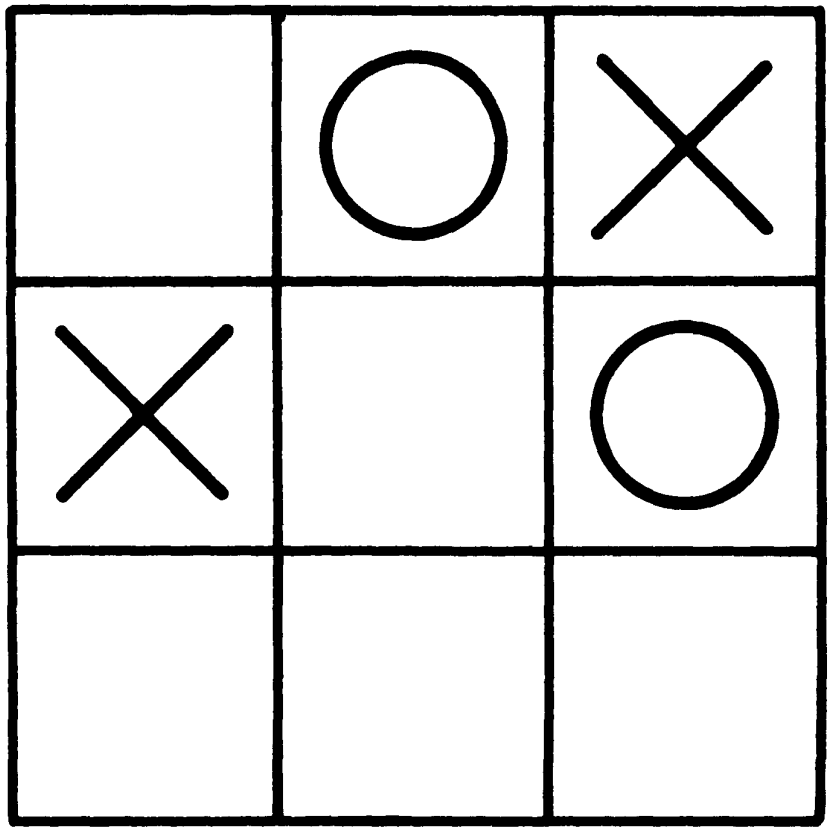

Figure 2.3: What is X's best move? What is O's best move?

Exercise 2.3. Which of the positions in Figure 2.5 are equivalent in torus tic-tac-toe? □

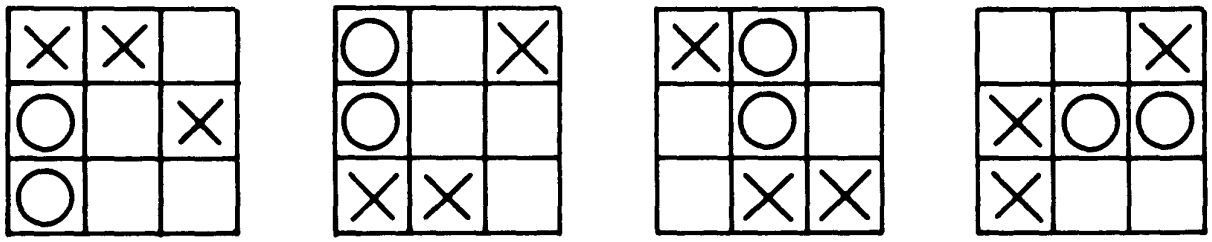

Figure 2.4: These four positions are equivalent in torus tic-tac-toe.

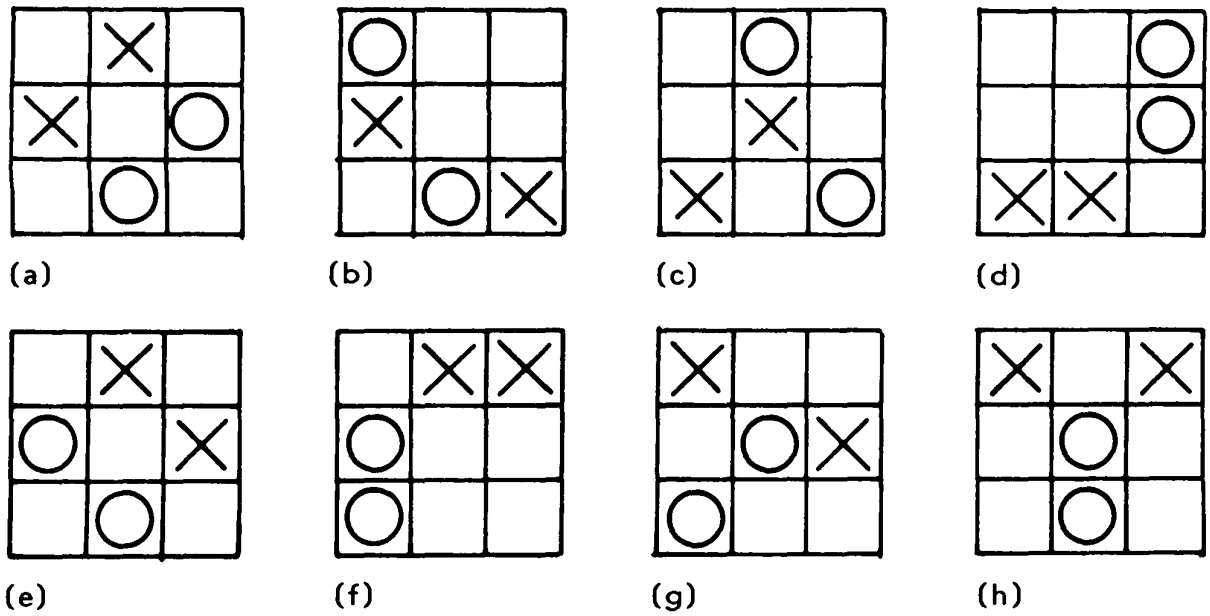

Figure 2.5: Which of these positions are equivalent in torus tic-tac-toe?

Exercise 2.4. In torus tic-tac-toe, how many essentially different opening moves does the first player have? How many different responses does her opponent have? Is either player guaranteed to win, assuming optimal play? (In traditional tic-tac-toe, either player can guarantee a draw.) □

Exercise 2.5. Chess on a torus is more challenging than tic-tac-toe. Consider the position shown in Figure 2.6. Which black pieces does the white knight threaten? Which black pieces threaten it? □

Exercise 2.6. Figure 2.7 shows some pieces on a torus chessboard. Which black pieces are threatened by *both* the white knight and the white queen? □

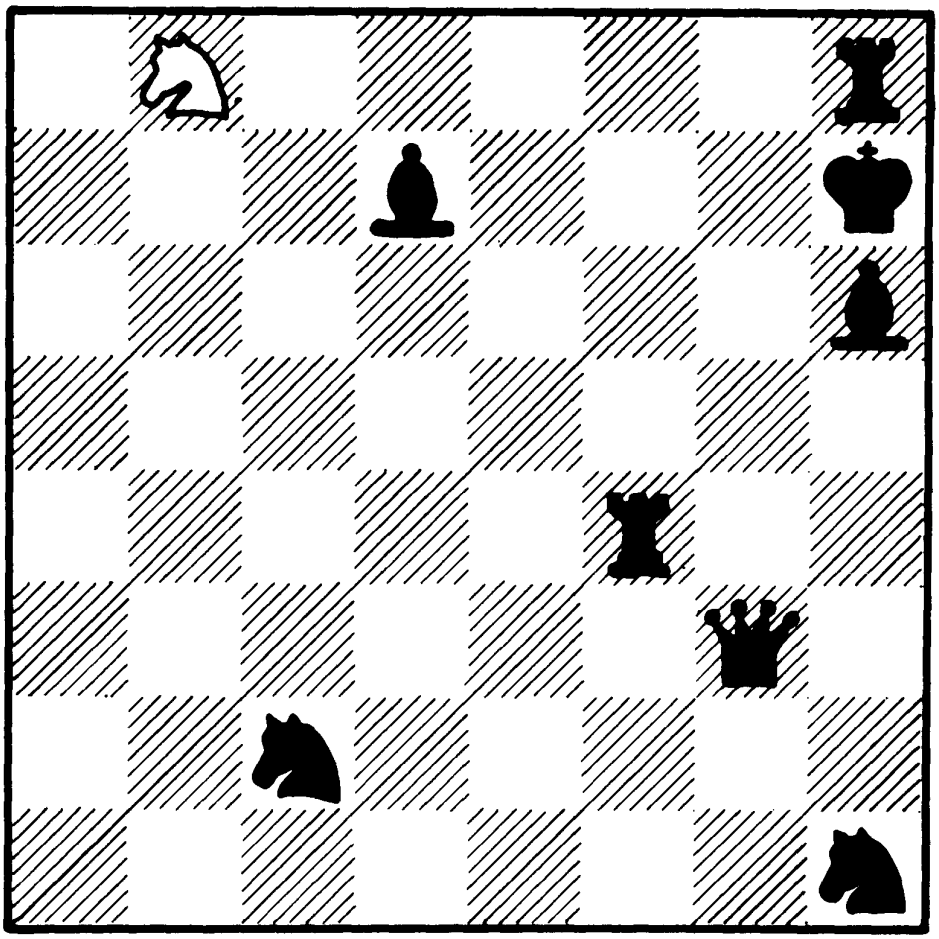

Figure 2.6: A chessboard becomes a torus when opposite sides are connected.

Exercise 2.7. Find a friend and play a few games of torus chess. The usual starting position just won't do for torus chess (try it and you'll see why). Instead either use the starting position of Figure 2.8, or make up a starting position of your own. All the pieces move normally except the pawns: a pawn moves one space forward, backward, to the left or to the right, and captures by moving one space on any diagonal. □

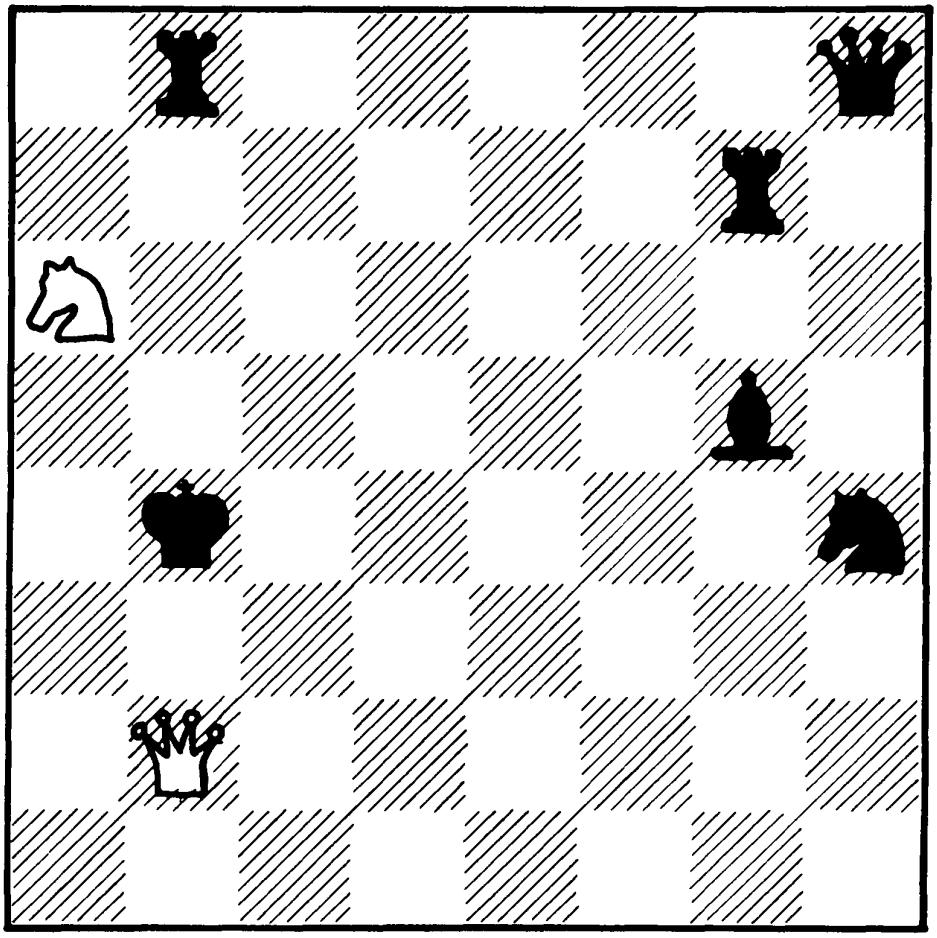

Figure 2.7: Which black pieces are threatened by *both* the white knight and the white queen?

Exercise 2.8. When a bishop goes out the upper right-hand corner of a torus chessboard, where does he return? □

Exercise 2.9. In torus chess, can a knight and a bishop simultaneously threaten each other? □

The flat torus we've been using for tic-tac-toe and chess is a

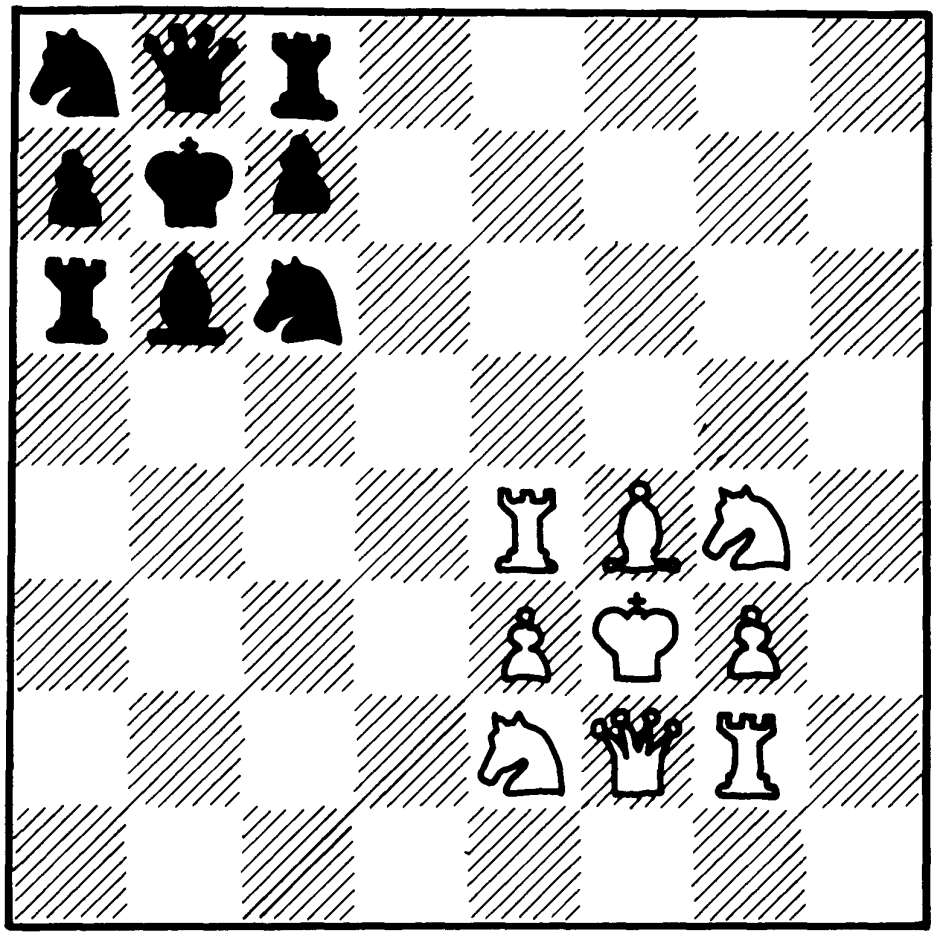

Figure 2.8: Here's one possible starting position for torus chess.

two-dimensional manifold, just like the two-dimensional manifolds in Figure 1.5. Like the two-manifolds of Figure 1.5, the flat torus has a finite area and no edges. *Unlike* those two-manifolds, the flat torus is defined abstractly--via gluing--instead of being drawn in three-dimensional space. The same trick works to define three-dimensional manifolds without resorting to pictures in four-dimensional space.

Our first three-dimensional manifold is analogous to the flat two-dimensional torus. It's called a **three-dimensional torus**, or **three-**

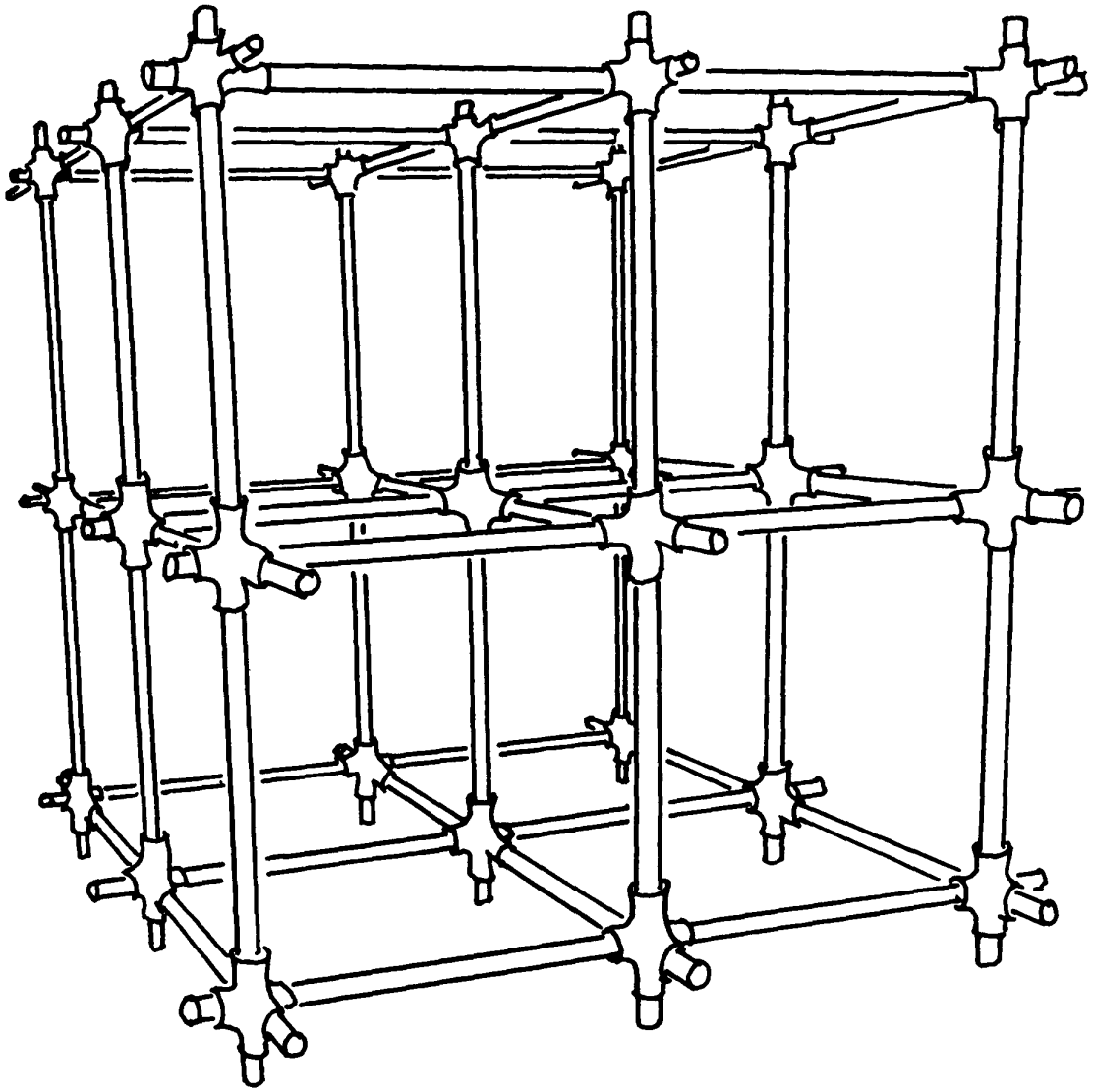

Figure 2.9: A section of jungle gym.

torus for short. To construct it, start with a solid block of space--the room you're in will do fine just so it's rectangular. Imagine the left wall glued to the right wall, not in the sense that you'd physically carry out the gluing, but in the sense that if you walked through the left wall you'd find yourself emerging from the right wall. Imagine the front wall glued to the back wall, and the

floor glued to the ceiling, in the same manner. You are now sitting in a three-dimensional torus! This three-torus has no edges, yet it's total volume is just the volume of the room you started with.

Exercise 2.10. What do you see when you look through the "walls" of the three-torus described above? For that matter, what do you see when you look through the "floor" or the "ceiling"? □

Imagine a three-torus made from a cube ten meters on a side. The cube contains a jungle gym consisting of a rectangular lattice of pipes (Figure 2.9); each segment of pipe is one meter long. When the cube's faces are glued to form a three-torus, the jungle gym continues uniformly across each face.

It would be fairly boring playing in this jungle gym by yourself. Sure, you could climb up a few meters or over a few meters, but your new location would be just like your old one. The visual effects would be interesting, though. You could look up ten meters, or over ten meters, and see yourself. You couldn't go meet yourself, though, since as soon as you started climbing towards your "other self", she would start climbing away from you in a (futile) effort to go meet *her* other self ten meters further away!

All in all, playing in this jungle gym would be a lot more fun with a friend. For example, your friend could wait for you while you climbed up ten meters to meet him. Of course, while you were climbing *up*, he could crawl ten meters *over*, so that when you got

to the rendezvous point he would be coming in from the side rather than just waiting there. Tag would be especially fun in a three-torus.

Exercise 2.11. One's imagination can go wild in a three-torus. For example, you could imagine flying around in real, three-dimensional biplanes. Or you could imagine a completely urbanized three-torus in which all north-south streets are one way northbound, and all east-west streets are one way eastbound, and all elevators go only up (you could still get wherever you wanted to go). Imagine other things you could do in a three-torus. □

Exercise 2.12. How could you play catch by yourself in a three-torus? □

In theory, if the universe is a three-torus we should be able to look out into space and see ourselves. Does the fact that astronomers have not done so mean that the real universe cannot be a three-dimensional torus? Not at all! The universe is only 10 or 20 billion years old, so if it were a very large three-dimensional torus--say 60 billion light-years across at its present stage of evolution--then no light would yet have had enough time to make a complete trip across. Another possibility is that we *are* in fact seeing all the way across the universe, but we just don't know it: when we look off into distant space we see things as they were billions of years ago, and billions of years ago our galaxy looked

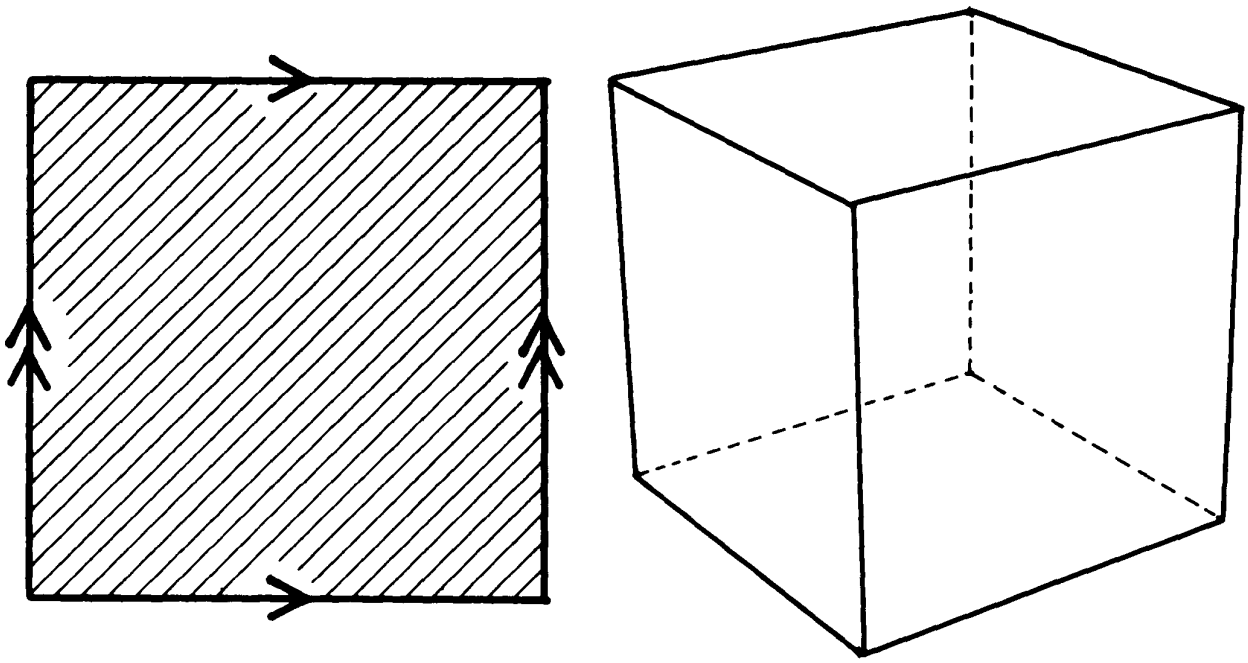

Figure 2.10: Representation of a flat two-dimensional torus (left) and a three-dimensional torus (right).

different than it does now. (This effect occurs because the light which enters a telescope today left its source billions of years ago, and has spent the intervening time travelling through intergallactic space.) In any case, we don't even know exactly what our galaxy looks like now, because we are inside it!

We conclude this chapter with a little visual notation (Figure 2.10). Henceforth a flat two-dimensional torus will be drawn as a square with arrows marked on its edges; you imagine the square's edges to be glued so that corresponding arrows match up. While it's possible to devise an analogous scheme for marking the faces of

a cube, it isn't very practical. So we'll represent a three-torus simply by drawing a cube and stating that opposite faces are considered glued. By the way, a two-manifold like the two-dimensional torus is called a surface even though it isn't the surface *of* anything.

3

Vocabulary

This chapter explains five concepts basic to the study of manifolds:

1. Topology vs. geometry.
2. Intrinsic vs. extrinsic properties.
3. Local vs. global properties.
4. Homogeneous vs. nonhomogeneous geometries.
5. Closed vs. open manifolds.

Don't worry about mastering these concepts right away! If you get the general idea now you can always refer back to this chapter later should the need arise. Besides, later chapters will reinforce the ideas introduced here.

Most of the examples in this chapter will be surfaces, but the concepts apply to three-manifolds as well.

Topology vs. Geometry

Imagine a surface made of thin, easily stretchable rubber. Bend, stretch, twist, and deform this surface any way you want (just don't tear it). As you deform the surface, it will change in many ways, but some aspects of its nature will stay the same. For example, the surface at the far left in Figure 3.1, deformed as it is, is still recognizable as a sort of sphere,[2] whereas the surface at the far right is recognizable as a deformed two-holed doughnut. The aspect of a surface's nature which is unaffected by deformation is called the **topology** of the surface. Thus the two surfaces on the left in Figure 3.1 have the same topology, as do the two on the right. But the sphere and the two-holed doughnut surface have different topologies: no matter how you try you can never deform one to look like the other (remember--violence such as ripping one surface open and regluing it to resemble the other is *not* allowed).

A surface's **geometry** consists of those properties which *do* change when the surface is deformed. Curvature is the most important geometrical property. Other geometrical properties include areas, distances and angles. An eggshell and a ping-pong ball have the same topology, but they have different geometries. (In this and subsequent examples the reader should idealize objects like eggshells and ping-pong balls as being truly two-dimensional, thus ignoring any thickness the real objects may possess.)

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[2] By a sphere we always mean just the surface, as opposed to a solid ball. Note that a sphere is intrinsically two-dimensional, while a solid ball is intrinsically three-dimensional.

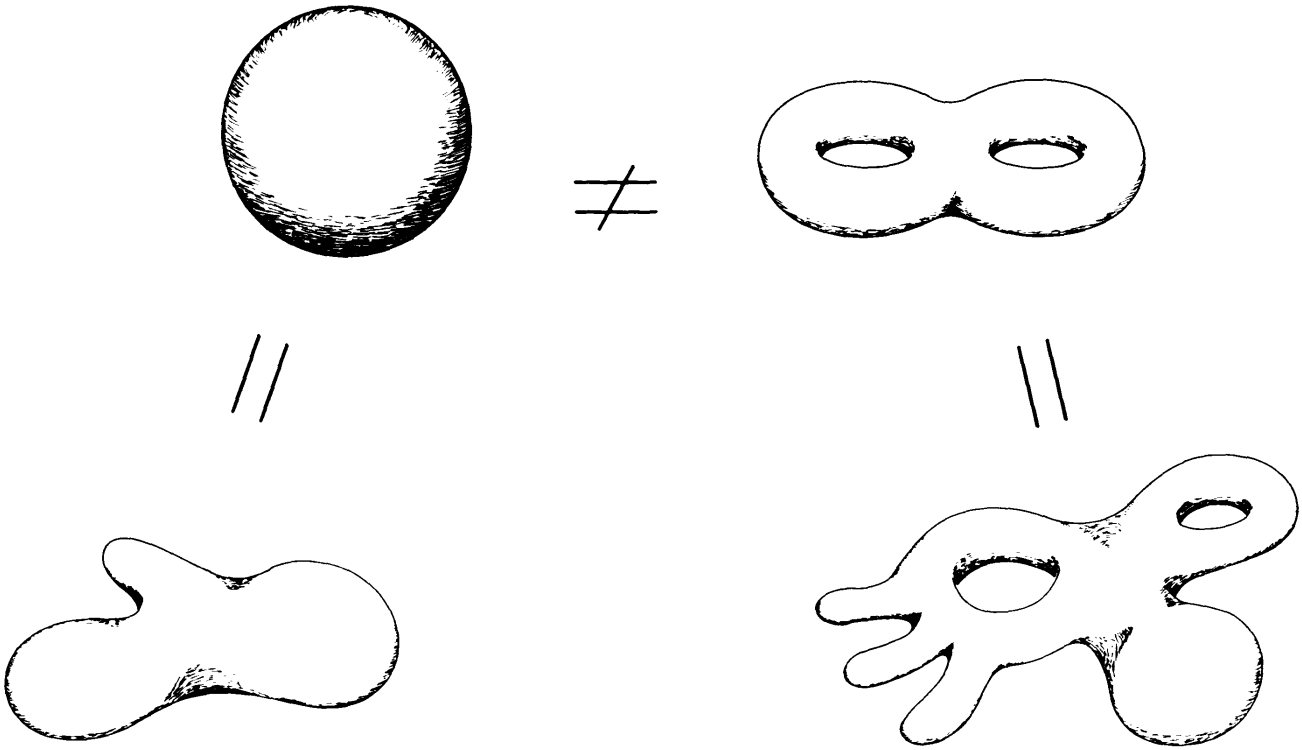


Figure 3.1: The two surfaces on the left are topologically indistinguishable, as are the two on the right.

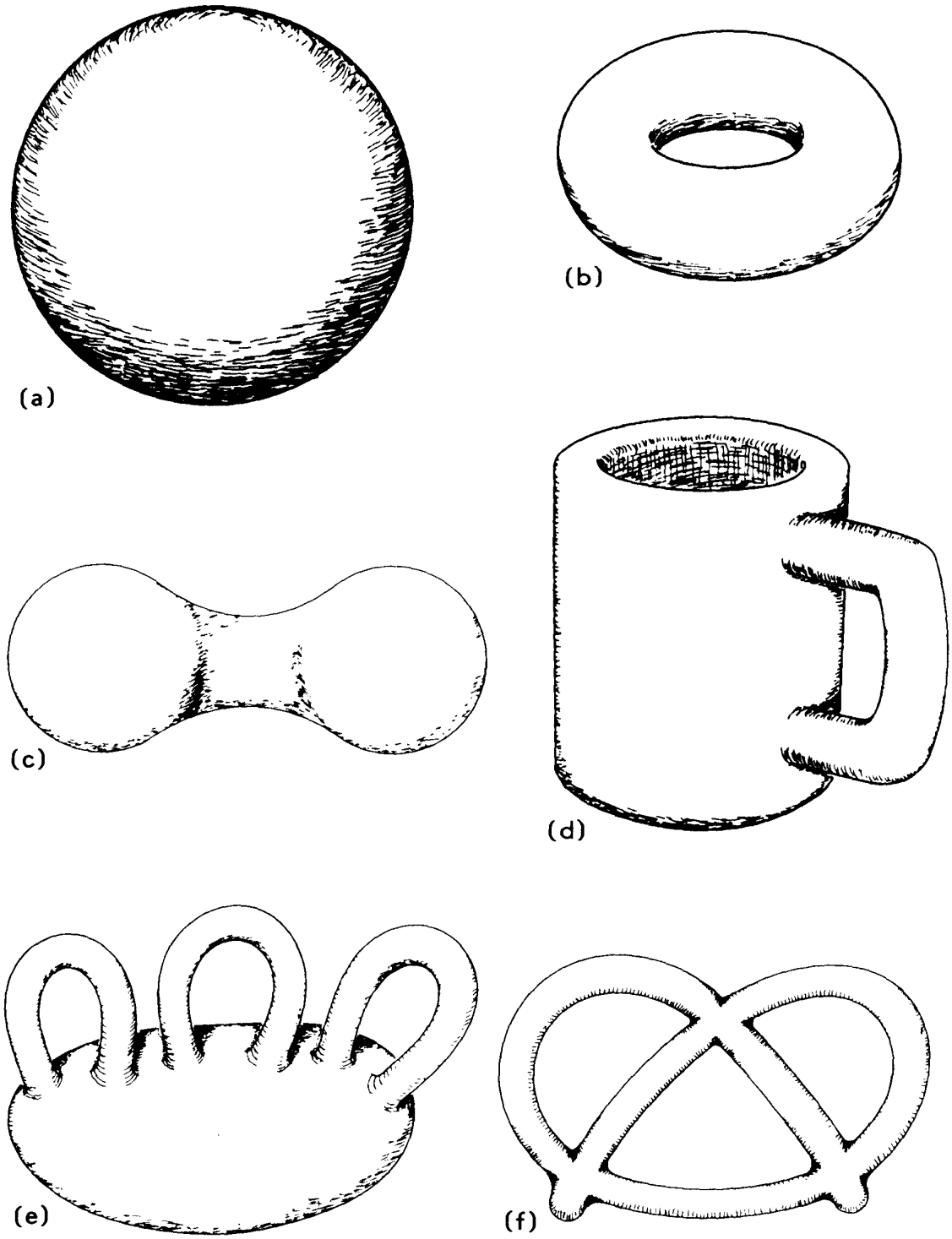


Figure 3.2: Which surfaces have the same topology?

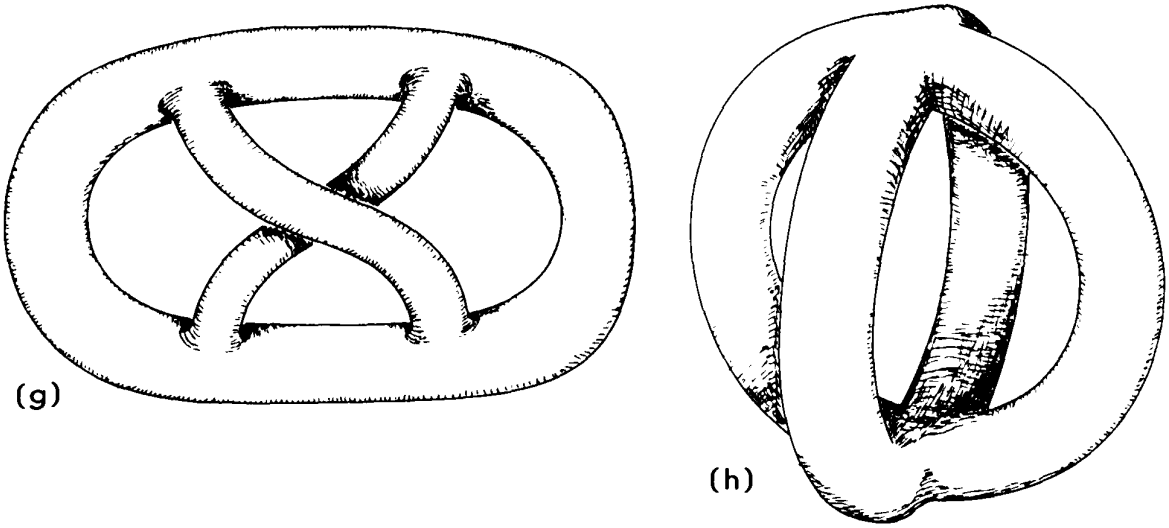


FIGURE 3.2: (Continued)

Exercise 3.1. Which of the surfaces in Figure 3.2 have the same topology? □

Exercise 3.2. In the story in Chapter 1, A Square discovered that in Flatland one can lay out two loops of thread which cross at only one point (namely downtown Flatsburgh). Did he discover a topological or a geometrical property of Flatland? □

If we *physically* glue the top edge of a square to its bottom edge, and its left edge to its right edge, then we will get a doughnut surface (Figure 3.3). The flat torus and the doughnut-surface torus have the same topology. They do not, however, have the same geometry. The doughnut surface is curved while the flat

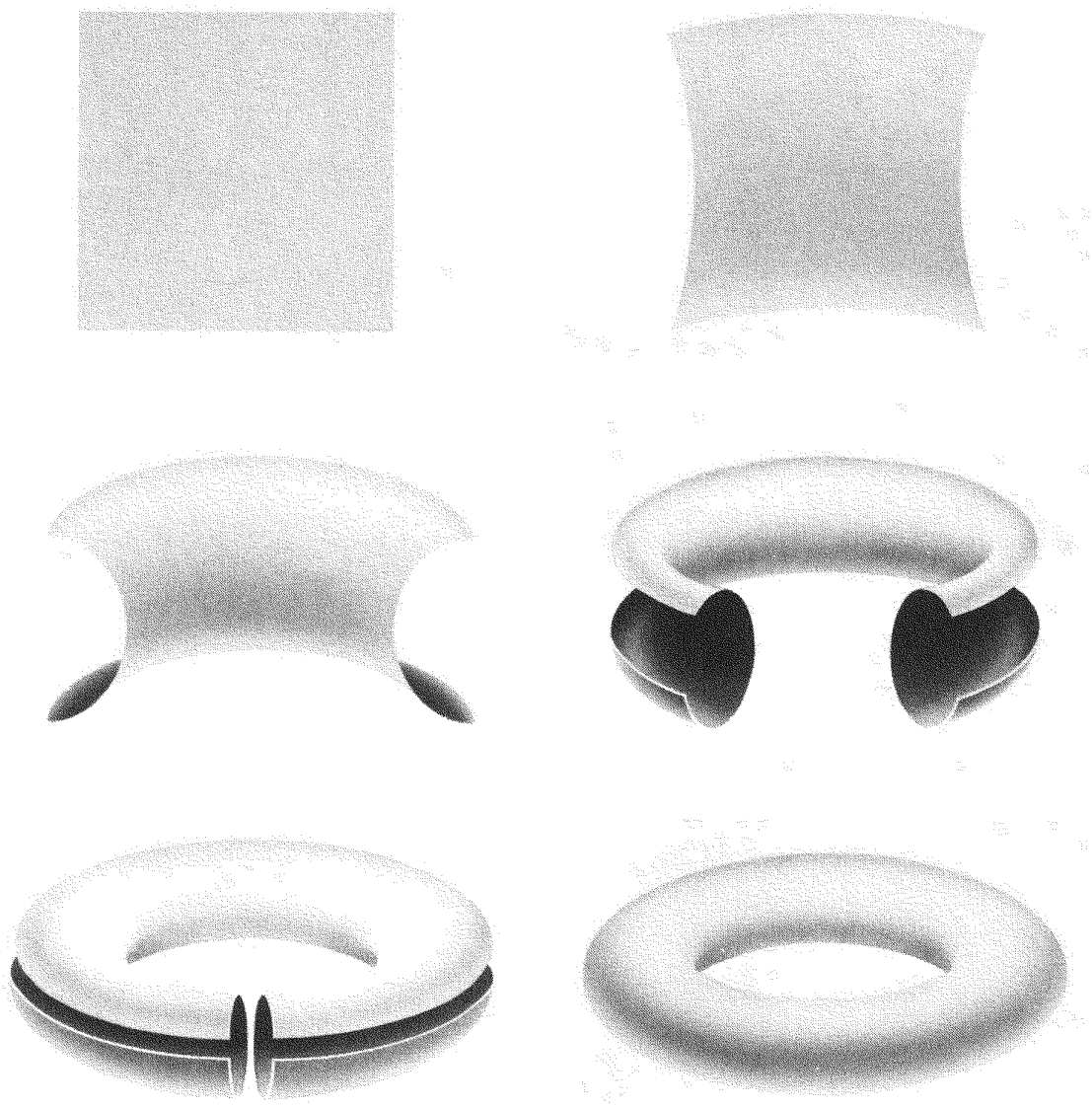


Figure 3.3: The flat torus and the doughnut surface have the same topology.

torus is obviously flat. In Figure 3.3 we had to deform the flat torus to get it to look like the doughnut surface.

The fact that the flat torus and the doughnut surface have the same topology explains why both are called tori ("tori", pronounced "tor-eye", is the plural of "torus"). It's only when we're interested in geometry that we distinguish the flat torus from the doughnut surface. In geometry the flat torus is vastly more important than the doughnut surface, so in this book, unless specified otherwise,

"torus" will mean "flat torus".

Intrinsic vs. Extrinsic Properties

Figure 3.4 shows how to put a twist in a rubber band. The twisted band is topologically different from the original untwisted one.

At least from *our* viewpoint the bands are topologically different. In contrast, imagine how a Flatlander living in the band itself would see the cut-twist-and-reglue procedure. His whole world is the band; he has no idea that three-dimensional space exists at all. Thus he has no way to detect the twist. He sees the band get cut, and then--after a pause--get restored exactly to its original condition!

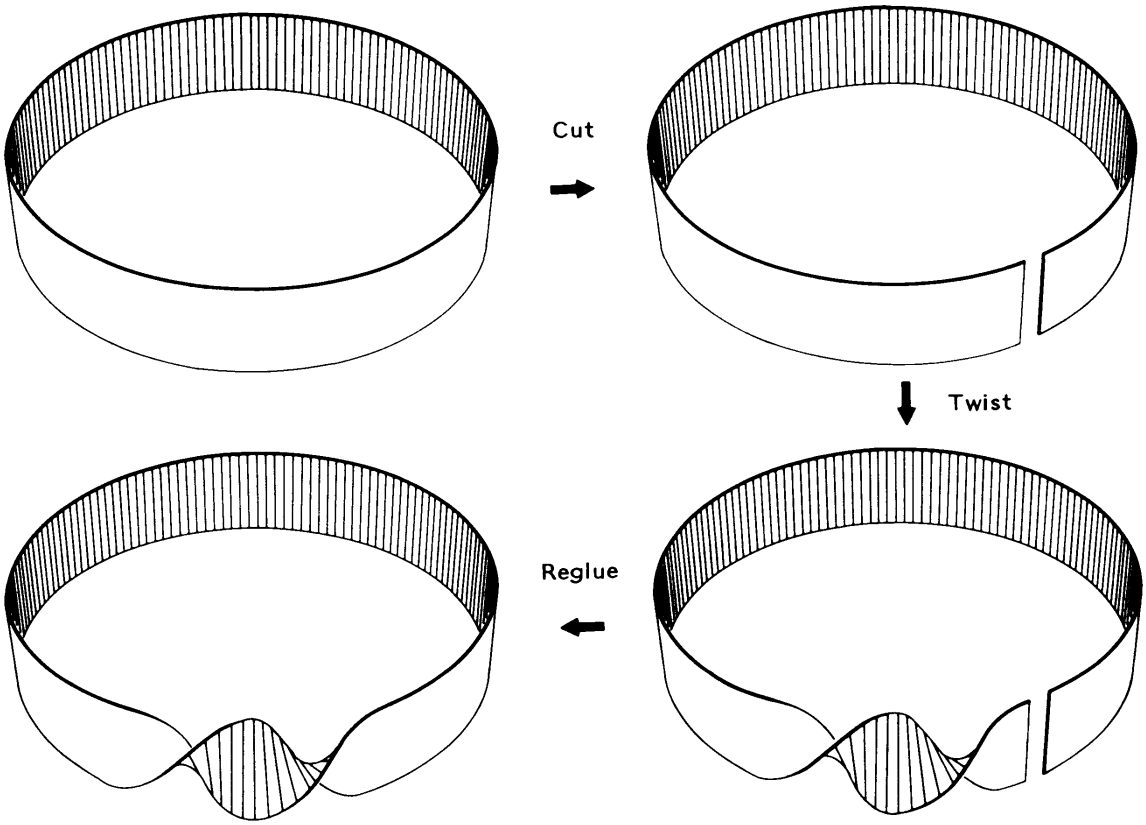


Figure 3.4: How to put a twist in a rubber band.

The **intrinsic topology** of the band has not changed, although its **extrinsic topology**--the way it's embedded in three-dimensional space--*has* changed.

In general, two surfaces have the same intrinsic topology if Flatlanders living in the surfaces cannot (topologically) tell one from the other. Two surfaces have the same extrinsic topology if one can be deformed *within three-dimensional space* to look like the other.

Exercise 3.3. All the surfaces in Figure 3.5 have the same intrinsic topology. Which have the same extrinsic topology as well? □

Exercise 3.4. Modify the cut-twist-and-reglue procedure of Figure 3.4 so that the intrinsic and extrinsic topology of the band both change. □

The intrinsic/extrinsic distinction also applies to the geometry of a surface. As an example, take a sheet of paper and bend it into a half-cylinder as shown in Figure 3.6. The **extrinsic geometry** of the paper has obviously changed. But the paper itself has not been deformed--its **intrinsic geometry** has not changed. In other words, a Flatlander living in a sheet of paper could not detect whether the paper was bent or not.

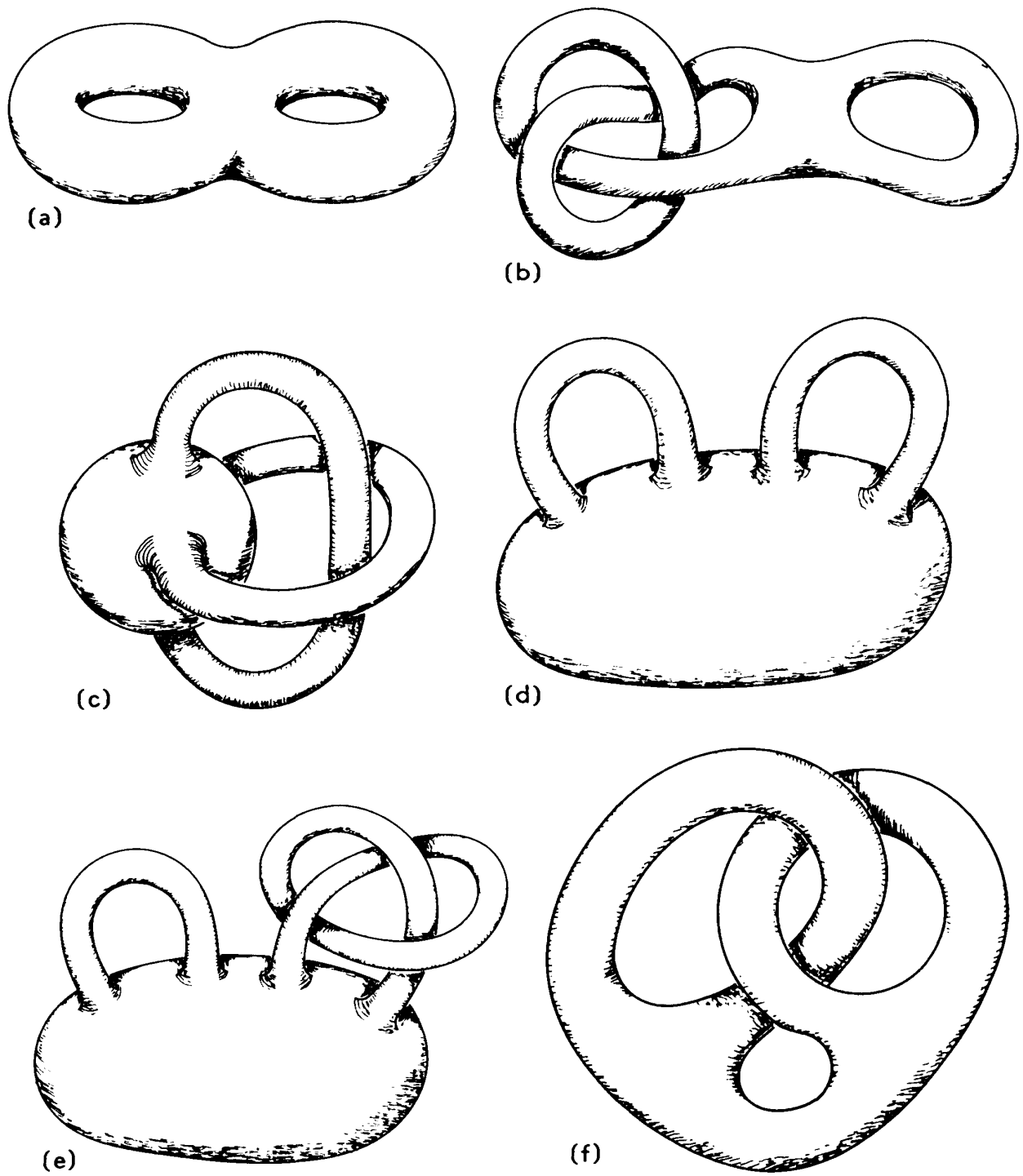


Figure 3.5: Which surfaces have the same extrinsic topology?

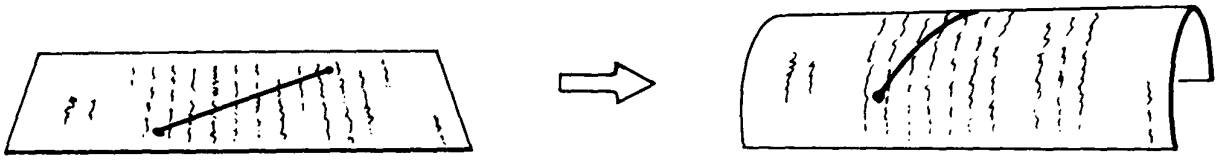


Figure 3.6: Bending a sheet of paper changes its extrinsic--but not its intrinsic--geometry.

Here's an experiment to illustrate the above idea: Mark two points on a (flat) sheet of paper, and draw a straight line connecting them. The line represents the shortest path between the points on the flat paper. Now roll the paper into a cylinder (tape or glue it in position if necessary). Get a piece of thread, and wrap it around the cylinder from one point to the other. The thread will lie directly over the line you drew, indicating that the shortest path on the cylinder is the same as the shortest path on the flat paper. This is not surprising, because the "flat" paper and the "curved" cylinder both have the same intrinsic geometry.

Exercise 3.5. You can roll a piece of paper into a cylinder without deforming the paper. Can you also roll it into a cone without deformation? Can you wrap it onto a basketball without deformation? What does this tell you about the intrinsic geometries of the paper, the cylinder, the cone, and the basketball? □

Figure 3.7 shows three surfaces with different intrinsic geometries. A Flatlander could compare these surfaces by studying

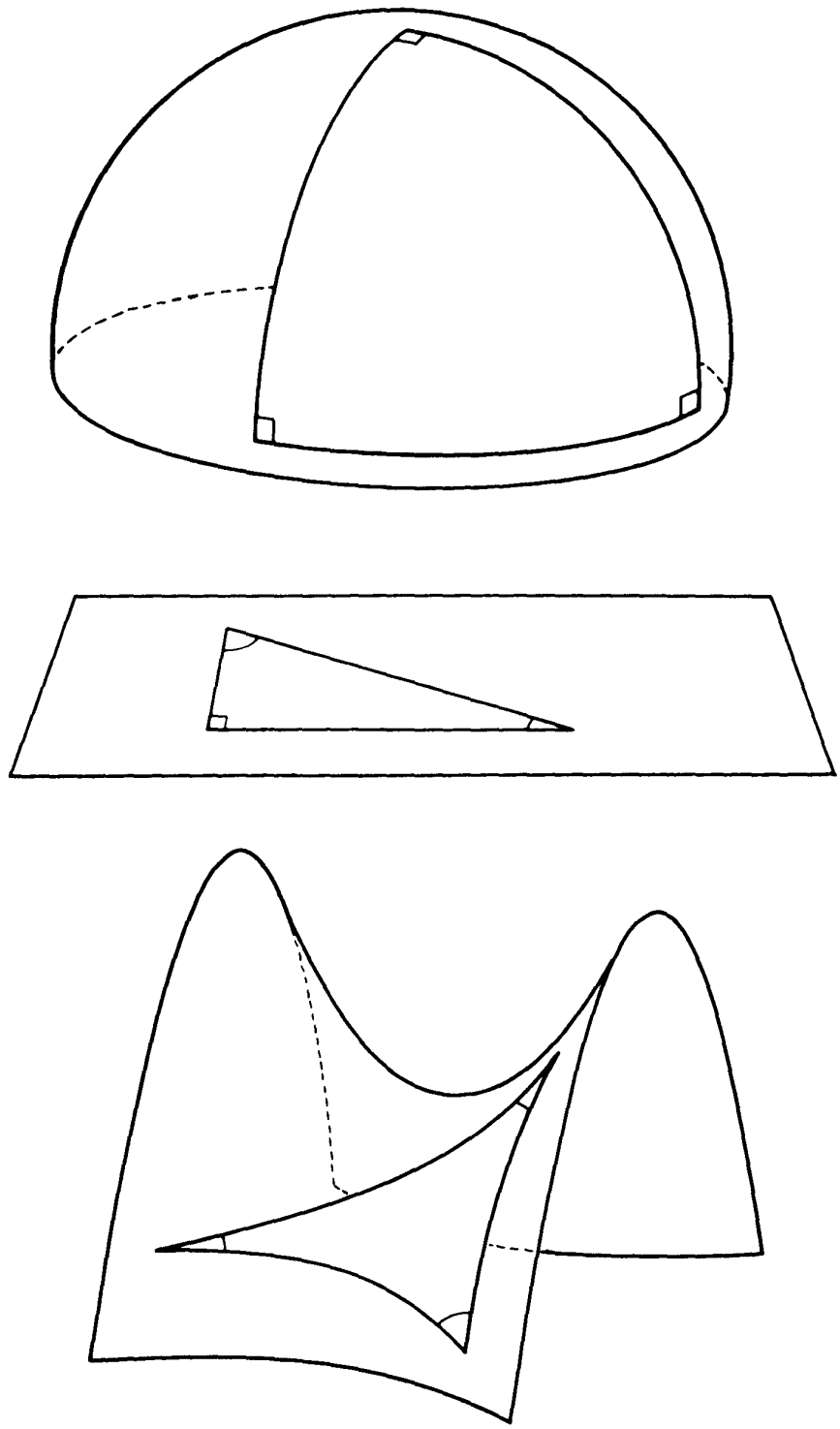


Figure 3.7: The hemisphere, the plane, and the saddle surface all have different intrinsic geometries.

the properties of triangles drawn on them. (The sides of a triangle are required to be intrinsically straight in the sense that they bend neither to the left nor to the right. A Flatlander finds an (intrinsically) straight line in a surface the same way we Spacelanders do in our universe, e.g. by pulling taut a piece of thread, or by seeing how a beam of light travels. An intrinsically straight line is called a **geodesic**.) On the hemisphere, the sum of the angles of any triangle is greater than 180°. For example, the triangle shown in the figure has all its angles equal to 90°, so its angle-sum is 90° + 90° + 90° = 270°. In the plane, on the other hand, every triangle has angle-sum exactly equal to 180°. And in the saddle surface, all triangles have angle-sum less than 180°. Thus a Flatlander could experimentally determine which surface he lived in: he need only lay out a triangle and measure its angles! These properties of triangles are treated in detail in Chapters 9 and 10.

The mathematician Gauss carried out precisely this experiment in our own three-dimensional universe. (Later chapters will explain how a three-dimensional manifold can be curved. Gauss, though, was interested in other obscure things, and probably *didn't* expect to discover the curvature of space). Gauss measured the angles in the triangle formed by the three mountain peaks Hohenhagen, Brocken and Inselsberg. To within the accuracy of his measurements he found space to be intrinsically flat, i.e. the angles added to 180°. However, the universe is so vast and the Earth is

so small that it would be impossible to detect any cosmic curvature on a terrestrial scale. Chapter 19 will treat the curvature of the universe in detail.

We Spacelanders can contemplate both the intrinsic and extrinsic properties of a surface. A Flatlander does not have this option. His two-dimensional universe is all that's real and perceptible to him, so he naturally adopts an intrinsic viewpoint. For example, a Flatlander raised on a torus would have a very good intuitive understanding of what a torus is like, assuming of course that a trip across the torus was short enough to be an everyday sort of thing. He'd know intuitively that if he were going into the city, and then out to visit his aunt, and then home again, that it would be quicker to keep going in the same direction than to retrace his route.

Exercise 3.6. Imagine living in a three-torus universe where, after visiting friends in one galaxy, and doing a little exploration in another, it's quickest to keep on going to get home rather than turning back. □

We humans perceive our universe intrinsically, so when we study three-manifolds, such as the three-torus, we naturally visualize them intrinsically too. Because surfaces will guide us in our study of three-manifolds, it will also be useful to think of surfaces intrinsically. To this end we make the convention that *all surfaces*

will be studied intrinsically, unless explicitly stated otherwise. Any extrinsic properties a surface may have will be ignored. (For example, we'll ignore the twist in the rubber band of Figure 3.4, and we'll ignore the bend in the sheet of paper of Figure 3.6.)

I'd like to insert a philosophical comment here. Even if the universe connects up with itself in funny ways (if it's a three-torus, for example), this *doesn't* mean that it curves around in some four-dimensional space. The essence of the intrinsic point of view is that

a manifold exists in and of itself,
and needn't lie in any higher-dimensional space.

Local vs. Global Properties

A surface or three-manifold has both local and global properties. **Local properties** are those observable within a small region of the manifold, whereas **global properties** require consideration of the manifold as a whole. Try out this definition on the following exercise. Note that a sphere and a plane differ both locally and globally, and both topologically and geometrically.

Exercise 3.7. A society of Flatlanders lives on a sphere. They

had always assumed they lived in a plane though, until one day somebody made one of the following discoveries. Which discoveries are local and which are global?

1. The angles of a triangle were carefully measured and found to be 61.2°, 31.7°, and 89.3°.
2. An explorer set out to the east and returned from the west, never deviating from a straight route.
3. As their civilization spread, the Flatlanders discovered the area of Flatland to be finite. □

The terms "local" and "global" are used most often in the phrases "local geometry" and "global topology". For example, a flat torus and a doughnut surface have the same global topology, but different local geometries. A flat torus and a plane, on the other hand, have the same local geometry but different global topologies. A three-torus has the same local geometry as "ordinary" three-dimensional space, but its global topology is different. The Flatlanders in Chapter 1 were discovering various global topologies for Flatland, but when Gauss surveyed the mountain peaks he was investigating the local geometry of our universe in the region of the Earth.

We can use the local/global terminology to restate the definition of a manifold. A two-dimensional manifold (i.e. a surface) is a space with the local topology of a plane, and a three-dimensional manifold is a space with the local topology of "ordinary" three-dimensional space. All two-manifolds have the same local topology, and all three-manifolds have the same local topology, but the local

topology of a two-manifold is different than that of a three-manifold. I should mention that three-manifolds serve as possible shapes for the universe precisely because their local topology matches that of ordinary space: we know almost nothing about the universe's global topology or its local geometry, but it's fair to assume that throughout the universe the local topology is just like the local topology of the "ordinary" space we occupy in the solar system.

Exercise 3.8. Define the concept of a one-dimensional manifold and give an example of one. □

Exercise 3.9. Compare an infinitely long cylinder to a plane. Do they have the same local geometry? (As usual we mean intrinsic local geometry, although you could also compare their extrinsic local geometries.) Do they have the same global topology? The same local topology? Which of the three types of discoveries listed in Exercise 3.7 could Flatlanders use to distinguish a cylinder from a plane? □

Exercise 3.10. Compare a three-torus made from a cubical room to one made from an oblong rectangular room. Do they have the same local topology? the same local geometry? the same global topology? the same global geometry? □

Homogeneous vs. Nonhomogeneous Geometries

A homogeneous manifold is one whose local geometry is the same

at all points. The local geometry of a **nonhomogeneous** manifold varies from point to point. A sphere is a homogeneous surface. The surface of an irregular blob is nonhomogeneous. A doughnut surface, while fairly symmetrical, is nonhomogeneous: it is convex around the outside but saddle-shaped near the hole. A flat torus, however, is homogeneous because it's flat at all points. The flat torus is more important in geometry than the doughnut surface precisely because it's homogeneous while the doughnut surface is not. Spheres are more important than surfaces of irregular blobs for the same reason. The sphere and the flat torus are the only homogeneous surfaces we have seen so far, but there will be plenty more. A major theme of this book is finding homogeneous geometries for manifolds which do not already have one.

Exercise 3.11. Is the three-torus a homogeneous three-manifold? □

Closed vs. Open Manifolds

Intuitively, **closed** means finite and **open** means infinite. Try out your intuition on the following exercise. You can check your answers in the back of the book.

Exercise 3.12. Which of the following manifolds are closed and which are open?

1. A circle
2. A line (the whole thing, not just a segment)
3. A two-holed doughnut surface

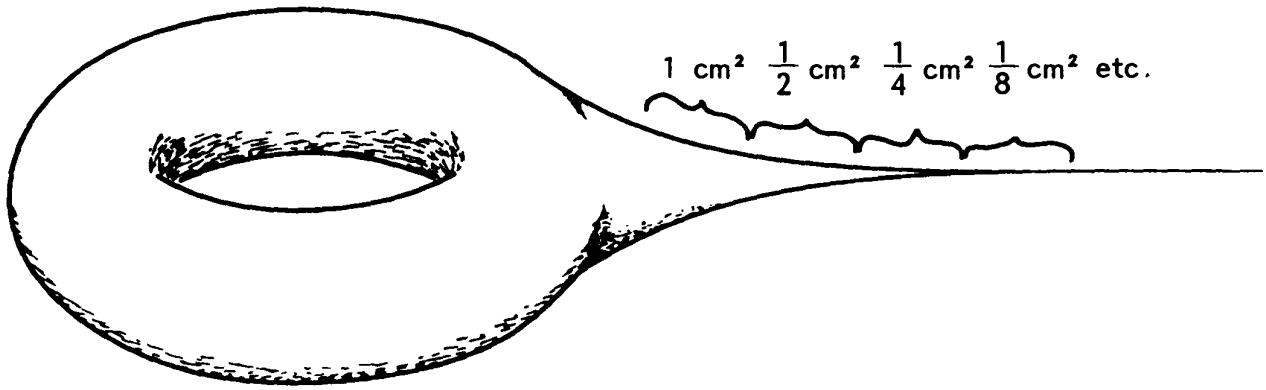


Figure 3.8: A doughnut surface with a "cusp" has a finite surface area even though the cusp is infinitely long.

4. A sphere
5. A plane
6. An infinitely long cylinder
7. A flat torus
8. Ordinary three-dimensional space
9. A three-torus. □

Unfortunately there are two complications to the simple idea of closed and open. One is that anything with an edge, such as a disk, is technically not even a manifold (it's a so-called manifold-with-boundary) and therefore does not count as either closed or open. Thus the terms "closed" and "open" imply that the manifold has no edges. This stipulation will not be an issue in this book, but in other books you may find that when an author makes a statement like "the flat torus is a closed surface", he is emphasizing not that the flat torus is finite, but that it has no

edges (unlike the square from which it was made, which does have edges).

The second complication is more interesting. It turns out that there are surfaces which are infinitely long, yet have only a finite area. A typical example is a doughnut surface with a so called "cusp" (Figure 3.8). The cusp is an infinitely long tube which gets narrower as it goes. The first centimeter of cusp has a surface area of 1 square centimeter (cm^2), the next centimeter of cusp has an area of $\frac{1}{2}$ cm^2, the next an area of $\frac{1}{4}$ cm^2, and so on. Thus, the total surface area of the cusp is $1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$ cm^2. What's important is not that the area of the cusp is precisely 2 cm^2, but that it's finite. For once you add in the (finite) area of the rest of the surface, you find that the surface as a whole has a finite area even though it's infinitely long. By convention a surface is classified as closed or open according to its distance across rather than its area, so the doughnut surface with a cusp is called open in spite of its finite area. After the following exercise we won't encounter any more cusps in this book.

Exercise 3.13. What would happen to A Square if he tried to take a trip down a cusp? □

This book deals mainly with closed manifolds, so from now on

"manifold" will mean "closed manifold",

unless explicitly stated otherwise.

4

Orientability

When our story left off in Chapter 1, our hero A Square and his fellow Flatlanders had just embarked on a Universal Survey of all of Flatland. The excitement was immense as the first survey party set out. But this excitement was nothing compared to the chaos that followed its return!

An old farmer living in an outlying agricultural district was the first one to run into the returning surveyors. He was going around a bend in the road and the surveyors were coming from the opposite direction. Fortunately no one was hurt in the collision. The farmer was a little annoyed that the surveyors didn't have the courtesy to keep to the proper side of the road, but his anger was quickly overcome by his joy at seeing them back safely, and by his interest in hearing their tales of adventure. He accompanied them into town.

As they approached the Flatsburgh City Limits, one of the surveyors noticed that the "Welcome to Flatsburgh" sign, the one that announced the Rotary Club meetings and all, had been replaced by a backwards version of the same thing. A "Welcome to Flatsburgh" sign, as it were.

"Those kids, always up to mischief," he chuckled.

"What's that you say?" asked the farmer, not sure he had heard properly.

"Oh, nothing. I was just amused at what some kids had done to that sign."

The farmer had no idea why the surveyor was so amused by the sign, but he decided not to make an issue of it.

The further the surveyors got into Flatsburgh, the more bewildered they became. *All* the signs were written backwards, and everyone, not just the old farmer, had taken to walking on the wrong side of the road. It was as if all of Flatsburgh had been mysteriously transformed into its mirror image while they were gone. Flatlanders in general tend to be superstitious, and for the surveyors this mirror reversal of Flatsburgh did not bode well.

Not that the citizens of Flatsburgh were any happier with the situation! They insisted that nothing the least bit unusual had happened in Flatsburgh while the surveyors were gone. It was the surveyors they found to be unusual, with their backwards writing and strange ideas about how Flatsburgh had been somehow transformed. In fact, they found the surveyors to be downright creepy, and, except for relatives and close friends, no one even

Figure 4.1: A mirror reversed Earth.

wanted to go near them.

As you might suspect, the Universal Survey was called off, and for the next three years no one went more than shouting distance from the civilized parts of Flatland . . .

(STORY TO BE CONCLUDED IN CHAPTER 5)

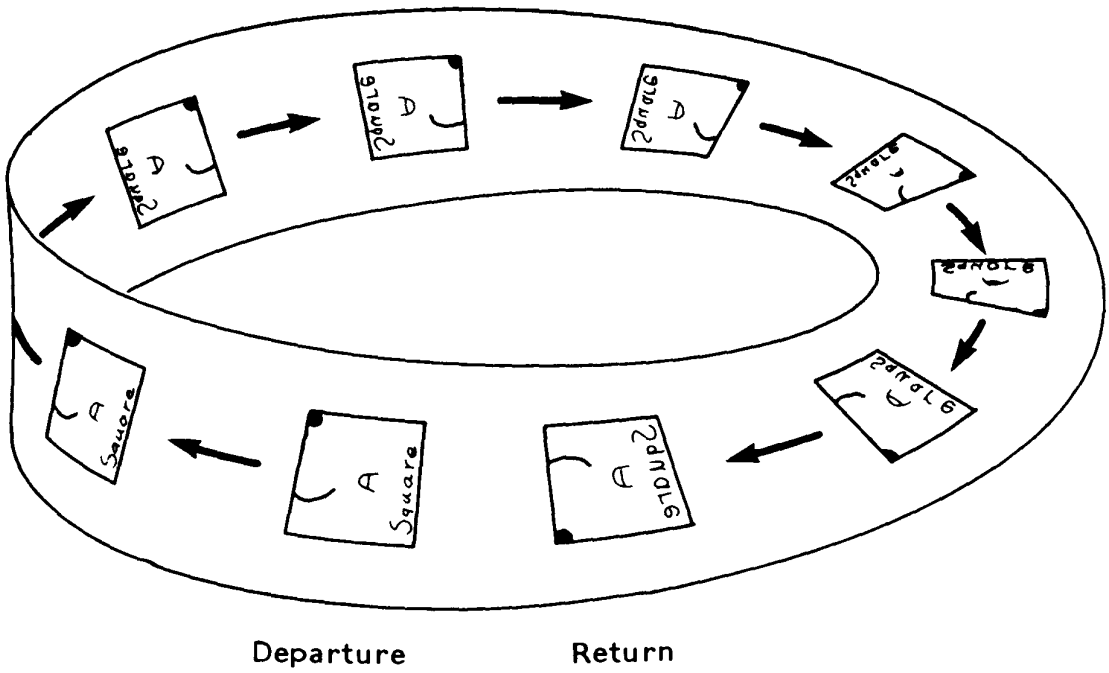


Figure 4.2: When A Square travels around a Möbius strip he comes back as his mirror image.

Exercise 4.1. Write a story in which you travel across the universe to an apparently distant galaxy, only to discover that you've made a complete trip around the universe and returned to our own galaxy. When you find the Earth, you're startled to see that it looks like Figure 4.1. What do you see when you land? Describe a walk through your hometown. What do people think of you? □

In the Flatland story, each surveyor came back to Flatland as his own mirror image. To see just how this occured, study Figure 4.2, which shows a swath of territory similar to the one the

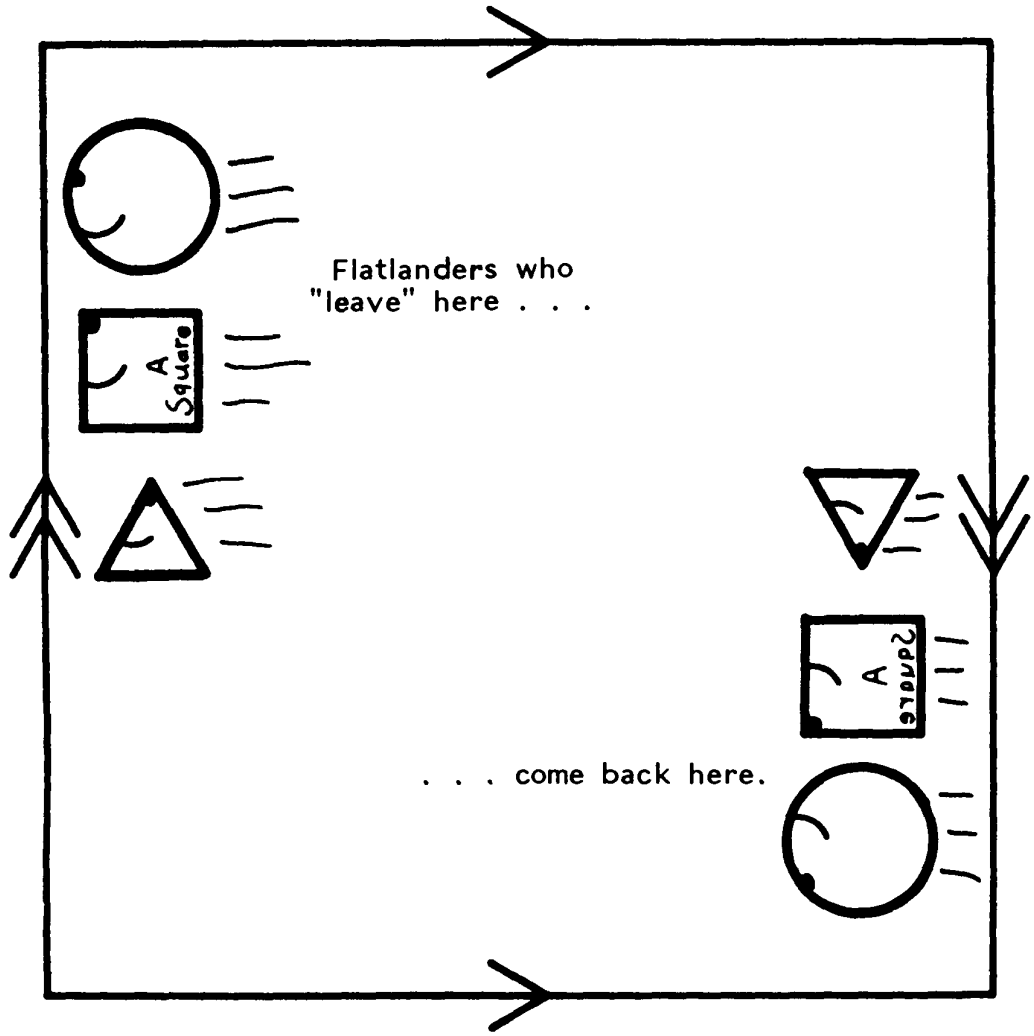


Figure 4.3: Glue the edges of this square so that the arrows match up and you'll get a Klein bottle. A Flatlander travelling off to the left comes back from the right as his mirror image.

surveyors traversed. This swath of territory is a **Möbius strip.** [A true Möbius strip has zero thickness. If you mistakenly imagine it to have a slight thickness--like a Möbius strip made from real paper--then you'll run into problems with A Square returning from his journey on the opposite side of the paper from which he

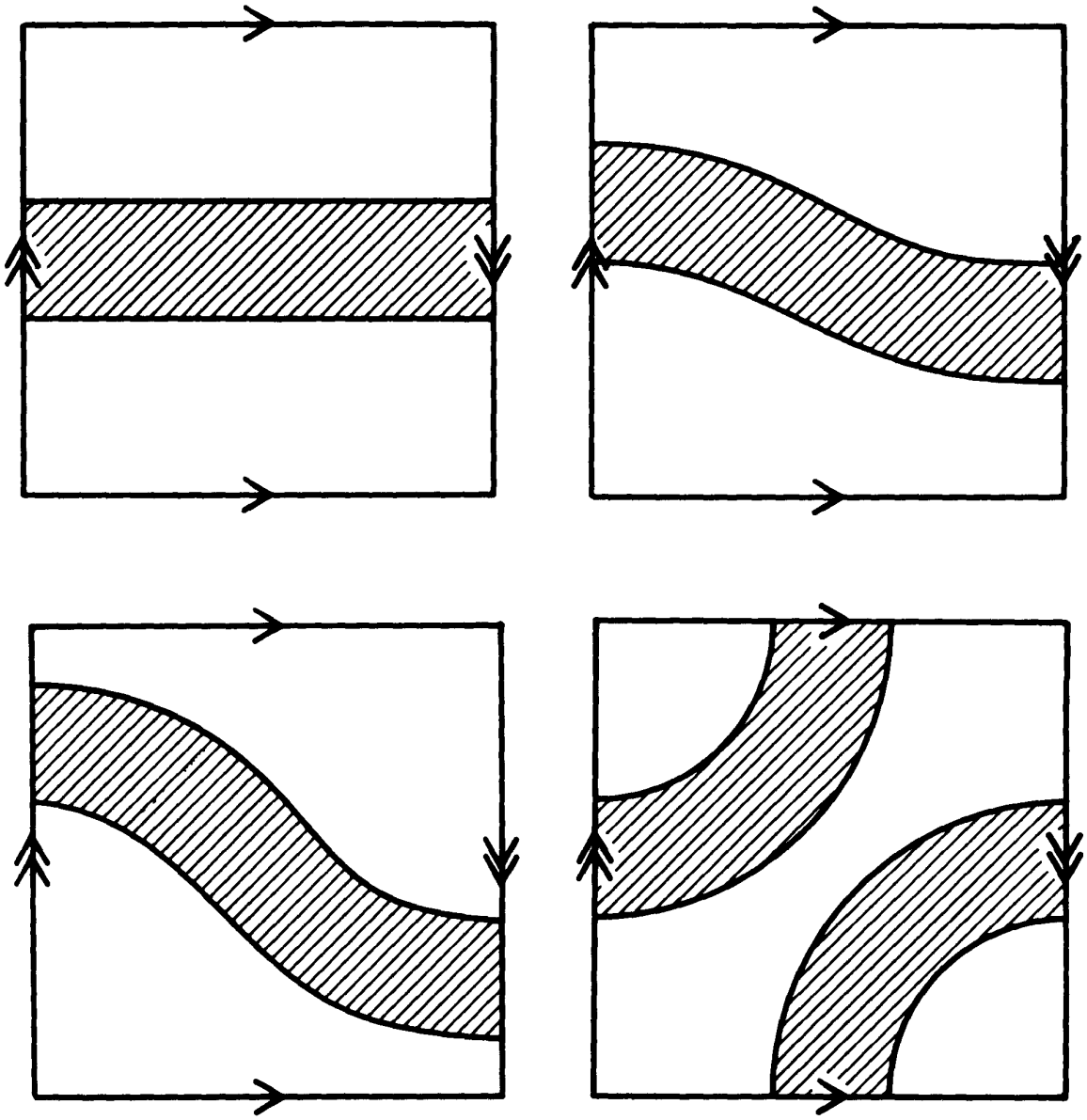


Figure 4.4: A Klein bottle contains many Möbius strips.

started. As long as the Möbius strip is truly two-dimensional (i.e. no thickness) this problem does not arise.]

The question is, in what sort of surface could a Flatlander traverse a Möbius strip? A Klein bottle is one example. You can make a Klein bottle from a square in almost the same way we made a

flat torus from a square. Only now the edges are to be glued so that the arrows shown in Figure 4.3 match up. As with the flat torus, I don't mean that these gluings should actually be carried out in three-dimensional space; I mean only that a Flatlander heading out across one edge comes back from the opposite one. The top and bottom edges are glued *exactly* as in the flat torus: when a Flatlander crosses the top edge he comes back from the bottom edge and that's all there is to it. The left and right edges, though, are glued with a "flip". When a Flatlander crosses the left edge he comes back from the right edge, but he comes back mirror reversed. A Klein bottle contains many Möbius strips (see Figure 4.4).

Exercise 4.2. Which of the positions in Figure 4.5 constitute a winning three-in-a-row in Klein bottle tic-tac-toe? □

Exercise 4.3. Imagine the chessboard in Figure 2.6 to be glued to form a Klein bottle rather than a torus. Which black pieces does the white knight threaten now? Which black pieces threaten it? □

There's a nice way to analyze positions in Klein bottle tic-tac-toe and chess. For example, say there's a Klein bottle tic-tac-toe game in progress. The position is as shown on the left side of Figure 4.6 and it's X's turn to move. Rather than hastily taking the upper right hand square, X pauses to carefully analyze the situation. He notes that the board's top edge is glued to its bottom

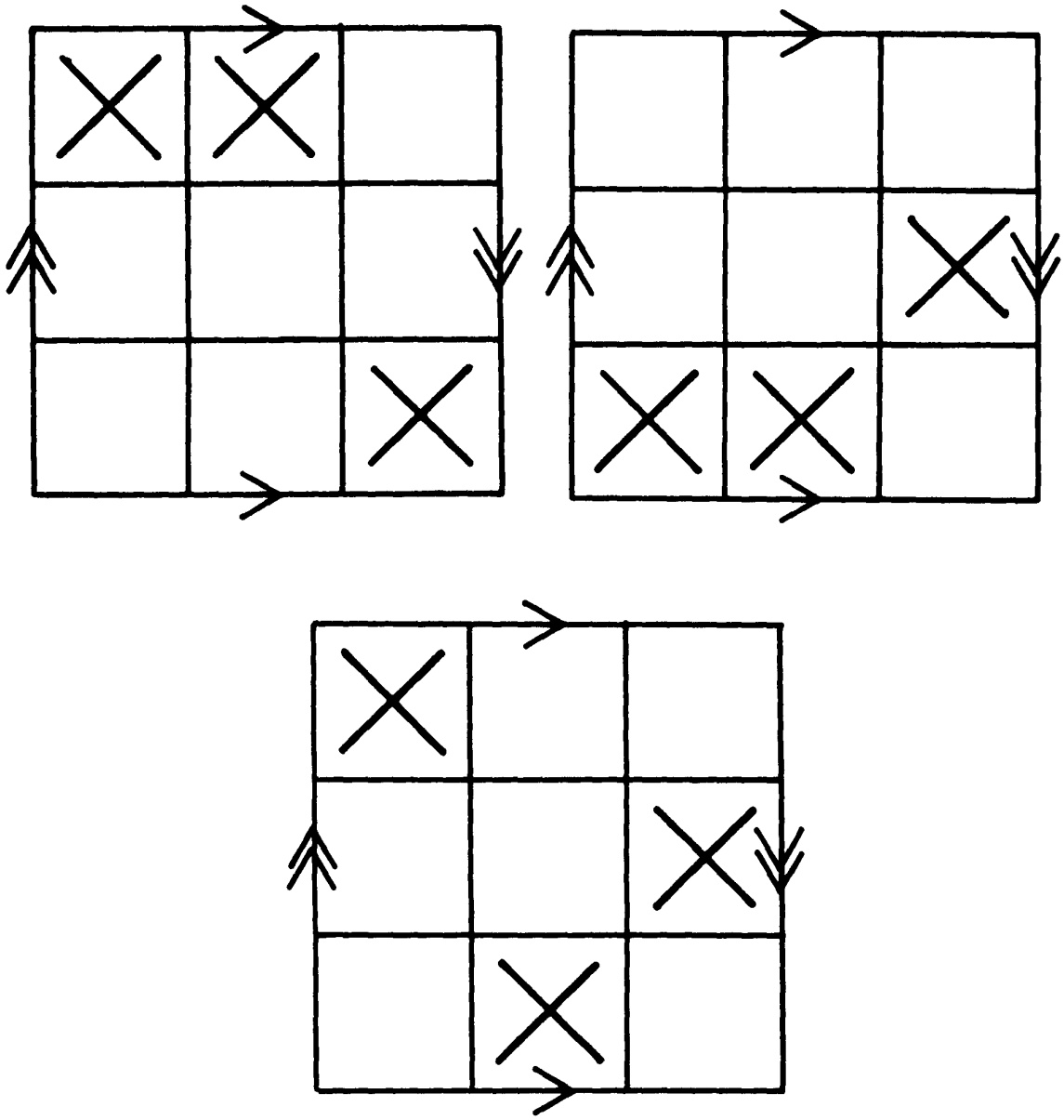


Figure 4.5: Which of these are winning positions in Klein bottle tic-tac-toe?

edge; therefore he draws a copy of the board above the original so that he can see more clearly how the top and bottom edges connect. He draws another copy below the original board for the same reason. He does the same on the left and the right, being careful

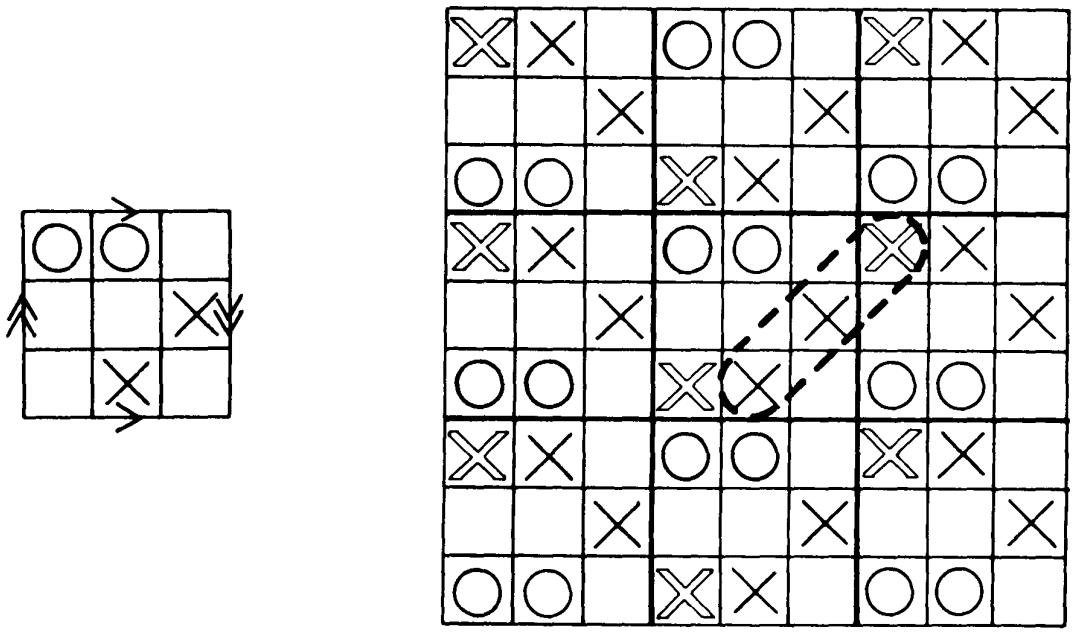


Figure 4.6: Analyzing a Klein bottle tic-tac-toe game.

to flip these copies top to bottom to account for the fact that he's playing tic-tac-toe on a Klein bottle and not on a torus. He continues this process, attaching new copies of the board to the old ones. In principle he could continue forever, but he knows that nine copies are always enough to see clearly what is going on. After examining the final picture, X gleefully takes the lower left hand corner and wins immediately.

Exercise 4.4. Use the above technique to find X's best move in each situation shown in Figure 4.7. □

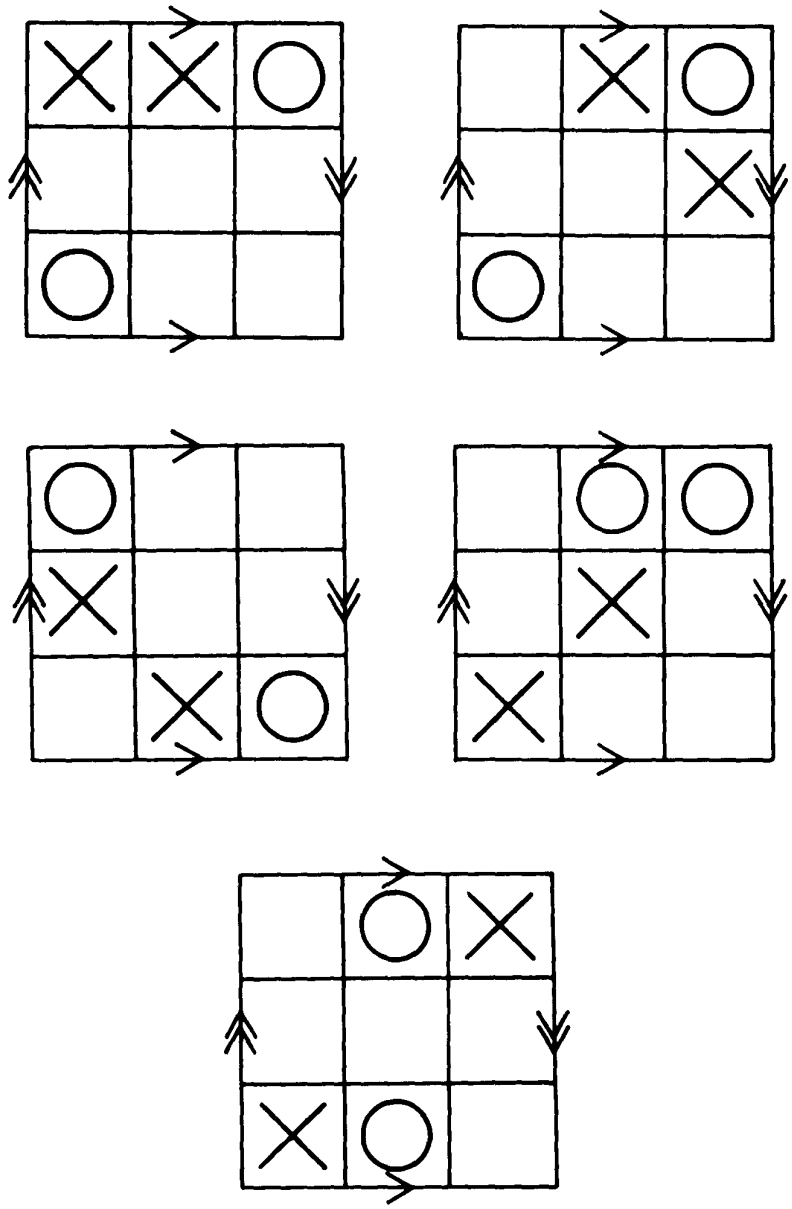


Figure 4.7: Find X's best move in each game.

Exercise 4.5. Find a friend and play a few games of Klein bottle tic-tac-toe. □

Exercise 4.6. Find a friend and play a few games of Klein bottle chess. □

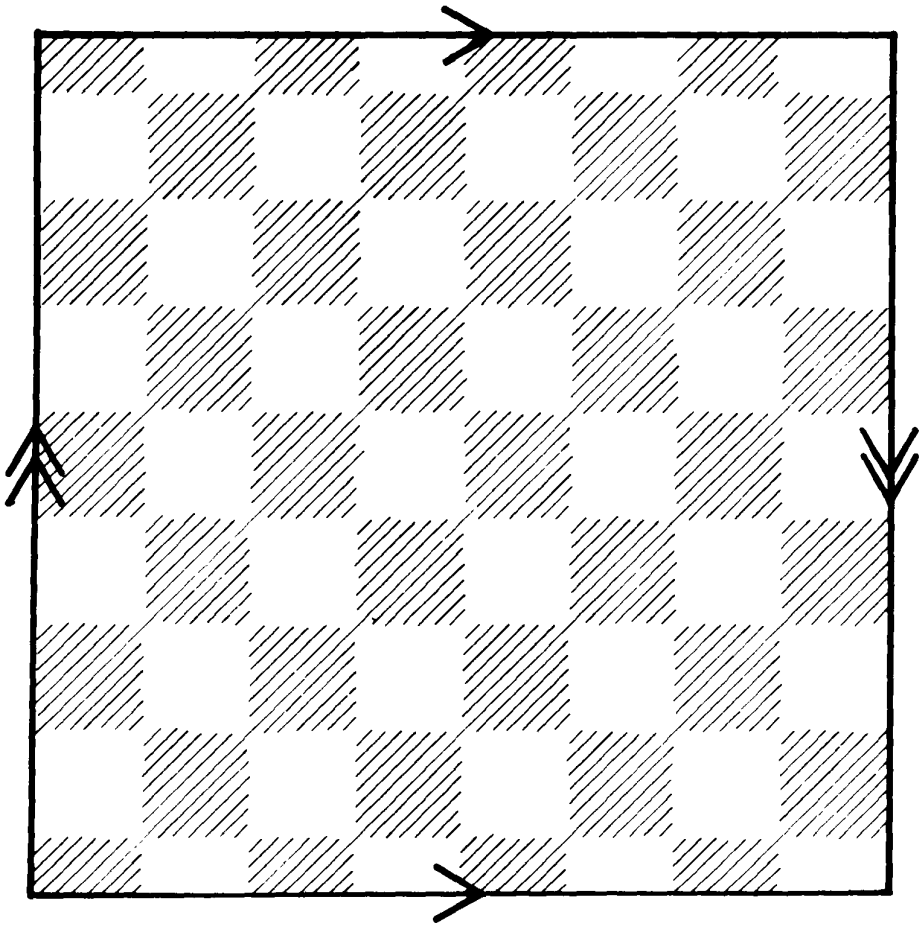


Figure 4.8: A new way to make a Klein bottle chessboard. The half-width row at the top gets glued to the half-width row at the bottom to become a normal row in the Klein bottle itself.

Exercise 4.7. When a bishop goes out the upper right hand corner of a Klein bottle chessboard, where does he return? Hint: Label the corners of the board and draw a picture like Figure 4.6. □

Exercise 4.8. In Exercise 4.3 you found that the knight and one of the bishops threatened each other simultaneously on the Klein bottle chessboard. How can this be? Shouldn't a knight threaten only pieces on an oppositely colored square, while a bishop

threatens only pieces on the same color square? Can a knight and a bishop ever threaten each other simultaneously if the chessboard is constructed as in Figure 4.8? Can a knight and a rook simultaneously threaten each other on this new board? □

The global topology of most manifolds must be understood intrinsically. But we can assemble a Klein bottle in three-dimensional space in order to apprehend its global topology more directly. Imagine making a Klein bottle from a square of rubber. Roll the square into a cylinder and glue the top edge to the bottom edge. That was the easy part. Now pass the cylinder through itself (as shown in Figure 4.9), and glue its ends together. The self-intersection is unpleasant, but there's no way to embed a Klein bottle in three-dimensional space without it. (As we'll see in Chapter 9, one *can* embed a Klein bottle in four-dimensional space with no self-intersection.) Note: Figure 4.9 was included to make the Klein bottle's global topology a little more real, but for most purposes it's better to picture the Klein bottle as a square with opposite edges glued appropriately.

The local geometry of the Klein bottle is everywhere flat. Thus, a Flatlander doing local experiments could not distinguish a Klein bottle from either a torus or a plane. It's important to note that the Klein bottle is flat not only in the region corresponding to the middle of the square, but also in the regions where the edges and corners meet. Figure 4.10 shows how the four corners fit together. (By the way, the "seams" created by the gluing should

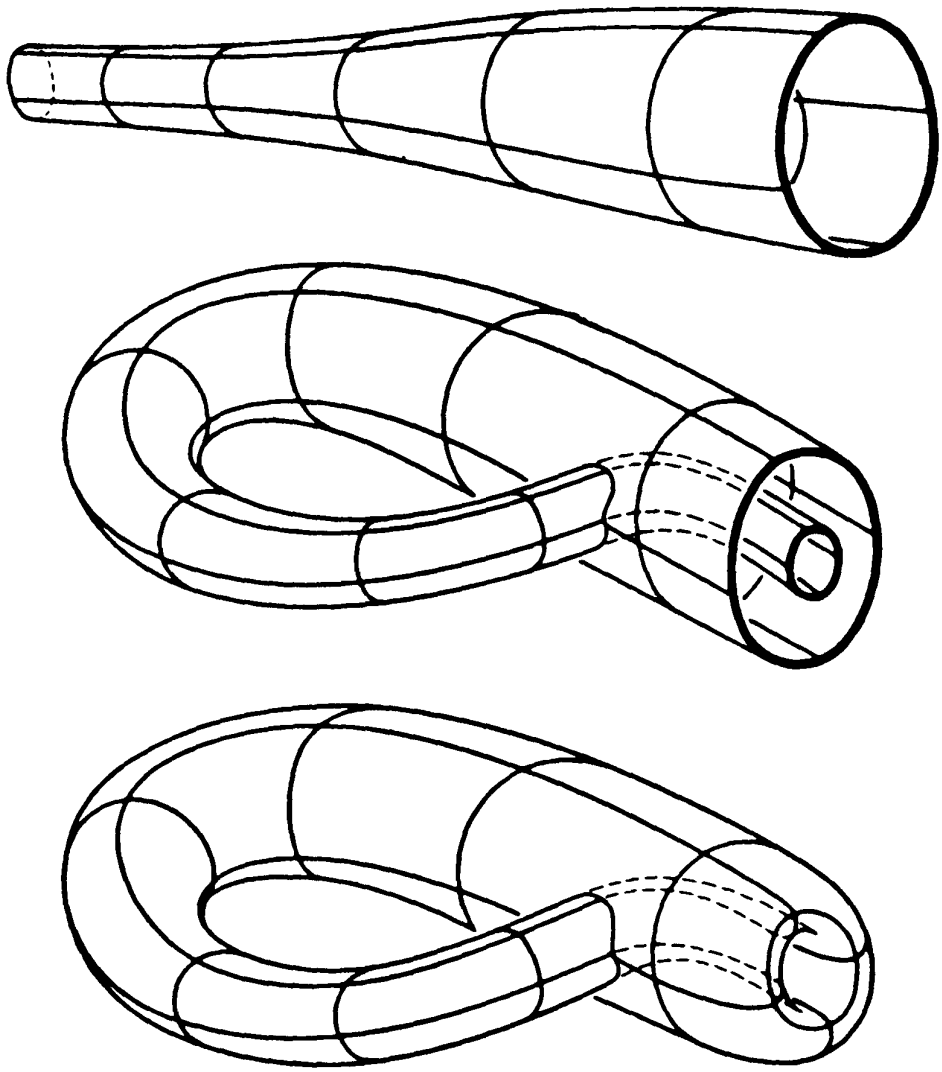


Figure 4.9: Gluing up a rubber Klein bottle.

all be erased. They are not part of the Klein bottle itself.) The Klein bottle is our third example of a homogeneous surface, the other two being the (flat) torus and the sphere.

A path in a surface or three-manifold which brings a traveller back to his starting point mirror-reversed is called an **orientation-**

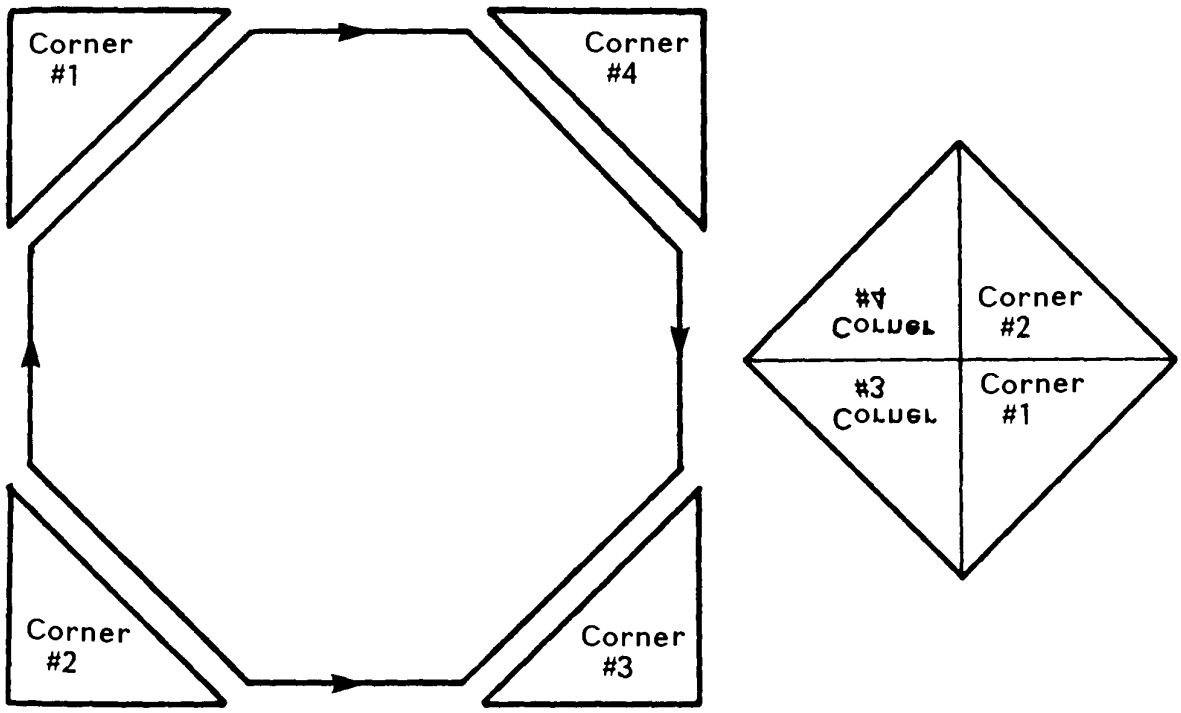


Figure 4.10: How the square's corners fit together in the Klein bottle.

reversing path. Manifolds which don't contain orientation-reversing paths are called **orientable**; manifolds which do are called **nonorientable.** Thus, a sphere and a torus are orientable surfaces. A Klein bottle is a nonorientable surface. The three-torus is an orientable three-manifold. But what about a nonorientable three-manifold?

We can make a nonorientable three-manifold in much the same way that we made the Klein bottle. Start with the block of space inside a room. Imagine the left wall glued to the right wall, and the floor glued to the ceiling, just like in the three-torus. Only

now imagine the front wall glued to the back wall with a side-to-side flip. If you walk through the front wall you'll return from the back wall mirror-reversed! Walk through it again and you'll come back in your usual condition.

Exercise 4.9. What do you see when you look through the back wall of this three-manifold? What about the other walls? □

Imagine this new three-manifold to contain a jungle gym like the one we built in the three-torus. This would be great fun to play in with a friend. Sometimes when you ran into your friend he would be right-handed, and other times you'd find him to be left-handed. You could play a special form of tag in which catching someone doesn't count unless you can guess which hand is his left hand. Obviously there's a big advantage for people who part their hair in the middle, and T-shirts with writing on them would be out of the question.

Exercise 4.10. Think up other fun things to do in a nonorientable three-manifold. You could, for example, steal one shoe from your friend during the night, take it around an orientation-reversing path, and quietly replace it before dawn. □

The **projective plane** is a surface which is locally like a sphere, but has a different global topology. It's made by gluing together the opposite points on the rim of a hemisphere (Figure 4.11).

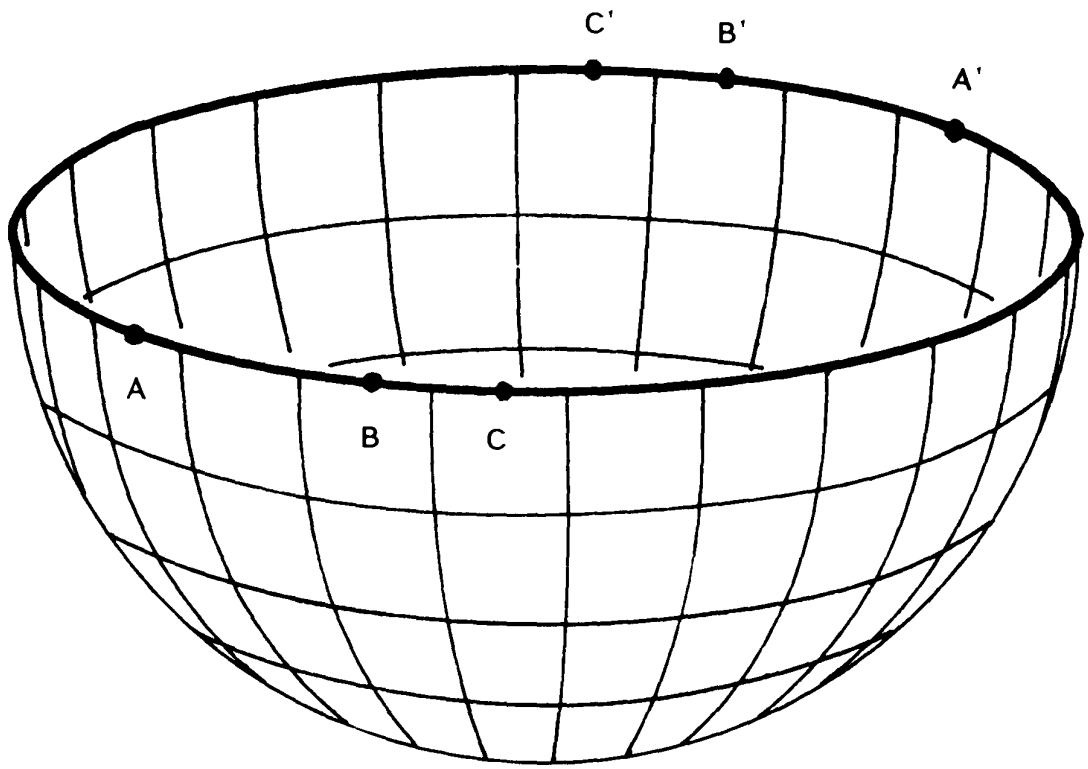


Figure 4.11: The projective plane is made by gluing together opposite points on the rim of a hemisphere.

Figure 4.12 shows what this gluing looks like locally, along a short section of the rim. We can show the gluing along any section of the rim we like, but we can't show the entire gluing at once because of its peculiar global properties. Thus you should concentrate on understanding how opposite sections of rim fit together, rather than trying to visualize the whole thing at once the way you'd visualize a sphere. The most important thing is that the hemisphere's geometry matches up perfectly when opposite sections of rim are glued, so the projective plane has the same local geometry as a sphere, even along the "seams" where the gluing

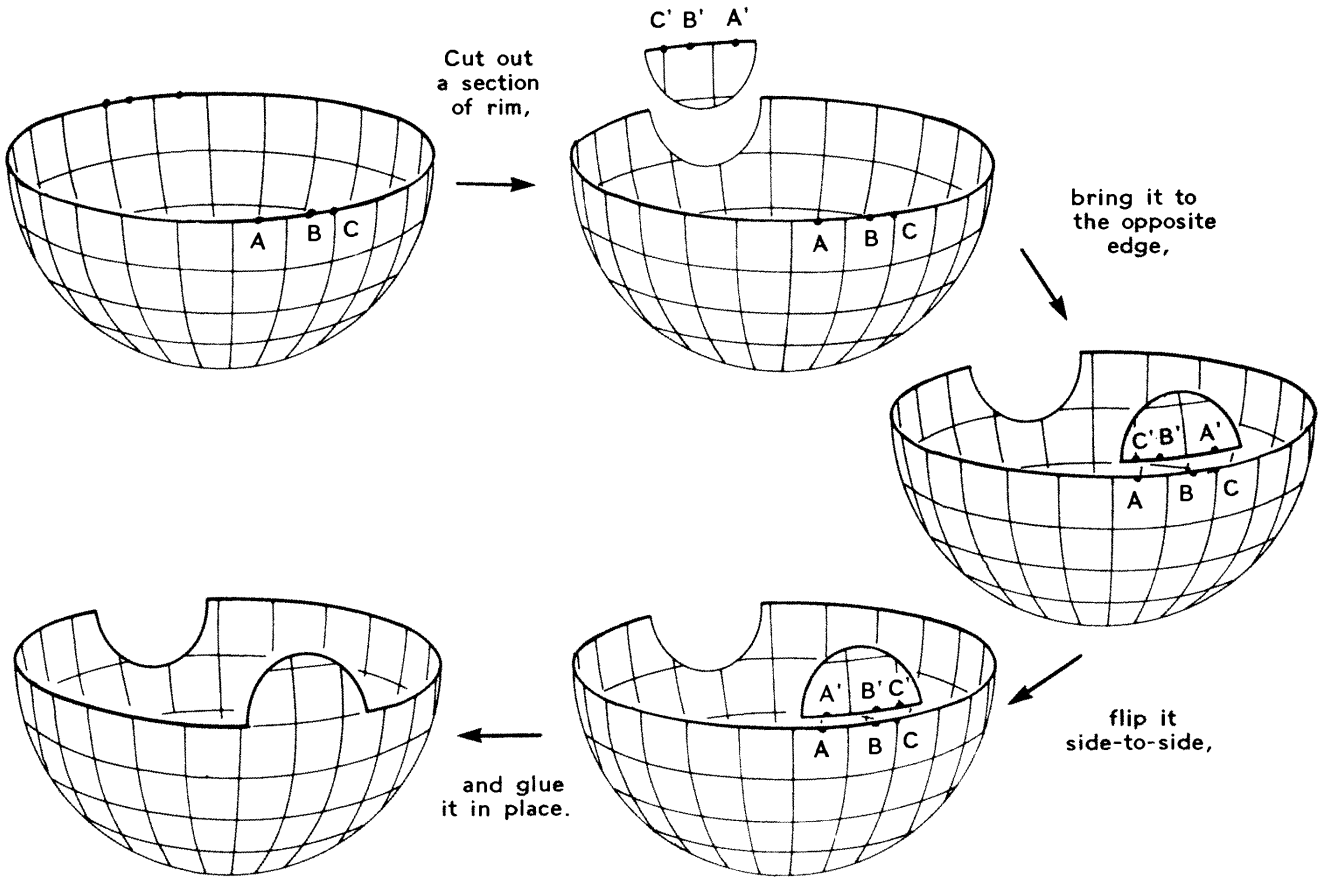


Figure 4.12: How to glue opposite sections of rim.

took place. The projective plane is our fourth homogeneous surface.

Exercise 4.11. Is the projective plane orientable? That is, if a Flatlander crosses the "rim", does he come back normal or mirror-reversed? □

Exercise 4.12. A Flatlander lives on a projective plane, which we visualize as a hemisphere with opposite rim points glued. The Flatlander's house is at the "south pole". One day he leaves his house and travels in a straight line (i.e. a geodesic) until he gets back home again. At what point along the route is he furthest from home? □

Exercise 4.13. A society of Flatlanders lives on a projective plane. They plan to build two fire stations. For maximal effectiveness the fire stations should be as far apart as possible. Where might the Flatlanders build them? (Be careful: opposite points on the rim of a hemisphere represent the same point in the projective plane.) How should three fire stations be positioned for maximal effectiveness? □

Exercise 4.14. A Flatlander knows he lives on either a sphere or a projective plane. How can he tell which it is? A second Flatlander knows he lives on either a projective plane or a Klein bottle; how can he decide? □

Exercise 4.15. So far we have seen four homogeneous surfaces: the sphere, the torus, the Klein bottle and the projective plane.

	orientable	nonorientable
curved local geometry		
flat local geometry		

Figure 4.13: The sphere, torus, Klein bottle and projective plane can fill this table. Which goes where?

Use them to fill in the table in Figure 4.13. □

If we are interested in only the topological properties of the projective plane, we can flatten the hemisphere into a disk, still remembering to glue opposite boundary points (Figure 4.14). The main advantage of doing this is that a disk is easier to draw than a hemisphere.

Still working topologically we can construct **projective three-space** by gluing opposite boundary points of a solid three-dimensional ball. We'll study its geometry in Chapter 14 when we study the geometry of the hypersphere.

Exercise 4.16. Is projective three-space orientable? If you cross the boundary, how do you come back? □

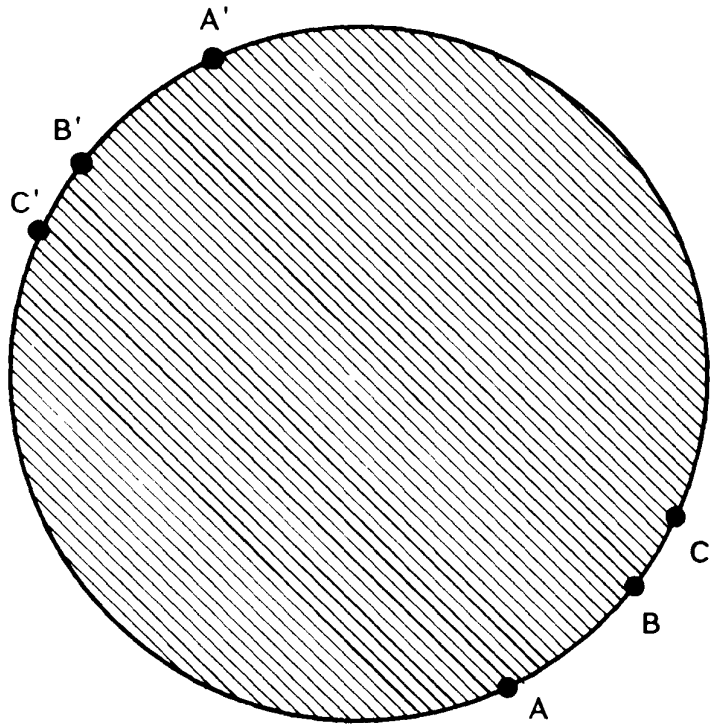


Figure 4.14: Topologically, a projective plane is a disk with opposite boundary points glued.

Exercise 4.17. Is orientability a local or a global property? Is it topological or geometrical? □

5

Connected Sums

Conclusion of the Flatland story:

The mirror-reversed surveyors adapted to their new condition more quickly than most had expected. The hardest part was learning to write properly, but even this became routine after a while. And with their increased competence came a greater acceptance on the part of the community. Things returned to normal.

In fact, as the years went by people even got a little adventurous. Almost every week somebody or another was heading out on an expedition. There were, of course, occasional incidents of explorers coming back mirror-reversed, but this was no longer a disaster. The reversed explorers were quickly rehabilitated. Besides, the reversal incidents were limited to those who passed through a certain "Reversing Region". The rest of Flatland seemed harmless enough, and trips there became quite common.

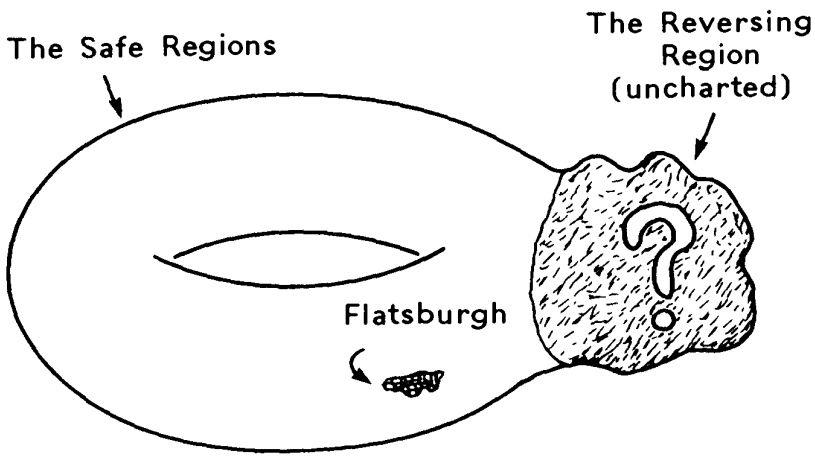


Figure 5.1: The safe regions of Flatland were charted first.

To protect travellers from accidental reversal, the Reversing Region was marked with clumps of stones spaced ten paces apart along its boundary. Once this was done, even the most timid Flatlanders enjoyed travelling about in the safe regions.

It wasn't long before an official survey of the safe regions was undertaken. The surveyors found that the safe regions resembled a doughnut surface (Figure 5.1), in accordance with one of the earliest proposed theories on the shape of Flatland.

Others were quick to point out, however, that this didn't mean that Flatland as a whole was a doughnut surface. For example, if the Reversing Region also resembled a doughnut surface, then Flatland would be a two-holed doughnut surface (Figure 5.2).

Curiosity overcame fear, and a survey of the Reversing Region was begun. Only the boldest of the surveyors volunteered for the job. It wasn't that they were afraid of getting reversed--that was common enough by now. They were afraid of getting reversed

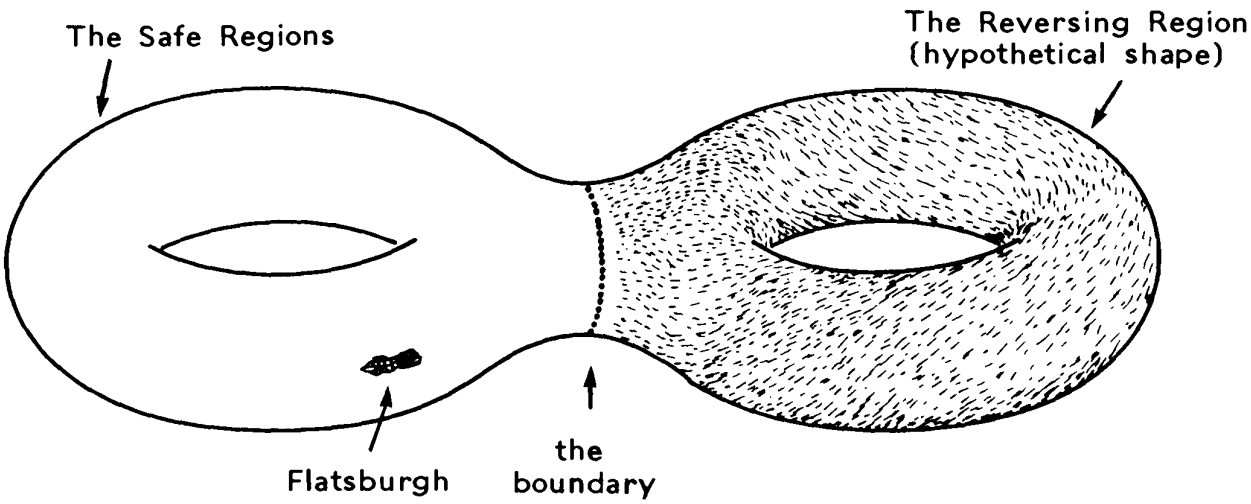


Figure 5.2: People were quick to point out that if the safe region and the Reversing Region each resembled a doughnut surface, then Flatland as a whole would be a two-holed doughnut surface.

twice! The popular consensus was that a second reversal would result in certain death. (There was a minority opinion that a second reversal would simply restore the victim to his original state, but this opinion didn't sell as well in the newspapers.)

The survey was divided into two stages. The purpose of the first stage was to get a rough idea of just how big the Reversing Region was, and to divide it into sectors to be mapped in detail during the second stage.

The first stage went smoothly, even though three of the surveyors came back mirror-reversed. But these reversed surveyors were brave enough to go back into the Reversing Region to help with the detailed surveying of the second stage. In fact, they even drew lots to see who was the bravest and would go back in first!

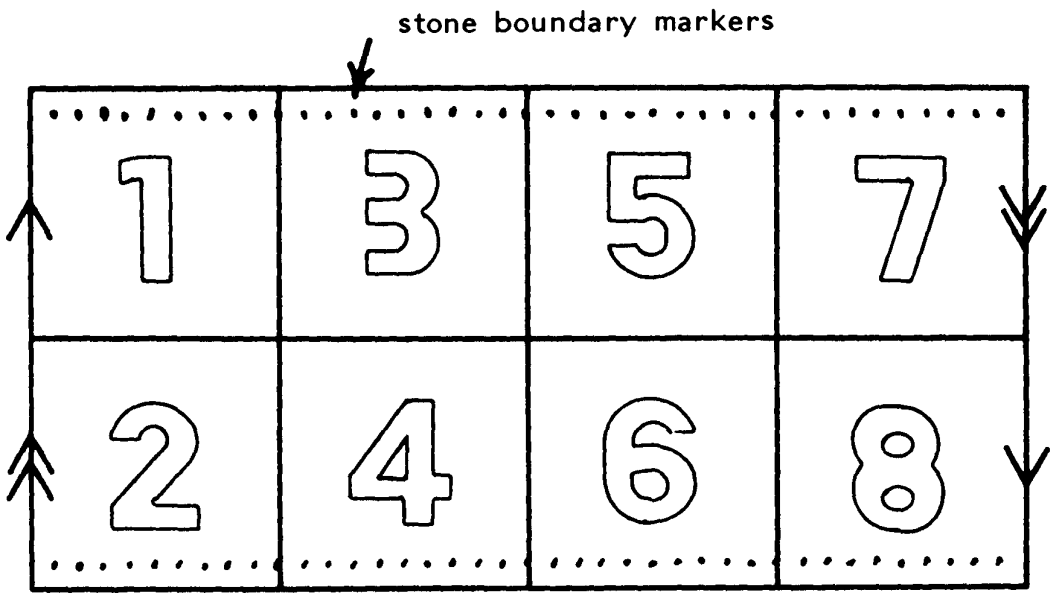


Figure 5.3: How the eight sectors pieced together.

After the completion of the first stage the Reversing Region was divided into eight sectors. During the second stage a separate team was sent to each sector to map it in detail. The whole operation had an air of Russian roulette, with each team wondering whether they were the ones mapping the dangerous sector that did the reversing. To everyone's surprise--and relief--all eight teams reported their respective sectors to be perfectly normal!

It was only when they compiled, consolidated, and compared the data from the different sectors that things got mysterious. They found the sectors connected up as shown in Figure 5.3. The mysterious thing was that Sector 1 connected to Sector 8, not Sector 7; it was Sector 2 that connected to Sector 7! They connected in such a confusing way!

Eventually confusion gave way to enlightenment. The Flatlanders realized that the mirror-reversal phenomenon wasn't so mysterious after all. It was simply that the space of Flatland connected up with itself in such a way that anyone taking a trip around the Reversing Region would come back with his left side where his right side was, and his right side where his left side was. The Flatlanders had discovered the Möbius strip!

This was an immense intellectual achievement. But it was a very practical achievement as well. The reversed surveyors were all sent on a trip around the Reversing Region to restore them to their original condition. Thereafter the Reversing Region was used mainly for pranks and other amusements.

Thus, the Universal Survey was complete: the surveyors had established beyond a doubt that Flatland consists of two regions, one a Mobius strip, and the other resembling a torus. The Flatlanders lived happily and peacefully forever after.

THE END

NOTE: All surfaces in this chapter will be considered topologically, so you may bend and twist them however you like!

As the Flatlanders pointed out in Figure 5.2, a two-holed doughnut surface bears a strong resemblance to two one-holed doughnut surfaces stuck together. In fact, we can make a two-holed doughnut surface from two one-holed ones by cutting a disk out of each and gluing together the exposed edges (see Figure 5.4). This operation is called a **connected sum.**

Exercise 5.1. What do you get when you form the connected sum of a two-holed doughnut surface and a one-holed doughnut surface? What is the connected sum of a six-holed doughnut surface and an eleven-holed one? □

Exercise 5.2. What do you get when you form the connected sum of a two-holed doughnut surface and a sphere? How about a Klein bottle and a sphere? A projective plane and a sphere? □

Exercise 5.3. The purpose of this exercise is to find out what you get when you cut a disk out of a projective plane; in Exercise 5.4 you will use this information to deduce what the connected sum of two projective planes is. Get some scratch paper, and work your way through the following steps, drawing a picture for each one.

1. Draw a (topological) projective plane as a disk with opposite edge points glued.
2. Remove a small disk from the center of the projective plane.
3. Cut what remains into two curved pieces as shown in Figure 5.5. Label the edges with arrows as shown.

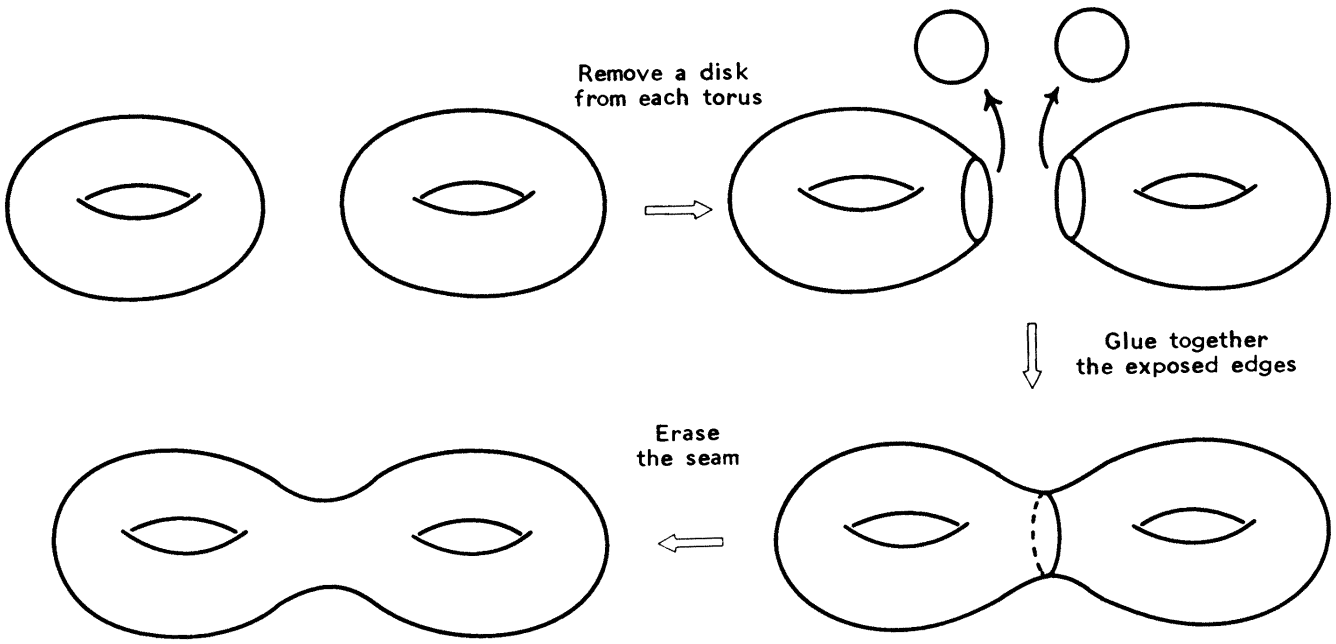


Figure 5.4: How to convert two one-holed doughnut surfaces into a single two-holed one.

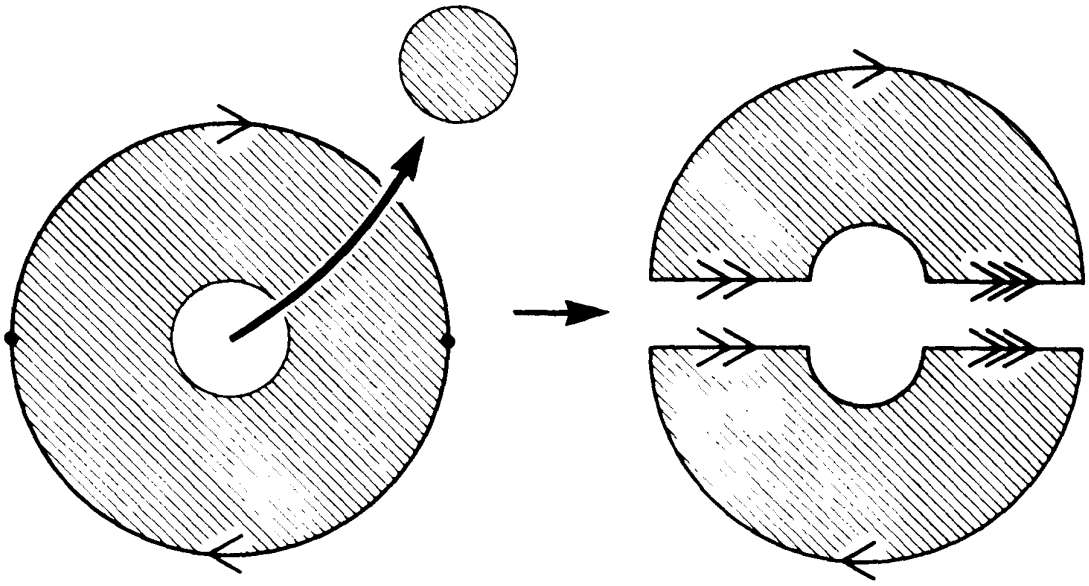


Figure 5.5: Remove a disk from a projective plane. Then cut what remains into two pieces as shown.

4. Straighten each curved piece into a rectangle. We're studying the *topological* properties of the projective plane, so it's OK to bend the pieces as if they were made of rubber. Just don't lose track of which arrows are on which edges.
5. Physically glue together the two long edges labelled with single arrows. You'll have to flip one piece over to do this.
6. Physically glue together the edges with the double and triple arrows. You'll have to do this in three-dimensional space--it can't be done in the plane.

The thing you end up with is topologically identical to the projective plane with the disk removed. What is its usual name? □

Exercise 5.4. What is the connected sum of two projective planes? Hint: You start with two projective planes, cut a disk out of each,

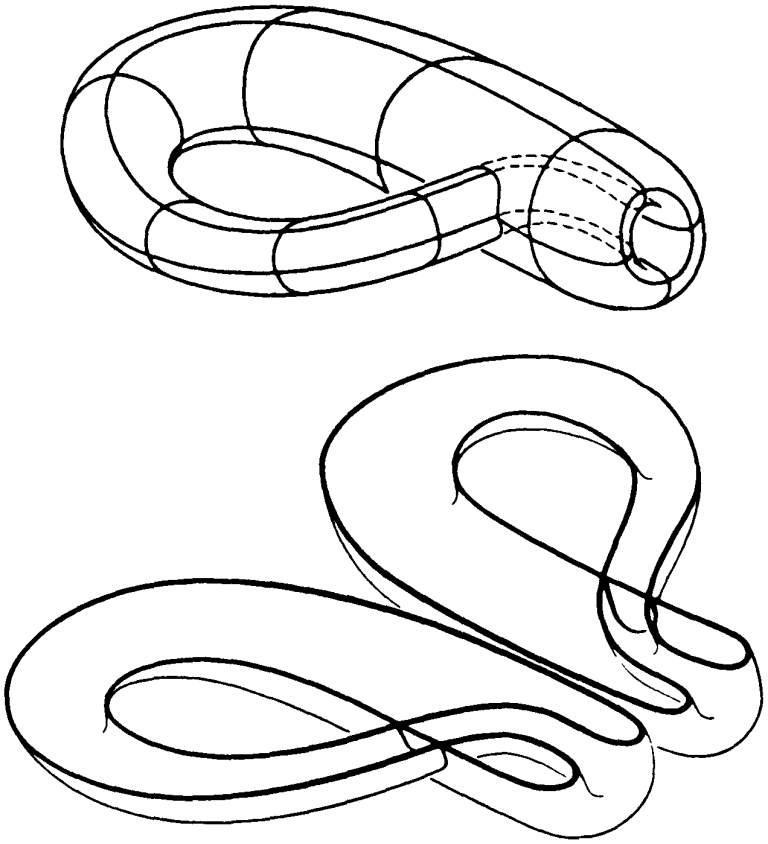


Figure 5.6: Cutting a Klein bottle in two.

and get two of the things you discovered in the previous exercise. Now refer to Figure 5.6 and the following limerick to help decide what you get when you glue the two things together.

A mathematician named Klein
Thought the Möbius strip was divine.
Said he, "If you glue
The edges of two
You'll get a weird bottle like mine." □

Exercise 5.5. In the story at the beginning of this chapter,

Flatland was the connected sum of what two surfaces? □

Most simple manifolds have shorthand names, usually written with a superscript to indicate their dimension. For example, the surfaces we've studied are

E^2	The (Euclidean) plane
S^2	The sphere
T^2	The torus
K^2	The Klein bottle
P^2	The projective plane
D^2	The disk

The abbreviations are pronounced "E-two", "S-two", "T-two", etc. By the way, there is no topological difference between a doughnut surface and a flat torus, so the abbreviation "T^2" may refer to either or both of them depending on the context. The three-manifolds we've seen are

E^3	"Ordinary" three-dimensional (Euclidean) space
T^3	The three-torus
D^3	A solid ball (i.e. a three-dimensional "disk")
P^3	Projective three-space

The nonorientable three-manifold we studied has a name, too; its name describes its structure as a so-called "product" and will be

revealed in the next chapter. Even one-dimensional manifolds have abbreviations, namely

E^1	The line
S^1	The circle
I	The interval, i.e. a line segment with both endpoints included.

The connected sum operation is abbreviated by a "#" symbol. For example, a two-holed doughnut surface is $T^2 \# T^2$ because it's topologically the connected sum of two tori. ($T^2 \# T^2$ is read "the connected sum of two tori" or simply "T-two connect-sum T-two".) Similarly, a three-holed doughnut surface is $T^2 \# T^2 \# T^2$. The topology of Flatland is succinctly written as $T^2 \# P^2$. One can even write equations with this notation. For example, in Exercise 5.4 you found that $P^2 \# P^2 = K^2$.

Exercise 5.6. State the results of Exercise 5.2 in the above notation. □

Sometime around the turn of the century, mathematicians discovered that *every* conceivable surface is a connected sum of tori and/or projective planes! (The sphere counts as a connected sum of zero tori and zero projective planes. I know this sounds hokey, but it is convenient. And it's not that unreasonable in light of

Exercise 5.2.) In other words, the table below provides a complete list of all possible surfaces. Any surface you might come up with is topologically equivalent to a surface in the table. The Klein bottle, for example, is equivalent to $P^2\#P^2$, which occurs as the third entry in the first row.

Number of Projective Planes

Number of Tori	0	1	2	3	...
0	S^2	P^2	$P^2\#P^2$	$P^2\#P^2\#P^2$	...
1	T^2	$T^2\#P^2$	$T^2\#P^2\#P^2$	$T^2\#P^2\#P^2\#P^2$	...
2	$T^2\#T^2$	$T^2\#T^2\#P^2$	$T^2\#T^2\#P^2\#P^2$	⋮	
3	$T^2\#T^2\#T^2$	$T^2\#T^2\#T^2\#P^2$	⋮	ETC.	
⋮	⋮	⋮	⋮	⋮	

Exercise 5.7. Find a surface in the table which is topologically equivalent to (a) $K^2 \# P^2$, (b) $K^2 \# T^2$, (c) $K^2 \# K^2$. □

Does this list contain duplications? For example, might $T^2 \# P^2$ really be the same surface as, say, $P^2 \# P^2 \# P^2$? Surprisingly enough, the list *does* contain duplications, and it's true that $T^2 \# P^2 = P^2 \# P^2 \# P^2$!

Exercise 5.8. In this exercise you discover that $T^2 \# P^2 = K^2 \# P^2$, and in Exercise 5.9 you'll use this information to deduce that $T^2 \# P^2 = P^2 \# P^2 \# P^2$. Convince yourself that each picture in Figure 5.7 is what its caption says it is. From Exercise 5.3 you

can deduce that gluing a disk to the edge of a Möbius strip converts the Möbius strip into a projective plane. Therefore T^2 # Möbius will become $T^2 \# P^2$ if you glue a disk to its edge, and K^2 # Möbius will become $K^2 \# P^2$ if you glue a disk to its edge. Since T^2 # Möbius and K^2 # Möbius are topologically the same (study Figure 5.8 to see why), $T^2 \# P^2$ and $K^2 \# P^2$ must also be the same. (Oddly enough, this does *not* mean that T^2 and K^2 are the same! Unlike for addition and multiplication of numbers, there is no "cancellation law" for connected sums.) □

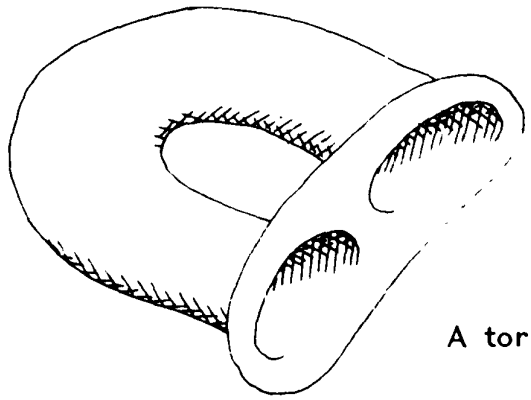
Exercise 5.9. Verify that $T^2 \# P^2 = P^2 \# P^2 \# P^2$ as claimed a couple paragraphs back. (Hint: This is easy! Simply combine Exercises 5.4 and 5.8.) □

Exercise 5.10. Assuming every surface is a connected sum of tori and/or projective planes, deduce that every surface is a connected sum of either tori *only* or projective planes *only*. That is, every surface is topologically equivalent to some surface on the following two column list:

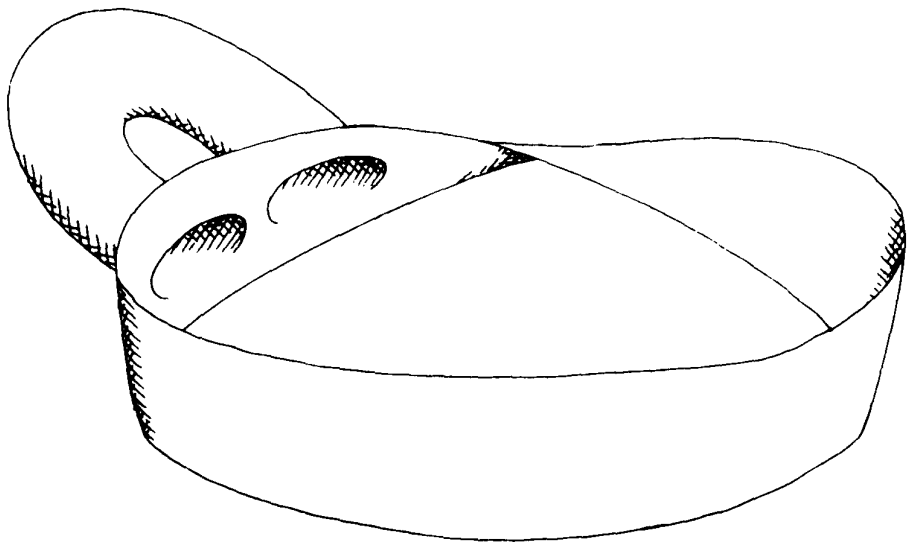
S^2

T^2	P^2
$T^2\#T^2$	$P^2\#P^2$
$T^2\#T^2\#T^2$	$P^2\#P^2\#P^2$
$T^2\#T^2\#T^2\#T^2$	$P^2\#P^2\#P^2\#P^2$
etc.	etc.

(Exercise 5.10 continues on page 83.)

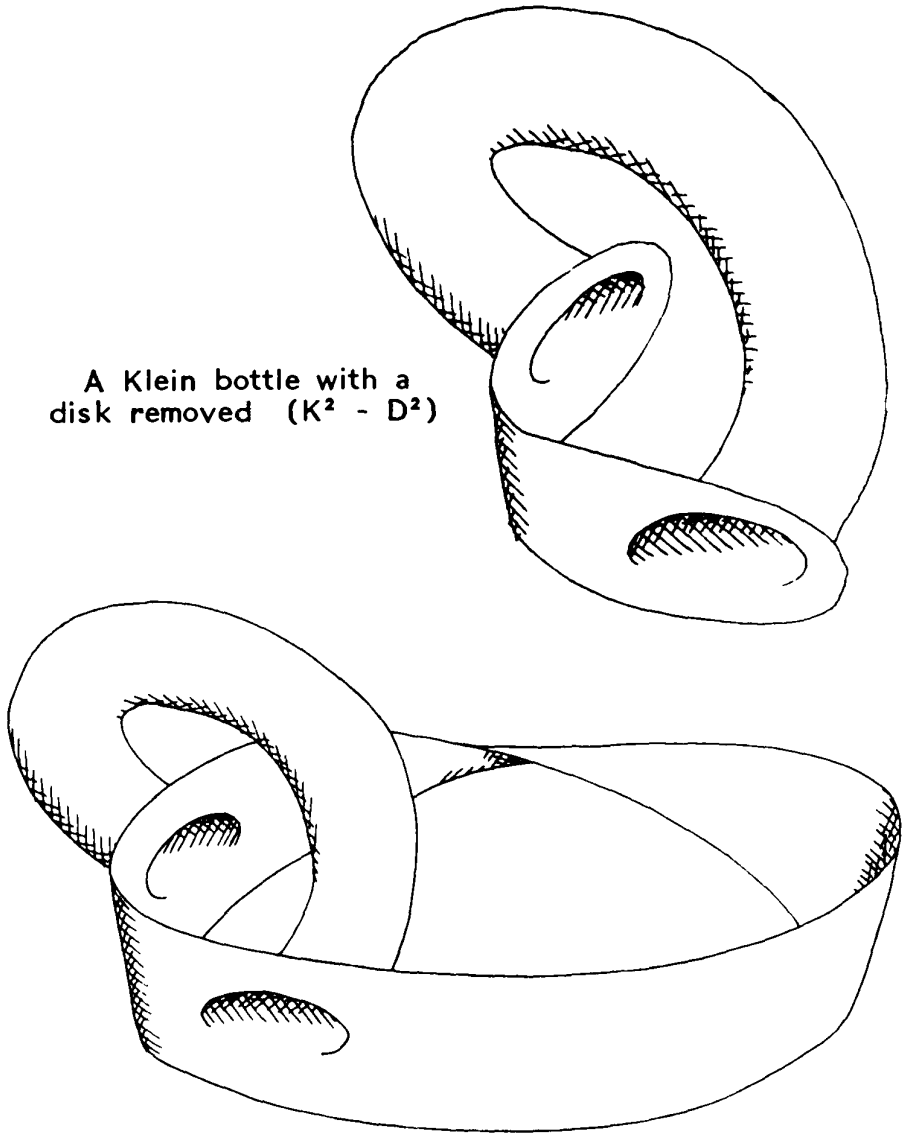


A torus with a disk removed
($T^2 - D^2$)



The connected sum of a torus
and a Möbius strip
(T^2 # Möbius)

Figure 5.7: Check that each picture is what its caption says it is.



The connected sum of a Klein
bottle and a Möbius strip
(K^2 # Möbius)

FIGURE 5.7: (Continued)

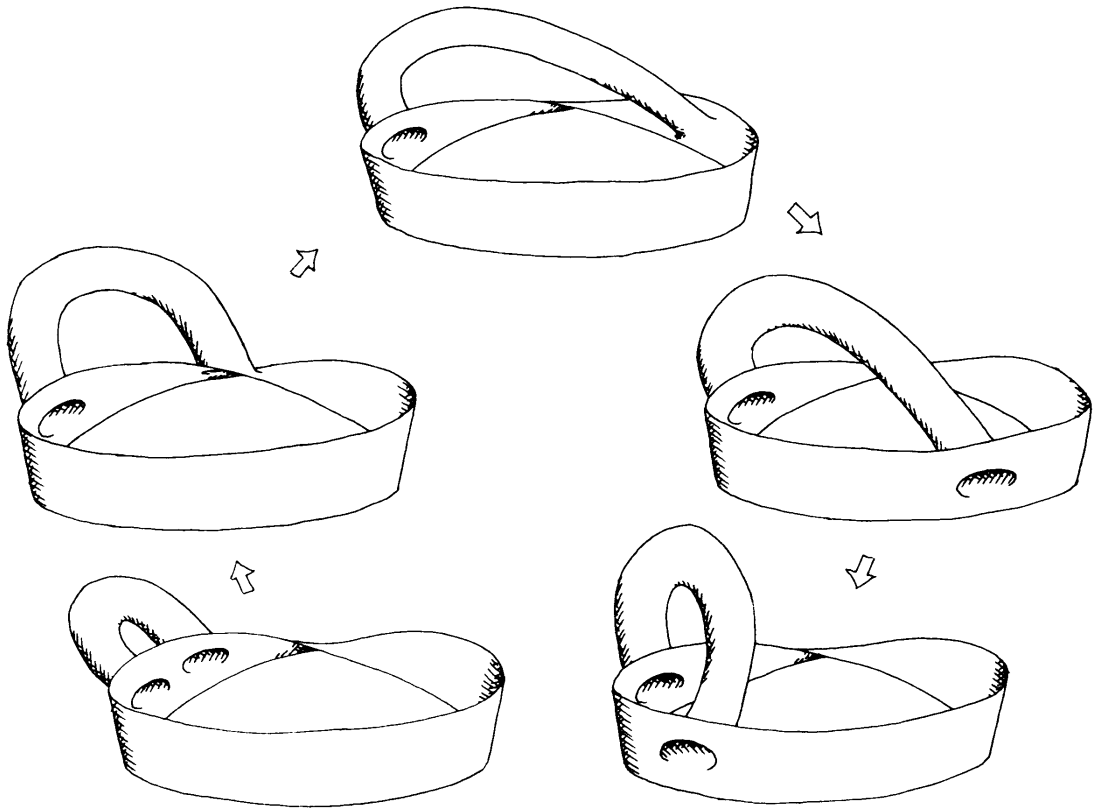


Figure 5.8: T^2 # Möbius and K^2 # Möbius can be deformed one into the other. Therefore they are topologically the same. *Real* rubber would of course break if you tried this sort of deformation, but in topology you needn't worry about such complications.

Where does $T^2 \# P^2$ (the surface in the Flatland story) appear on this list? Where is $T^2 \# K^2$? $P^2 \# S^2$? $S^2 \# S^2$? Which surfaces on the list are orientable?

In Chapter 12 we'll see that all the surfaces on this new list really are different. □

Exercise 5.11. Match each surface in Column A to a topologically equivalent surface in Column B.

Column A	*Column B*
$T^2\#S^2$	$P^2\#P^2$
K^2	$K^2\#P^2$
$S^2\#S^2\#S^2$	$S^2\#S^2$
$P^2\#T^2$	$P^2\#P^2\#P^2\#K^2$
$K^2\#T^2\#P^2$	T^2

□

One can also talk about a connected sum of two three-manifolds (you remove a solid ball from each and glue the remaining three-manifolds together along the exposed spherical boundary).

6

Products

A cylinder is the **product** of a circle and an interval. It qualifies as such because (see Figure 6.1) it is both

1. A bunch of intervals arranged in a circle, and
2. A bunch of circles arranged (in this case stacked) in an interval.

More concisely, a cylinder is the product of a circle and an interval because it is both (1) a circle of intervals and (2) an interval of circles. It is abbreviated as $S^1 \times I$ (pronounced "a circle cross an interval" or simply "S-one cross I").

A torus is a second example of a product. It's the product of one circle (drawn dark in Figure 6.2) with another (drawn light). This is because the torus is a circle of circles in two different ways: both as a dark circle of light circles and as a light circle of dark circles. This fact is abbreviated by the equation $T^2 = S^1 \times S^1$ (read "a torus is a circle cross a circle" or simply

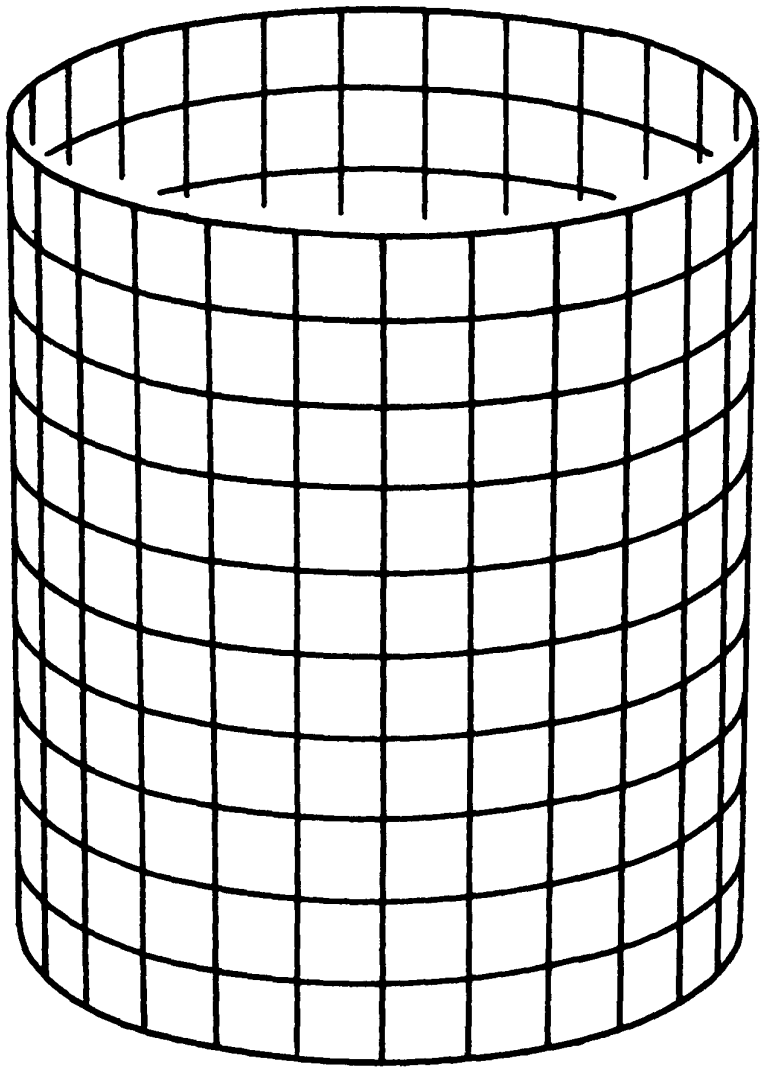


Figure 6.1: A cylinder is the product of a circle and an interval. (Recall that an "interval" is a line segment with both endpoints included.)

"T-two equals S-one cross S-one"). The torus in Figure 6.2 is drawn as a doughnut surface, which is OK since we are interested only in its topology.

[By the way, the torus is the only closed surface (with no edges) that is a product. The reason is that a two-dimensional

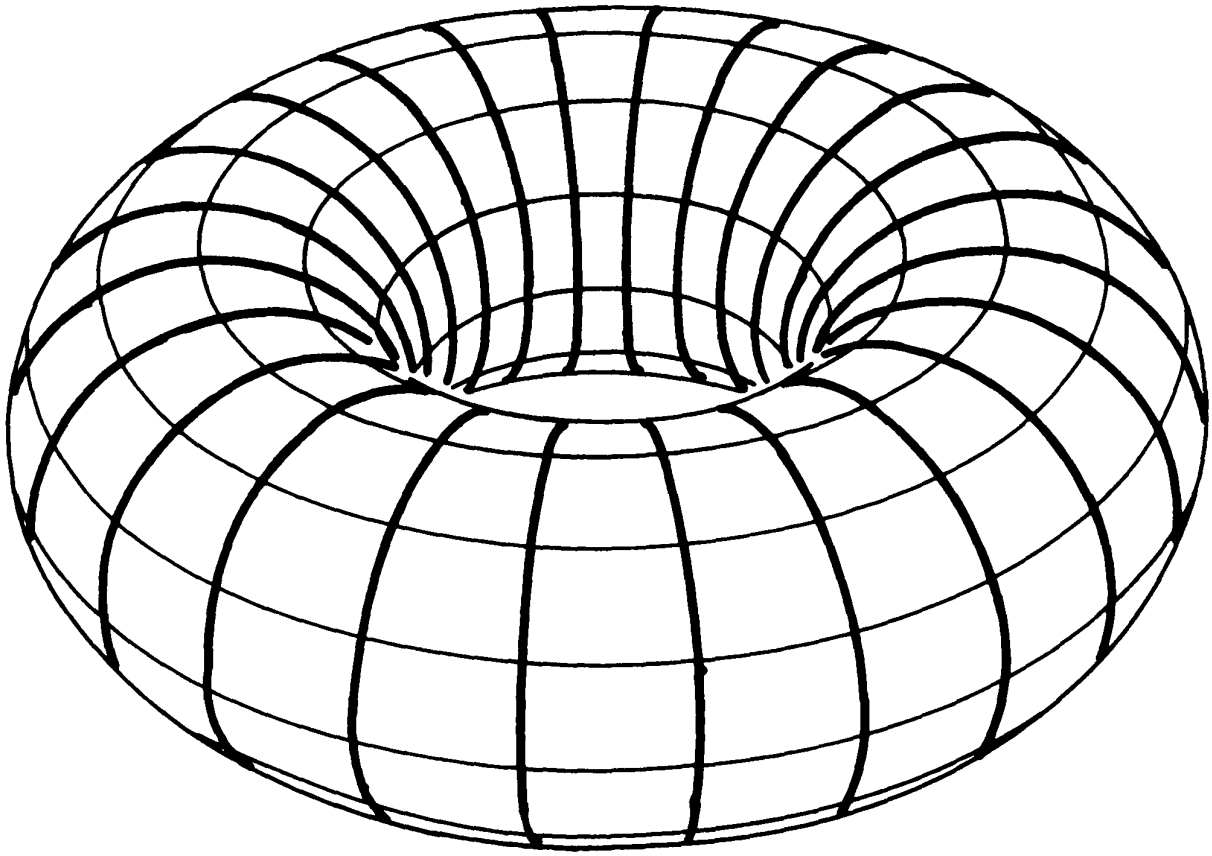


Figure 6.2: A torus is a circle cross a circle.

product must be the product of two one-dimensional things, and the circle is the only one-dimensional thing available that has neither endpoints (like an interval) nor infinite length (like a line); thus $S^1 \times S^1$ is the only two-dimensional product having neither an edge nor an infinite area.]

Exercise 6.1. What are the usual names for each of the following products:

1. $I \times I$
2. $E^1 \times E^1$

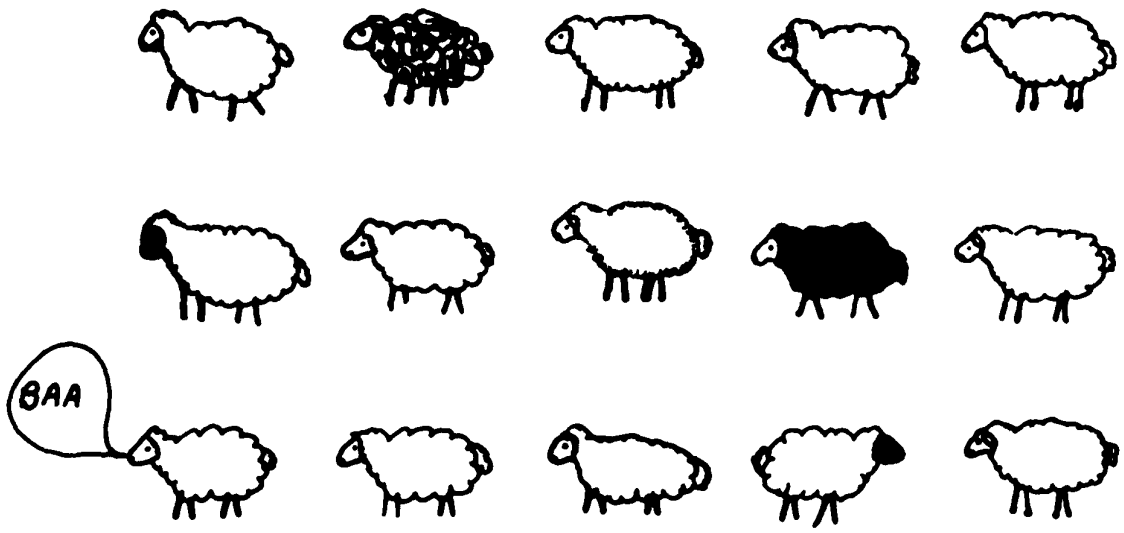


Figure 6.3: Fifteen is the product of three and five.

3. $S^1 \times E^1$

4. $E^1 \times I$. □

Exercise 6.2. Is the Möbius strip a product? □

Exercise 6.3. What's $D^2 \times S^1$? Work topologically. □

There is a connection between products of manifolds and products of numbers in the sense of multiplication. For example, 15 is the product of 3 and 5, and the 15 sheep in Figure 6.3 form both 3 rows of 5 and 5 columns of 3.

Our first example--the cylinder--is a product not only in the topological sense but in the geometrical sense as well. To be

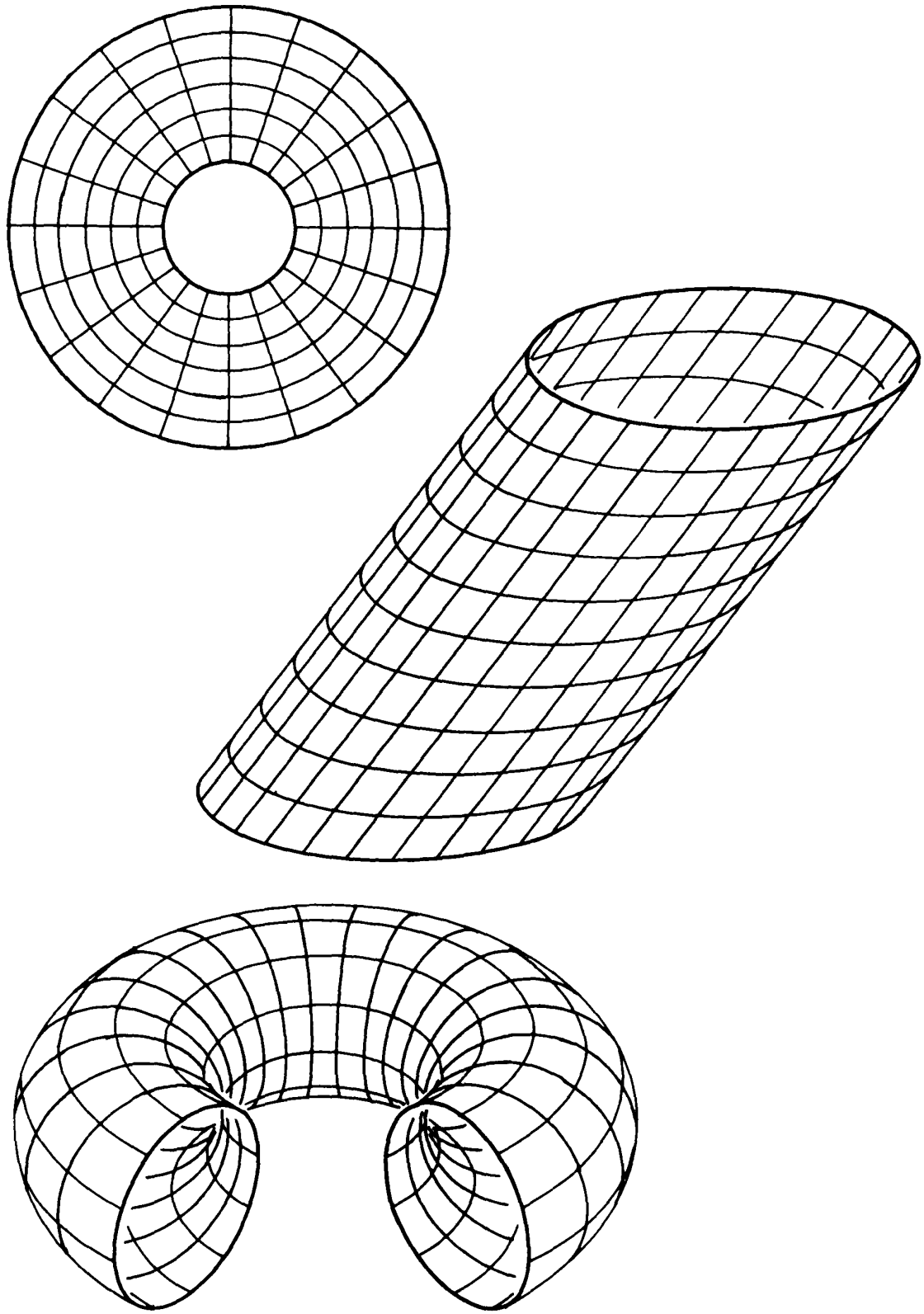


Figure 6.4: Three versions of $S^1 \times I$ which are topological--but not geometrical--products.

precise, it qualifies as a **geometrical product** because it satisfies the following three conditions:

1. All the circles are the same size.
2. All the intervals are the same size.
3. Each circle is perpendicular to each interval.

Exercise 6.4. Figure 6.4 shows three renditions of $S^1 \times I$ which are topological--but not geometrical--products. Which of the above three conditions is violated in each case? □

Exercise 6.5. Draw one version of $I \times I$ which is a geometrical product and another version which is only a topological product. □

The doughnut surface in Figure 6.2 is not a geometrical product because the light circles are not all the same size. In fact, we can't draw a geometrical $S^1 \times S^1$ in three-dimensional space at all. The best we can do is to draw a cylinder as in Figure 6.1 and imagine that the top is glued to the bottom. The gluing converts each vertical interval into a circle, thereby converting $S^1 \times I$ into $S^1 \times S^1$. We know this new product is a geometrical one because

1. All the original circles are the same size,
2. All the intervals that get converted into circles are the same size, and
3. Each circle is perpendicular to each interval.

But we also know that this cylinder with its ends glued together is a flat torus. Therefore a geometrical $S^1 \times S^1$ is a flat torus!

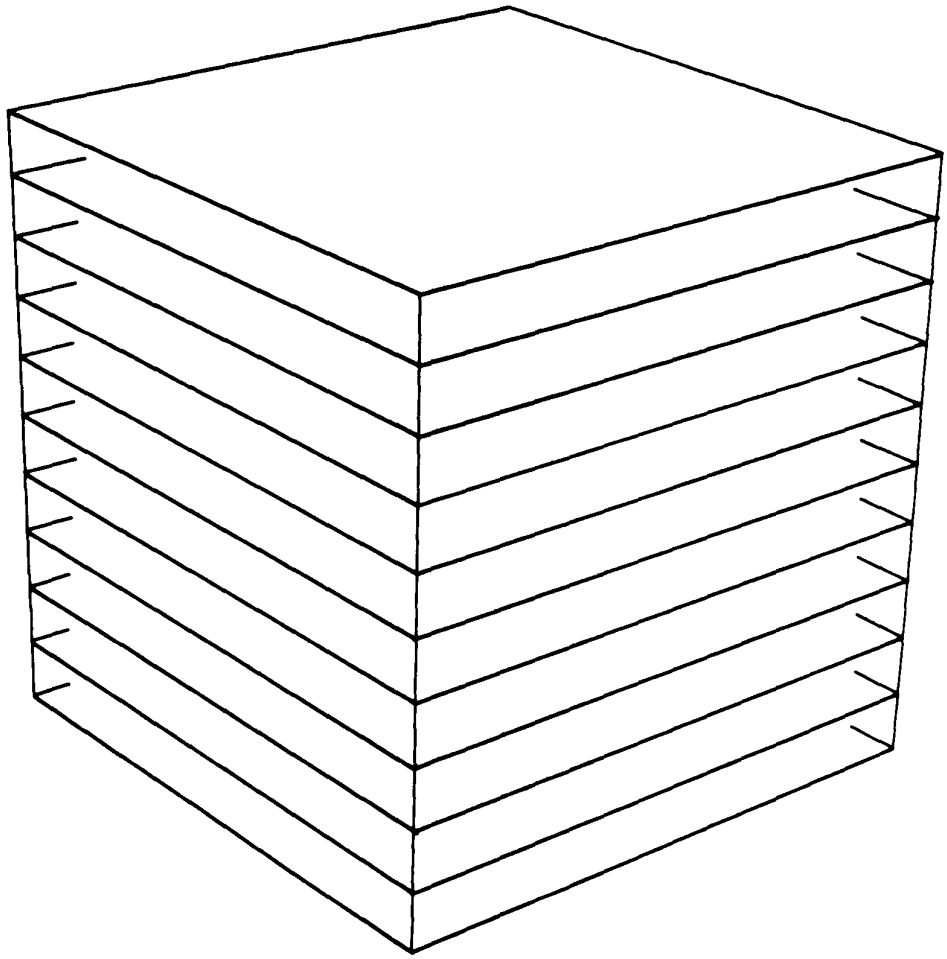


Figure 6.5: A three-torus is the product of a two-torus and a circle.

Our main reason for studying products is to better understand three-manifolds. It's true that the vast majority of three-manifolds *aren't* products, but many of the simplest and most interesting ones are. For example, the three-torus is the product of a two-torus and a circle (in symbols $T^3 = T^2 \times S^1$). Here's how to see it. Recall that a three-torus is a cube with opposite faces glued. Imagine this cube to consist of a stack of horizontal layers as shown in Figure 6.5. When the cube's sides get glued, each

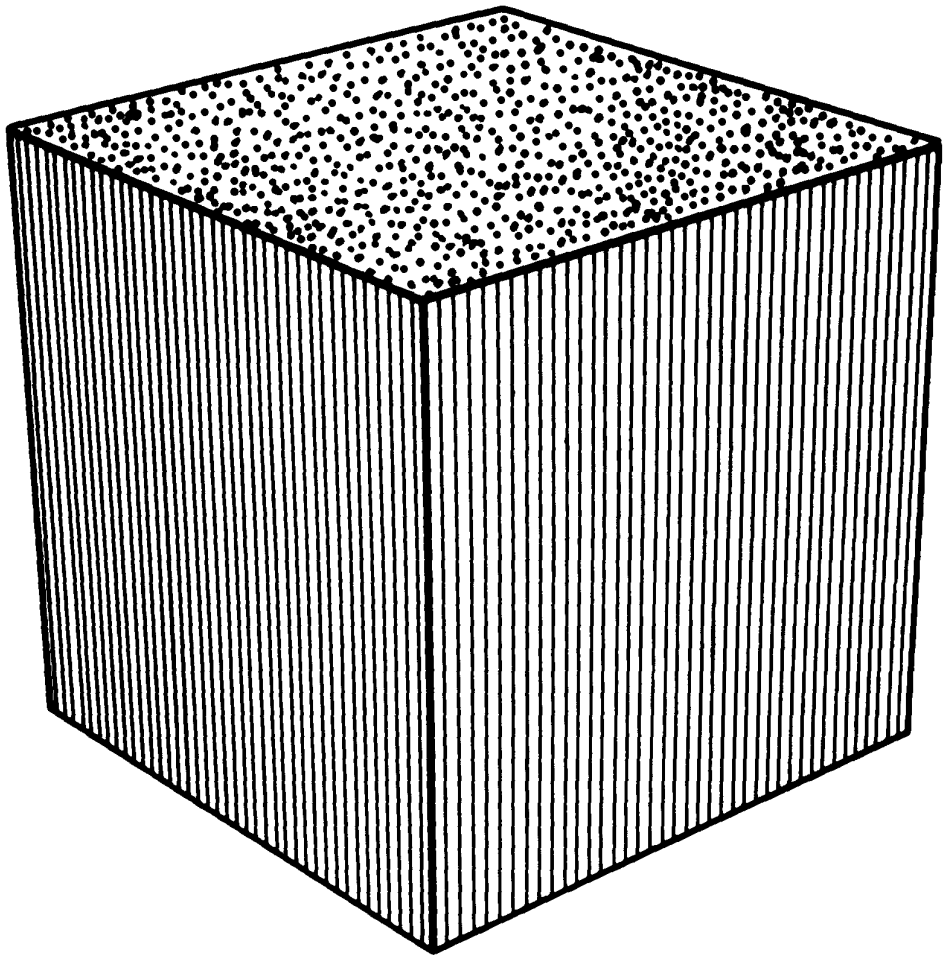


Figure 6.6: To see that T^3 is a torus of circles, first imagine the cube to be filled with spaghetti standing on end.

horizontal layer gets converted into a torus--a flat torus in fact. At this stage we have a stack of flat tori. When the cube's top is glued to its bottom, this stack of tori is converted into a circle of tori. We still have to check that T^3 is a torus of circles as well. To do this, imagine the cube to be filled not with horizontal layers, but with vertical intervals, like lots of spaghetti standing on end (Figure 6.6). When the cube's sides get glued this square of intervals becomes a torus of intervals, and when the top and bottom

are glued it becomes a torus of circles as required. Thus the three-torus is both a circle of tori and a torus of circles, so it is the product of a torus and a circle.

The three-torus is, in fact, a geometrical product because

1. All the horizontal tori are the same size (Figure 6.5).
2. All the vertical circles are the same size (Figure 6.6).
3. Each torus is perpendicular to each circle.

Exercise 6.6. In Chapter 4 we made a nonorientable three-manifold by gluing a room's front wall to its back wall with a side-to-side flip, while gluing the floor to the ceiling, and the left wall to the right wall, in the usual way. This nonorientable three-manifold is a product. What's it the product of? (Hint: The relevant pictures look just like Figures 6.5 and 6.6, only the gluings are different.) Is this a geometrical product? □

It's time for a brand new three-manifold with a brand new local geometry! The manifold is $S^2 \times S^1$ (read "a sphere cross a circle" or "S-two cross S-one"), but before investigating it, let's pause for a moment to see how a Flatlander might deal with $S^1 \times S^1$.

A Flatlander can't visualize $S^1 \times S^1$ directly, so he uses a mental trick. He first imagines $S^1 \times I$ as in Figure 6.7, and then he imagines the inner (circular) edge to be glued to the outer (circular) edge. If he wants a geometrical product, he must in addition pretend that all the circles are the same size! At first it's

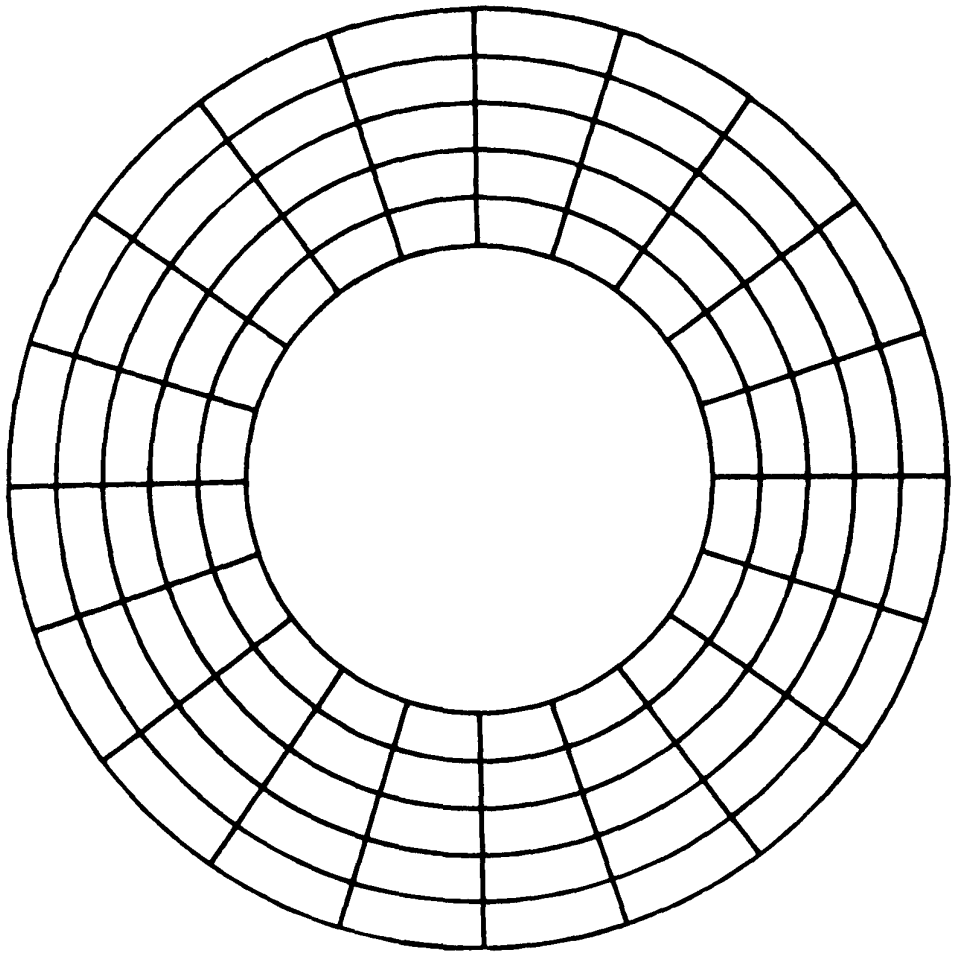


Figure 6.7: $S^1 \times I$ as imagined by a Flatlander.

hard for him to pretend that these circles are the same size when in his mental image they clearly are not; but doing so does help him understand $S^1 \times S^1$, so he puts up with the contradiction until he eventually gets an intuitive feeling for what $S^1 \times S^1$ is *really* like.

Actually, the hardest thing for a Flatlander to accept is that a thread pulled taut between the points shown in Figure 6.8 would follow the apparently curved arc of one of the circles. In the more

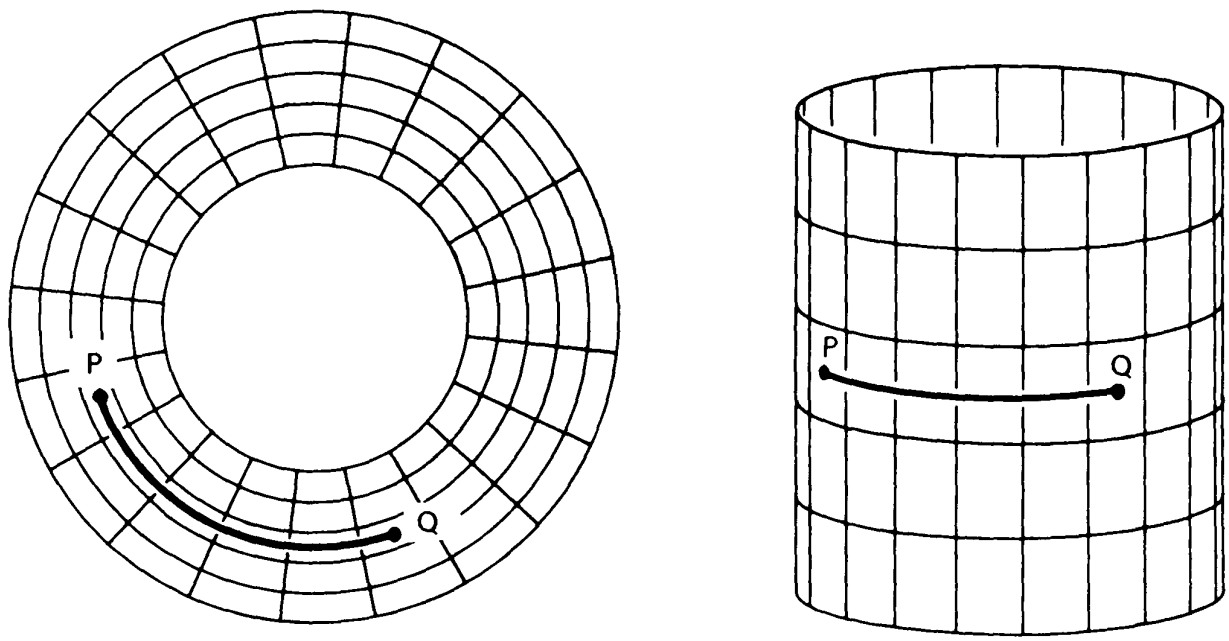


Figure 6.8: A thread pulled taut between P and Q follows the path shown.

refined drawing of a Spacelander (shown on the right in Figure 6.8) it's obvious that the thread *would* follow the circle, but to the Flatlander it seems that the thread would want to "cut across the middle" somehow.

We Spacelanders picture $S^2 \times S^1$ the same way Flatlanders picture $S^1 \times S^1$. First we visualize $S^2 \times I$ as a thickened spherical shell (Figure 6.9). To convert this shell to $S^2 \times S^1$ we imagine the inner spherical boundary to be glued to the outer spherical boundary. If we want a geometrical product things get tougher: we must pretend that the various spherical layers in the spherical shell are all the same size! (A four-dimensional being would have

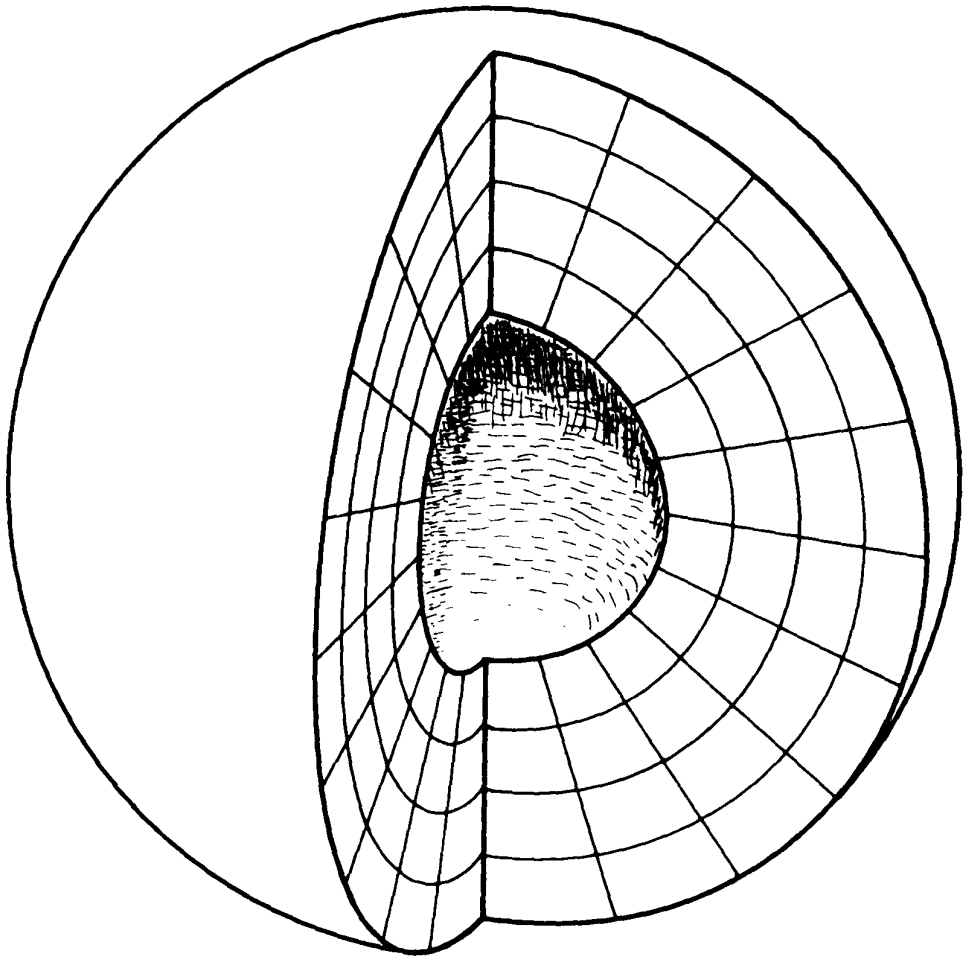


Figure 6.9: Visualize $S^2 \times I$ as a ball with a hollow center.

no trouble drawing a picture in which the layers really are the same size, but for us Spacelanders it isn't so easy.)

Figure 6.10 shows two interesting surfaces contained in a geometrical $S^2 \times S^1$. The grey one is a spherical "cross-section" of $S^2 \times S^1$. A thread pulled taut between any two points on this sphere will follow the sphere's curved surface! This would be perfectly obvious to a four-dimensional being: in her drawing

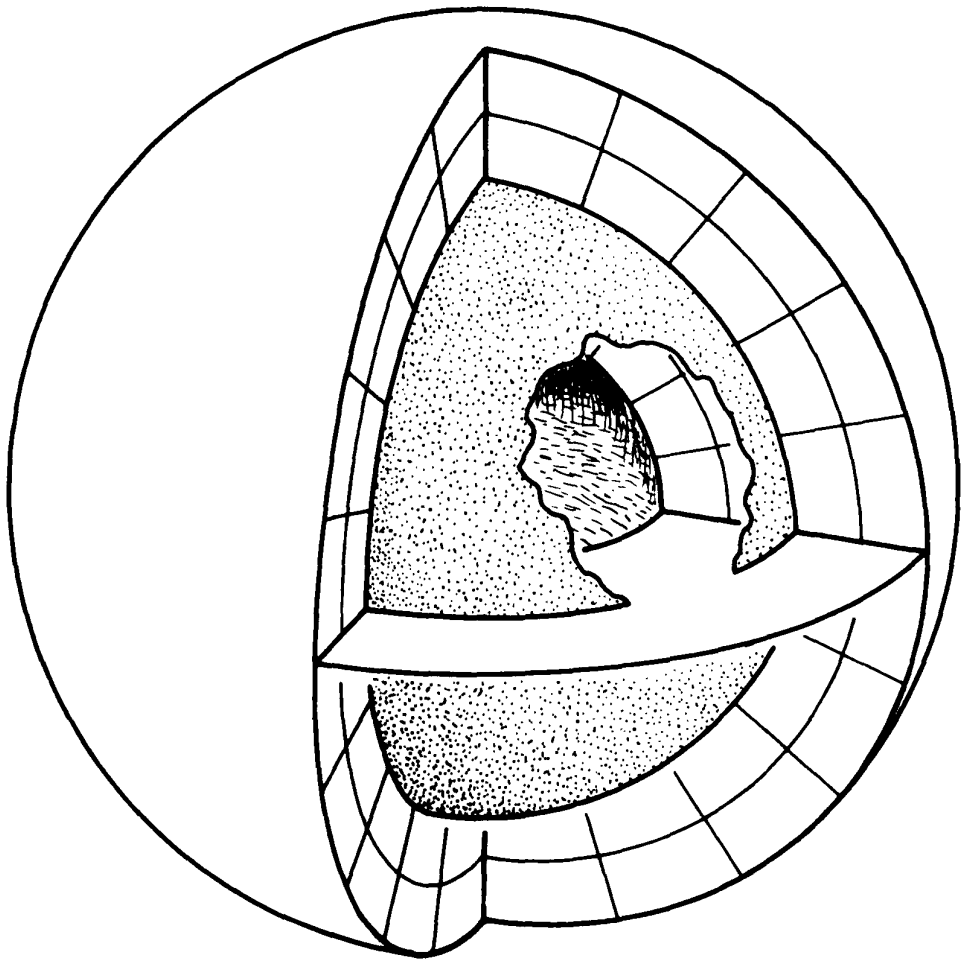


Figure 6.10: A sphere cross a circle is what you get by imagining the inner surface of $S^2 \times I$ to be glued to the outer surface. The gluing converts the horizontal ring-like surface into a torus. The grey sphere, of course, remains unaffected by the gluing.

there'd be no reason for the thread to bend to one side of the sphere or the other (compare Figure 6.8(b)). But to us naive Spacelanders it seems (incorrectly!) that the thread ought to "cut across the middle" somehow. If Gauss lived in $S^2 \times S^1$ and surveyed a triangle formed by three points on this sphere, he'd find the sum of the angles of the triangle to be greater than 180°--

just like the sum of the angles of any spherical triangle (recall Figure 3.7).

The white surface shown in Figure 6.10 is a flat torus. It looks just like a Flatlander's conception of a flat torus as based on Figure 6.7. If Gauss were to measure a triangle in this surface, he'd find its angles to add up to 180°.

We've seen that in $S^2 \times S^1$ different triangles may have different angle sums, depending upon how they're situated. Thus, $S^2 \times S^1$ is an example of a three-manifold which is homogeneous but *not* isotropic. A **homogeneous** manifold is one whose local geometry is everywhere the same. An isotropic manifold is one in which the geometry is the same *in all directions.* $S^2 \times S^1$ is not isotropic because some two-dimensional slices have the local geometry of a sphere, while other slices have the local geometry of a plane, as we have discovered in the preceding two paragraphs. (The distinction between homogeneous and isotropic manifolds is not readily apparent to most Spacelanders because of the peculiar fact that there are no surfaces which are homogeneous but not isotropic.)

Exercise 6.7. Find a nonorientable three-manifold which is a product and has the same local geometry as $S^2 \times S^1$. Hint: What might such a manifold be the product of? □

We now have an infinite number of three-manifolds at our disposal! Specifically, we can take each surface on the list in

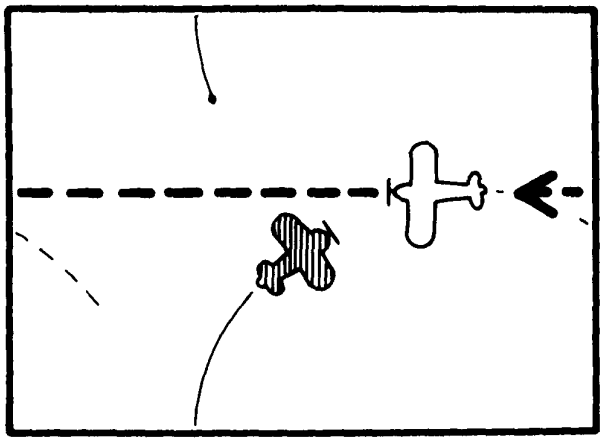
Exercise 5.10 and construct the product of that surface and a circle. For example, we could construct a five-holed doughnut surface cross a circle. Topologically we picture this manifold as a thickened five-holed doughnut surface with its inside glued to its outside, just as we pictured a sphere cross a circle ($S^2 \times S^1$) as a thickened sphere with its inside glued to its outside. We won't be able to understand the geometries of these product three-manifolds until after we've investigated the geometries of surfaces (cf. Chapters 10, 11, and 18).

7

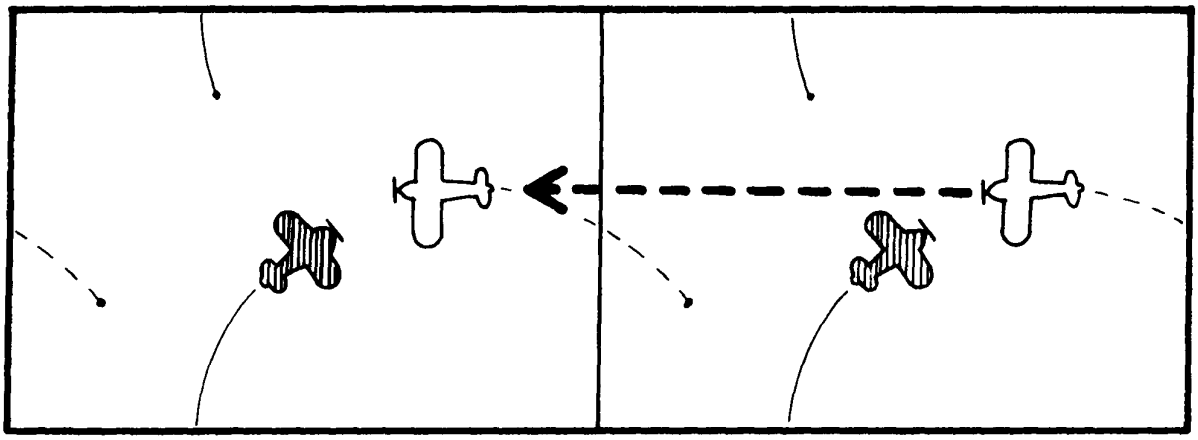
Flat Manifolds

What would you see if you lived in a closed three-manifold? This chapter answers that question for three-manifolds, such as the three-torus, which have the local geometry of "ordinary" three-dimensional space (i.e. flat three-manifolds).

First let's consider what a two-dimensional biplane pilot sees as she flies about in the flat torus sky of Chapter 3. If she looks towards an "edge of the screen", she'll find she's looking at herself from behind! (See Figure 7.1.) She can also see herself by looking along a diagonal, as illustrated in Figure 7.2. In fact she can see herself in infinitely many directions! The flat torus sky looks to her like a plane with infinitely many copies of her in it! (Figure 7.3.) Of course if an enemy biplane is chasing her she'll see infinitely many copies of it as well. These pictures are analogous to the pictures we used in Chapter 4 to analyze Klein bottle tic-tac-toe games (Figure 4.6).



The pilot can
see herself.

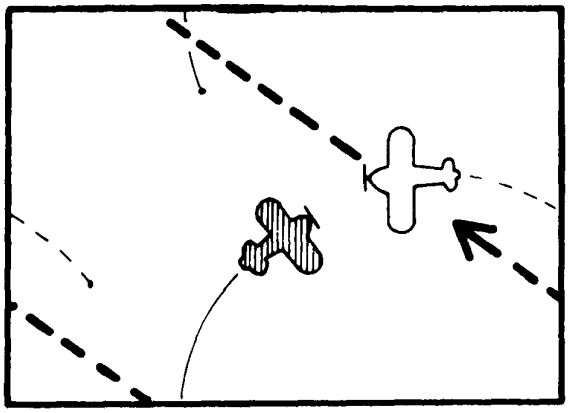


How the situation
appears to her.

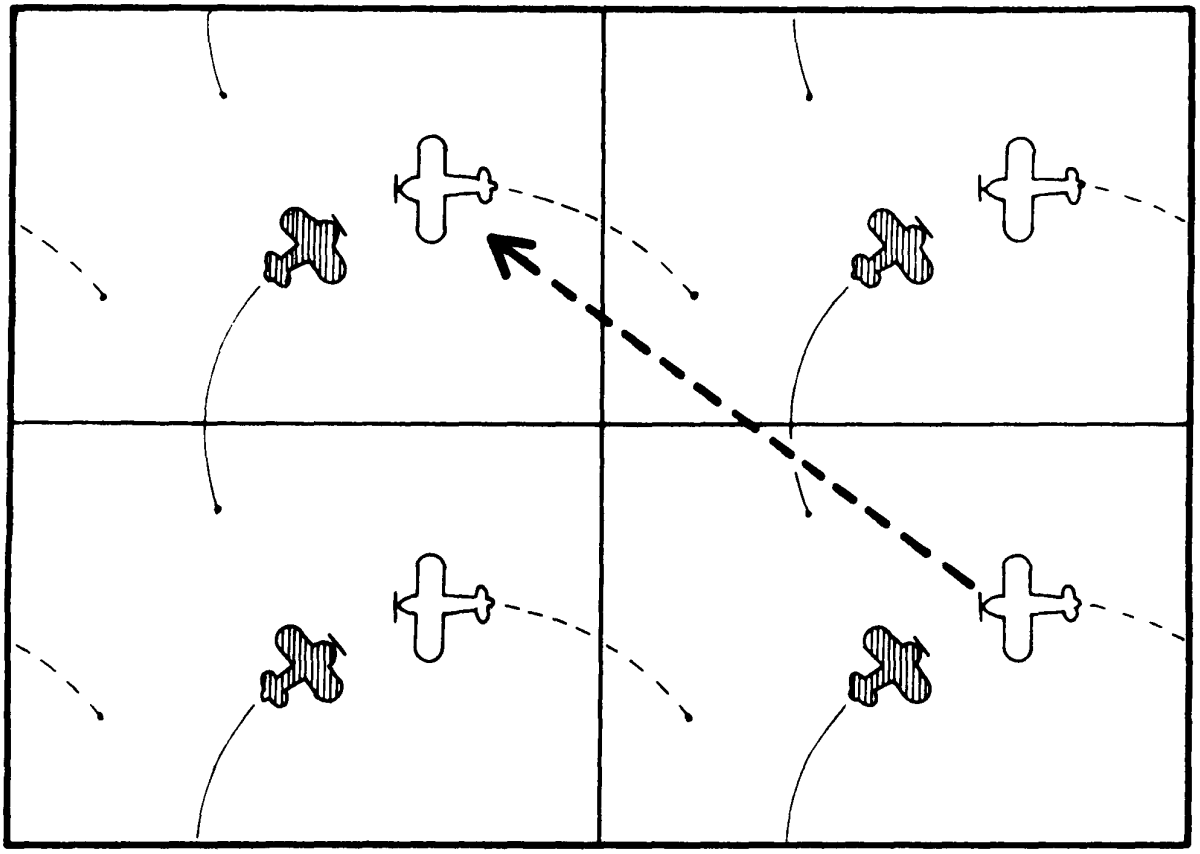
Figure 7.1: A biplane pilot in a flat torus.

Exercise 7.1. Draw a picture (analogous to Figure 7.3) showing how the sky would appear to a biplane pilot flying in a Klein bottle. □

Exercise 7.2. Analyze the *torus* tic-tac-toe game of Exercise 2.2 using the method of Figure 4.6. □



She sees herself by
looking along a diagonal.



How it appears to her.

Figure 7.2: The biplane pilot sees herself in a different direction.

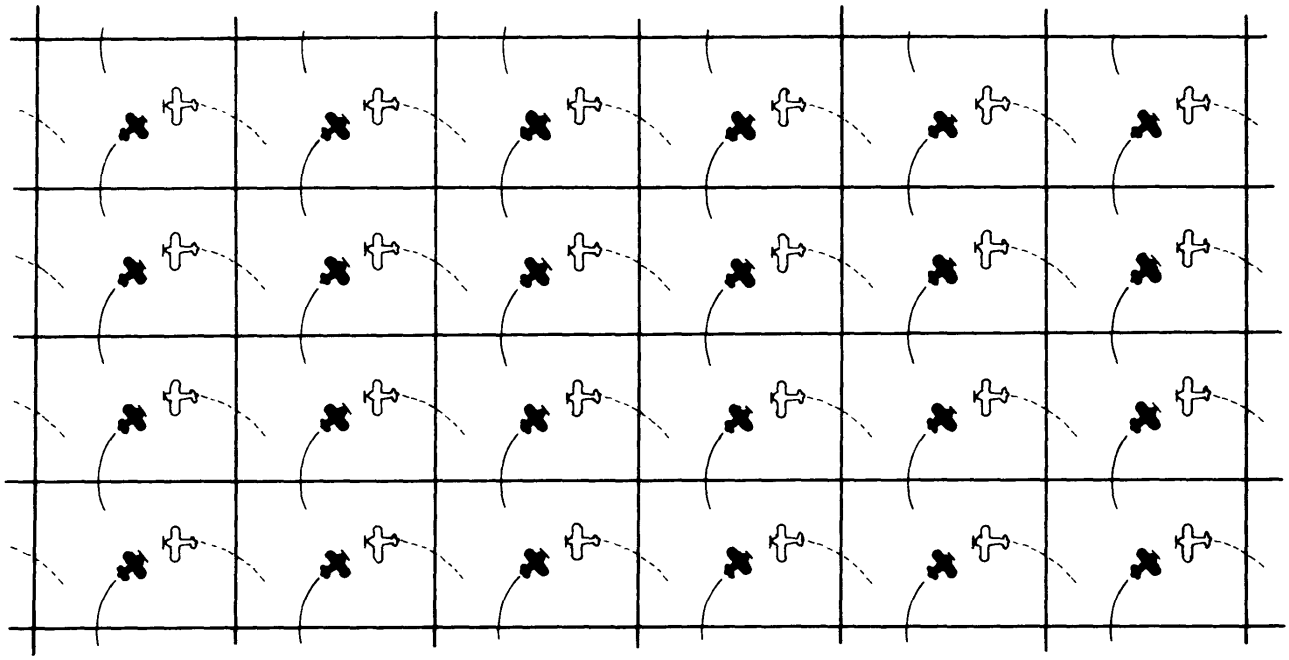


Figure 7.3: The biplane pilot's view of her world.

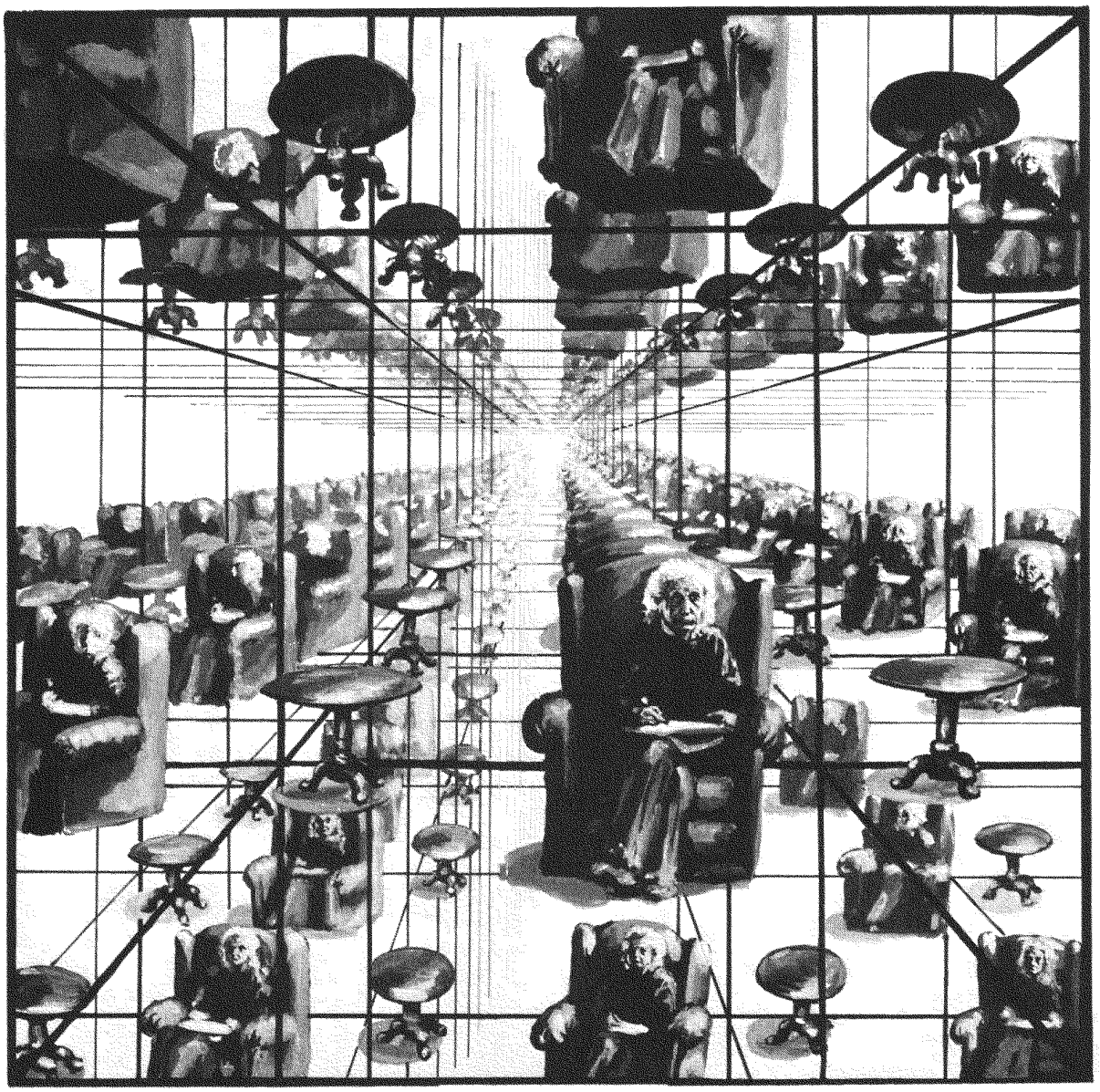


Figure 7.4: When opposite walls of a room are "glued" to make a three-torus, the view looks like this.

Now let's make some analogous drawings of three-manifolds. Figure 7.4 shows the view inside the three-torus described in Chapter 2. This three-torus consists of a living room with its left

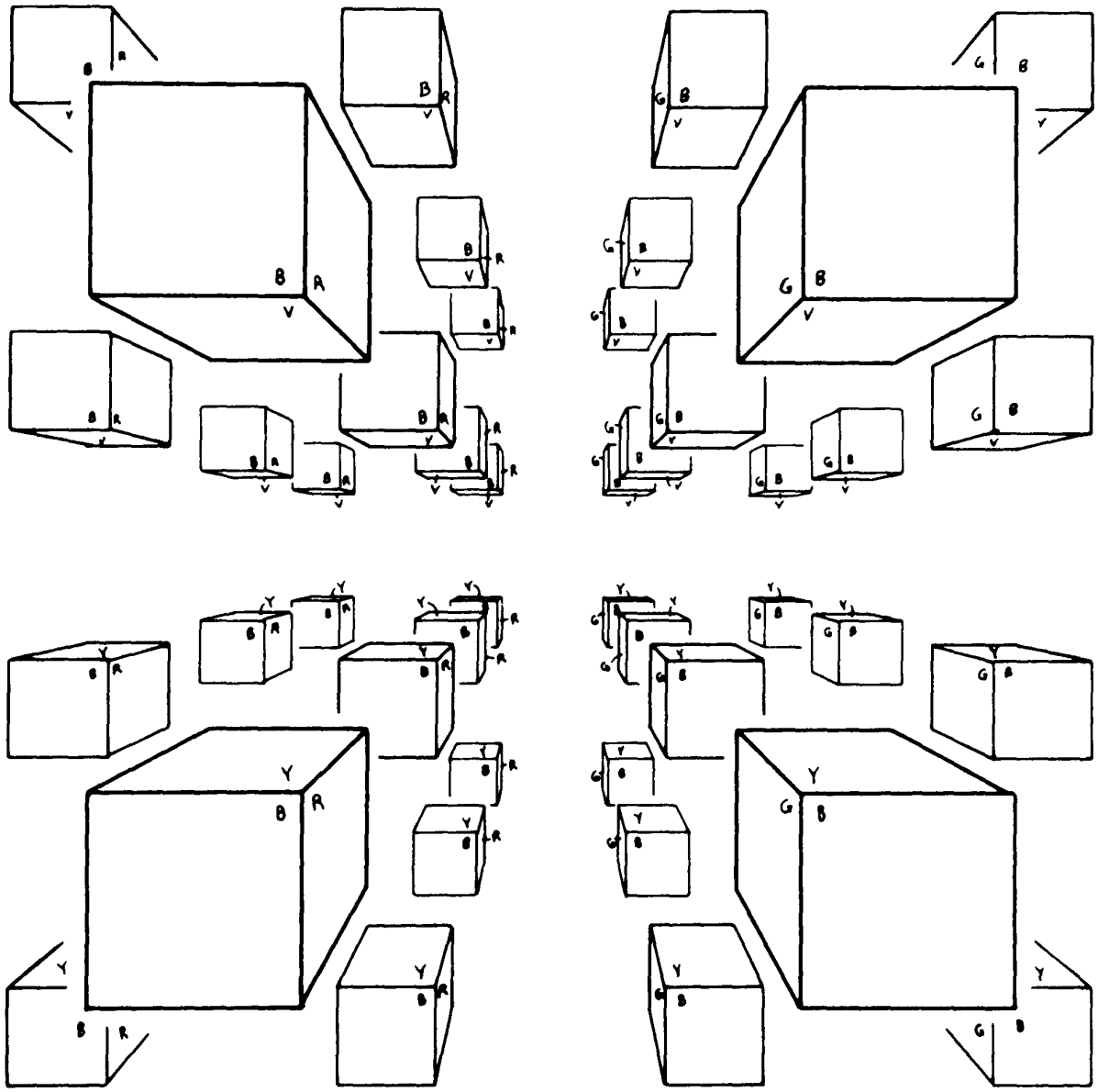


Figure 7.5: The view inside a three-torus containing a single small cube.

wall glued to its right wall, its front wall glued to its back wall, and its floor glued to its ceiling.

Figure 7.5 shows the view inside a different three-torus: this one is made from a cube containing a smaller cube with colored faces (Figure 7.6).

DO-IT-YOURSELF COLORING

Get ahold of some colored pens or pencils and use them to color the faces of the little cubes in Figures 7.5 and 7.6 according to the code R=Red, O=Orange, Y=Yellow, G=Green, B=Blue, V=Violet. You'll eventually need to color Figures 7.7, 7.10, 7.11, and 7.12, and the figures for Exercise 7.3, according to the same code.

It's fun to imagine the view inside other three-manifolds made from the (big) cube of Figure 7.6. For example, consider making $K^2 \times S^1$ from this cube. ($K^2 \times S^1$ is the nonorientable three-manifold described in Chapter 4--it's made by gluing a cube's top and bottom, and left and right sides, in the usual way, but gluing the front to the back with a side-to-side flip.) Color Figure 7.7 as indicated by the code to see the view inside $K^2 \times S^1$. *IMPORTANT NOTE: On the little cube opposite faces have complementary colors: red is opposite green, blue is opposite orange, and yellow is opposite violet.* Imagine yourself to be sitting on one of the cubes in Figure 7.7. When you look up, down, to the left or to the right you see other colored cubes positioned just as the cubes in Figure 7.5 are. But the cubes immediately in front of you and immediately behind you appear to have undergone a side-to-side mirror reversal

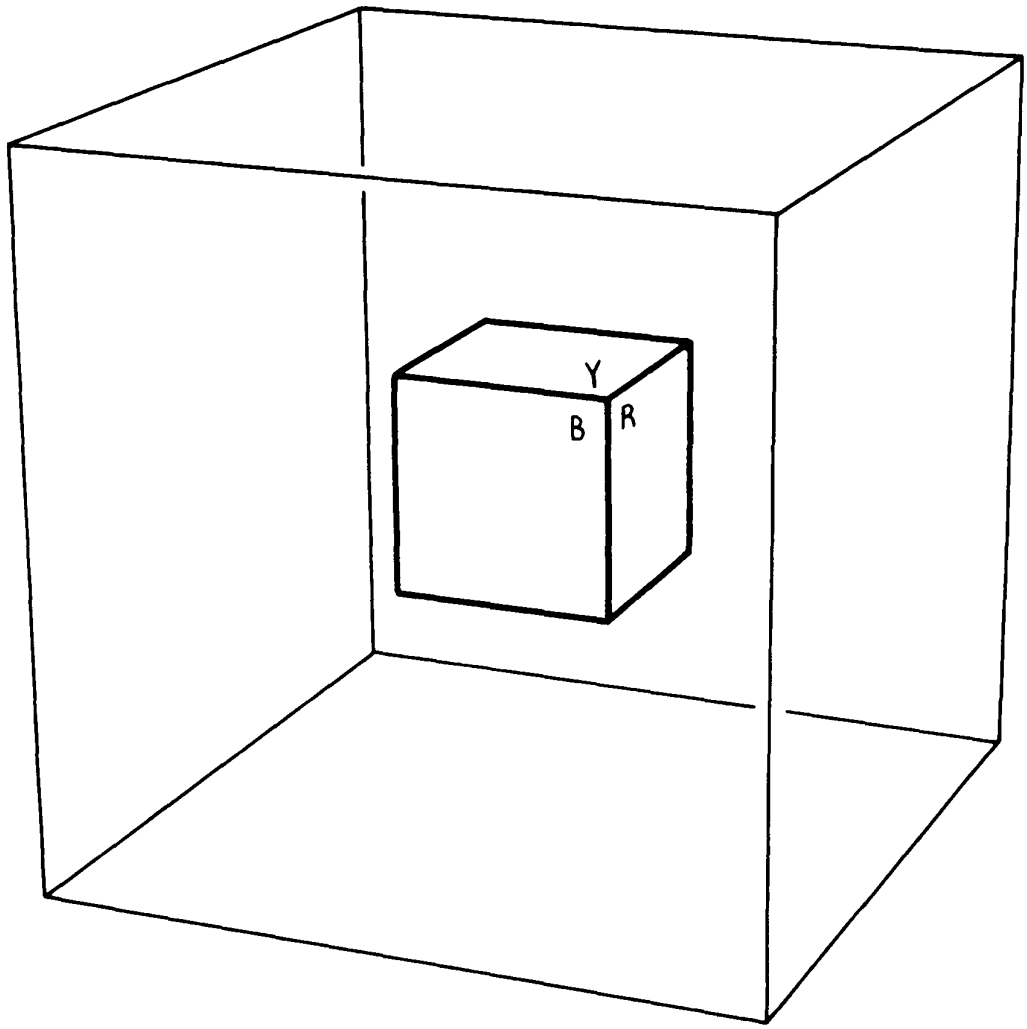


Figure 7.6: If you glue opposite faces of this (big) cube, the view inside the resulting three-torus will be as shown in Figure 7.5.

(thus interchanging the red and green faces); this is because the front and back faces of the big cube were glued with a side-to-side flip. The view in $K^2 \times S^1$ is analogous to the drawing of a Klein bottle sky you made in Exercise 7.1. You might want to pause for a moment to think this through.

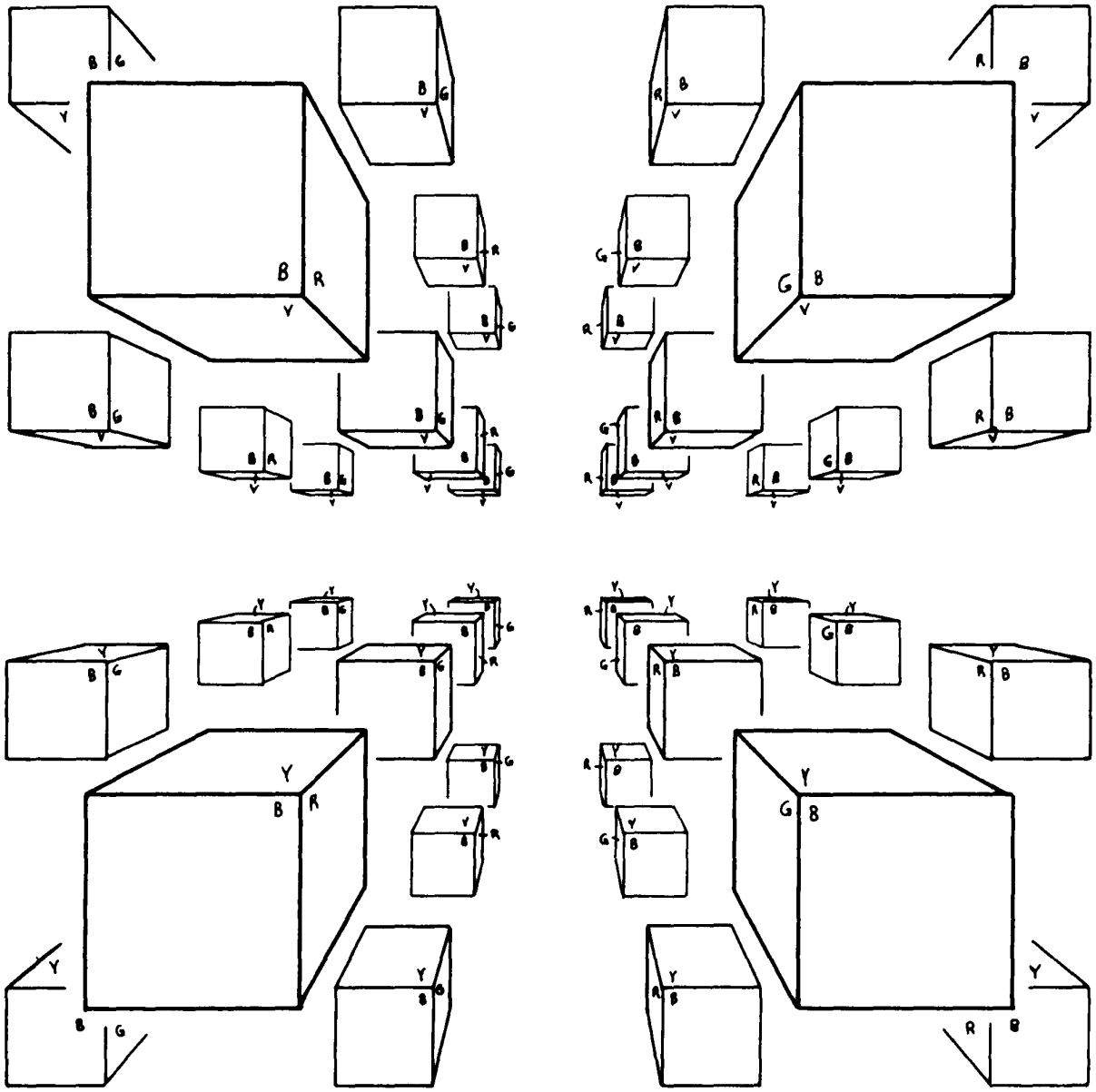


Figure 7.7: The view inside $K^2 \times S^1$.

Exercise 7.3. Here's a new three-manifold, the "quarter turn manifold". Start with a cube like the one in Figure 7.6, glue its front, back, left and right sides as if you were making a three-torus, but then glue its top to its bottom with a quarter turn

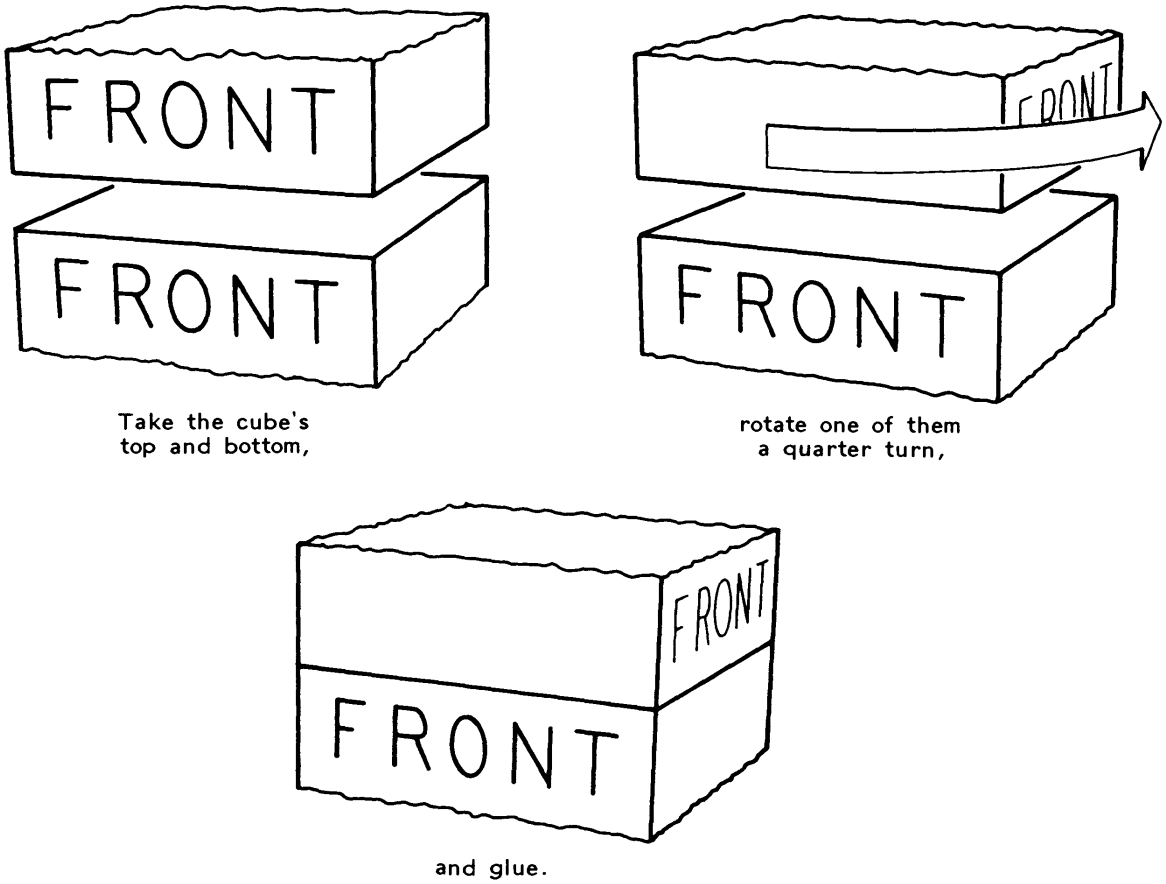


Figure 7.8: How to glue the top of a cube to the bottom with a quarter turn (local intrinsic picture).

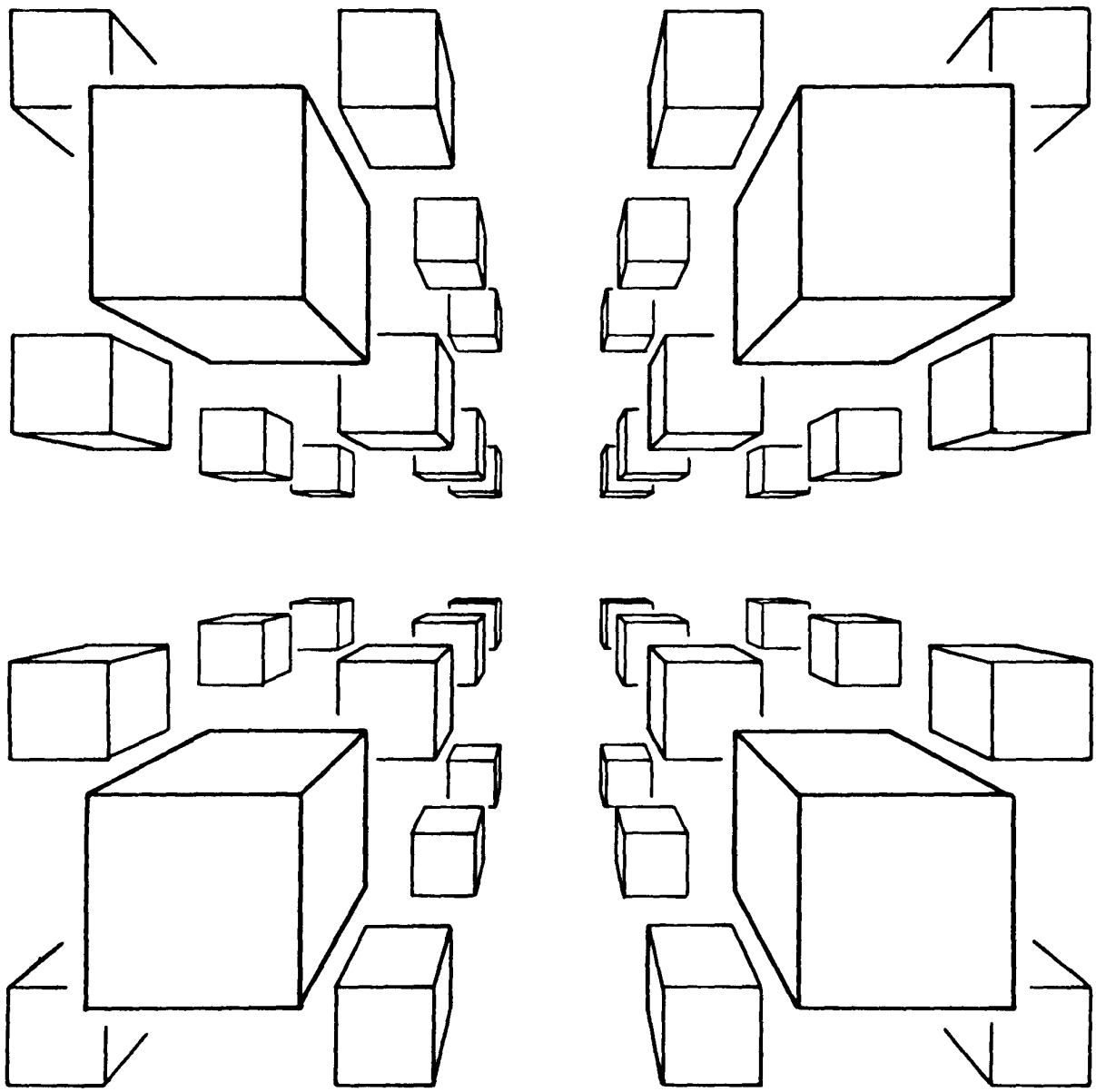


Figure 7.9: Make several copies of this figure and color them to illustrate the view inside each manifold of Exercise 7.3.

(Figure 7.8 shows how). Make a copy--by hand or by photocopy machine--of Figure 7.9 and color it to show the view inside this manifold.

You might also want to illustrate the views in a "half turn manifold" and a "three quarters turn manifold". Compare the view in the three quarters turn manifold to the view in the original quarter turn manifold; can you intrinsically tell one from the other? □

In the preceding exercise you colored the view inside the quarter turn manifold based on a knowledge of how the cube's faces were glued. You can also reverse this process: given a colored illustration of the view inside a manifold, you can deduce what the gluings are which produce it. For example, in Figure 7.7,

1. Each little cube is the mirror image--via a left-to-right reversal--of the cube immediately behind it. This tells you that the big cube's front and back faces are glued with a side-to-side flip.
2. Each little cube is identical to the cube above it. This tells you that the big cube's top and bottom faces are glued normally.
3. Similarly, each little cube is identical to the cube to its left. This tells you that the left and right faces are glued normally.

If you apply similar reasoning to your colored illustration of the view inside the quarter turn manifold, you can "rediscover" that the cube's top and bottom are glued with a quarter turn.

Exercise 7.4. Color Figure 7.10 as indicated by the code. How should the (big) cube of Figure 7.6 be glued to produce this view?

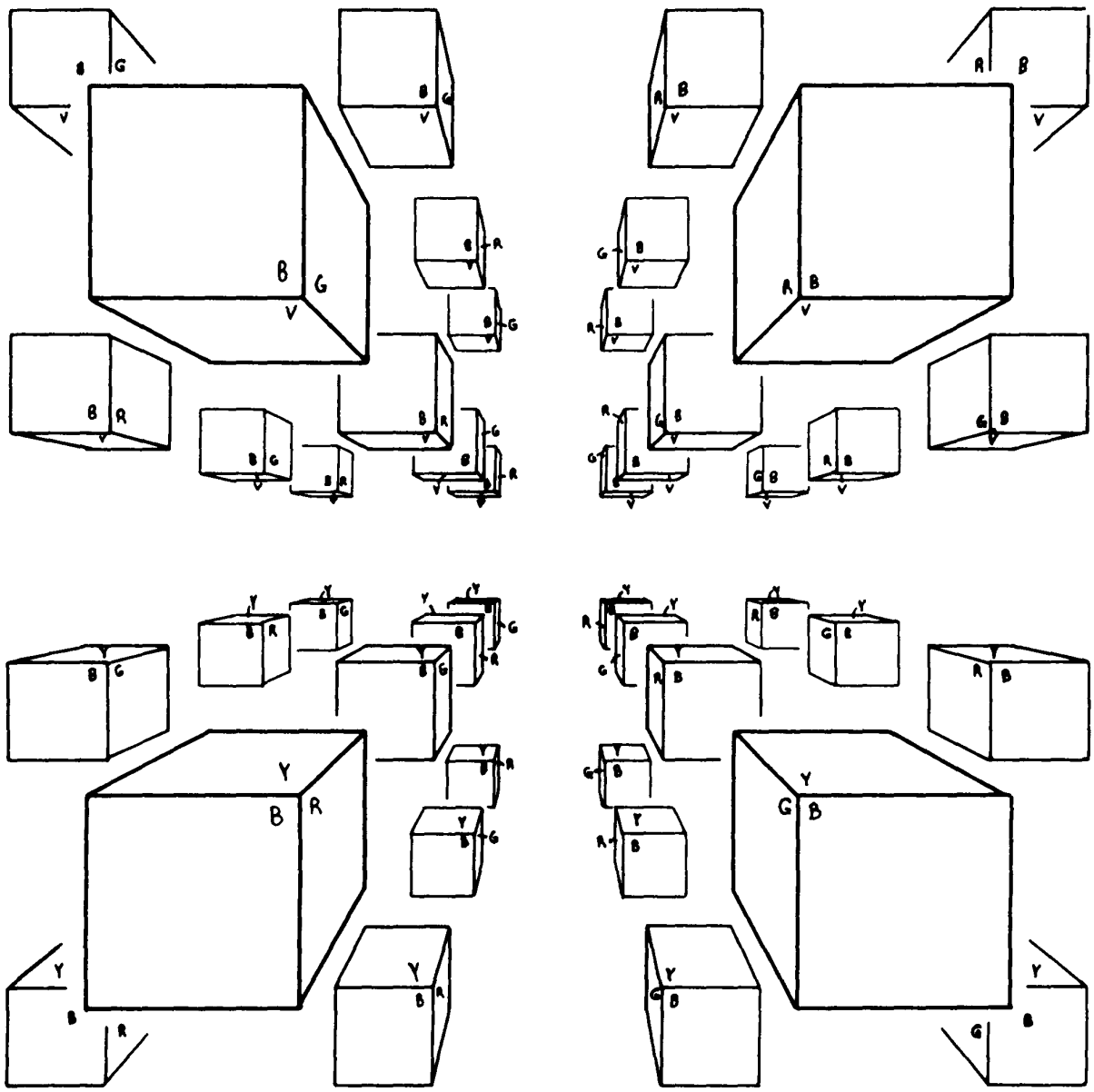


Figure 7.10: What gluing of the big cube produces this view?

For more practice, color Figures 7.11 and 7.12, and decide what gluings will produce these views. (All three of these manifolds will be important in Chapters 17 and 18 as examples of what's known as a "Klein bottle bundle over a circle".) □

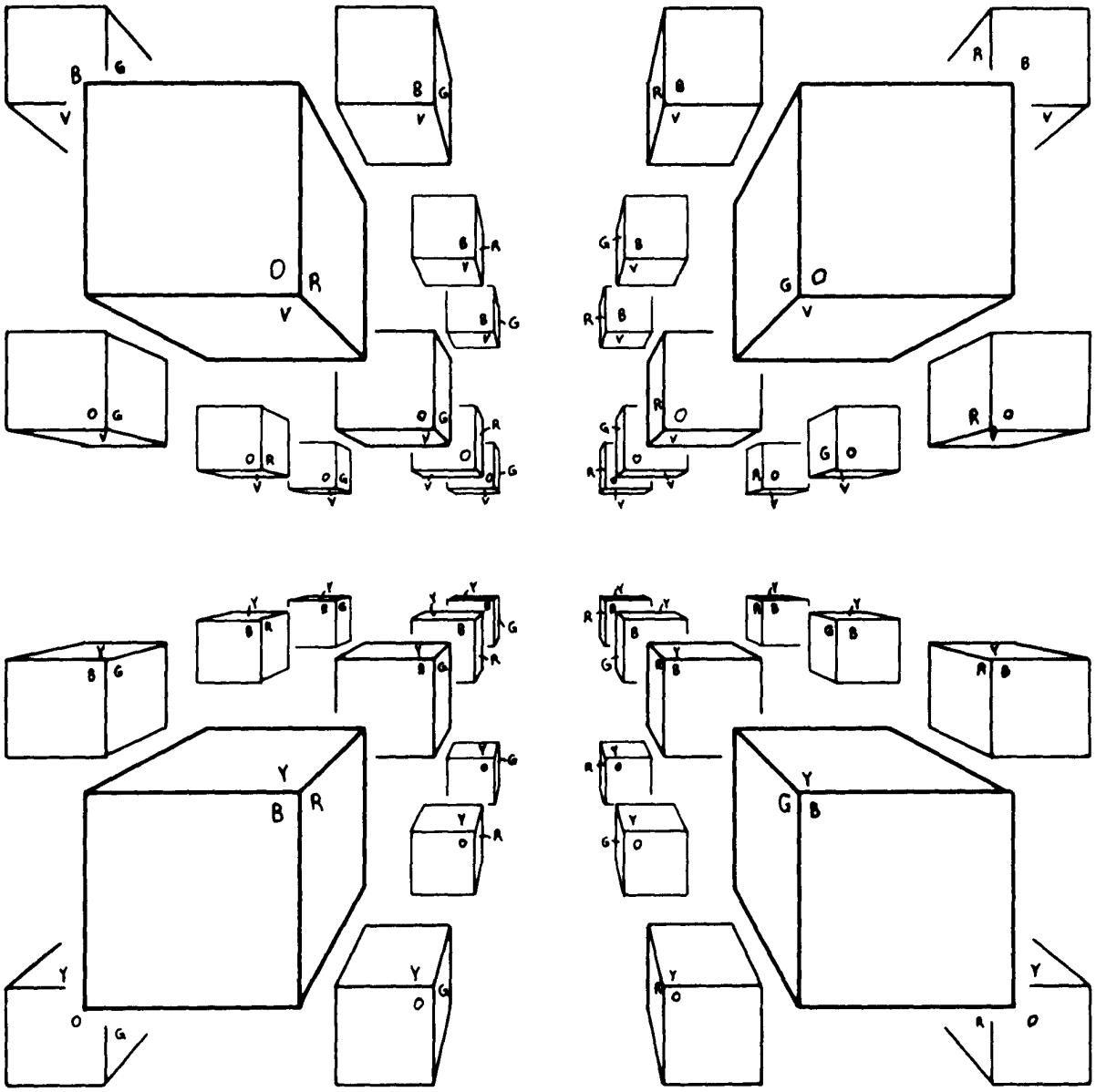


Figure 7.11: What gluing of the big cube produces this view?

Exercise 7.5. Which of the manifolds from Exercises 7.3 and 7.4 are orientable and which are nonorientable? (Recall that to determine orientability you have to check whether crossing a face of the cube can bring you back from the opposite face mirror reversed.) □

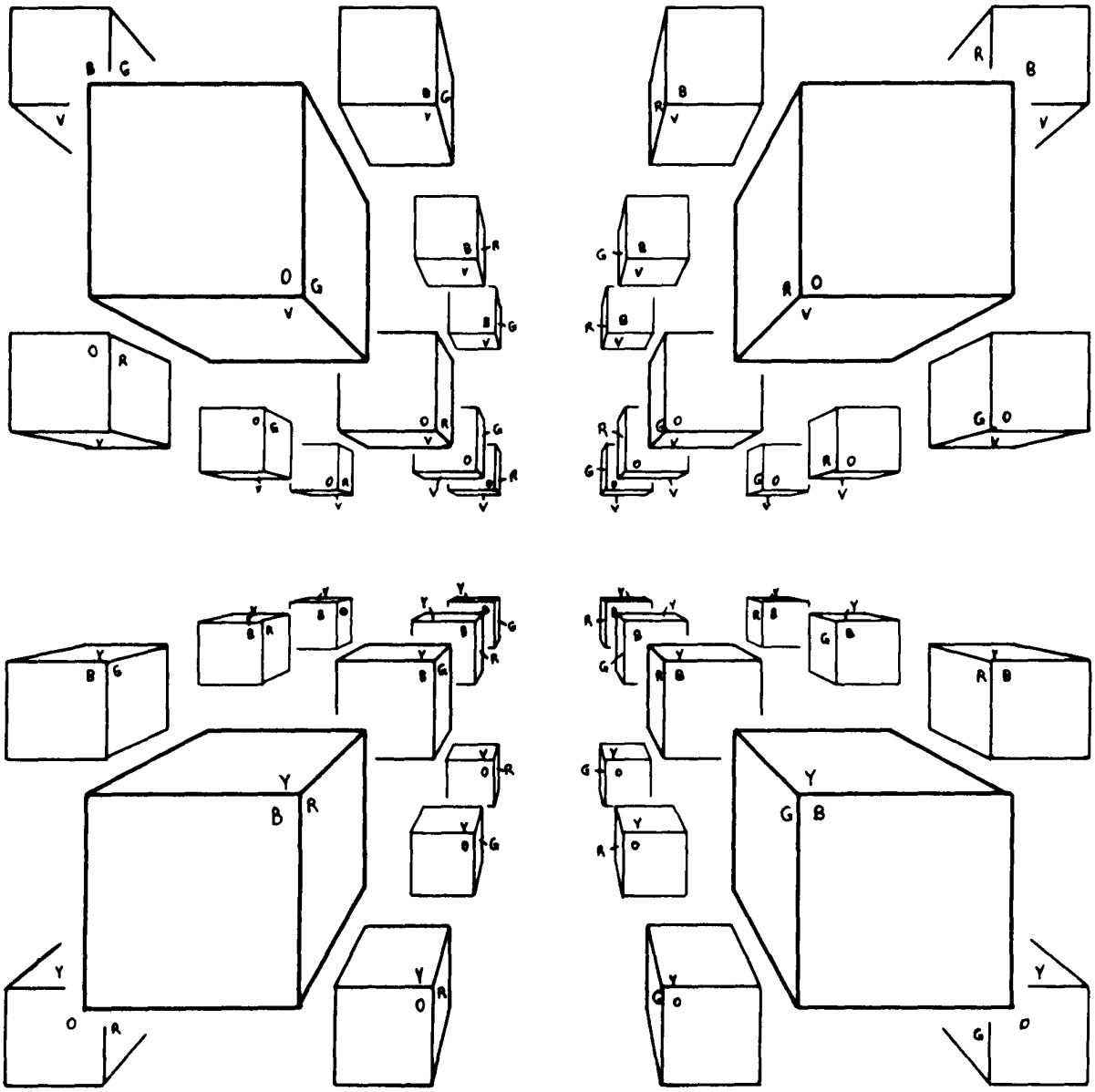


Figure 7.12: What gluing of the big cube produces this view?

Two comments are in order concerning these three-manifolds made from cubes. First, each has the local geometry of ordinary three-dimensional Euclidean space. In other words, each is flat. Second, if you suddenly found yourself in one of these manifolds--

but you didn't know which one it was--you could easily check for orientability by raising your right hand. If all the other copies of yourself raised their right hands you'd know the manifold was orientable; if any of them raised their left hands you'd know it was nonorientable. With more detailed observations you could determine the nature of the manifold exactly (as in Exercise 7.4, only now you yourself take the place of the colored cube).

Not all three-manifolds are made from cubes, and not all surfaces are made from squares. For example, consider a regular hexagon whose opposite edges are abstractly glued. If you are willing to deform the hexagon you can physically carry out the gluings in three-dimensional space (Figure 7.13). You will find that the surface has the same global topology as a torus. Because the surface is made from a hexagon and has the topology of a torus, it is called a **hexagonal torus**. Its local geometry is flat because the hexagon is flat (geometrically you should think of the hexagonal torus as a hexagon with abstractly glued edges, rather than as a doughnut surface).

Exercise 7.6. Imagine a biplane pilot flying about in a hexagonal torus. Draw a picture analogous to Figure 7.3 showing how the hexagonal torus appears to the pilot. What if a second biplane is flying around with her? □

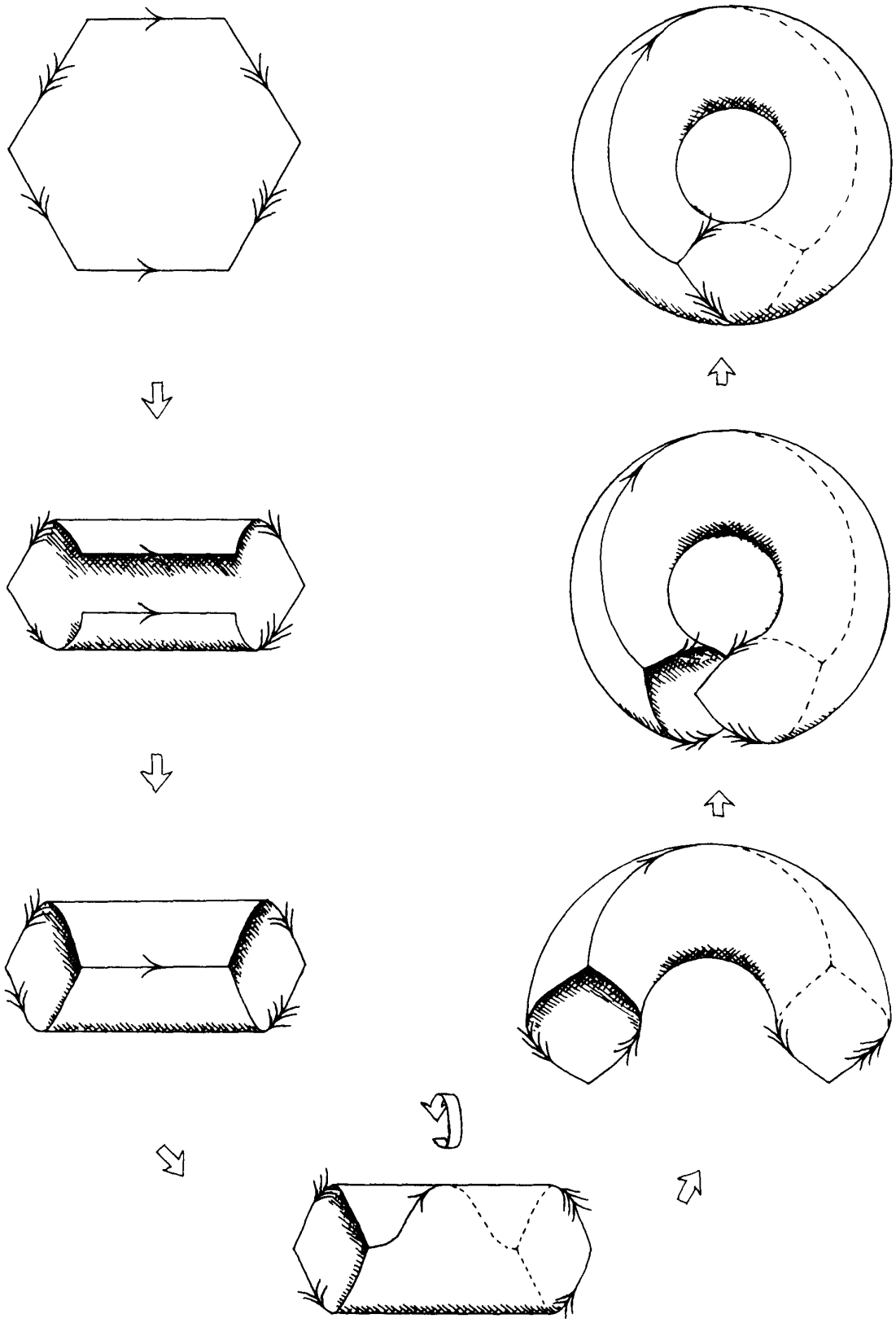
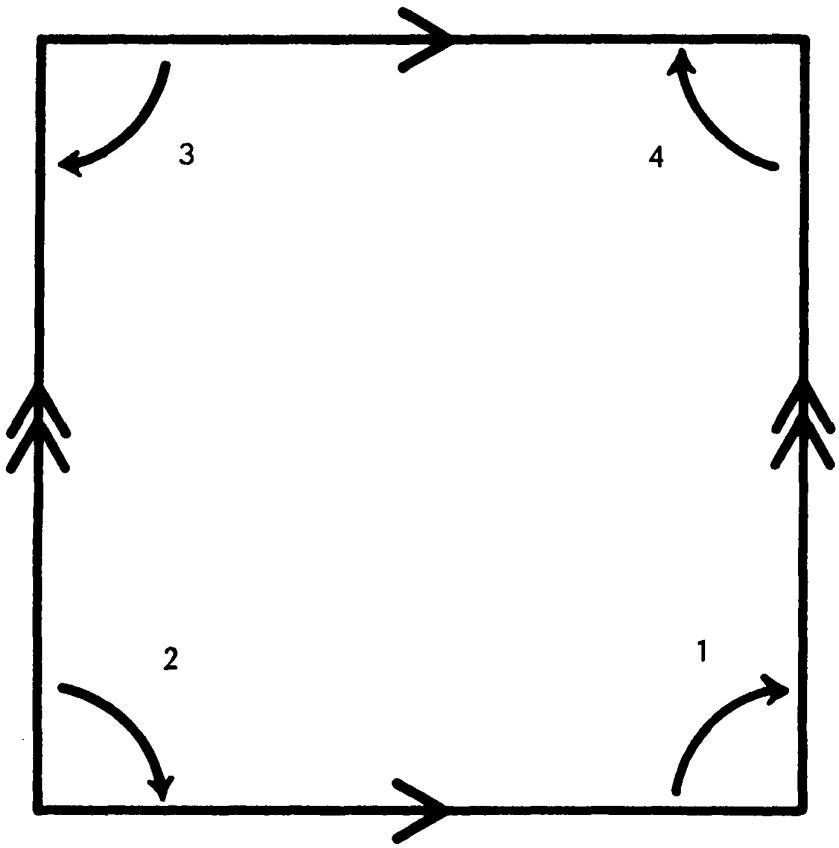


Figure 7.13: To physically glue together opposite edges of a hexagon, you must deform the hexagon into the shape of a doughnut surface. A hexagon with abstractly glued edges therefore has the same global topology as a torus.

For the most part the hexagonal torus is very similar to the usual flat torus. One way that it's different, however, is that all six corners of the hexagon do not get glued to a single point. Compare Figure 7.13 to Figure 3.3. In Figure 3.3 all four corners of the square meet at a single point in the surface, whereas in Figure 7.13 the hexagon's six corners meet in two groups of three corners each.

Later on it will be important to know how a polygon's corners fit together, and in many cases we won't be able to physically carry out the gluings in three-dimensional space. Fortunately we can always tell how the corners fit simply by studying the polygon itself. For example, imagine a Flatlander in the ordinary flat torus who decides to go for a walk around the point at which the square's four corners meet. (Intrinsically he could never locate such a point, but for the sake of argument pretend he is taking a walk around just that point.) Ignore the lower picture in Figure 7.14, and follow his progress in the upper picture. He begins in the lower right hand corner of the square. He passes through the right edge of the square into the lower left hand corner. From there he moves through the bottom edge into the upper left hand corner, then through the left edge into the upper right hand corner, and finally through the top edge to get home. By following his progress in the square we have deduced that all four corners meet at a single point. In contrast, a Flatlander going for a walk in a hexagonal torus would visit only three corners before returning home, either corners 1, 2 and 3 or corners a, b and c,



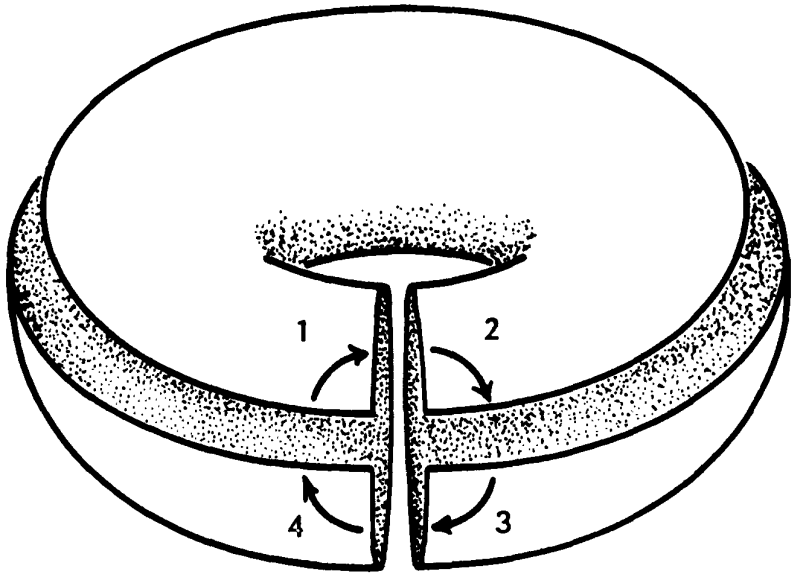


Figure 7.14: A Flatlander walking in a small circle can visit all four corners.

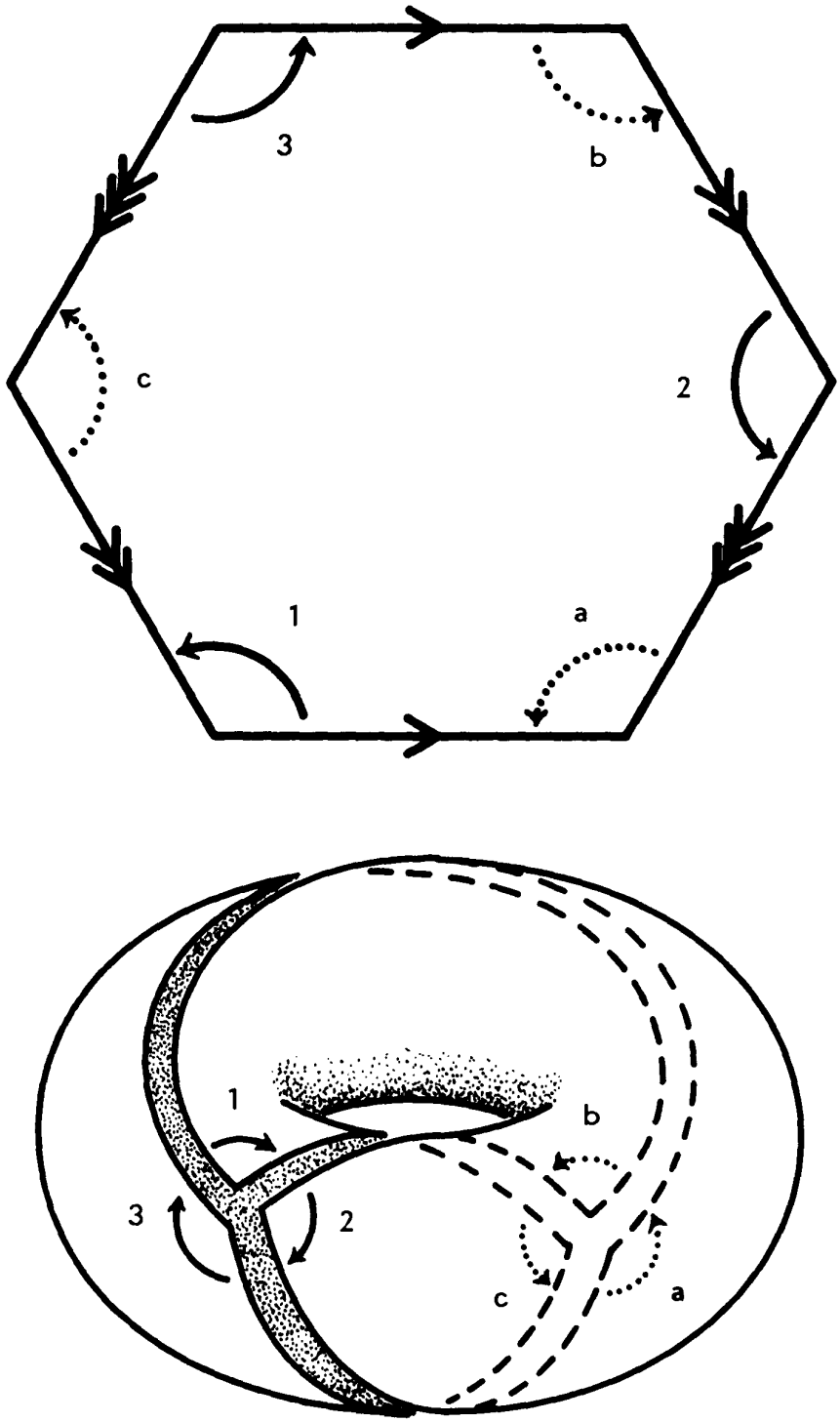


Figure 7.15: A Flatlander walking in a small circle visits a group of three corners.

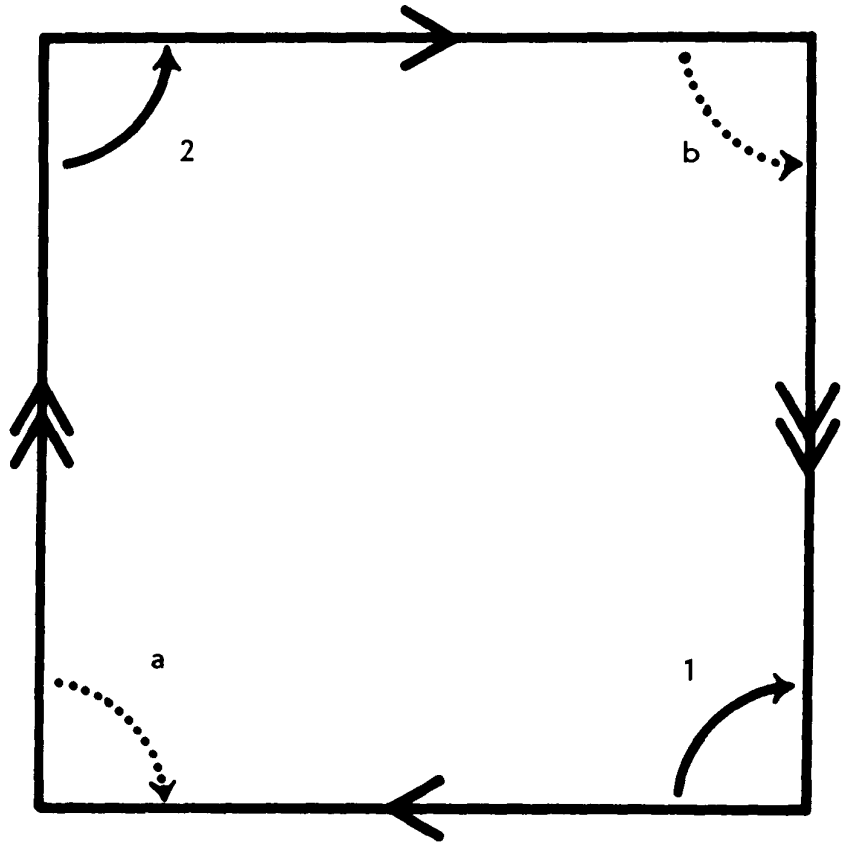


Figure 7.16: Glue each pair of opposite sides with a flip. A Flatlander walking in a small circle can visit only two corners.

depending on where he started (see Figure 7.15). Now let's use this "walking around corners" technique to study a new surface. Consider a square whose edges are glued as indicated by the arrows in Figure 7.16 (note: *both* pairs of edges are glued with a flip). A Flatlander starting in the lower right hand corner passes through the right edge into the *upper* left hand corner; from there he passes through the top edge to get home. Similarly, a Flatlander starting in the lower left hand corner goes to the upper right hand corner and then home. This tells us that the square's

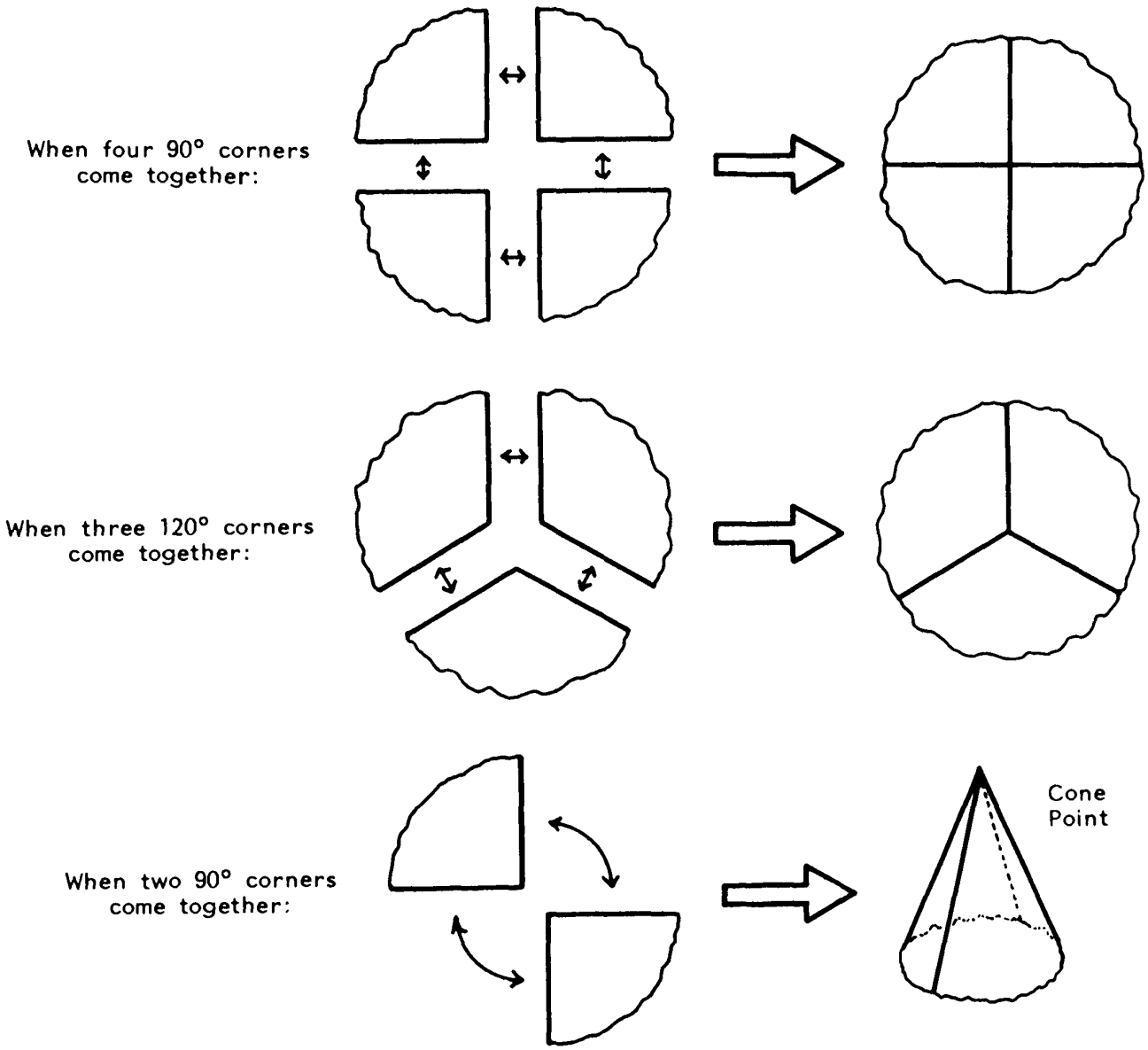


Figure 7.17: If the corners add up to less than 360° you'll get a "cone point".

corners meet in two groups of two corners each. But something funny is going on here: two corners won't fit together properly! Figure 7.17 shows that when two 90° corners come together you get

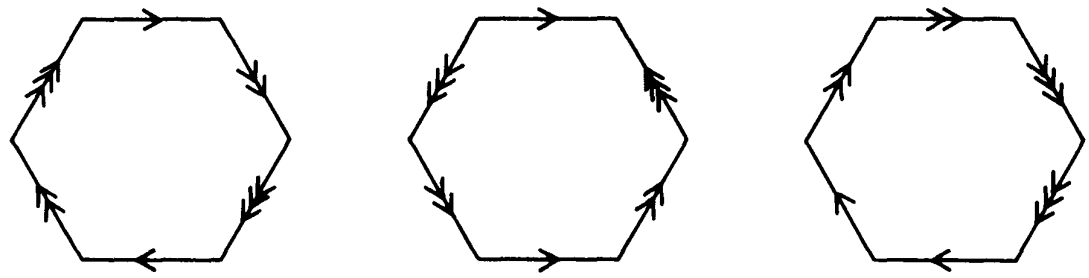


Figure 7.18: Which of these surfaces have cone points?

a "cone point". In general you will get a cone point whenever the sum of the angles at a point is less than 360°.

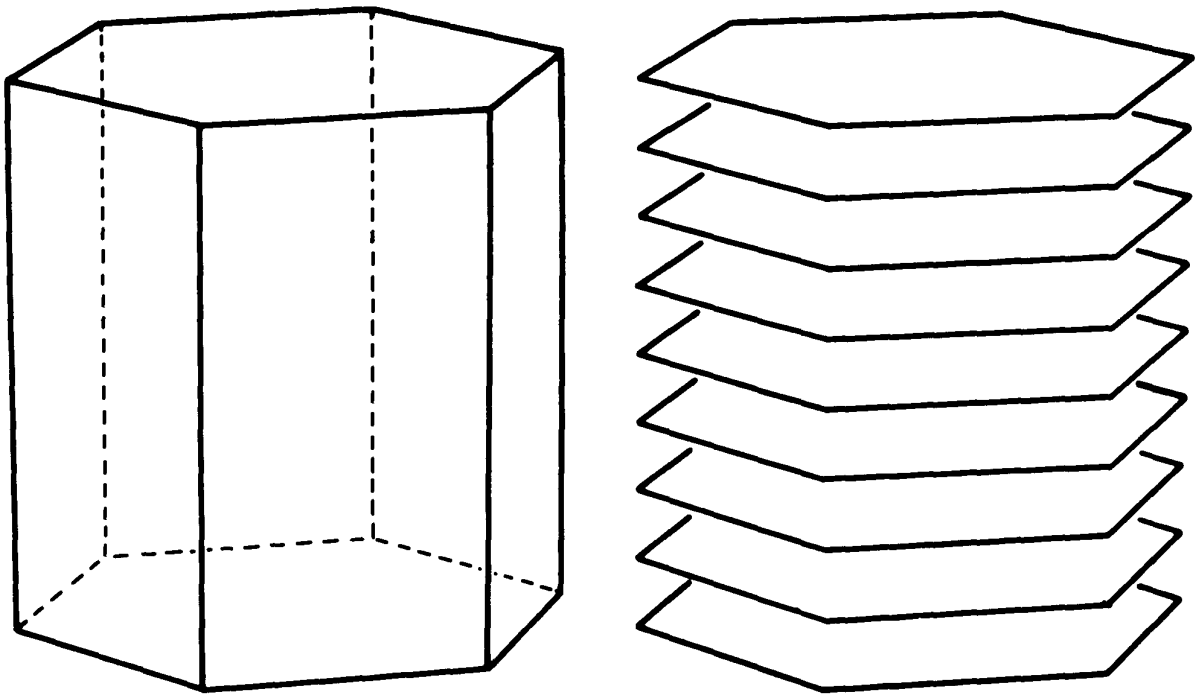
Exercise 7.7. Figure 7.18 shows three surfaces, each a hexagon with edges glued as indicated by the arrows. Use the "walking around corners" technique described above to decide how each hexagon's corners fit together. Which of these surfaces have cone points? □

Exercise 7.8. Which of the surfaces in the preceding exercise are orientable? □

Exercise 7.9. Draw a picture analogous to Figure 7.3 showing a biplane pilot's impression of the second surface in Figure 7.18. □

Exercise 7.10. Do a hexagonal torus and an ordinary flat torus have the same local topology? the same local geometry? the same global topology? the same global geometry? □

At this point I'd like to clear up a definitional matter. When we say a surface has a flat geometry we mean that its local geometry is



(a) The basic prism

(b) The prism thought of as a hexagon cross an interval

Figure 7.19: A hexagonal prism is useful for making three-manifolds.

flat (i.e. Euclidean) at *all* points. Cone points are not allowed. For example, the surface defined in Figure 7.16 is not considered flat, even though it has a flat local geometry at all points except two. In contrast, a hexagonal torus is considered flat because it has a flat local geometry at all points, without exception.

Exercise 7.11. Which of the surfaces of Exercise 7.7 are flat? □

The remaining three-manifolds in this chapter will be made from a hexagonal prism like the one shown in Figure 7.19(a). When you

glue the opposite side faces of this prism each horizontal layer (see Figure 7.19(b)) becomes a hexagonal torus and you get a hexagonal torus cross an interval. If you then glue the bottom of the prism to the top you'll wind up with a hexagonal torus cross a circle, which is a new form of three-dimensional torus. It's called a **hexagonal three-torus.**

Exercise 7.12. Do the hexagonal three-torus and the usual three-torus have the same local geometry? the same global topology? the same global geometry?

Imagine yourself inside a hexagonal three-torus. You see other copies of yourself. How are they arranged? □

Exercise 7.13. Use a hexagonal prism to construct a "1/3 turn manifold" and a "1/6 turn manifold" analogous to the quarter turn manifold of Exercise 7.3. If you lived in one of these manifolds, how would the other copies of yourself be arranged? □

At first glance it's not at all obvious that the hexagonal three-torus and the usual three-torus are topologically the same (i.e. have the same global topology). Similarly, in Chapter 5 it was a challenge to understand that $P^2 \# P^2 \# P^2$ and $T^2 \# P^2$ were the same. This leaves one in doubt as to whether other seemingly different manifolds might also be topologically the same. Around the turn of the century mathematicians invented various tools to deal with this problem. For example, the Euler number

(Chapter 12) can conclusively decide whether or not two surfaces are the same topologically. More complicated tools are required for three-manifolds, such as the "homology groups" and the "fundamental group" (see Chapters 5 and 8 of Armstrong's *Basic Topology*, or any algebraic topology book). Fortunately in practice manifolds which look different almost always are.

8

Orientability Vs. Two-Sidedness

A Möbius strip is nonorientable. In addition, the Möbius strip in Figure 8.1 has only one side, as Escher's ants are demonstrating. In general a surface lying in a three-manifold is called one-sided if a (three-dimensional!) ant can go for a walk along it and come back on the opposite side of the surface from which she started. It's often thought that nonorientability and one-sidedness are just two aspects of the same property, but this is not the case, as you can see from the examples in Exercise 8.1.

A surface (lying in a three-manifold) which is not one-sided is called two-sided, because in this case the surface has two separate sides, and a (three-dimensional) ant walking on the surface can never get from one side to the other.

This chapter is dedicated to Bob Messer, who showed me a two-sided Möbius strip at a time when I hadn't the slightest idea that such a thing could possibly exist.

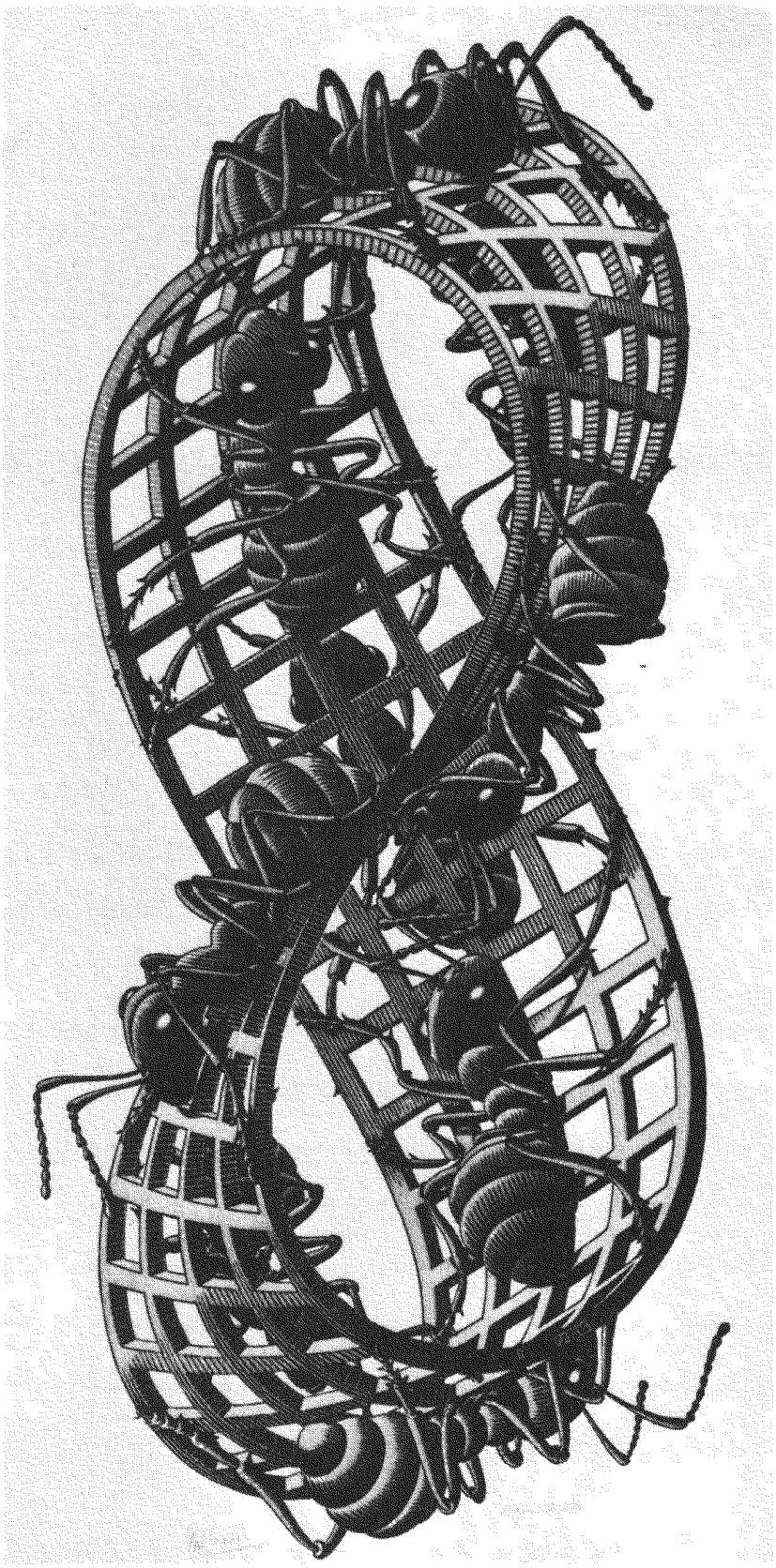


Figure 8.1: Escher's ants demonstrate that this Möbius strip has only one side. (*Möbius Strip II* by M.C. Escher, copyright © Beeldrecht, Amsterdam/VAGA, New York. Collection Haags Gemeentemuseum--The Hague. 1985.)

Exercise 8.1. Figure 8.2 shows four ways to construct a three-manifold by gluing opposite faces of a cube. In each case the cube contains a square which gets glued to form a surface in the three-manifold. For each example identify the surface and say whether it's orientable or nonorientable, and whether it's one-sided or two-sided. To determine orientability you have to check whether a (two-dimensional) Flatlander can go for a walk *in* the surface and return to his starting point mirror reversed. To determine "sidedness" you have to check whether a (three-dimensional) ant can go for a walk *on* the surface and return to her starting point on the opposite side of the surface.

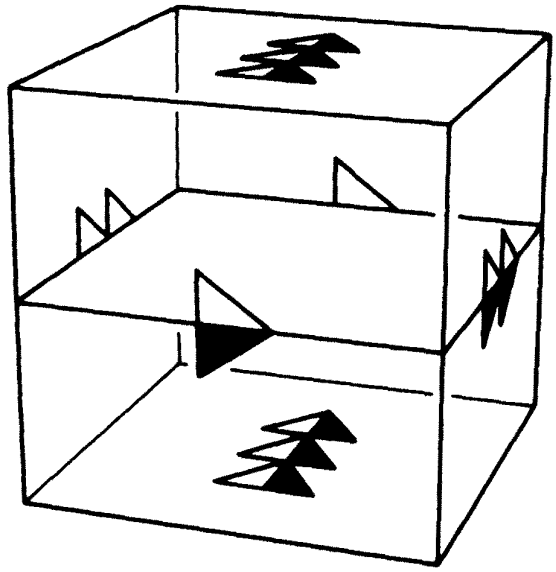
Use the four examples to fill in the following table:

	orientable	nonorientable
one-sided		
two-sided		

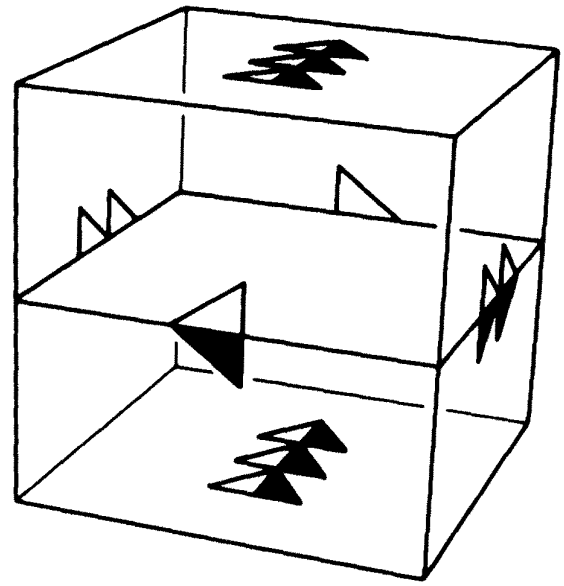
□

Exercise 8.2. Is sidedness an intrinsic or an extrinsic property of a surface? What about orientability? Are all Klein bottles nonorientable? Are they all one-sided? □

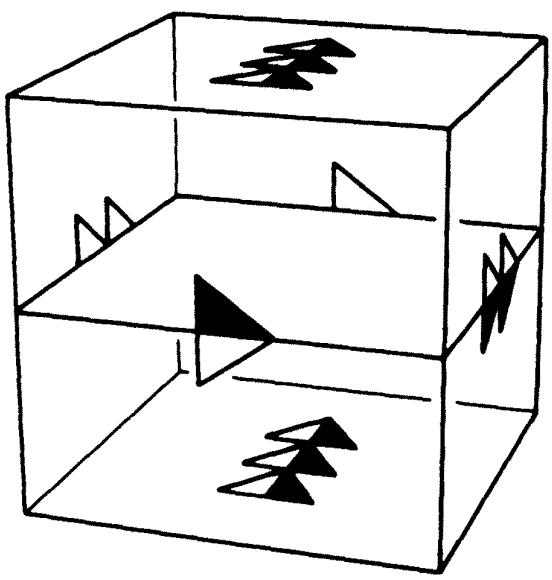
Exercise 8.3. Construct a three-manifold containing a two-sided Mobius strip. Just for fun, draw some red ants on one side of the



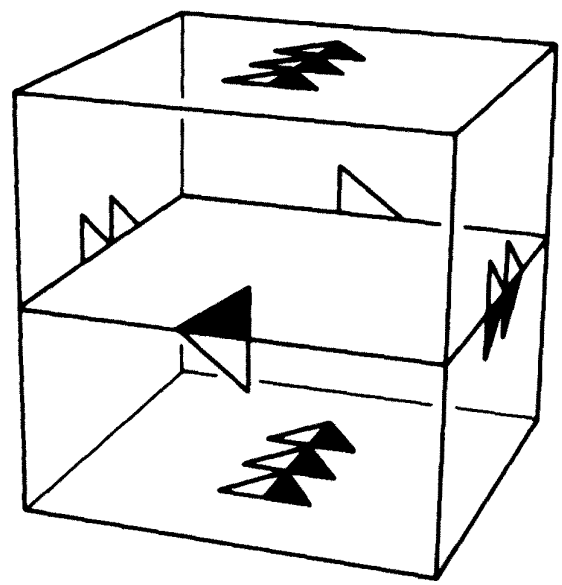
(1) Glue opposite faces in the usual way to form a 3-torus.



(2) Glue the top to the bottom, and the left side to the right side, in the usual way, but glue the front to the back with a side-to-side flip. This 3-manifold is $K^2 \times S^1$.



(3) Glue the top to the bottom, and the left side to the right side, in the usual way, but glue the front to the back with a top-to-bottom flip. This 3-manifold is also $K^2 \times S^1$.



(4) Glue the top to the bottom, and the left side to the right side, in the usual way, but glue the front to the back with both a side-to-side flip *and* a top-to-bottom flip. The two flips combined are the same as a 180° rotation, and the manifold is the half-turn manifold of Exercise 7.3.

Figure 8.2: Four three-manifolds, each containing a surface.

Möbius strip and some black ants on the other. Note that the red ants and black ants can never meet (at least not without crawling around the edge of the Möbius strip!). □

One's initial impression that nonorientability and one-sidedness are just different aspects of a single property is not entirely groundless. *In an orientable three-manifold,* every two-sided surface is orientable and every one-sided surface is nonorientable. To see why, pretend that you are standing with a friend on a surface in an orientable three-manifold. Your friend decides to go for a walk. Because you're in an orientable three-manifold you know he can't possibly come back mirror reversed. Assume for the moment that the surface you're standing on is one-sided, and that your friend comes back from his walk on the side opposite from where you're standing. Also assume that the surface is transparent so you can look through and see the bottoms of his feet. (In this manifold gravity is directed towards the surface, so neither of you fall off!) Because your friend is on the other side of the surface, his footprints appear mirror reversed (Figure 8.3). If you study just his footprints (ignore his three-dimensional body) you'll see that they trace out an orientation reversing path in the surface. Therefore this one-sided surface is nonorientable.

If, on the other hand, the surface were two-sided, then your friend would come back from his walk on the same side of the surface he started on. Because the three-manifold is orientable,

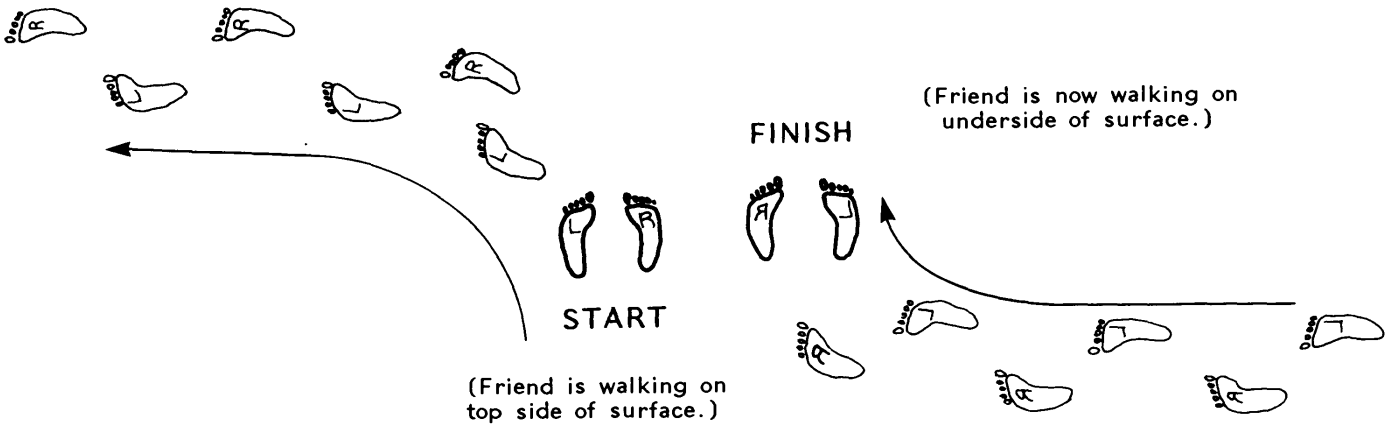


Figure 8.3: Going for a walk on a one-sided surface in an orientable three-manifold. At the end of the walk your friend *isn't* mirror-reversed; but because he's now standing on the other side of the surface, his footprints appear mirror-reversed to you.

he'd be his normal self rather than his mirror image. In particular, his footprints would not be mirror reversed. This shows that no matter where he walks, his footprints will never trace out an orientation reversing path in the surface. In other words, this two-sided surface is orientable.

Part II
Geometries on Surfaces

9

The Sphere

In a sequel to *Flatland*, an octagon by the name of Mr. Puncto surveys some rather large triangles and finds that each has angles which add up to slightly more than 180°! This discovery excites Mr. Puncto, and, after double checking his data, he makes the discovery public. Unfortunately neither the civil authorities nor the scientific establishment share his excitement. They suspect he is merely inventing excuses to explain some errors in his measurements, and they dismiss him from his job. The true explanation is that the Flatlanders are living not in a plane but on a sphere, and on a sphere the angles of a triangle really do add up to more than 180°, as we shall soon see. This incident and much more is described in the book *Sphereland* by Dionys Burger. I heartily recommend *Sphereland* to all readers of the present book (just don't be put off by the rather dull summary of *Flatland* at the beginning).

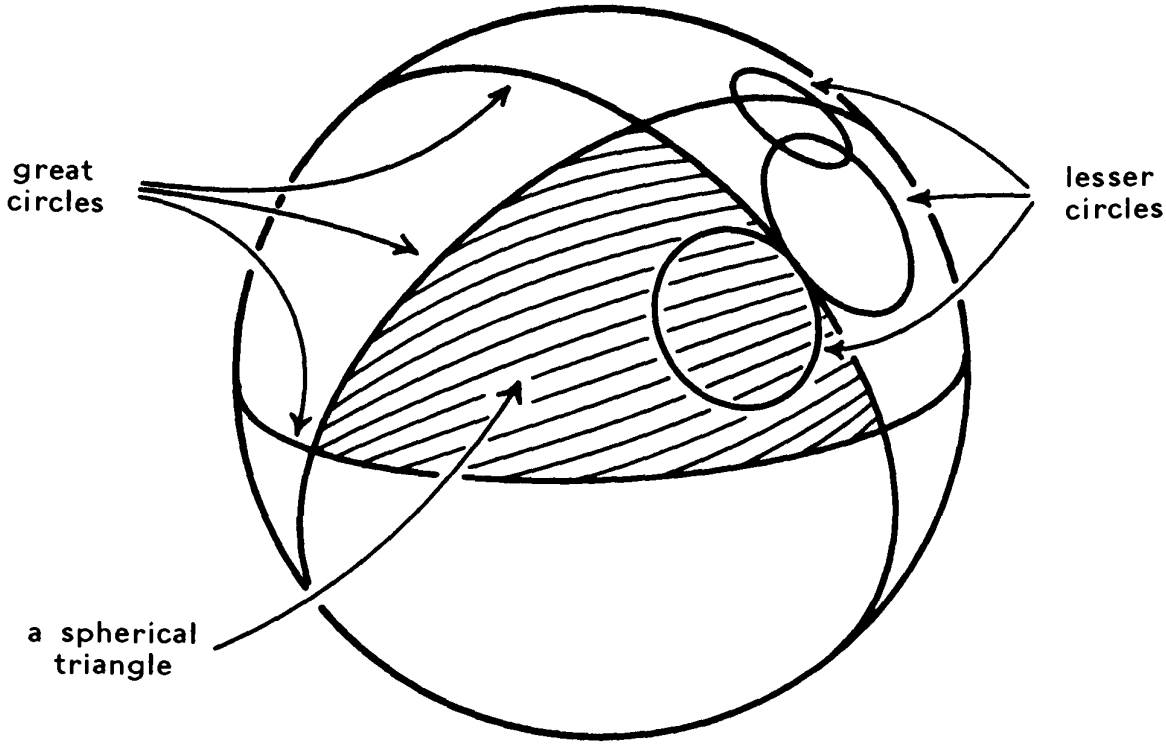


Figure 9.1: On any sphere, the **great circles** are those circles which are as big as possible. A great circle appears straight to a Flatlander on the sphere. By contrast, any lesser circle appears to bend to one side or the other.

A triangle drawn on a sphere is called a **spherical triangle.** Each side of a spherical triangle is required to be a geodesic; that is, it is required to be intrinsically straight in the sense that a Flatlander on the sphere would perceive it as bending neither to the left nor to the right. A side of a spherical triangle is thus an arc of a so-called great circle (see Figure 9.1).

From now on we will *measure all angles in radians,* to facilitate easier comparison of angles and areas (in a minute you'll see how and why we want to do this). Recall that π radians = 180°, $\pi/2$ radians = 90°, etc. Except when specified otherwise, we will

henceforth assume that all spheres are unit spheres, i.e. they all have radius one.

Exercise 9.1. For each spherical triangle in Figure 9.2 compute (1) the sum of its angles in radians, and (2) its area. To compute the areas, use the fact that the unit sphere has area 4π. For example, the first triangle shown occupies 1/8 of the sphere, so its area is $(4\pi)/8 = \pi/2$.

Find a formula relating a spherical triangle's angle-sum to its area. This formula appeared in 1629 in the section "De la mesure de la superfice des triangles et polygones sphericques, nouvellement inventee Par Albert Girard" of the book *Invention nouvelle en L'Algebre* by Albert Girard.

You should try to find the formula before reading on, because the following paragraphs give it away. □

Exercise 9.2. What is the area of a spherical triangle whose angles in radians are $\pi/2$, $\pi/3$ and $\pi/4$? What is the area of a spherical triangle with angles of 61°, 62° and 63°? □

Even though there is no overwhelming need for a proof of the formula you just discovered, I would like to include one anyhow because it is so simple and elegant. (It is not, however, the sort of thing you're likely to stumble onto on your own. I struggled for hours without being able to prove the formula at all.)

First we have to know how to compute the area of a "double lune". A **double lune** is a region on a sphere bounded by two

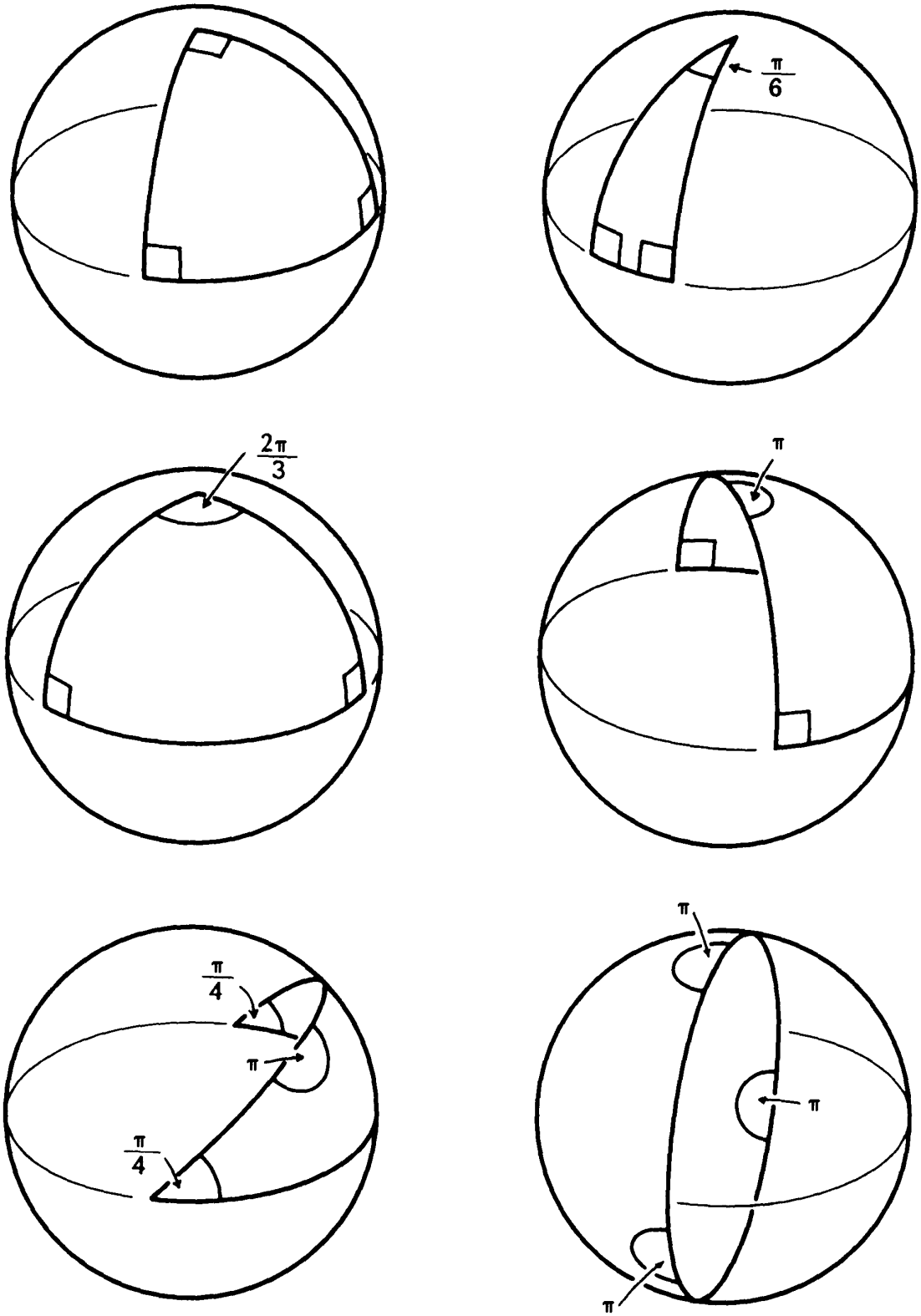


Figure 9.2: Some assorted spherical triangles. Three of the triangles are "degenerate" in the sense that each has one or more angles equal to π. The last triangle occupies an entire hemisphere, and its three sides all lie on the same great circle.

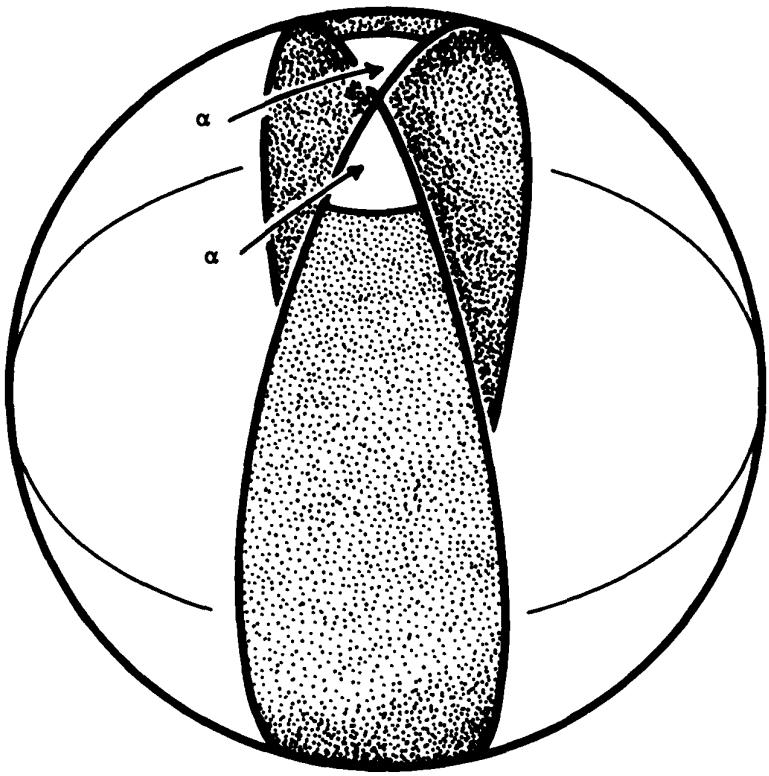


Figure 9.3: A double lune with angle α.

great circles, as shown in Figure 9.3. The largest the angle α can ever be is π, at which point the double lune fills up the entire sphere. So if α is, say, $\pi/3$, then we reason that since $\pi/3$ is 1/3 the greatest possible angle π, the double lune must fill up 1/3 the area of the entire sphere, namely $(1/3)(4\pi) = 4\pi/3$. Using the same reasoning, we get that the area of a double lune with angle α is $(\alpha/\pi)(4\pi) = 4\alpha$. You can check this formula for some special cases, e.g. $\alpha = \pi/2$ or $\alpha = \pi$.

Now we'll find a formula for the area of a spherical triangle with angles α, β and γ. First extend the sides of the triangle all the way around the sphere to form three great circles, as shown in

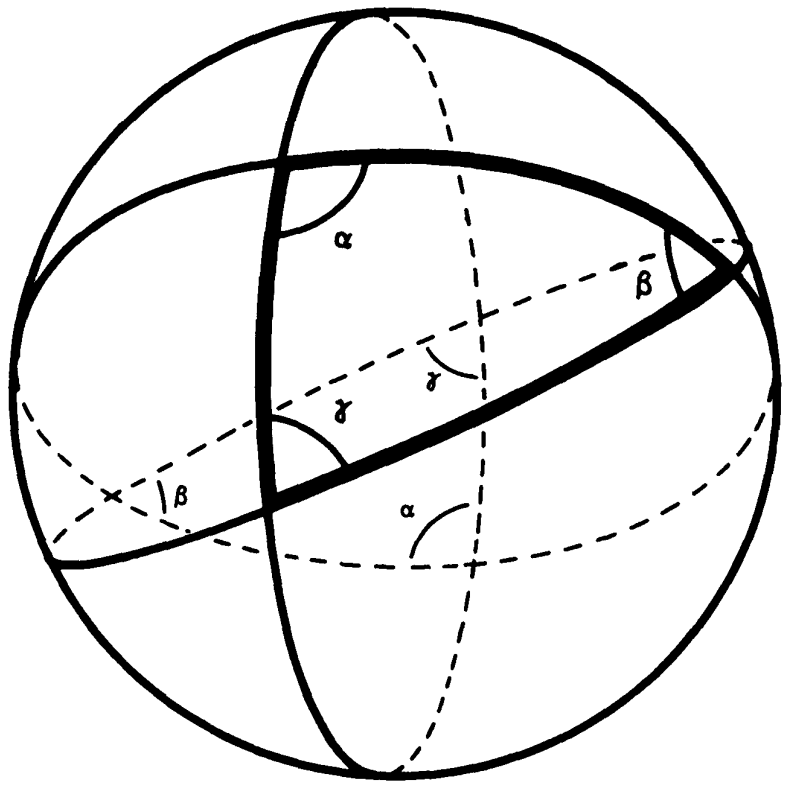


Figure 9.4: Extend the edges of the spherical triangle, and the resulting great circles will form an "antipodal triangle" on the back side of the sphere.

Figure 9.4. An "antipodal triangle", identical to the original, is formed on the back side of the sphere. Figure 9.5 shows three possible ways to shade in double lunes. These double lunes have respective angles α, β and γ, and therefore their areas are 4α, 4β and 4γ.

Now look what happens if we shade in all three double lunes simultaneously (Figure 9.6). All parts of the sphere get shaded in at least once, and the original and antipodal triangles each get shaded in three times (once for each double lune). So. . .

$$\begin{pmatrix}\text{area of}\\ \text{first}\\ \text{double}\\ \text{lune}\end{pmatrix} + \begin{pmatrix}\text{area of}\\ \text{second}\\ \text{double}\\ \text{lune}\end{pmatrix} + \begin{pmatrix}\text{area of}\\ \text{third}\\ \text{double}\\ \text{lune}\end{pmatrix} = \overbrace{\begin{pmatrix}\text{area of}\\ \text{entire}\\ \text{sphere}\end{pmatrix}}^{\left[\begin{matrix}\text{everything}\\ \text{was shaded}\\ \text{in once}\end{matrix}\right]} + 2\overbrace{\begin{pmatrix}\text{area of}\\ \text{original}\\ \text{triangle}\end{pmatrix}}^{\left[\begin{matrix}\text{each triangle}\\ \text{was shaded}\\ \text{in two more}\\ \text{times}\end{matrix}\right]} + 2\begin{pmatrix}\text{area of}\\ \text{antipodal}\\ \text{triangle}\end{pmatrix}$$

$$4\alpha + 4\beta + 4\gamma = 4\pi + 2A + 2A$$

$$4(\alpha + \beta + \gamma) = 4(\pi + A)$$

$$\alpha + \beta + \gamma = \pi + A$$

$$(\alpha + \beta + \gamma) - \pi = A$$

which is just what we wanted to prove! In words, this formula says that the sum of the angles of a spherical triangle exceeds π by an amount equal to the triangle's area.

Exercise 9.3. The formula $(\alpha+\beta+\gamma) - \pi = A$ applies only to triangles on a sphere of radius one. How must you modify the formula to apply to triangles on a sphere of radius two? What about radius three? Write down a general formula for triangles on a sphere of radius r. □

Exercise 9.4. A society of Flatlanders lives on a sphere whose radius is exactly 1000 meters. A farmer has a triangular field with perfectly straight (i.e. geodesic) sides and angles which have been carefully measured as 43.624°, 85.123° and 51.270°. What is the area of the field? Don't forget to convert the angles to radians. (Bonus Question: How accurately do you know the field's area? That, plus or minus what percent?) □

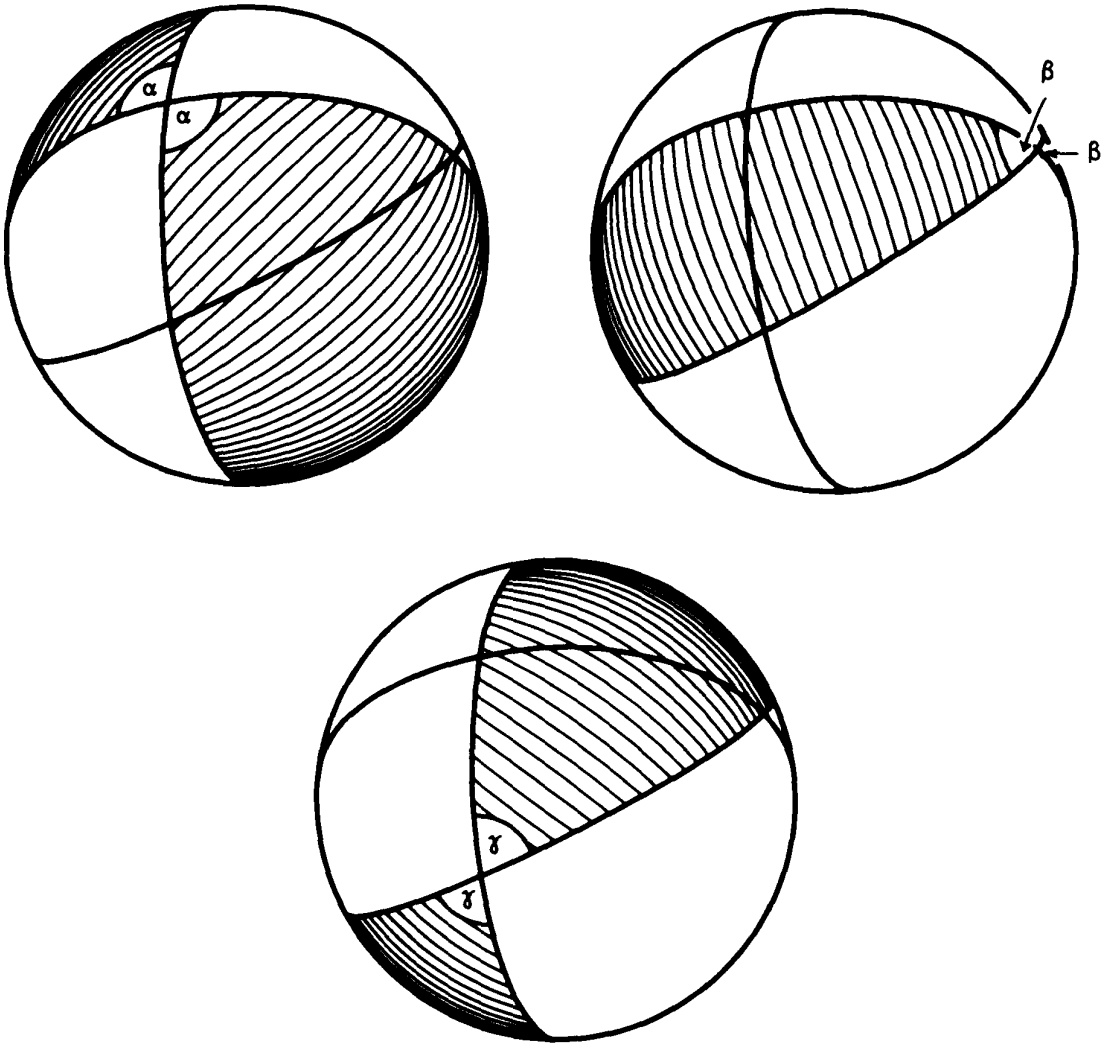


Figure 9.5: Three ways to shade in double lunes.

Exercise 9.5. A society of Flatlanders lives on a sphere. They carefully survey a triangle and find its angles to be 60.0013°, 60.0007° and 60.0011°, while its area is 5410.52 square meters. What is the area of the entire sphere? □

Exercise 9.6. Estimate the angle-sum of each of the following spherical triangles. Hint: First make a rough estimate of the area

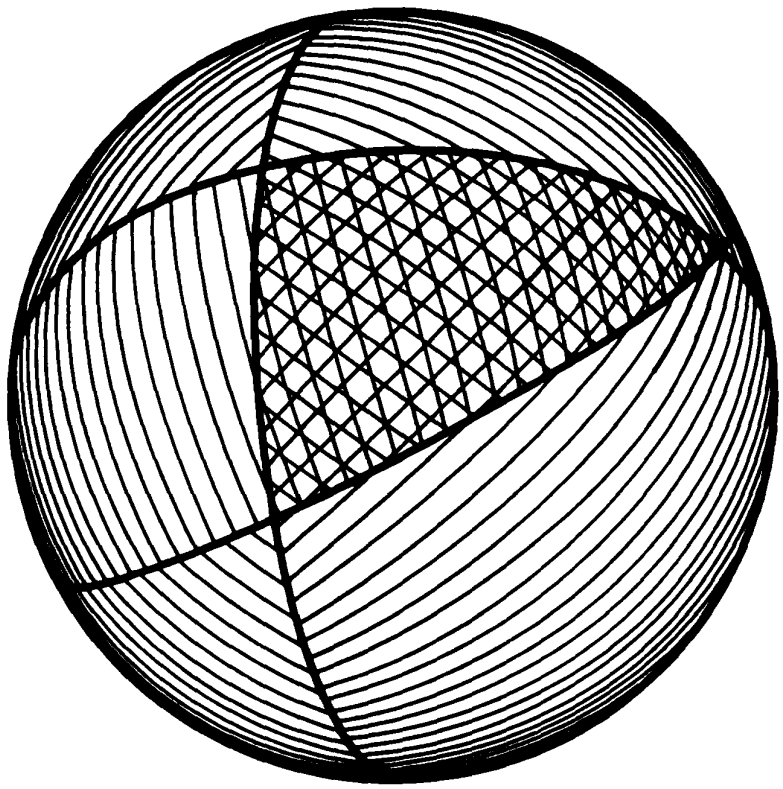


Figure 9.6: Look what happens when we shade in all three double lunes at once.

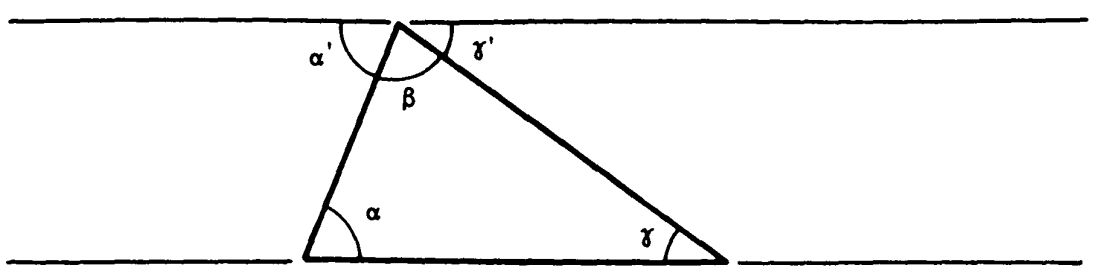
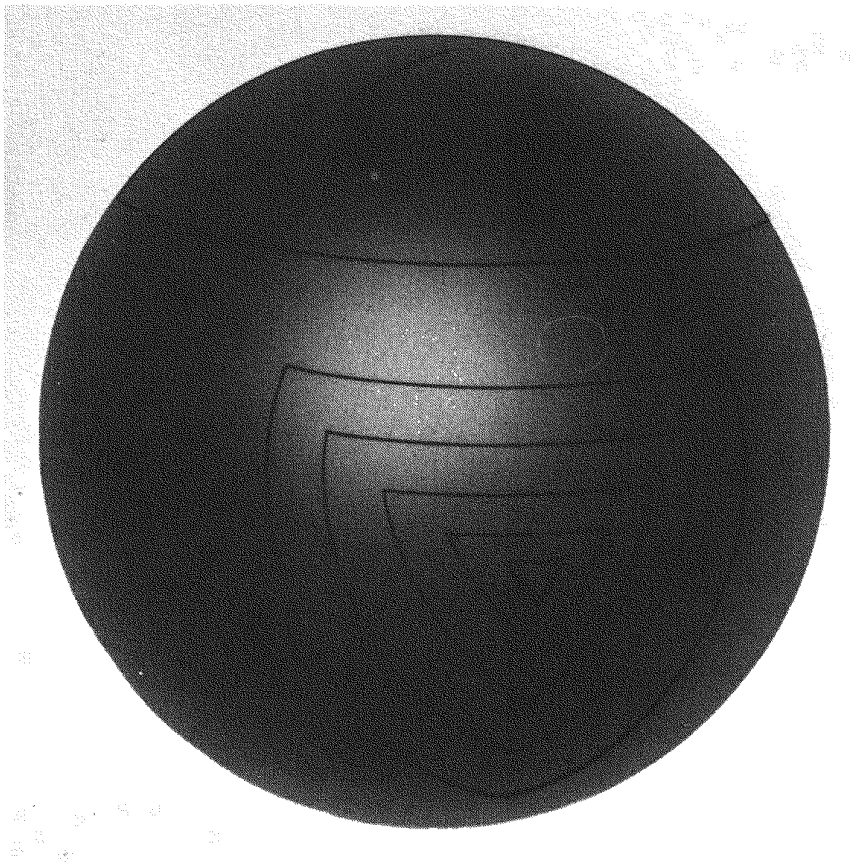


Figure 9.7: A quick proof that the sum of the angles in a Euclidean triangle is π: (1) $\alpha' + \beta + \gamma' = \pi$, (2) $\alpha = \alpha'$ and (3) $\gamma = \gamma'$, therefore (4) $\alpha + \beta + \gamma = \pi$.



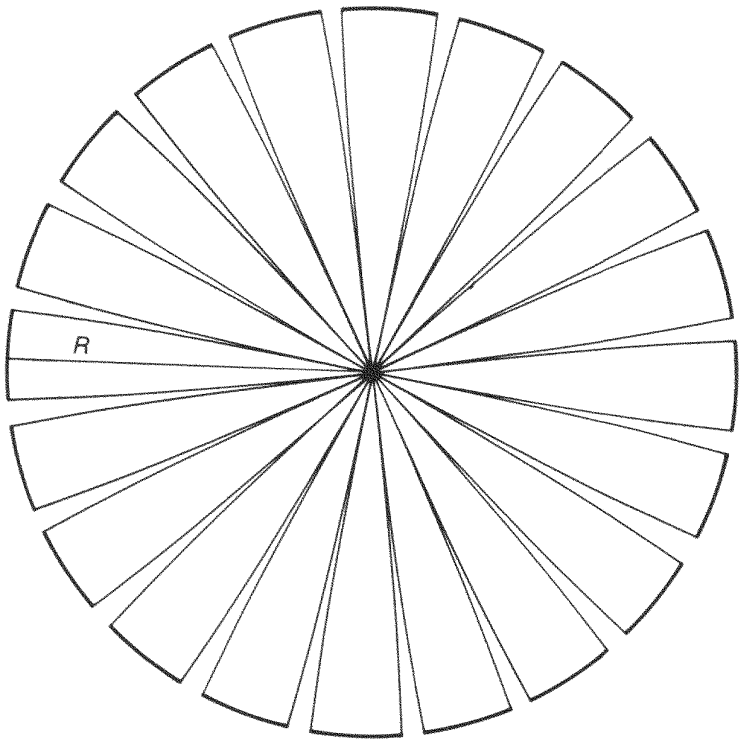


Figure 9.8: A piece of a sphere splits open when flattened.

enclosed by each triangle, and then apply the formula from Exercise 9.3. Because you have to guess the areas of the triangles, you will get only approximate, not exact, answers. The radius of the Earth is roughly 6400 km.

1. The triangle formed by Providence, Newport and Westerly, R.I. These cities are roughly 50 km apart.
2. The triangle formed by Houston, El Paso and Amarillo, TX. These cities are roughly 1000 km apart.
3. The triangle formed by Madras, India; Tokyo, Japan; and Leningrad, USSR. These cities are roughly 7000 km apart.

□

We've now seen the first major way in which the geometry of a sphere differs from the geometry of a plane. Namely, the sum of the angles of a spherical triangle exceeds π by an amount proportional to the triangle's area, whereas the sum of the angles of a Euclidean (= flat) triangle equals π exactly (study Figure 9.7 for a proof of this last fact).

A piece of a sphere rips open when flattened onto a plane (Figure 9.8). This shows that a circle on a sphere has a smaller circumference and encloses less area than a circle of the same radius in a plane. I should stress that the radius of a circle on a sphere is measured along the sphere itself--the way a Flatlander would measure it. Figure 9.9 shows that on a sphere a circle's circumference can actually shrink, even though the circle's (intrinsically measured) radius is still increasing.

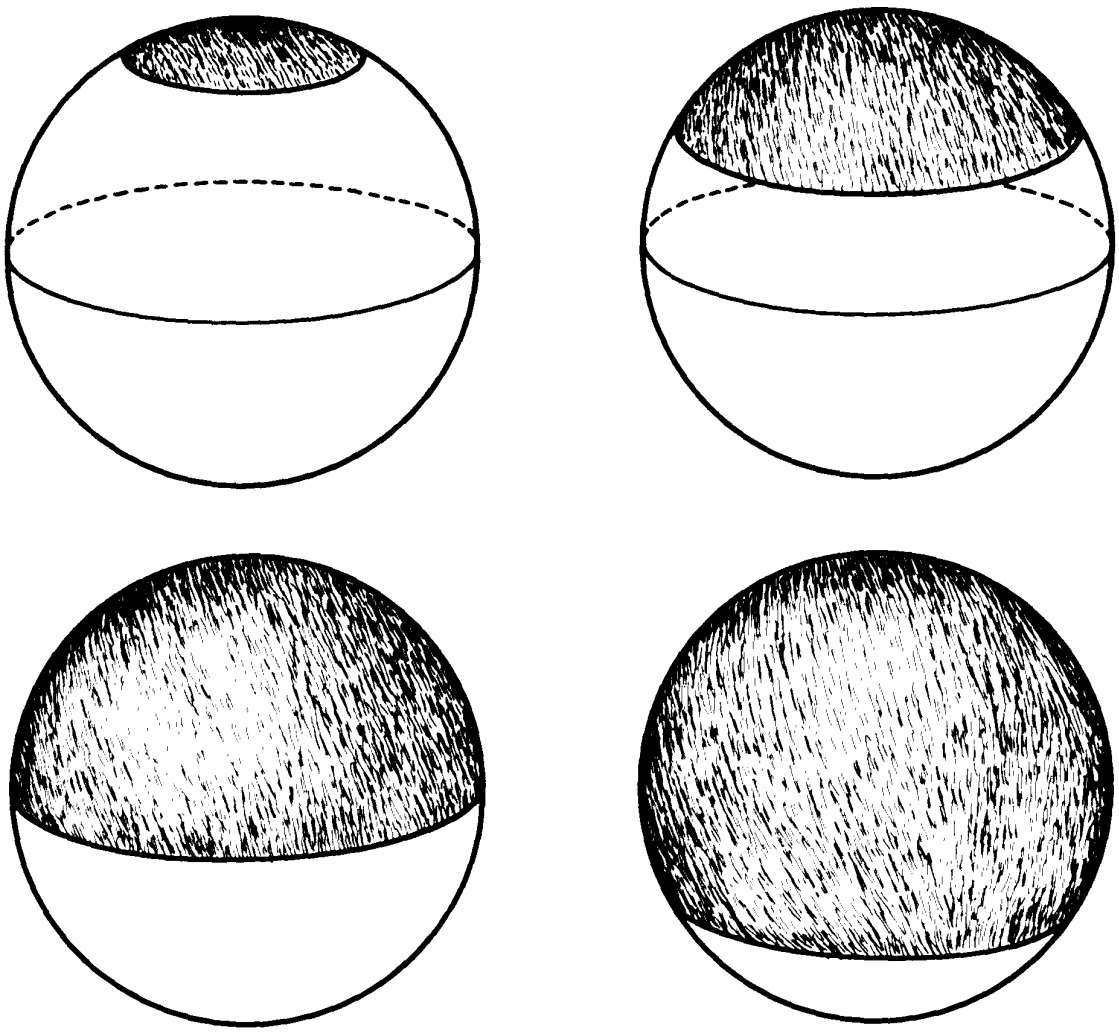


Figure 9.9: The circle's circumference first increases, but then decreases once the circle is past the equator.

Exercise 9.7. *(Review exercise)* What other surface has the local geometry of a sphere? Could Flatlanders living on this other surface tell that they weren't on a sphere? □

10
The Hyperbolic Plane

There are three types of homogeneous two-dimensional geometries, as illustrated in Figure 10.1. Two of these are already familiar. The first geometry shown is the familiar geometry of a sphere. This geometry is often called **elliptic geometry**, and is said to have **positive curvature.** The second geometry is the familiar geometry of the Euclidean plane. It is called **Euclidean geometry** and is said to have **zero curvature.** The third geometry shown is, loosely speaking, a saddle shaped geometry. It is less familiar than elliptic or Euclidean geometry, but certainly no less important. It is called **hyperbolic geometry** and is said to have **negative curvature.** The **hyperbolic plane** (abbreviated $\mathbf{H}^2$) is an infinite plane which has hyperbolic geometry (= constant negative curvature) everywhere, just as the Euclidean plane is an infinite plane which has Euclidean geometry (= zero curvature) everywhere. It is a fact of nature, though, that there can be no infinite plane with elliptic geometry

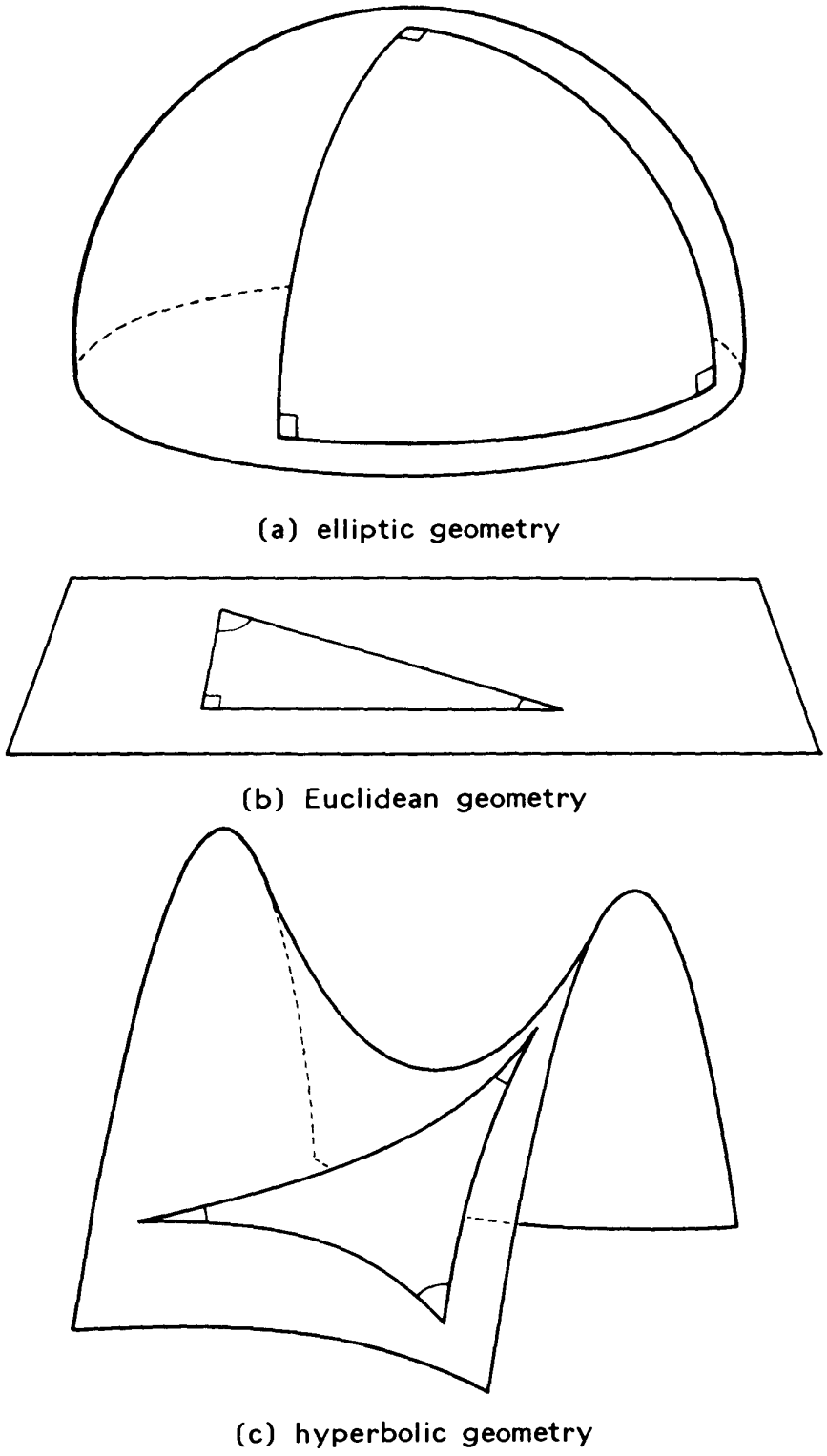


Figure 10.1: The three homogeneous two-dimensional geometries.

(= constant positive curvature) everywhere--it will, without fail, close back onto itself to form either a sphere or a projective plane.

While the saddle shape provides a good local picture of the hyperbolic plane, it fails to convey what the hyperbolic plane is like on a larger scale. To get a feel for the larger scale nature of the hyperbolic plane, make yourself some hyperbolic paper as described in Exercise 10.1. Handling the hyperbolic paper will do for your understanding what words and pictures cannot.

Exercise 10.1. *How to make hyperbolic paper.* (Thanks to Bill Thurston for this idea.) You'll need a large number of equilateral triangles, at least a hundred or so. The most efficient way to make them is to make several photocopies of Figure 10.3. Tape seven triangles together like the seven shaded triangles in Figure 10.2. Then add more triangles to the pattern in such a way that each vertex of the original figure is itself surrounded by seven triangles. Continue in this fashion as long as you want. The surface you get will be very floppy, but this is how it should be. The larger the surface, the floppier and more accurate it will be. You can't go wrong so long as every vertex is surrounded by exactly seven triangles. A construction hint: Don't cut triangles apart only to tape them back together again! For example, rather than starting with seven separate triangles, you can start by inserting a single triangle into a block which has been slit and creased as shown in the inset to Figure 10.2. □

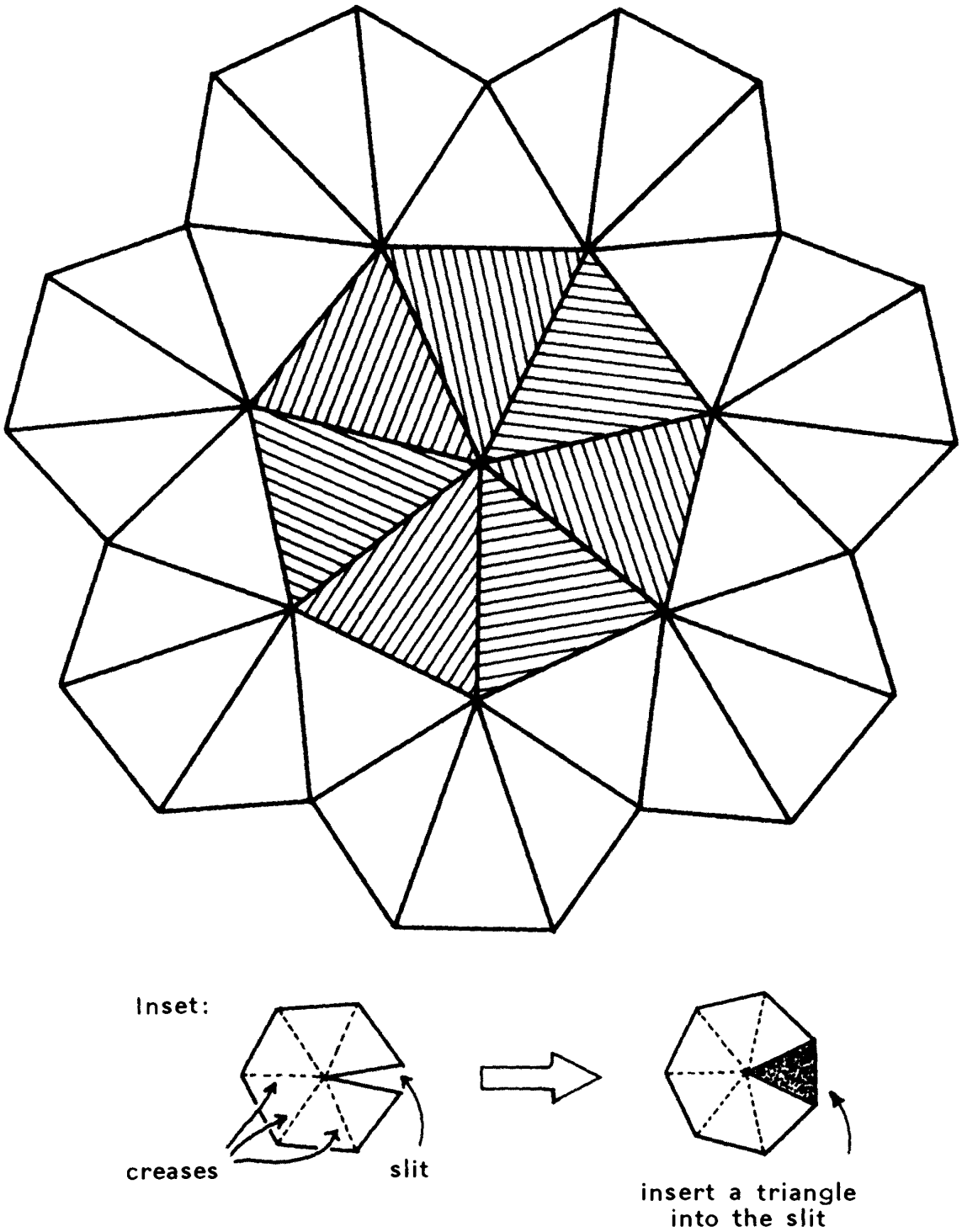


Figure 10.2: *(See Exercise 10.1)* Tape equilateral triangles together so that seven meet at each vertex. The inset shows a good way to get started.

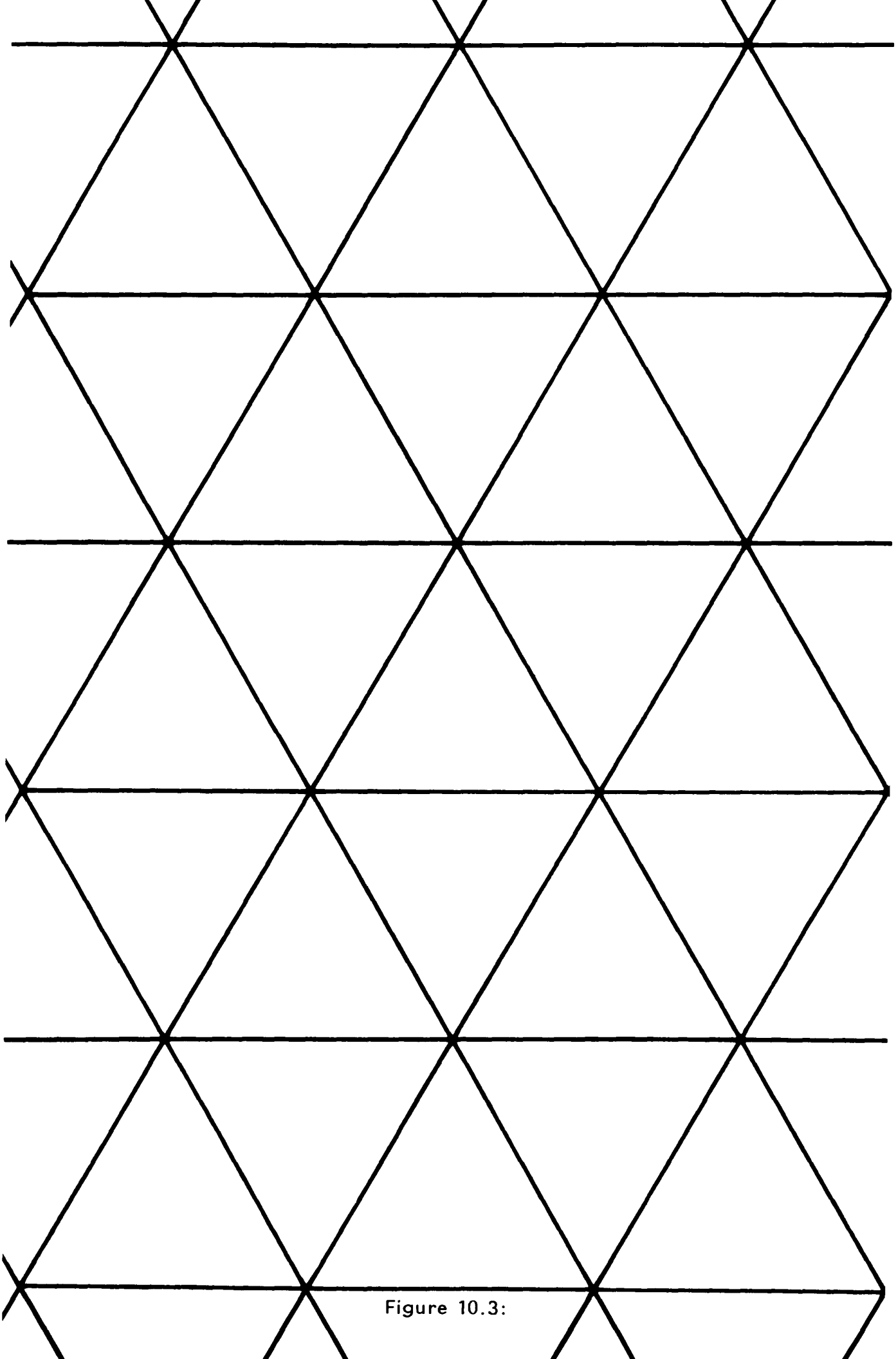


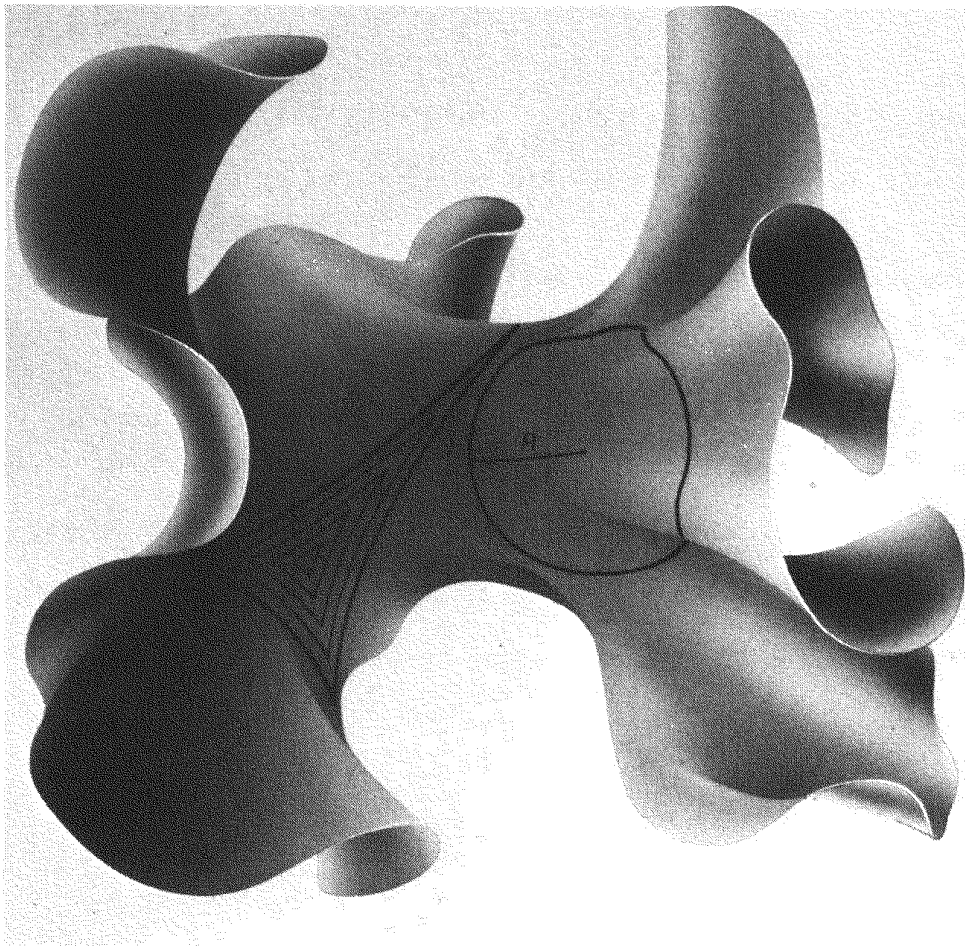
Figure 10.3:

Exercise 10.2. What happens when you tape equilateral triangles together five per vertex instead of seven per vertex? Try it! □

Note that regions near the edge of your hyperbolic paper are indistinguishable from regions near the center. This demonstrates that the hyperbolic plane is homogeneous.

Elliptic and hyperbolic geometry tend to have contrasting properties; Euclidean geometry tends to serve as the borderline between them. Recall that a piece of a sphere rips open when flattened (Figure 9.8). In contrast, a piece of a hyperbolic plane wrinkles and overlaps when flattened (Figure 10.4).

More important for us will be the question of angles. In elliptic geometry the angles of a triangle add up to more than π radians (180°); on a sphere of radius one the angles exceed π by an amount exactly equal to the triangle's area $[(\alpha+\beta+\gamma) - \pi = A]$. In hyperbolic geometry the angles of a triangle add up to less than π radians; on the standard hyperbolic plane the angles are less than π by an amount exactly equal to the triangle's area $[\pi - (\alpha+\beta+\gamma) = A]$. [But just as there can be smaller or larger spheres on which $(\alpha+\beta+\gamma) - \pi$ is proportional but not equal to the given triangle's area, so too can there be "larger" or "smaller" hyperbolic planes in which $\pi - (\alpha+\beta+\gamma)$ is proportional but not equal to the triangle's area. In this book, unless specified otherwise, "sphere" will refer to the sphere of radius one, for which $(\alpha+\beta+\gamma) - \pi = A$, and "hyperbolic plane" will refer to the standard



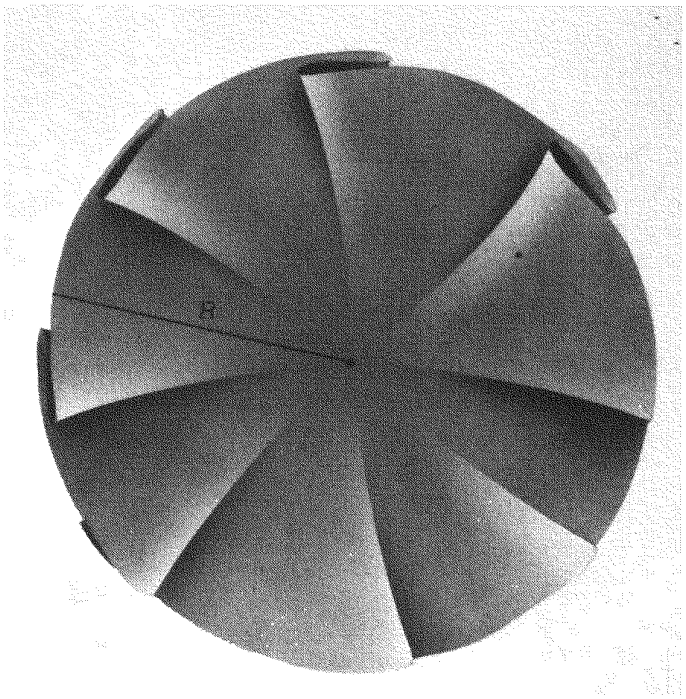


Figure 10.4: A piece of a hyperbolic plane wrinkles and overlaps when flattened.

hyperbolic plane for which $\pi - (\alpha+\beta+\gamma) = A$.] In the next chapter you will see why we are so interested in angles.

Exercise 10.3. In the hyperbolic plane, what is the area of a triangle whose angles are $\pi/3$, $\pi/4$ and $\pi/6$? □

It is a curious fact that no triangle in the hyperbolic plane can have an area greater than π. As the sides of a triangle get longer its angles get pointier, but even as the sides get infinitely long and the angles go to zero, the area never exceeds π. This is consistent with the formula $\pi - (\alpha+\beta+\gamma) = A$. I'll say more about this phenomenon in Chapter 15.

11

Geometries on Surfaces

The left side of Figure 11.1 shows the three surfaces you studied in Exercise 7.7. The first has cone points because the hexagon's corners are glued together in groups of only two. The second has a flat geometry with no cone points because the hexagon's corners are glued in groups of three. The third has the opposite of a cone point because all six corners of the hexagon are glued together. (You may want to review the material preceding Exercise 7.7. See in particular Figure 7.17.)

Both cone points and their opposites can be eliminated by utilizing the peculiar properties of the sphere and the hyperbolic plane. On a sphere, for example, larger hexagons have larger angles (just as larger triangles have larger angles). So to get rid of the cone points in the first example we need only put the hexagon on a sphere and let it grow until its angles are big enough that the cone points disappear. This occurs when each angle is

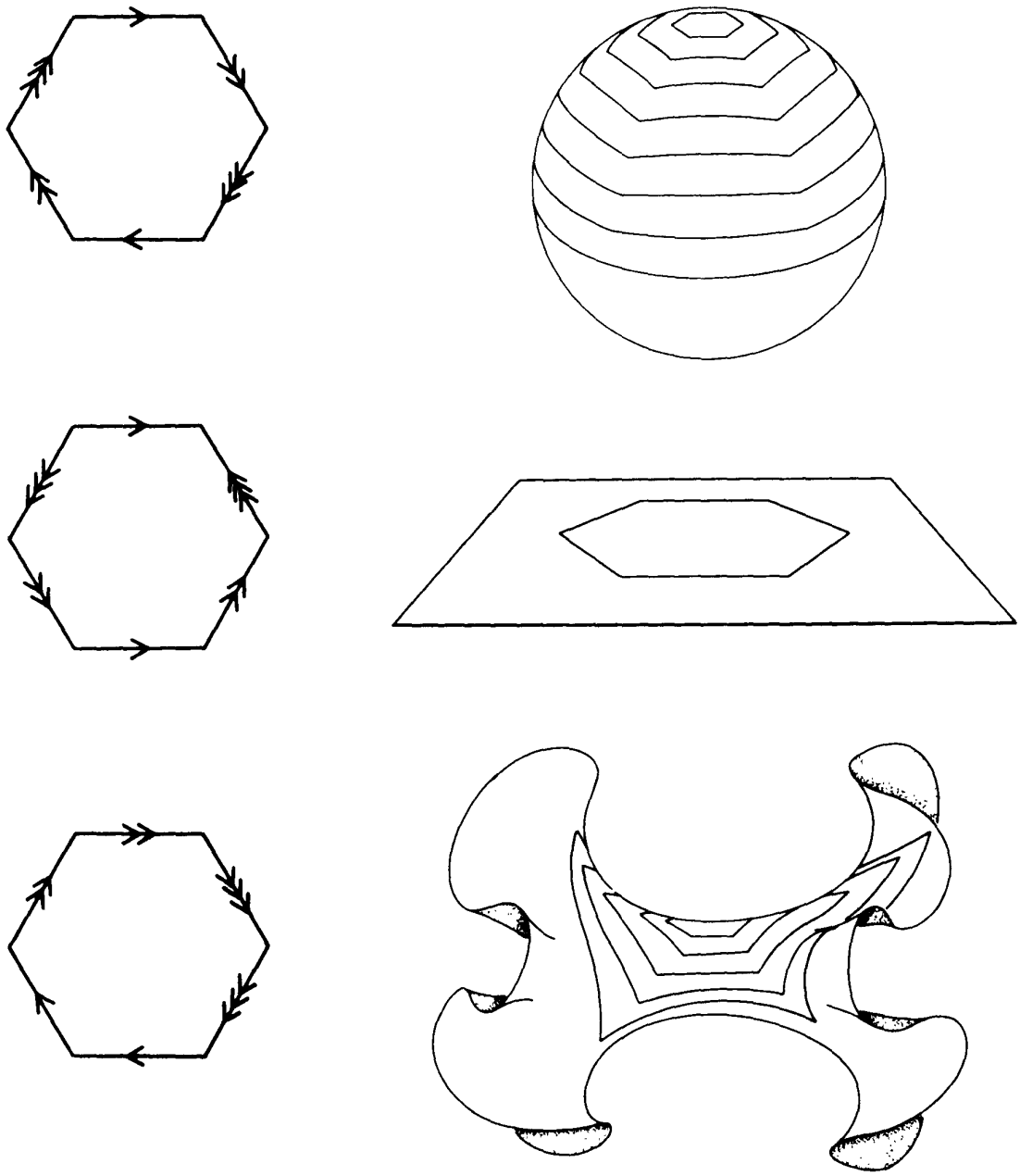


Figure 11.1: Every surface can be given a homogeneous geometry.

180° and the hexagon fills an entire hemisphere. In the course of eliminating the cone points we have given the surface a homogeneous elliptic geometry. (By the way, do you recognize this surface?)

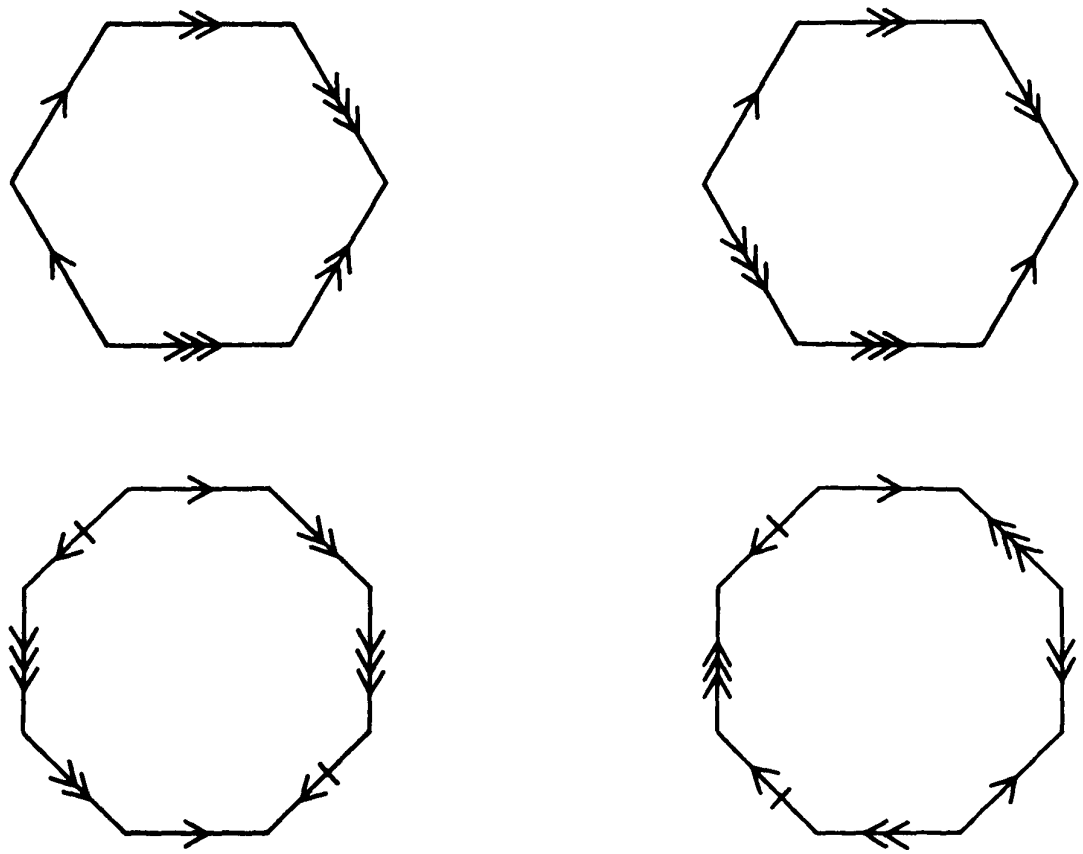


Figure 11.2: Find a homogeneous geometry for each of these surfaces.

To get rid of the opposite of a cone point in the last example we have to put the hexagon on a hyperbolic plane, where larger polygons have smaller angles. If we let the hexagon grow until each angle shrinks to 60°, then the opposite-of-a-cone-point will disappear. In the process the surface will acquire a homogeneous hyperbolic geometry.

Exercise 11.1. The surface in Figure 7.16 has two cone points. How can they be eliminated? What homogeneous geometry does this surface acquire? By the way, what surface is this? □

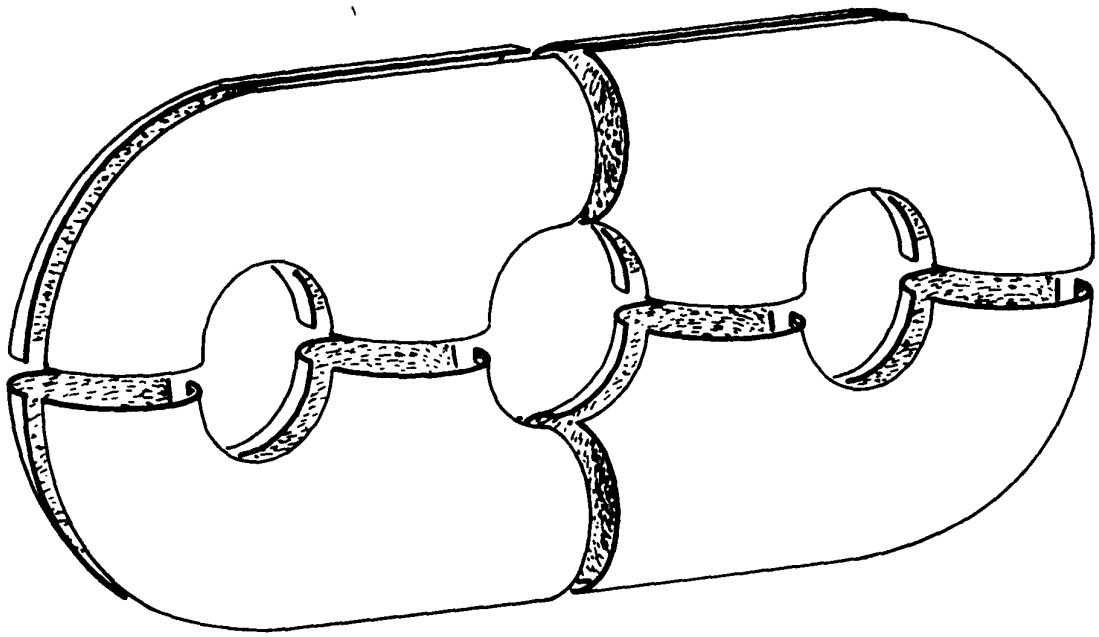


Figure 11.3: A three-holed doughnut surface can be divided into eight hexagons whose corners meet in groups of four.

Exercise 11.2. For each surface in Figure 11.2 use the "walking around corners" technique of Chapter 7 to determine how the polygon's corners fit together and whether the surface has cone points. Which of the surfaces can be given elliptic geometry, which can be given Euclidean geometry, and which can be given hyperbolic geometry? □

It turns out that *every* surface can be given some homogeneous geometry. You can modify the above technique to find homogeneous geometries for surfaces which are drawn in three dimensions rather than as glued polygons. For example, say you want to find a homogeneous geometry for a three-holed doughnut surface. First

cut the surface into hexagons whose corners meet in groups of four (Figure 11.3). Deform each hexagon to be regular (like the hexagons on the left in Figure 11.1). Regular hexagons have 120° corner angles--which are too big to fit together in groups of four--so put each hexagon in the hyperbolic plane and let it expand until its angles decrease to 90°. Each hexagon now has both (1) a homogeneous hyperbolic geometry, and (2) corner angles of 90°. Now reglue the hexagons to reassemble the surface. Don't imagine physically regluing them in three-dimensional space--this can't be done without deforming them and ruining their hyperbolic geometry. Instead imagine each to be reglued to its previous neighbors in the abstract sense that a Flatlander leaving one hexagon reappears in a neighboring one. This abstractly reassembled surface has the same global topology as the original three-holed doughnut. But unlike the original three-holed doughnut, whose local geometry varied irregularly from point to point, this new surface has the homogeneous local geometry of the hyperbolic plane. The hexagons' 90° corners fit perfectly in groups of four, so there are no cone points.

Exercise 11.3. Draw a picture showing how to cut a four-holed doughnut surface ($T^2\#T^2\#T^2\#T^2$) into hexagons whose corners meet in groups of four. Do the same for a two-holed doughnut surface ($T^2\#T^2$). How many hexagons do you get when you cut up an n-holed doughnut surface? □

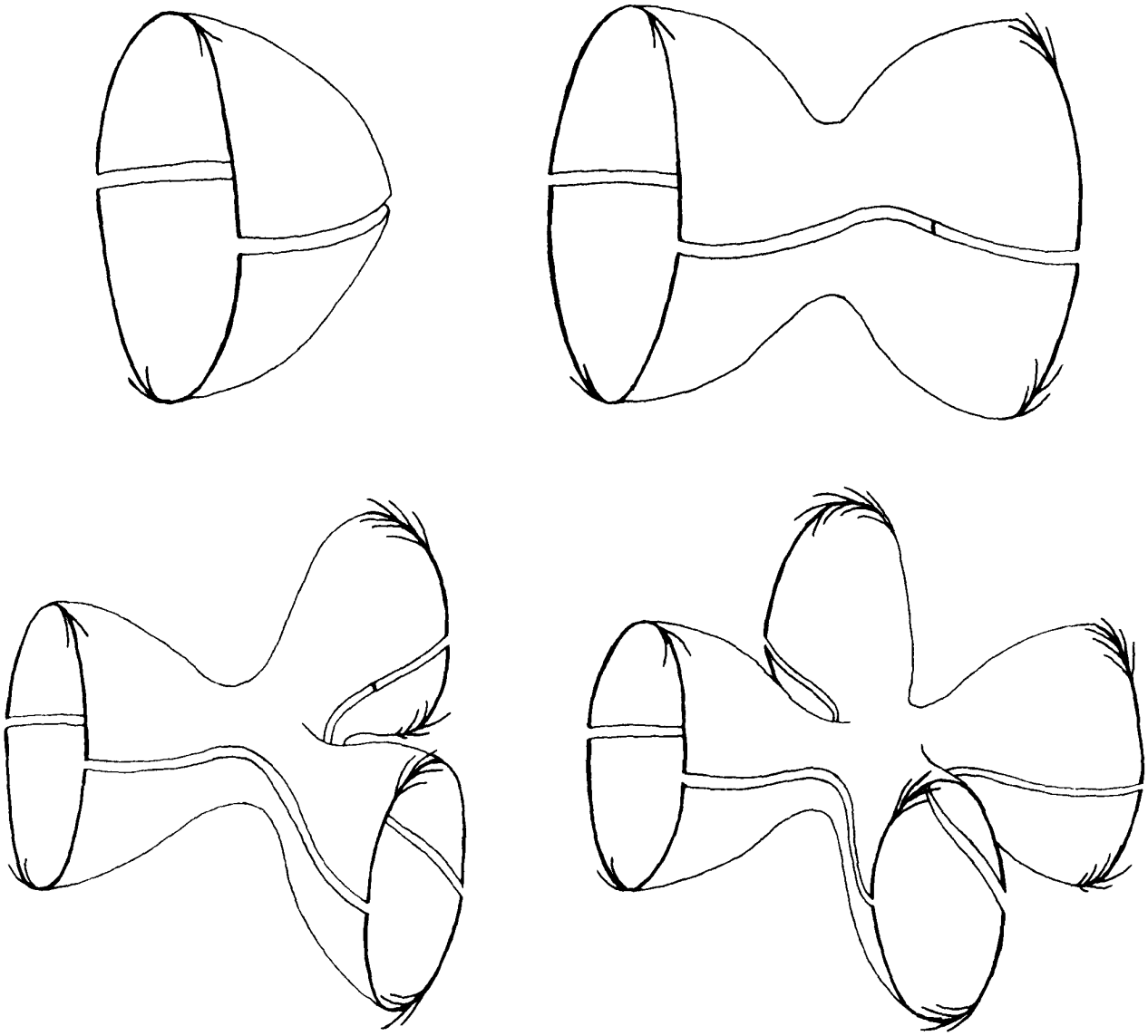


Figure 11.4: The connected sum of n projective planes can be divided into two $2n$-gons whose corners meet in groups of four.

Any doughnut surface with at least two holes can be cut into hexagons whose corners meet in groups of four. Any such surface can, therefore, be given a hyperbolic geometry by the above technique. A one-holed doughnut surface cannot be given a

hyperbolic geometry, but it can be given a Euclidean geometry (flat torus). A zero-holed doughnut surface, i.e. a sphere, also cannot be given a hyperbolic geometry, but its usual elliptic geometry is homogeneous. As stated above, every surface can be given some type of homogeneous geometry. In Chapter 12 we'll see that no surface can be given more than one type.

Exercise 11.4. The connected sum of two projective planes can be cut into two squares (Figure 11.4). Similarly, $P^2\#P^2\#P^2$ can be cut into two hexagons, $P^2\#P^2\#P^2\#P^2$ can be cut into two octagons, etc. In each case the polygon's corners meet in groups of four (two corners on one side of a rim are glued to the two corners on the opposite side). Which of these surfaces can be given which homogeneous geometry? □

Exercise 11.5. Recall that every surface is topologically equivalent to a surface on the following list:

S^2	T^2	$T^2\#T^2$	$T^2\#T^2\#T^2$	$T^2\#T^2\#T^2\#T^2$	. . . etc.
	P^2	$P^2\#P^2$	$P^2\#P^2\#P^2$	$P^2\#P^2\#P^2\#P^2$	. . . etc.

Place each surface on the list into an appropriate box in the table of Figure 11.5 according to its orientability and which homogeneous geometry it can be given. □

Except for the sphere and the projective plane, which have elliptic geometry, and the torus and the Klein bottle, which have Euclidean (flat) geometry, *all* surfaces can be given hyperbolic geometry. When a surface can be given a certain homogeneous geometry, one

	orientable	nonorientable
elliptic		
Euclidean		
hyperbolic		

Figure 11.5

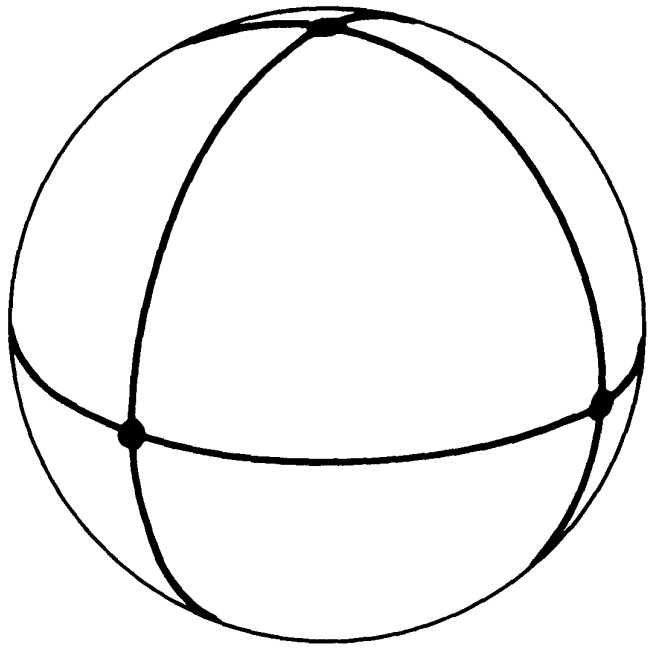
says that the surface "**admits**" that geometry. Thus the sphere and the projective plane admit elliptic geometry, the torus and the Klein bottle admit Euclidean geometry, and all other surfaces admit hyperbolic geometry.

12

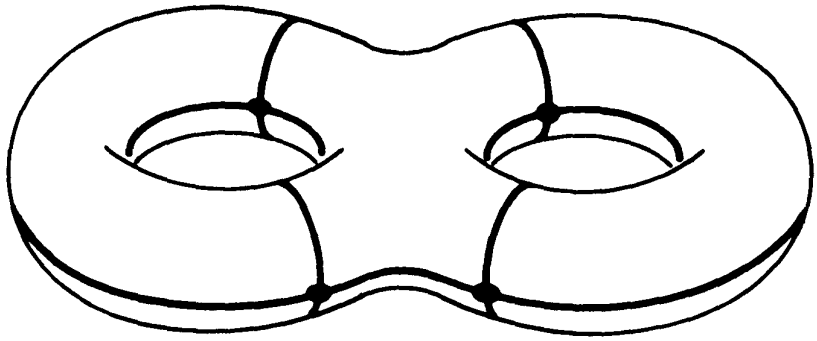
The Gauss-Bonnet Formula and the Euler Number

Every surface has an **Euler number,** an integer which contains essential information about the surface's global topology. ("Euler" is pronounced "oiler".) The Euler number is easy to compute, and it immediately predicts which homogeneous geometry the surface will admit: surfaces with positive Euler number admit elliptic geometry, surfaces with zero Euler number admit Euclidean geometry, and surfaces with negative Euler number admit hyperbolic geometry. In fact, the Euler number is so powerful that if you know a surface's Euler number and you know whether it's orientable or not, then you can immediately say what global topology the surface has! The Gauss-Bonnet formula relates a surface's Euler number to its area and curvature. ("Bonnet" is a French name, so the "t" is silent and the stress is on the second syllable.)

The Euler number is defined in terms of something called a cell-division, so we'll start by saying just what **cells** and **cell-divisions**

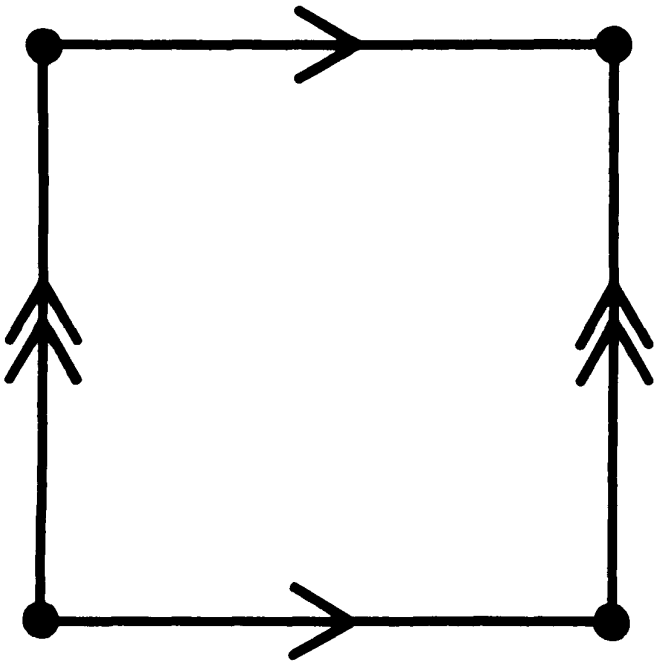


(a) a cell-division of S^2

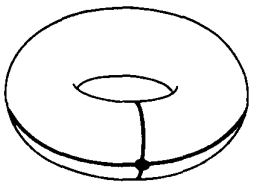


(b) a cell-division of $T^2 \# T^2$

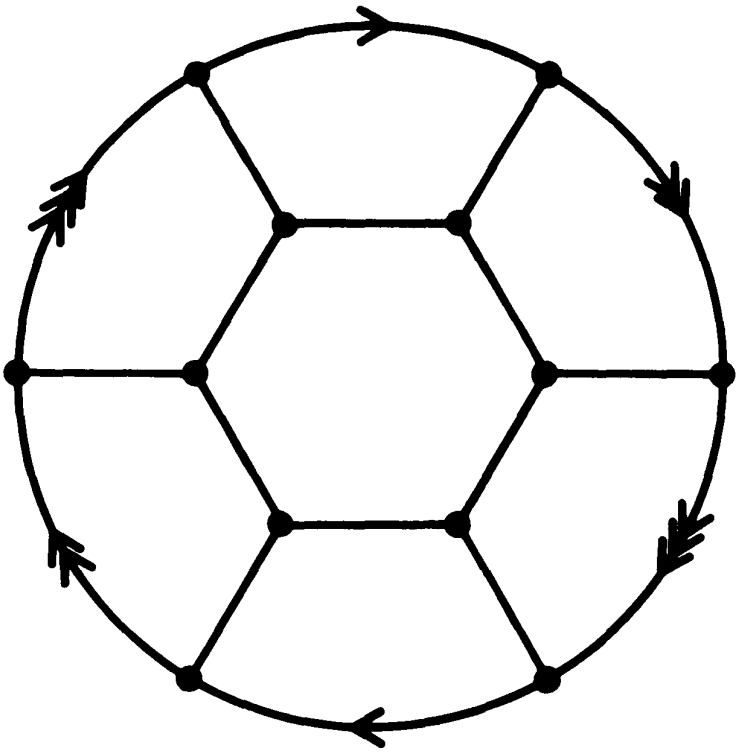
Figure 12.1: Some typical cell-divisions.



inset:



(c) a cell-division of T^2



(d) a cell-division of P^2

are. A zero-dimensional cell is a point (usually called a vertex). A one-dimensional cell is topologically a line segment (usually called an edge). And a two-dimensional cell is topologically a polygon (usually called a face, even if it isn't the face *of* anything). A cell-division is what you get when you divide a surface into cells. It's similar to the decomposition of a surface into polygons used in Chapter 11 (cf. Figure 11.3), only now we're interested in the vertices and edges as well as the faces.

Figure 12.1 shows some typical cell-divisions. The first is a cell-division of S^2 consisting of six vertices, twelve edges, and eight triangular faces. The second is a cell-division of $T^2\#T^2$ consisting of eight vertices, sixteen edges, and six faces (the two faces in the middle--one on top and one on bottom--are topologically octagons, while the two on each end are topologically squares). The third is a cell-division of T^2. It contains *one* vertex--not four--because the four corners of the square meet at a single point (see inset, and recall Figure 3.3). Similarly, it contains only two edges, because the square's four edges are glued together in pairs. Finally, this cell-division obviously contains only one face. The last drawing in Figure 12.1 shows a cell-division of P^2.

Exercise 12.1. Count the number of vertices, edges and faces in the cell-division of P^2 given in Figure 12.1(d). Be careful: vertices and edges on the disk's circumference are glued together in pairs. □

From now on

v = the number of vertices in a cell-division

e = the number of edges in a cell-division

f = the number of faces in a cell-division.

So, for example, the above cell-division of S^2 has $v = 6$, $e = 12$ and $f = 8$, while the cell-division for $T^2 \# T^2$ has $v = 8$, $e = 16$ and $f = 6$.

By the way, we will usually assume that the edges in a cell-division are geodesics.

To "discover" the Gauss-Bonnet formula and the Euler number we'll need a couple facts about polygons. A polygon with n sides is called an n-gon.

Exercise 12.2. In the Euclidean plane the angles of any triangle add up to exactly π radians. Use Figure 12.2 (ignore the formulas) to deduce that the angles of a Euclidean n-gon add up to $(n - 2)\pi$ radians. Thus the angles of a quadrilateral add up to 2π, the angles of a pentagon add up to 3π, etc. □

Exercise 12.3. The area of a triangle on a unit sphere is the sum of its angles minus π [in symbols $A = (\alpha+\beta+\gamma) - \pi$]. Deduce that the area of an n-gon on a unit sphere is A = (sum of all angles) $- (n - 2)\pi$. (Hint: The area of an n-gon is the sum of the areas of $n - 2$ triangles, as per Figure 12.2.) By referring to the previous exercise you can interpret this formula

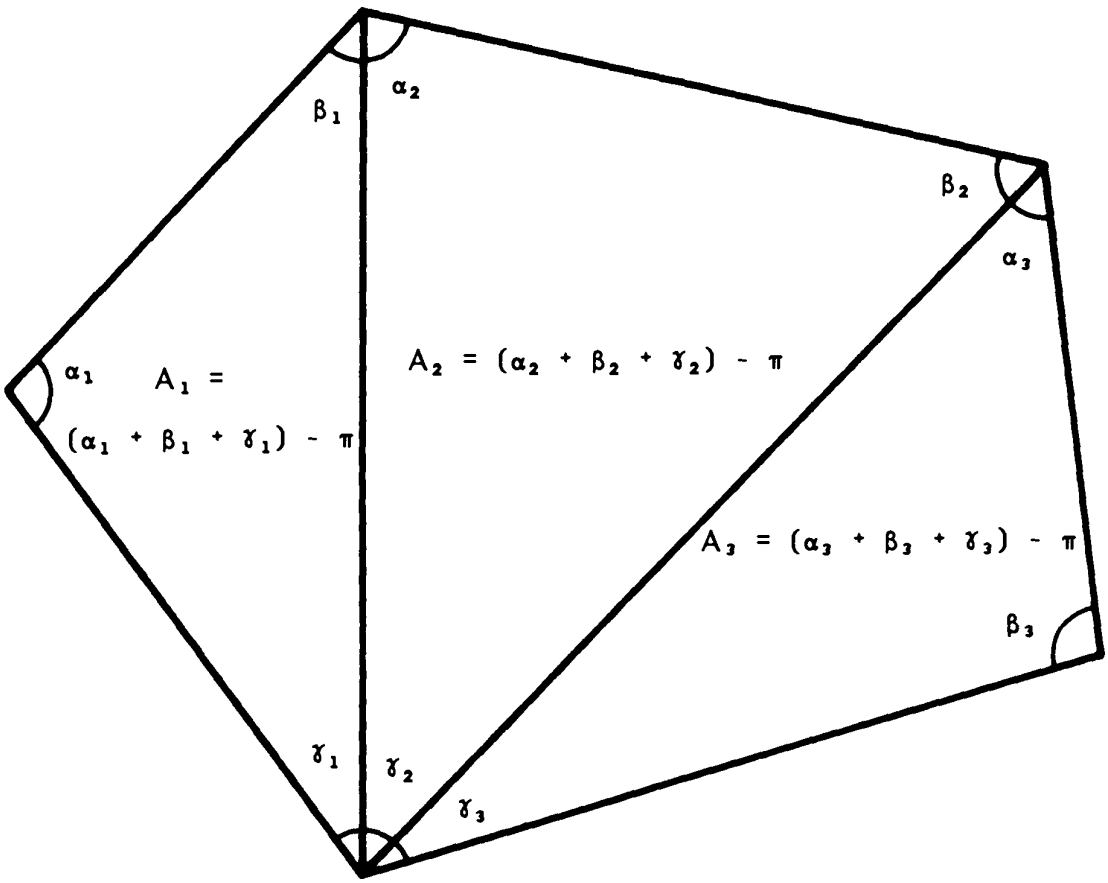


Figure 12.2: An n-gon can be divided into n - 2 triangles.

as saying that the area of a spherical n-gon equals the difference between what the angles actually are and what they would have been if the n-gon were flat. □

Exercise 12.4. Find a formula for the area of an n-gon in the hyperbolic plane. □

It is now fairly easy to "discover" the Gauss-Bonnet formula and the Euler number. We'll do it for a sphere first. Give the sphere

a cell-division--any cell-division will do. The plan is to express the area of the sphere as the sum of the areas of the faces of the cell-division, and then use the formula $A = (\text{sum of angles}) - (n - 2)\pi$ to get a new expression for the area of each face. Then we'll follow our noses and see what happens.

I'll first write out the computation in its entirety, then I'll go back and explain each of the steps. (For an extra challenge, you might want to see how much of the computation you can figure out on your own *before* you read the explanations.)

$$1)\quad \text{Total Area} = \begin{bmatrix}\text{area of}\\ \text{first face}\end{bmatrix} + \begin{bmatrix}\text{area of}\\ \text{second face}\end{bmatrix} + \cdots + \begin{bmatrix}\text{area of}\\ \text{last face}\end{bmatrix}$$

$$2)\quad = \left[\begin{pmatrix}\text{sum of angles}\\ \text{of first face}\end{pmatrix} - (n_1-2)\pi\right] + \cdots + \left[\begin{pmatrix}\text{sum of angles}\\ \text{of last face}\end{pmatrix} - (n_k-2)\pi\right]$$

$$3)\quad = \left[\begin{pmatrix}\text{sum of angles}\\ \text{of first face}\end{pmatrix} + \cdots + \begin{pmatrix}\text{sum of angles}\\ \text{of last face}\end{pmatrix}\right] - (n_1 + n_2 + \ldots + n_k)\pi + (2+2+\ldots+2)\pi$$

$$4)\quad = \begin{bmatrix}\text{sum of all angles in}\\ \text{the entire cell-division}\end{bmatrix} - (n_1 + n_2 + \ldots + n_k)\pi + (2 + 2 + \ldots + 2)\pi$$

$$5)\quad = 2\pi v - 2\pi e + 2\pi f$$

$$6)\quad = 2\pi(v - e + f)$$

Step-by-step Explanation:

Step 1 Write the area of the surface as the sum of the areas of the faces of the cell-division.

Step 2 Use the formula $A = (\text{sum of angles}) - (n - 2)\pi$ (from Exercise 12.3) to get a new expression for the area of each face.

Step 3 Regroup terms.

Step 4 Note that adding up all the angles of all the faces is the same as adding up all the angles in the entire cell-division.

Step 5 There are three things going on at this step.

Substep A. The sum of all the angles in a cell-division is $2\pi v$. This is because there are v vertices, and the sum of the angles surrounding each vertex is 2π. For a specific example, consider the cell-division of S^2 shown in Figure 12.1(a). We could have computed the sum of all its angles as

$$(\text{sum of angles in each face}) \times (\text{number of faces})$$
$$= (\pi/2 + \pi/2 + \pi/2) \times 8 = (3\pi/2) \times 8 = 12\pi,$$

but instead we compute it as

$$(\text{sum of angles surrounding each vertex}) \times (\text{number of vertices})$$
$$= 2\pi \times v = 2\pi \times 6 = 12\pi.$$

Substep B. Note that $n_1 + n_2 + \ldots + n_k = 2e$, where e is the total number of edges in the cell-division. Each "n" represents the number of edges of a certain face. By adding up all the n's we are adding up all the edges in the entire cell-division. However, each edge is the boundary of two faces, so each edge gets counted twice. This is

why $n_1 + n_2 + \ldots + n_k$ equals 2e rather than just e. A specific example might clarify things. The cell-division of S^2 shown in Figure 12.1(a) has 8 faces, each of which has 3 edges. Therefore it might seem that the cell-division has $8 \times 3 = 24$ edges. But each edge was counted twice--really there are only 12 edges.

Substep C. The last part of Step 5 is the substitution $2 + 2 + \ldots + 2 = 2f$, where f is the number of faces in the cell-division. This is easy to understand: back in line (1) there was one term for each face in the cell-division, so now the sum $2 + 2 + \ldots + 2$ must contain one "2" for each face.

Step 6 Factor out 2π.

The formula Area $= 2\pi(v - e + f)$ is a stunning conclusion! We know that a unit sphere has area 4π, so the formula says that $v - e + f$ *must* be two, *no matter what cell-division we choose!*

Exercise 12.5. Check that $v - e + f = 2$ for the cell-division of Figure 12.1(a). Then draw any other cell-division of S^2 (regular or irregular) and check that $v - e + f = 2$ for it as well. □

The derivation of the formula $A = 2\pi(v - e + f)$ did not rely on the fact that the surface in question was a sphere. It required

only that the surface have the *local* geometry of a unit sphere. Therefore the computations apply equally well to a projective plane of radius one. The projective plane is made from a hemisphere, which has only half the area of a sphere, so in this case the formula $A = 2\pi(v - e + f)$ becomes $2\pi = 2\pi(v - e + f)$ and we conclude that $v - e + f$ must equal one for *every* cell-division of P^2!

Exercise 12.6. Check that $v - e + f = 1$ for the cell-division of P^2 shown in Figure 12.1(d) and referred to in Exercise 12.1. Then draw some other cell-division of P^2 and check that $v - e + f = 1$ for it too. □

The formula $A = 2\pi(v - e + f)$ is the **Gauss-Bonnet formula** for surfaces with elliptic geometry. The quantity $v - e + f$ is the **Euler number;** it is invariably denoted by the greek letter χ (spelled "chi" and pronounced "kai"). Thus $\chi \equiv v - e + f$, and the Gauss-Bonnet formula for elliptic geometry can be written $A = 2\pi\chi$. Because $\chi \equiv v - e + f = 2$ for every cell-division of S^2, we say that "the Euler number of a sphere is two", and write $\chi(S^2) = 2$. Similarly, a projective plane has Euler number one and we write $\chi(P^2) = 1$.

Exercise 12.7. Modify the derivation of $A = 2\pi\chi$ to discover an analogous Gauss-Bonnet formula for surfaces with hyperbolic geometry. □

Exercise 12.8. Use the cell-division of Figure 12.1(b) to compute $\chi(T^2\#T^2)$. To check your answer, recompute $\chi(T^2\#T^2)$ using a different cell-division (for instance, you could divide $T^2\#T^2$ into four hexagons as in Exercise 11.3). □

Exercise 12.9. What is the area of a two-holed doughnut with (standard) hyperbolic geometry? Hint: Combine your answers from the preceding two exercises. □

Exercise 12.10. Compute the Euler number of a three-holed doughnut. What is the area of a three-holed doughnut with hyperbolic geometry? □

There is also a Gauss-Bonnet formula for surfaces with Euclidean geometry. In Euclidean geometry the angles of an n-gon add up to exactly $(n - 2)\pi$ (recall Exercise 12.2), so for any cell division

$$\begin{pmatrix}\text{sum of angles}\\ \text{of first face}\end{pmatrix} + \dots + \begin{pmatrix}\text{sum of angles}\\ \text{of last face}\end{pmatrix} = (n_1-2)\pi + \dots + (n_k-2)\pi$$

$$\begin{pmatrix}\text{sum of all angles in}\\ \text{the entire cell-division}\end{pmatrix} = (n_1 + n_2 + \dots + n_k)\pi - (2 + 2 + \dots + 2)\pi$$

$$2\pi v = 2\pi e - 2\pi f$$

$$2\pi(v - e + f) = 0$$

where the steps follow as in the earlier derivation of the elliptic Gauss-Bonnet formula. Since $0 = 2\pi(v - e + f) = 2\pi\chi$, it follows that a surface with Euclidean geometry must have Euler number zero!

Exercise 12.11. Compute the Euler number of a torus using the cell-division of Figure 12.1(c). Then recompute it using a different cell-division of your own design. Compute $\chi(K^2)$ using two different cell-divisions. □

The three Gauss-Bonnet formulas can be summarized as $kA = 2\pi\chi$, where $k = -1$, 0 or $+1$ according to whether the surface has hyperbolic geometry (*negative* curvature), Euclidean geometry (*zero* curvature) or elliptic geometry (*positive* curvature). Later in the chapter we'll see more general forms of the Gauss-Bonnet formula, but even now we can deduce that no surface admits more than one homogeneous geometry: the "curvature" of the geometry ($k = -1, 0, +1$) must have the same sign as the Euler number. Intuitively, the surface's global topology determines which homogeneous local geometry "fits" it.

Exercise 12.12. In previous exercises you have computed $\chi(S^2)$, $\chi(T^2)$, $\chi(T^2\#T^2)$ and $\chi(T^2\#T^2\#T^2)$. What is the pattern? What do you think $\chi(T^2\#T^2\#T^2\#T^2)$ is? □

It's easy to derive a general formula for the Euler number of the connected sum of n tori. In Exercise 11.3 you found that an n-holed doughnut can be divided into 4n - 4 hexagons whose corners meet in groups of four. Once we figure out how many edges and vertices such a cell-division has it will be easy to compute the Euler number.

1. There are $6 \times (4n - 4) \div 2 = 12n - 12$ edges. *Reason:* Each of the 4n - 4 hexagons contributes six edges, for a total of 24n - 24, but we have to divide by two to compensate for the fact that each edge has been counted twice because it's the border of two hexagons.

2. There are $6 \times (4n - 4) \div 4 = 6n - 6$ vertices. *Reason:* Each of the 4n - 4 hexagons contributes six vertices, for a total of 24n - 24, but we have to divide by four to compensate for the fact that each vertex has been counted four times because it's the corner of four hexagons.

The Euler number of the connected sum of n tori is $\chi = v - e + f = (6n - 6) - (12n - 12) + (4n - 4) = 2 - 2n$ in agreement with the pattern established in Exercise 12.12.

Exercise 12.13. The connected sum of n projective planes can be divided into two $2n$-gons whose corners meet in groups of four (Figure 11.4). Compute the Euler number. Complete the table in Figure 12.3. □

orientability

Euler number	orientable	nonorientable
2	S^2	
1		P^2
0	T^2	
-1		
-2	$T^2 \# T^2$	
-3		
-4	$T^2 \# T^2 \# T^2$	
-5		
-6	$T^2 \# T^2 \# T^2 \# T^2$	
-7		
-8	$T^2 \# T^2 \# T^2 \# T^2 \# T^2$	
.		
.	etc.	
.		

Figure 12.3

A priori it could be very difficult to decide whether two manifolds are topologically the same. In fact, the problem of deciding whether two *three-dimensional* manifolds are topologically

the same is so difficult that in spite of decades of work by many people, no one has yet found a foolproof procedure to do it. (There are many *practical* procedures to tell two three-manifolds apart, but each of these procedures can be "fooled" by some appropriate pair of similar-looking manifolds.) In this light it is amazing that, according to the table in Figure 12.3, we can conclusively identify a surface just by computing its Euler number and deciding whether it's orientable or not! For example, say a certain surface has Euler number -4 and is orientable, then by consulting the table we can conclude that the surface must be topologically equivalent to $T^2\#T^2\#T^2$. Similarly, a nonorientable surface with Euler number -2 must be $P^2\#P^2\#P^2\#P^2$.

Exercise 12.14. For each surface in Figure 11.1, compute its Euler number (be careful counting those edges and vertices!), determine its orientability, and use the table in Figure 12.3 to identify the surface. Do the same for the surfaces in Figure 11.2. □

We'd like to apply the Gauss-Bonnet formula $kA = 2\pi\chi$ to all homogeneous surfaces, not just those with "standard" elliptic geometry (the geometry of a sphere of radius one), "standard" hyperbolic geometry, or Euclidean geometry. Consider, for example, how we might apply the formula to a sphere of radius three. A sphere of radius three has the same Euler number as a sphere of radius one, but its area is nine times as great.

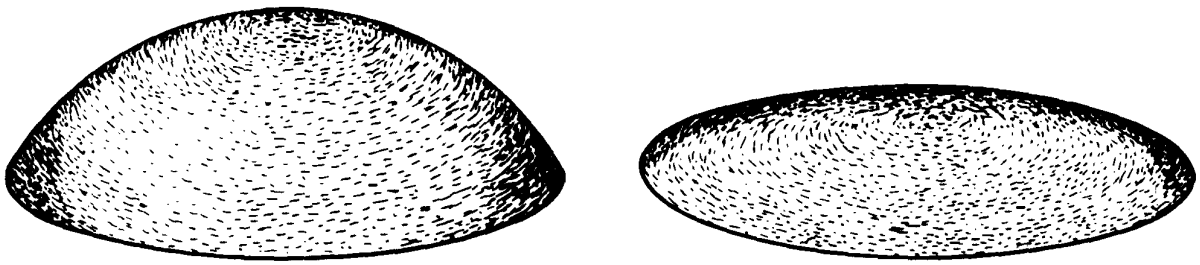


Figure 12.4: A piece of a small sphere is more curved than a piece of a large sphere.

Therefore its "curvature" k must be only one-ninth as great if the formula $kA = 2\pi\chi$ is still to hold. This is not unreasonable: larger spheres certainly *look* less curved than smaller spheres (Figure 12.4). In general, we define the curvature of a sphere of radius r to be $k \equiv 1/r^2$. A projective plane with the local geometry of a sphere of radius r also has curvature $k \equiv 1/r^2$.

Exercise 12.15. If distance is measured in meters, what units is curvature measured in? Do the units of kA match the units of $2\pi\chi$?

□

Exercise 12.16. Compute the curvature, area and Euler number of a projective plane of radius two meters, and check that the Gauss-Bonnet formula holds. (The area of a *sphere* of radius r is $4\pi r^2$.)

□

We can rephrase the above definition of curvature by saying that if we start with a sphere of radius one and expand all distances by a factor of r ($r > 1$ means the sphere actually expands, $r < 1$

means it shrinks) then the resulting sphere will have curvature $k \equiv 1/r^2$. This rephrased definition is awkward, but, unlike the first definition, it avoids the concept of "radius" and can be adapted to the hyperbolic plane. The standard hyperbolic plane has curvature $k \equiv -1$. If we expand all distances in the standard hyperbolic plane by a factor of r, then the curvature of the resulting hyperbolic plane is defined to be $k \equiv -1/r^2$. Any surface with the local geometry of such a hyperbolic plane also has curvature $k \equiv -1/r^2$. For example, if $T^2\#T^2$ is given a standard hyperbolic geometry its curvature is -1, its area is 4π, and the Gauss-Bonnet formula reads $(-1)(4\pi) = 2\pi(-2)$. But when the surface is enlarged by a factor of five, then the curvature becomes -1/25, the area becomes $25 \times 4\pi = 100\pi$, and the Gauss-Bonnet formula reads $(-1/25)(100\pi) = 2\pi(-2)$ (correct again!).

Exercise 12.17. A society of Flatlanders lives in a universe with the global topology of $T^2\#T^2\#T^2$ and a homogeneous local geometry of constant curvature $-.00001$ (meters)$^{-2}$. What is the area of their universe? □

We can refer to any surface with a homogeneous geometry as a "surface of constant curvature k", where the exact value of k depends on the geometry of the surface in question. We're talking about a surface with elliptic geometry when k is positive, with Euclidean geometry when k is zero, and with hyperbolic geometry

when k is negative. The closer k is to zero, the flatter the surface is, and the further k is from zero (in either direction), the more curved (either positively or negatively) the surface is. The curvature k is usually called the **Gaussian curvature.** We can use this terminology to summarize our results as

The Unified Gauss-Bonnet Formula
for Surfaces of Constant Curvature

If a surface has area A, Euler number χ, *and constant Gaussian curvature k, then*

$$kA = 2\pi\chi.$$

Exercise 12.18. A society of Flatlanders lives on a surface of area 1,984,707 square meters and constant Gaussian curvature -3.1658×10^{-6} meters^{-2}. What is the global topology of their surface? □

Exercise 12.19. Which is more curved, a $P^2\#P^2\#P^2$ with area 6 square meters, or a $T^2\#T^2$ with area 9 square meters? □

You might wonder how Flatlanders measure curvature. They can do it by measuring the area and angles of a triangle. Recall that

$A_\Delta = (\alpha + \beta + \gamma) - \pi$ for a triangle on a sphere of radius one, $A_\Delta = \pi - (\alpha + \beta + \gamma)$ for a triangle on the standard hyperbolic plane, and $\alpha + \beta + \gamma = \pi$ for a triangle on a flat surface. These three formulas can be summarized as $kA_\Delta = (\alpha + \beta + \gamma) - \pi$, where $k = -1$, 0 or +1 according to the geometry. Not surprisingly, this formula is valid for triangles on a surface of any constant Gaussian curvature k. In fact for $k > 0$ the formula is equivalent to the formula $A_\Delta = r^2[(\alpha + \beta + \gamma) - \pi]$ you derived in Exercise 9.3. (By the way, throughout this chapter "A_Δ" denotes the area of a triangle while "A" denotes the area of a whole surface.)

Exercise 12.20. A Flatlander survey team has measured the angles of a triangle as 34.3017°, 62.5633° and 83.1186°, and they have measured its area as 2.81 km². Assuming their universe is homogeneous, what is its Gaussian curvature? (Don't forget to convert the angles to radians.)

The Flatlanders later discover that their universe is orientable and has an area of roughly 250,000 km². Deduce the global topology. □

The Gauss-Bonnet formula can be generalized to apply to *non*-homogeneous surfaces whose Gaussian curvature varies irregularly from point to point. The idea is that positive and negative curvature cancel, and the net total curvature equals $2\pi\chi$. On a doughnut surface, for example, the positive curvature cancels the

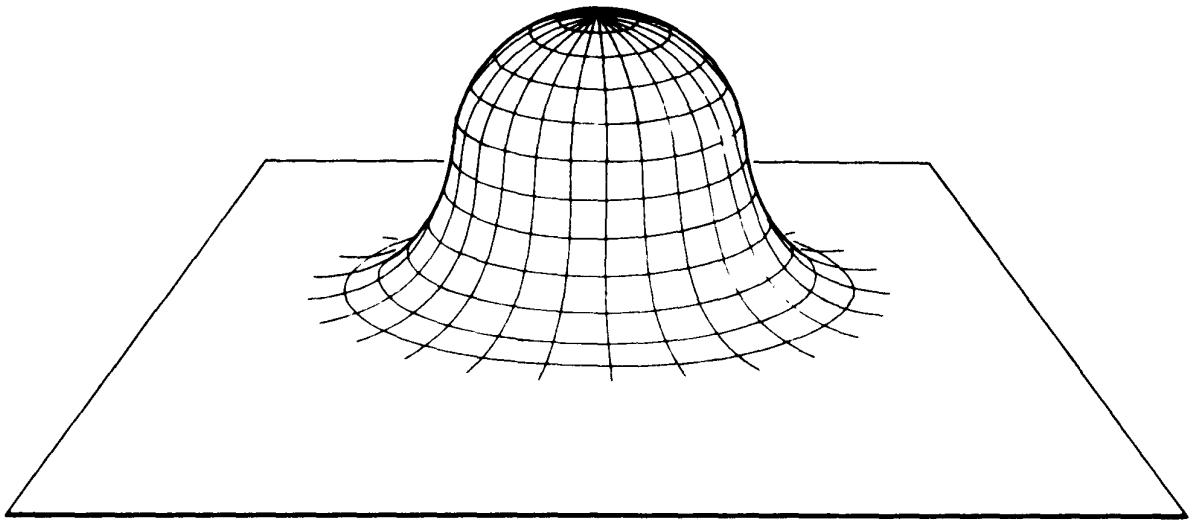


Figure 12.5: When you make a blip in a surface you create a small region of positive curvature. Where is the compensating negative curvature?

negative curvature exactly. (The outer, convex portion of a doughnut surface has positive curvature, while the portion around the hole is negatively curved.) No matter how you deform a surface, you can never change its total curvature! When you create positive curvature in one place, you invariably create an equal amount of negative curvature someplace else.

Exercise 12.21. When you make a blip on a surface you create a small region of positive curvature (Figure 12.5). Where is the compensating region of negative curvature? □

One can use the language of calculus to state this latest Gauss-Bonnet formula precisely. The total curvature is represented as

the integral of the curvature over the surface, and the Gauss-Bonnet formula reads

$$\int k \, dA = 2\pi\chi$$

[Mathematically experienced readers can easily justify the steps in the following proof. The basic idea is to (1) use a sufficiently fine cell-division to approximate the integral as a sum, (2) generalize the formula $kA_\Delta = (\alpha + \beta + \gamma) - \pi$ to apply to n-gons (cf. Figure 12.2) and substitute it in, and (3) proceed *exactly* as in the derivation of the elliptic Gauss-Bonnet formula.

$$\begin{aligned}
\int k \, dA &= \Sigma \, k_i A_i \\
&= \Sigma \, [(\alpha_i + \beta_i + \ldots + \xi_i) - (n_i - 2)\pi] \\
&= \Sigma(\text{all angles}) - \pi(\Sigma \, n_i) + \pi(\Sigma \, 2) \\
&= 2\pi v - 2\pi e + 2\pi f \\
&= 2\pi\chi.
\end{aligned}$$

Note that when k is constant this formula reduces to $kA = 2\pi\chi$.]

Part III

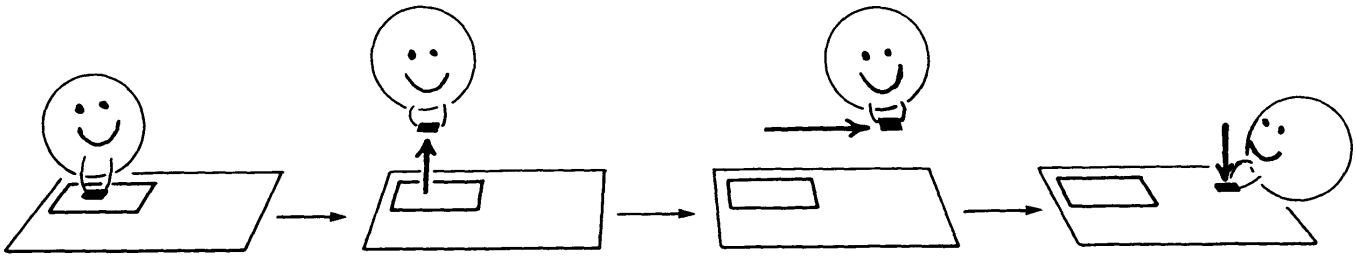
Geometries on Three-Manifolds

13

Four-Dimensional Space

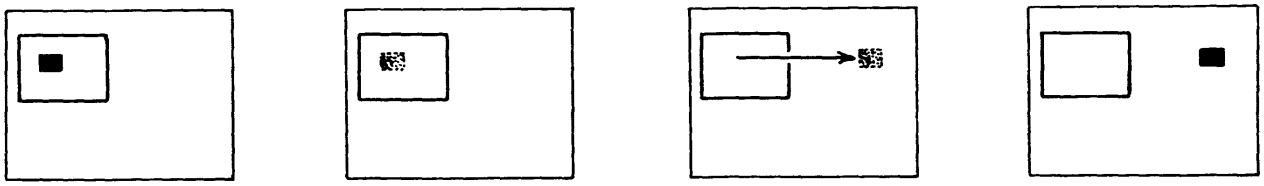
In Edwin Abbott's book *Flatland*, A Square's two-dimensional world happens to be embedded in a three-dimensional space. The climax of the book occurs when a sphere from this three-dimensional space comes to visit A Square and tell him about the world of three dimensions. Not surprisingly, A Square finds the sphere's explanations completely unenlightening. So "the sphere, having in vain tried words, resorts to deeds" (*Flatland*, p. 77). He goes over to A Square's locked cupboard, picks up a tablet, moves it over a little ways, and plunks it back down into the plane of Flatland (Figure 13.1).

Naturally enough, A Square is horrified. As far as he can tell the tablet somehow dematerialized, passed through the wall of his locked cupboard in this condition, and then rematerialized on the other side of the room. This was very, very disconcerting.



The sphere picks up the tablet, moves it over, and lowers it back down.

Figure 13.1: How the sphere removed the tablet from the cupboard.



The tablet dematerializes, passes through the wall of the cupboard in its dematerialized state, and rematerializes on the opposite side of the room.

Figure 13.2: A Square's erroneous impression of the incident.

Not that we can blame poor A Square. Throughout his whole life he's experienced only forward/backward and left/right motions, so it's not surprising that he has trouble dealing with what we Spacelanders would call up/down motions. When the tablet moves away from the cupboard without moving forwards, backwards, to the right, or to the left, it's only natural for him to assume that it stayed put but somehow became ethereal enough to subsequently pass through the wall (Figure 13.2).

Imagine how you would feel if some strange creature, after rambling on for a while about four-dimensional space, were to remove a pitcher of juice from your fridge and place it on your table--without opening the refrigerator door! The explanation, of course, is that the creature lifts the pitcher *"up"* into the fourth dimension, passes it *"over"* the refrigerator wall, and lowers it back *"down"* into our three-dimensional space. Compare Figure 13.3 to Figure 13.2.

In the next chapter we'll use four-dimensional space to define the hypersphere. The hypersphere requires four dimensions for its definition just as an ordinary sphere requires three dimensions. First, though, we'll take a look at some other four-dimensional phenomena, and also deal with a few philosophical questions.

Other Four-Dimensional Phenomena

On p. 11 of *Geometry, Relativity and the Fourth Dimension*, R. Rucker tells of a mystic named Zöllner who thought that spirits

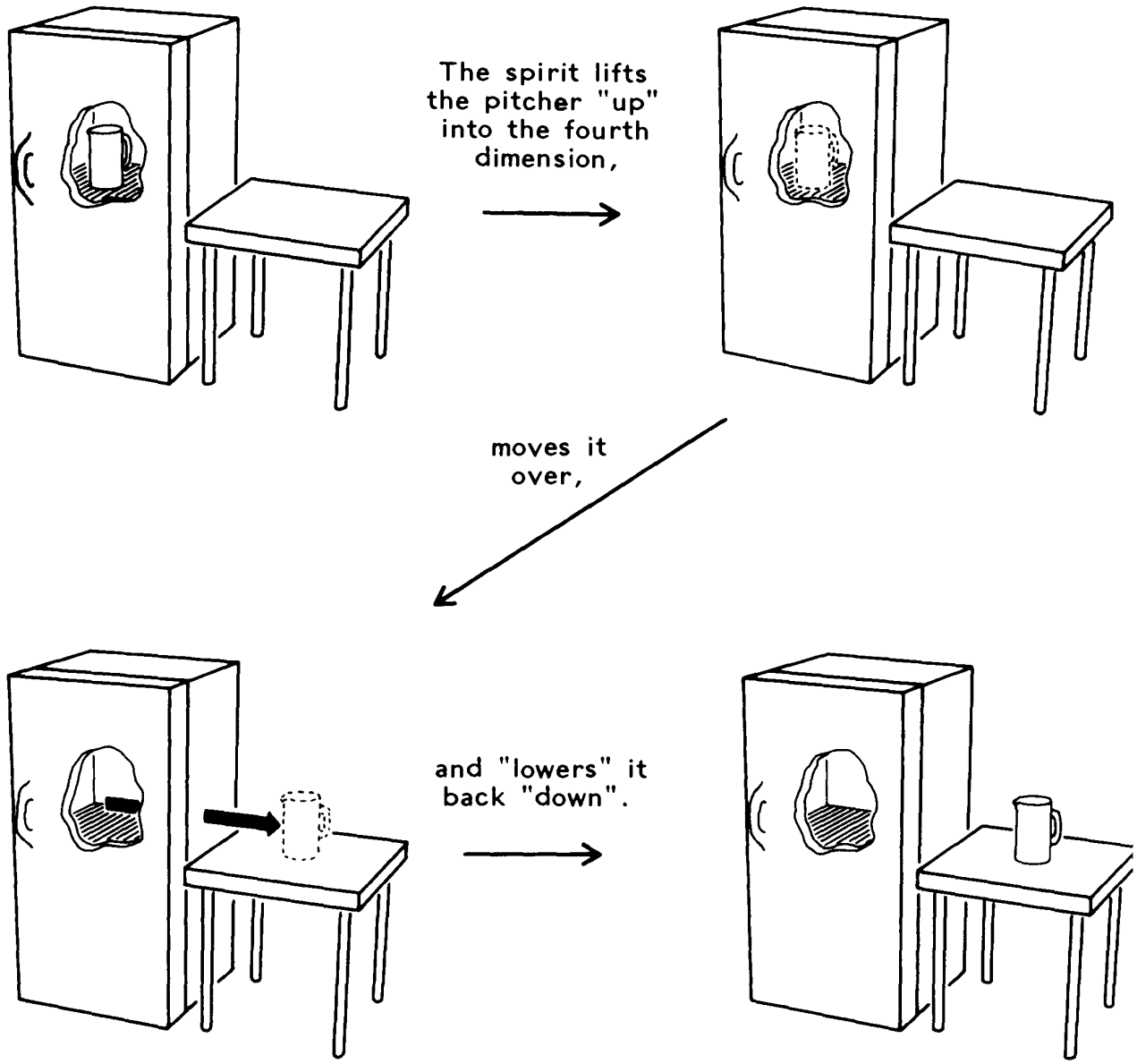


Figure 13.3: How the four-dimensional creature removes the juice pitcher from the refrigerator without opening the door.

were four-dimensional beings capable of snatching up three-dimensional objects in much the same way that the sphere snatched up A Square's tablet:

> . . . Prof. Zöllner was also concerned with getting the spirits to do something that would provide a lasting and incontrovertible proof of their four-dimensionality. His idea was a good one. He had two rings carved out of solid wood, so that a microscopic examination would confirm that they had never been cut open. The idea was that spirits, being free to move in the fourth dimension, could link the two rings without breaking or cutting either one. In order to ensure that the rings had not been carved out in a linked position, they were made of different kinds of wood, one alder, one oak. Zöllner took them to a seance and asked the spirits to link them, but unfortunately, they didn't.

Figure 13.4 shows how the spirits were supposed to link the two rings.

Exercise 13.1. How could four-dimensional spirits untie a knotted loop of rope such as the one in Figure 13.5? □

Exercise 13.2. In Chapter 4 we could embed a Klein bottle in E^3 only by allowing it to intersect itself (recall Figure 4.9). Explain how to embed a Klein bottle in E^4 with no self-intersection. ("Ordinary" four-dimensional space is abbreviated E^4, just as ordinary three-dimensional space is E^3, a plane is E^2, and a line is E^1.) □

Before moving on I would like to mention one more four-dimensional curiosity: a sphere can be knotted in four-dimensional space just like a circle can be knotted in three-dimensional space. Give some thought to how A Square imagines a knotted circle (Figure 13.6) and then try to understand the knotted sphere by analogy (Figure 13.7).

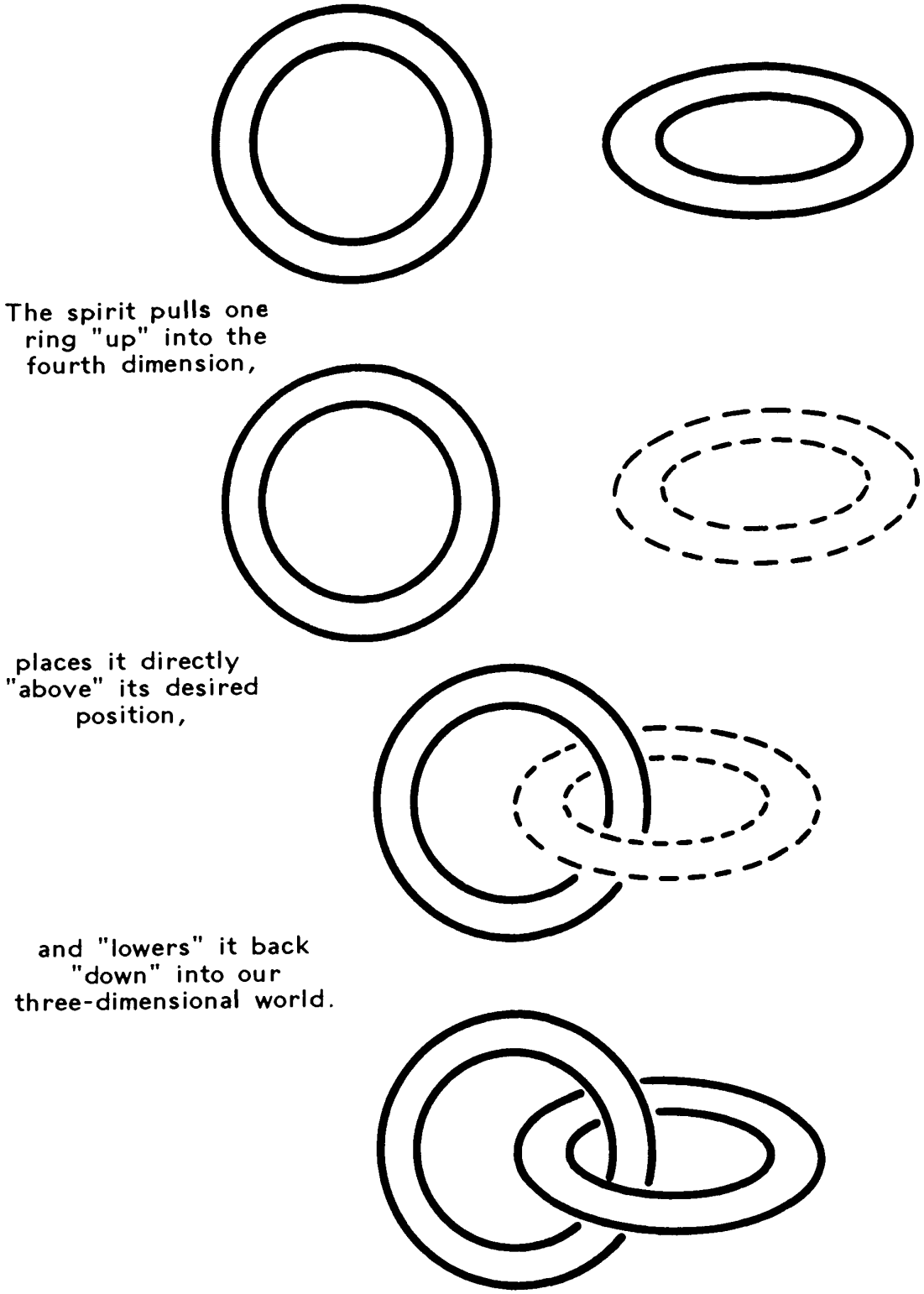


Figure 13.4: A four-dimensional spirit could link two rings together.

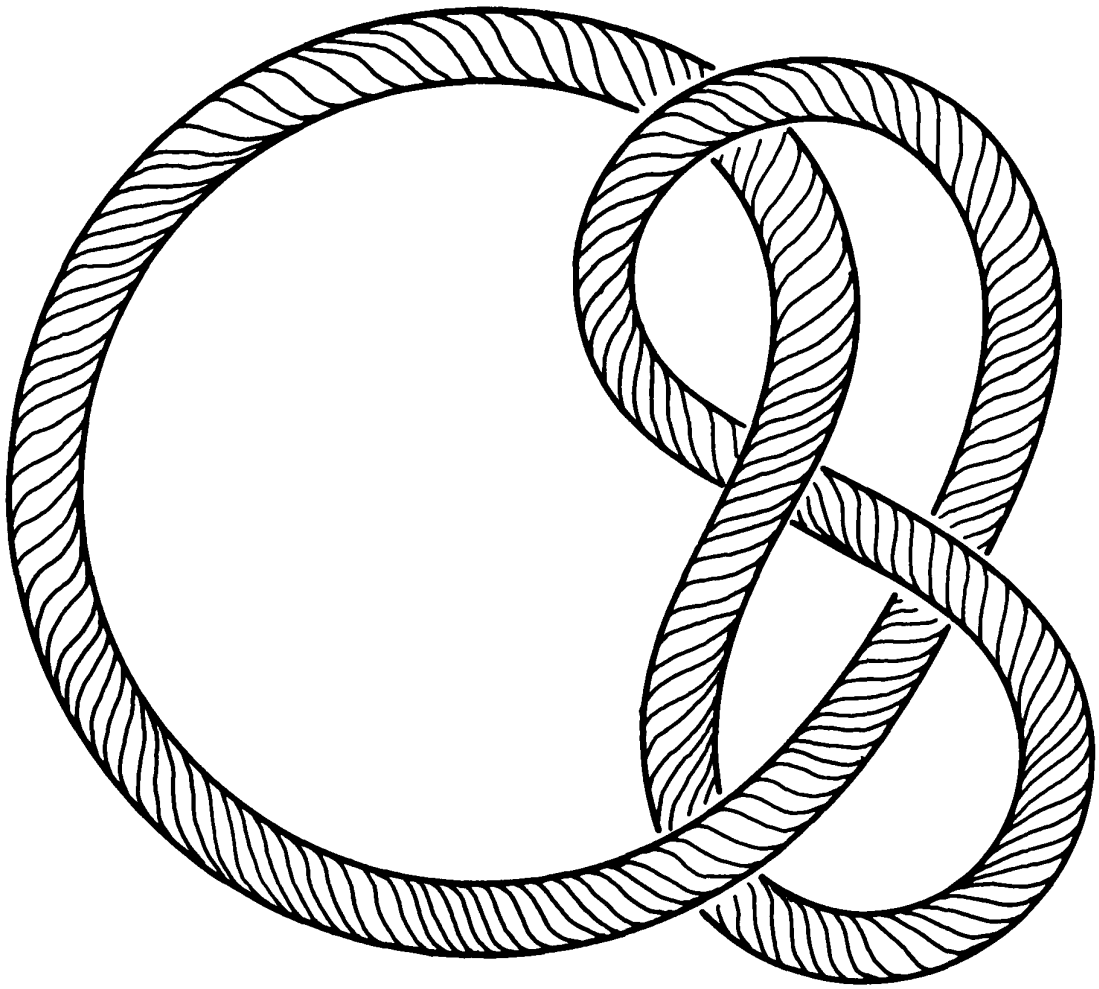


Figure 13.5: How could a four-dimensional spirit untie this "figure-eight" knot?

Philosophical Comments

(1) Many authors claim that you cannot visualize four-dimensional space. This simply isn't true. (It is true that you must visualize E^4 differently than E^3.) My personal opinion is that your mind is as *capable* of visualizing four dimensions as three. The reason three dimensions is so much easier in practice is that the real universe is three-dimensional: from the day you were born you've

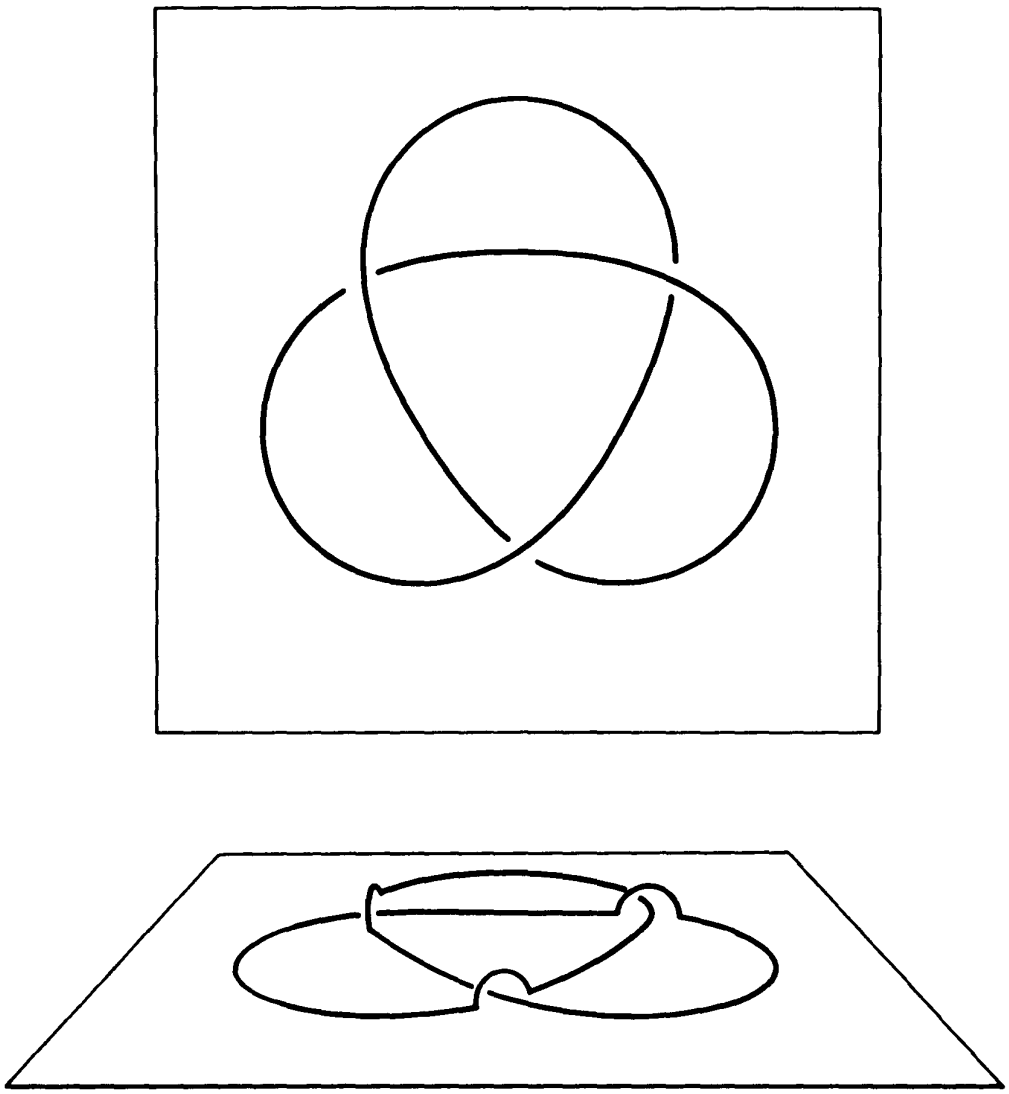


Figure 13.6: To imagine a knotted circle in E^3, A Square first draws a (topological) circle which crosses itself. At each crossing point he then imagines one piece of the circle to pass over the other piece in the third dimension. We Spacelanders can easily visualize the resulting knotted circle.

been getting practice in understanding three dimensions. At first visualizing four dimensions is difficult and tiring--just as newborn babies no doubt find three dimensions confusing at first. With practice it becomes easier.

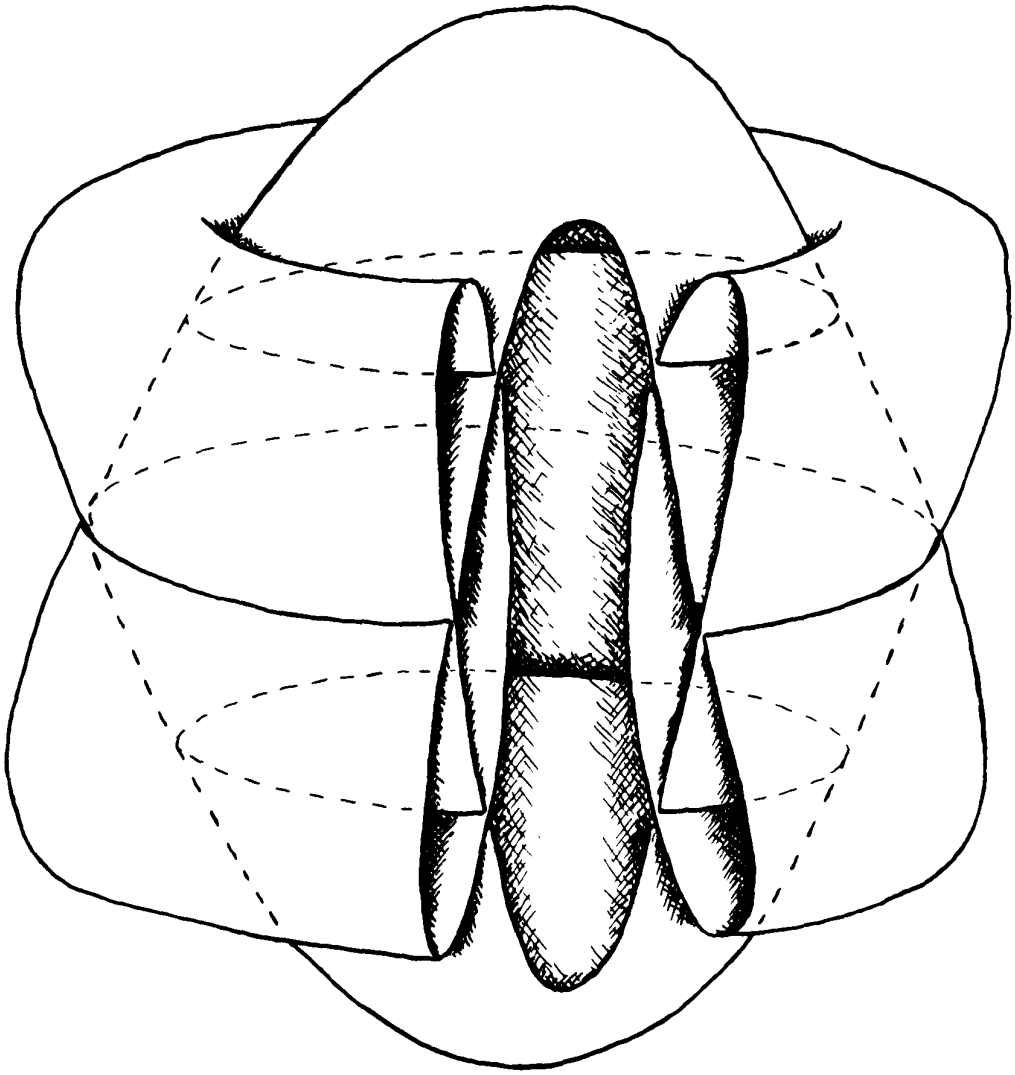


Figure 13.7: To imagine a knotted sphere in E^4, we Spacelanders first draw a (topological) sphere which intersects itself. Along each circle of intersection we then imagine one sheet to pass over the other in the fourth dimension.

(2) Physicists often combine three-dimensional space and one-dimensional time into a four-dimensional entity called **spacetime.** I originally intended to describe this idea in detail here, but I don't think I can improve on the discussion given in Chapter 4, "Time as a Higher Dimension", of R. Rucker's *Geometry, Relativity and the*

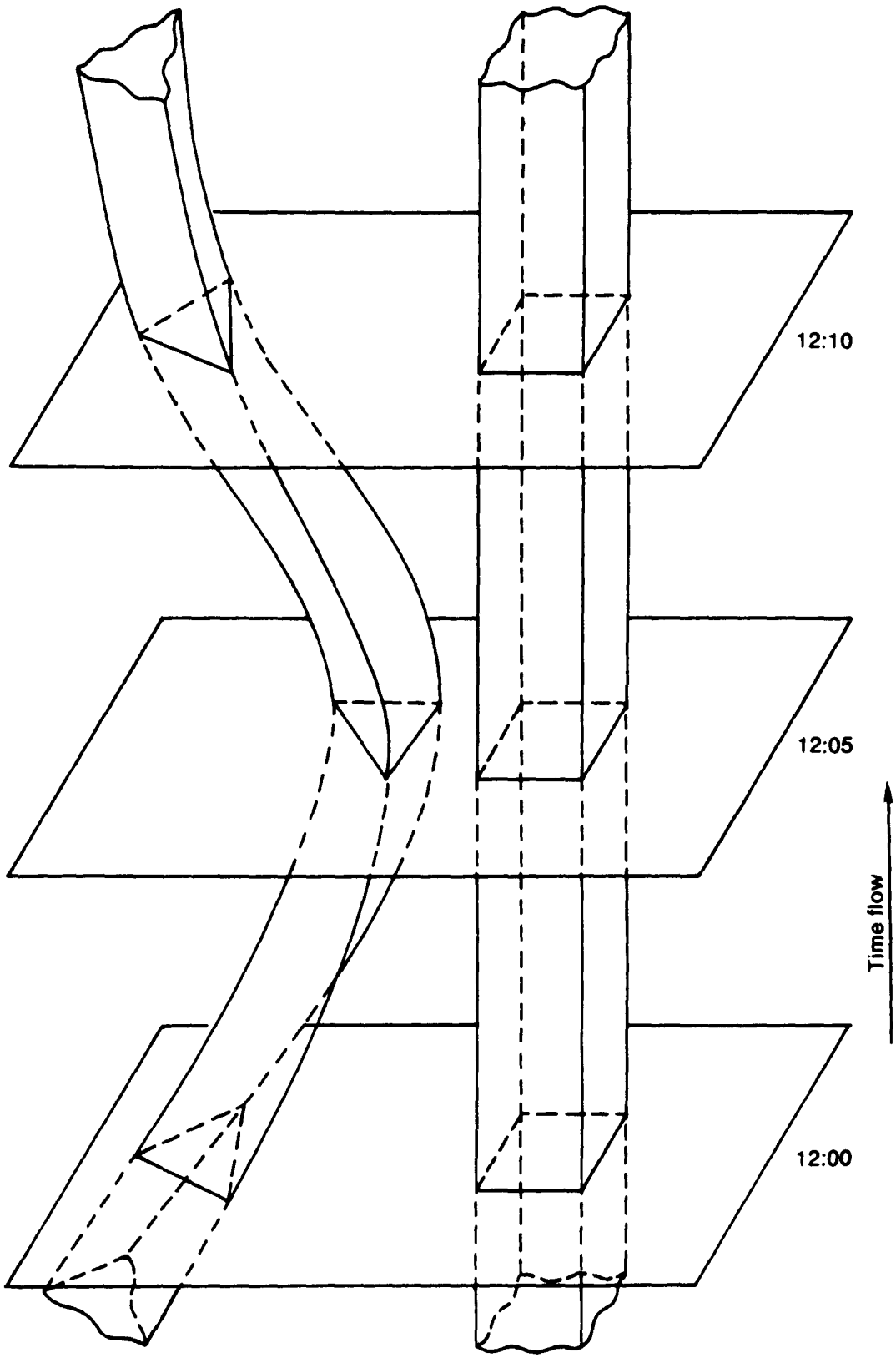


Figure 13.8: Figure 78 from R. Rucker's *Geometry, Relativity and the Fourth Dimension*.

Fourth Dimension. To give you the general idea right here and now, I've reproduced Figure 78 from that chapter (it appears here as Figure 13.8), along with the following explanation:

> To get a good mental image of space-time, let us return to Flatland. Suppose that A. Square is sitting alone in a field. At noon he sees his father, A. Triangle, approaching from the west. A. Triangle reaches A. Square's side at 12:05, talks to him briefly, and then slides back to where he came from. Now, if we think of time as being a direction perpendicular to space, then we can represent the Flatlanders' time as a direction perpendicular to the plane of Flatland. Assuming that "later in time" and "higher in the third dimension" are the same thing, we can represent a motionless Flatlander by a vertical worm or trail and a moving Flatlander by a curving worm or trail, as we have done in Figure 78.
>
> We can think of these 3-D space-time worms as existing timelessly. . . .

Rucker goes on to explain the implications this idea has for consciousness, the perception of time, and the nature of reality. This is thought-provoking stuff--I recommend it highly. (If you're in a hurry, you can easily read that one chapter independently of the rest of the book.)

In Chapter 14 we'll stick to imagining four dimensions in a purely spatial way--time won't enter the picture at all.

(3) A final metaphysical comment: The four-dimensional space we imagine is purely a mathematical abstraction. I make no claim that it exists physically like the three-dimensional universe does.

14
The Hypersphere

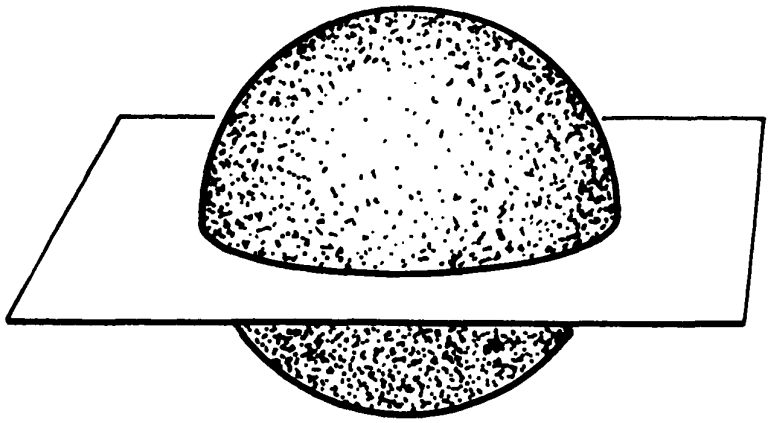
Roughly speaking, a sphere is like a circle, only one dimension bigger. A hypersphere, or three-sphere, is the analogous three-manifold one dimension bigger than a sphere. Compare the formal definitions of the circle, the sphere, and the hypersphere:

1. The unit circle (= one-sphere = S^1) is the set of points in E^2 which are one unit away from the origin. Anything topologically equivalent to it is called a **topological circle** and anything geometrically equivalent is called a **geometrical circle**.
2. The unit sphere (= two-sphere = S^2) is the set of points in E^3 which are one unit away from the origin. Anything topologically equivalent to it is called a **topological two-sphere** and anything geometrically equivalent is called a **geometrical two-sphere**.

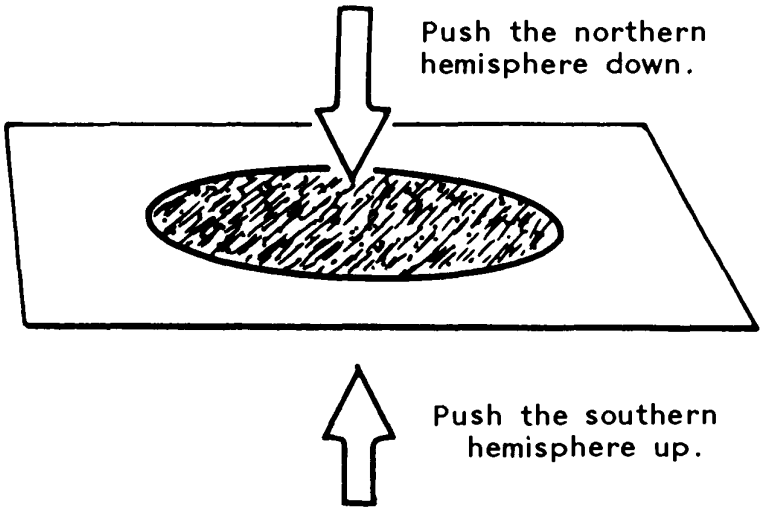
3. The unit hypersphere (= three-sphere = S^3) is the set of points in E^4 which are one unit away from the origin. Anything topologically equivalent to it is called a **topological three-sphere** and anything geometrically equivalent is called a **geometrical three-sphere.**

A Warning on Terminology: Our two-sphere is defined in three-dimensional space, where it is the boundary of a three-dimensional ball. This terminology is standard among mathematicians, but not among physicists. So don't be surprised if you find people calling the two-sphere a three-sphere. They're interested in the dimension of the space that the two-sphere happens to be in, while we're interested in the intrinsic dimension of the two-sphere itself. (A two-sphere is still intrinsically two-dimensional even if it's sitting in E^4, like the knotted two-sphere of Figure 13.7.) Similar comments apply to the three-sphere. Also note carefully the distinction between "sphere" and "ball" as used above. Some people use "sphere" to mean "ball", but not us. We do, however, use "disk" and "ball" interchangeably ("disk" sounds better in two dimensions and "ball" sounds better in three, but the concept is essentially the same).

How can one visualize a three-sphere? To answer this question, let's take a look at how A Square might visualize a two-sphere. Assume for the moment that he's interested only in its topology and



The Original Two-Sphere



The Flattened Version

Figure 14.1: A Square visualizes the two-sphere topologically as two superimposed disks.

not its geometry. He can make things easy for himself by flattening the sphere into the plane of Flatland, as shown in Figure 14.1. The northern and southern hemispheres each become a disk in Flatland. The two disks are superimposed and joined together along their circular boundary (the equator). Thus, the task of visualizing the two-sphere has been reduced to the task of visualizing two superimposed disks. A Square has to remember, though, that the "crease" at the equator doesn't exist in the real two-sphere--it's merely an artifact of the flattening process. This method of visualizing the two-sphere is called the **double disk method**, as is the analogous method of visualizing the three-sphere (see Exercise 14.1).

Exercise 14.1. Imagine a three-sphere topologically as two superimposed solid balls in E^3. These balls are joined together along their spherical boundary. (The equator of S^3 is a two-sphere!) □

Exercise 14.2. To recover the geometry of the two-sphere, A Square imagines one disk bending upward into the third dimension and the other disk bending downward. Modify your mental image from Exercise 14.1 to let one solid ball bend "upward" into the fourth dimension, and the other ball bend "downward". □

Note: Some readers may understand the three-sphere's global topology more easily by imagining it as two *non*superimposed solid balls whose surfaces are abstractly glued together (Figure 14.2).

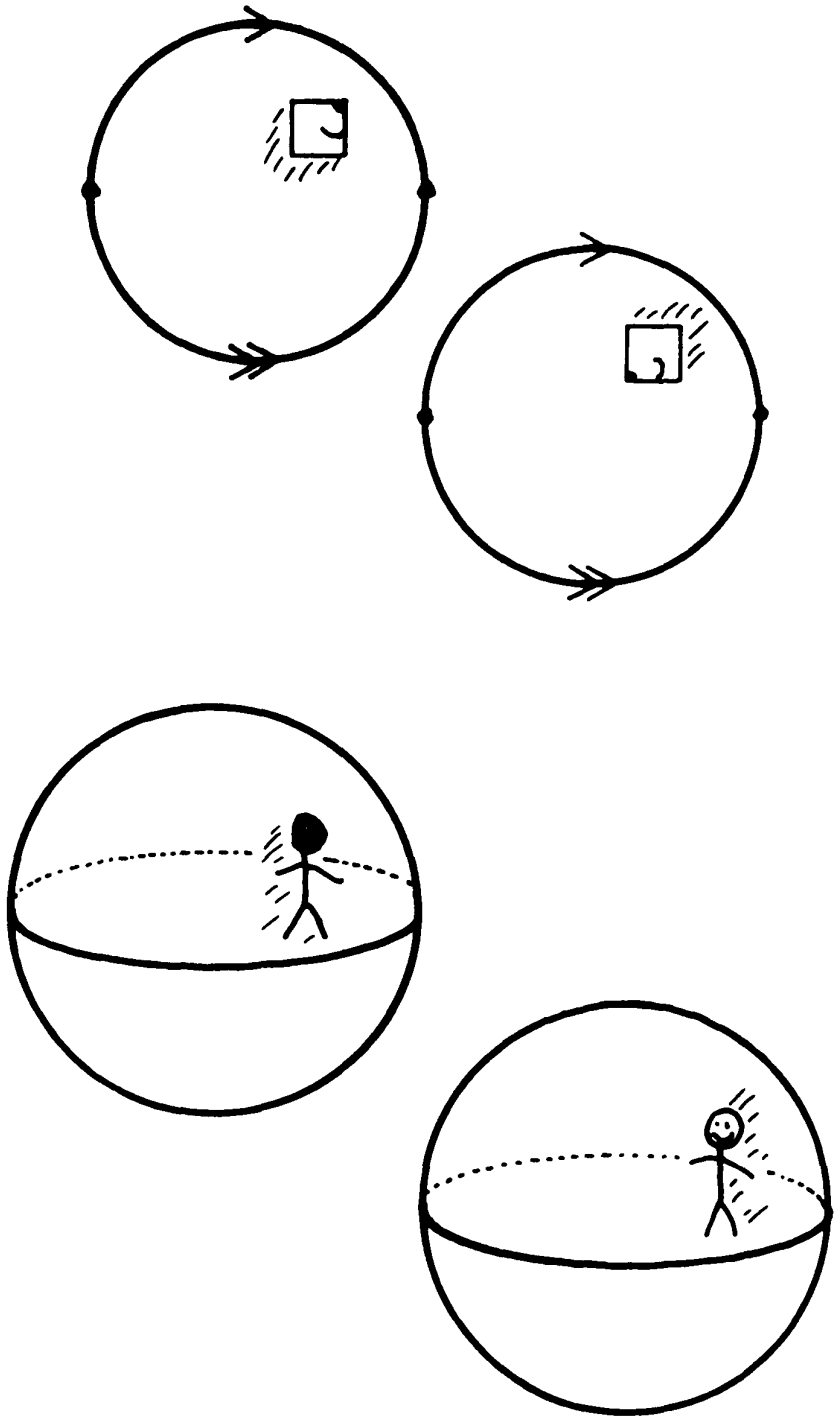


Figure 14.2: A two-sphere may be represented topologically as two disks with edges glued together. Similarly, a three-sphere may be represented topologically as two solid balls with surfaces glued together.

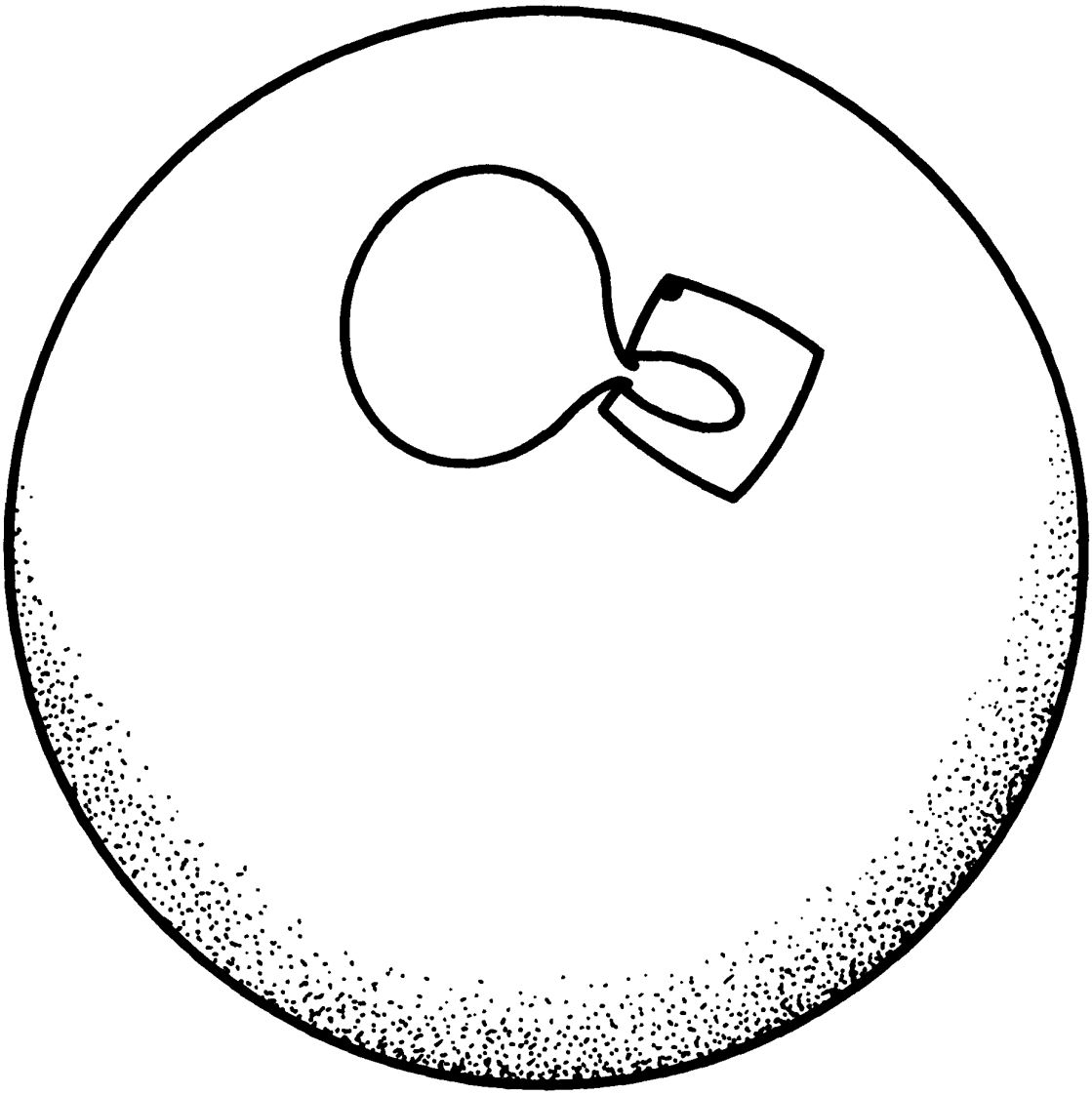


Figure 14.3: A Square inflates a balloon on a two-sphere.

Exercise 14.3. Pretend you live in a fairly small three-sphere. What will eventually happen to you if you keep blowing air into an easily stretchable balloon? (Hint: What will eventually happen to A Square in Figure 14.3?) □

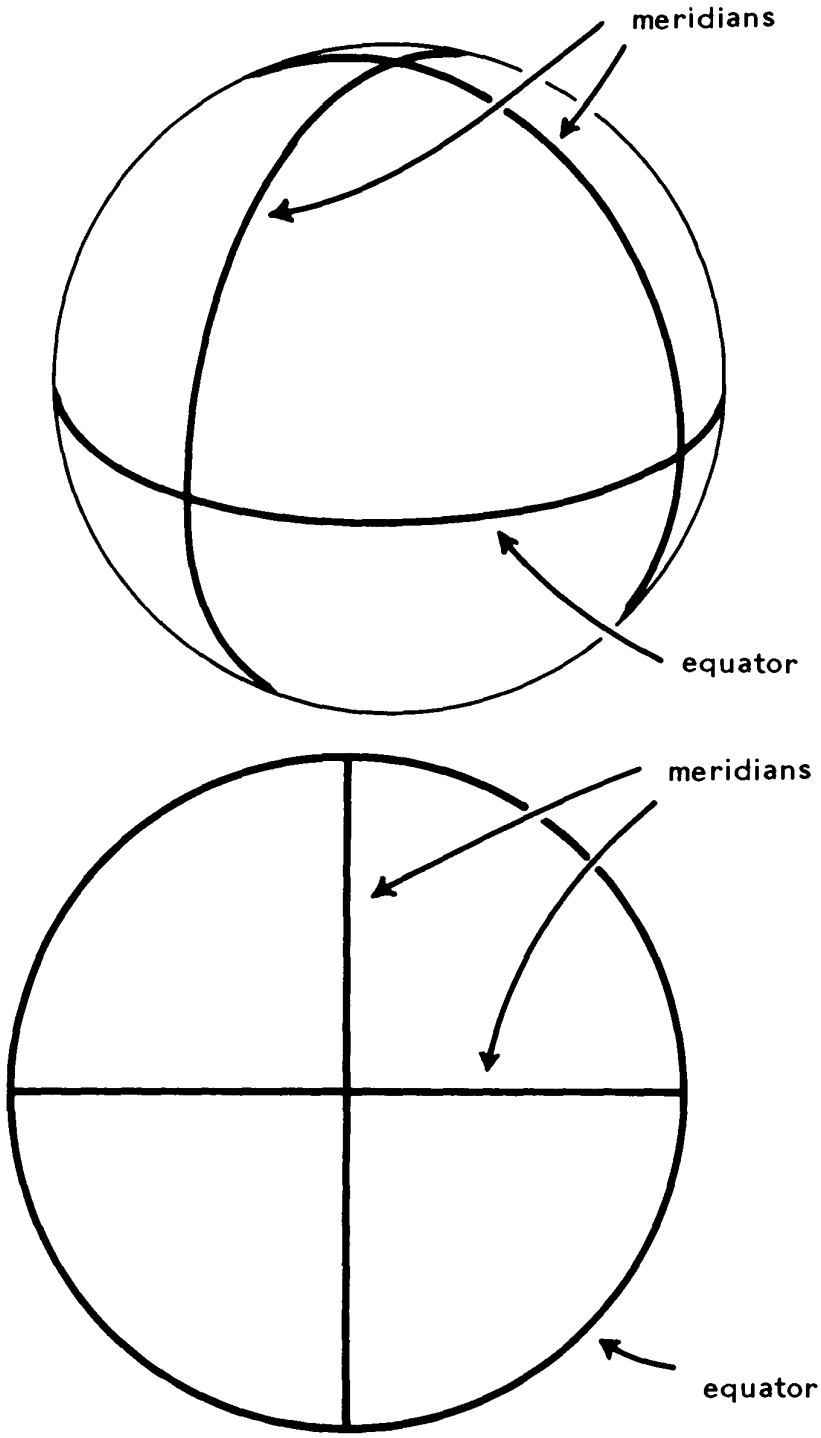


Figure 14.4: The first drawing is a Spacelander's view of three great circles on a two-sphere. The second drawing is a top view of the two-sphere after it has been flattened into a horizontal plane. Only the equator is still a circle. Each meridian has been flattened into a line segment.

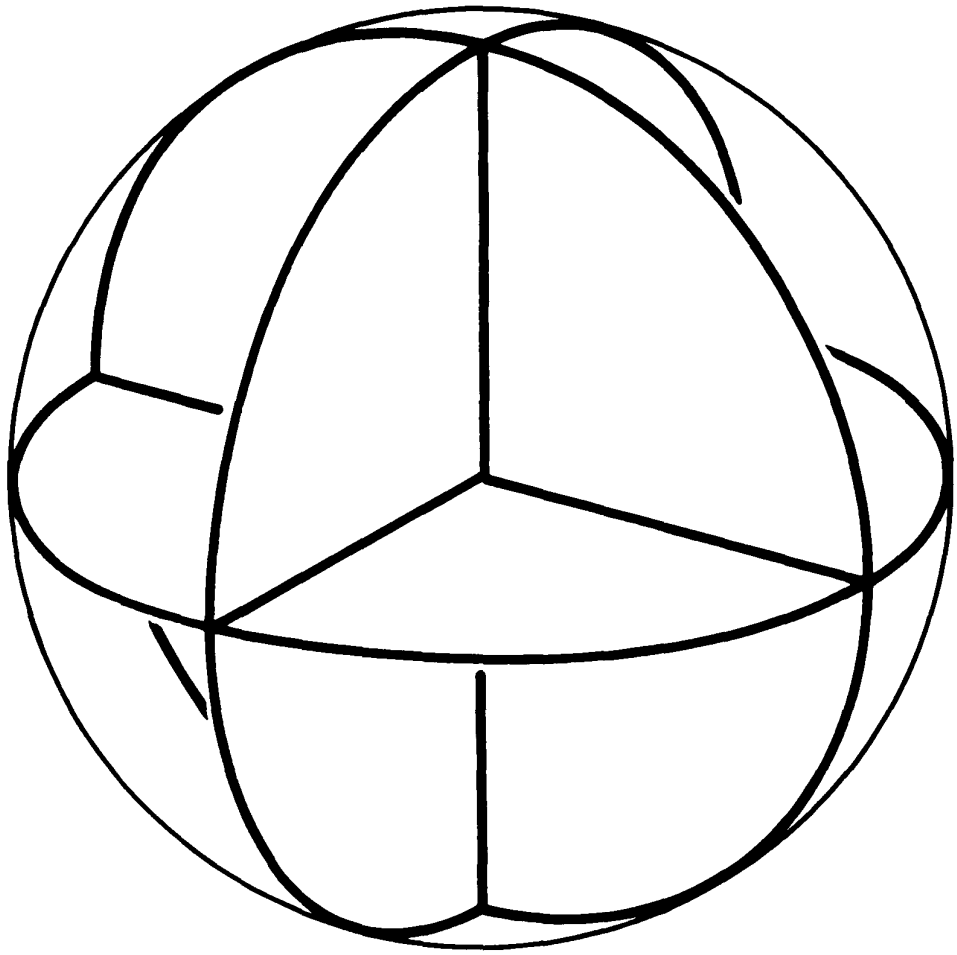


Figure 14.5: This drawing represents four "great two-spheres" in a hypersphere. It is completely analogous to the second drawing in Figure 14.4. The surface of the ball is the three-sphere's "equator". Each disk is the flattened remains of a two-sphere, so imagine each disk as two disks, one bending "upward" and the other "downward" into the fourth dimension.

Figure 14.4 shows three great circles on a two-sphere (they happen to be the intersection of the two-sphere with each of the coordinate planes). We Spacelanders visualize these great circles quite easily because we can draw the two-sphere in three-dimensional space. Flatlanders, on the other hand, draw the two-sphere flattened into their plane. The equator is still a circle, but

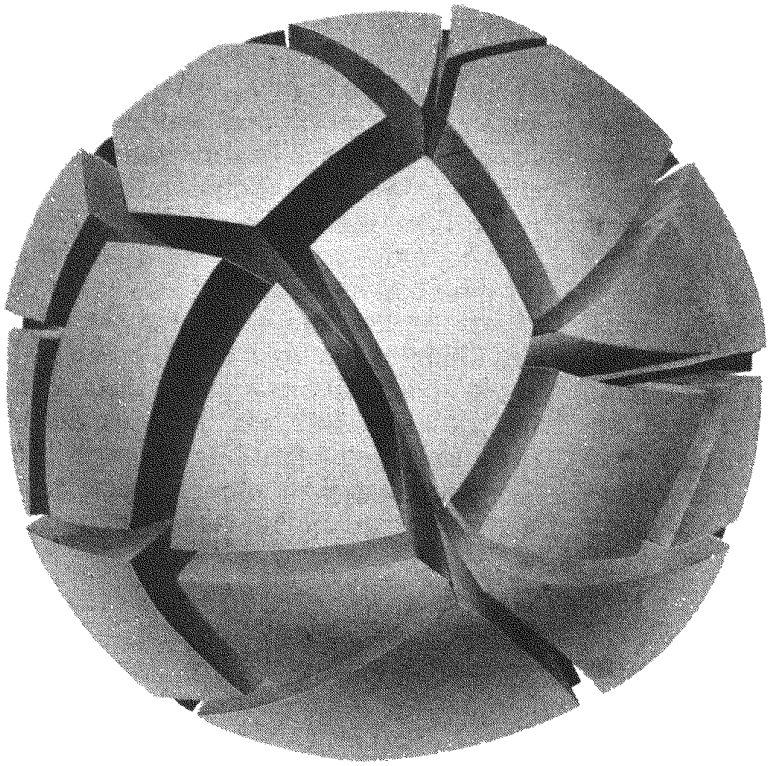


Figure 14.6: If we do not allow it to stretch, a piece of a three-sphere will split open in Euclidean space. Note that every cross-section of this split open ball is a split open disk like the one in Figure 9.8.

the meridians look like line segments. The Flatlanders must remember that each apparent line segment is really two line segments, one arching upward into the third dimension and the other arching downward. Two line segments together form a perfect geometrical circle.

Figure 14.5 shows four "great two-spheres" on a three-sphere (they happen to be the intersection of the three-sphere with each of the coordinate hyperplanes). It's hard for us Spacelanders to visualize the great two-spheres because the three-sphere they lie in

has been flattened down into our three-dimensional space. Only the "equatorial" two-sphere still looks like a sphere. Each of the others now looks like a disk. Each is really two disks, of course, one arching upward into the fourth dimension and the other arching downward. Two disks together form a perfect geometrical sphere.

The preceding two paragraphs show how every slice of a three-sphere is a two-sphere, just as every slice of a two-sphere is a circle.

If we do not allow it to stretch, a piece of a three-sphere will split open in Euclidean space (Figure 14.6) just as a piece of a two-sphere splits open in the plane (Figure 9.8).

Polyhedra in a three-sphere have larger angles than do polyhedra in Euclidean space (Figure 14.7). It turns out that polyhedra in "hyperbolic space" (Chapter 15) have smaller angles than do polyhedra in Euclidean space. We'll utilize these facts to find homogeneous geometries for certain three-manifolds (Chapter 16), just as we found homogeneous geometries for surfaces (Chapter 11).

Say our universe is a three-sphere. We can measure its curvature by measuring the curvature of a great two-sphere. And we can measure the curvature of a great two-sphere by measuring the angles and area of a triangle lying on it. If we don't care which great two-sphere we are measuring--and we don't because they are all the same--then we can measure any triangle we want.

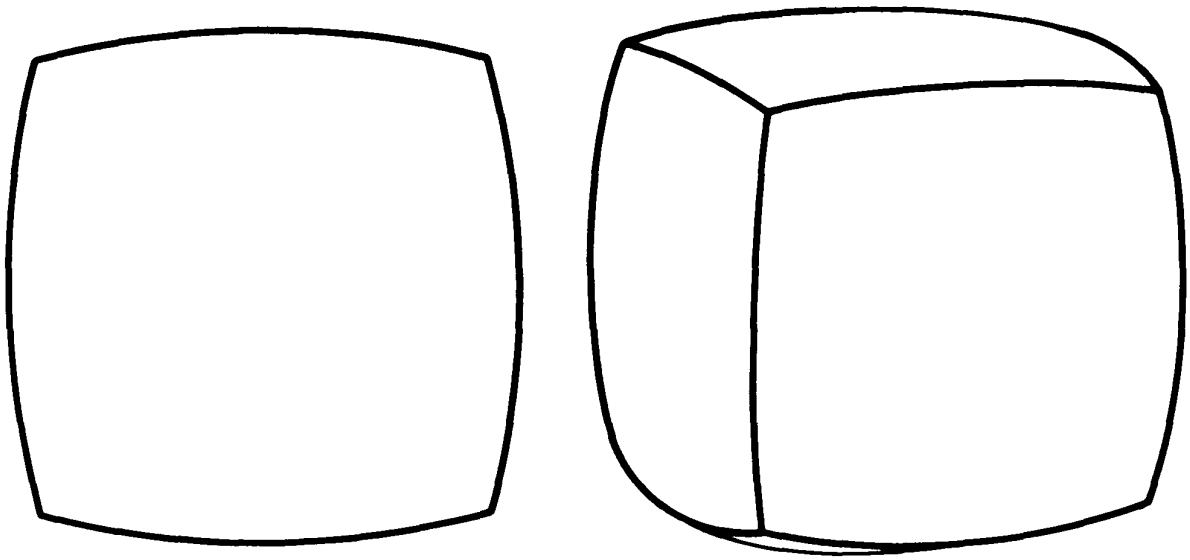


Figure 14.7: Polyhedra in a three-sphere have larger angles than do polyhedra in ordinary Euclidean space, just as polygons on a two-sphere have larger angles than do polygons in the Euclidean plane. Note: When representing a spherical polygon in the Euclidean plane it is customary to make the sides bulge so that the angles come out right. On the sphere itself, of course, the polygon's sides do not bulge--they are perfect geodesics. Similarly, when representing a polyhedron from the three-sphere it is customary to make the sides bulge so that the edge and corner angles come out right, even though the sides do not bulge in the three-sphere itself.

Exercise 14.4. Gauss measured the angles of a triangle roughly 100 km on a side, probably using instruments accurate to, say, 10 minutes of arc. What is the smallest curvature he could detect? What is the radius of a two-sphere with this curvature? The radius of any great two-sphere is the same as the radius of the three-sphere it lies in, so the answer to the previous question represents the radius of the largest S^3-universe whose curvature Gauss could detect. Note that small three-spheres have large curvature and large three-spheres have small curvature, so it's easier to detect

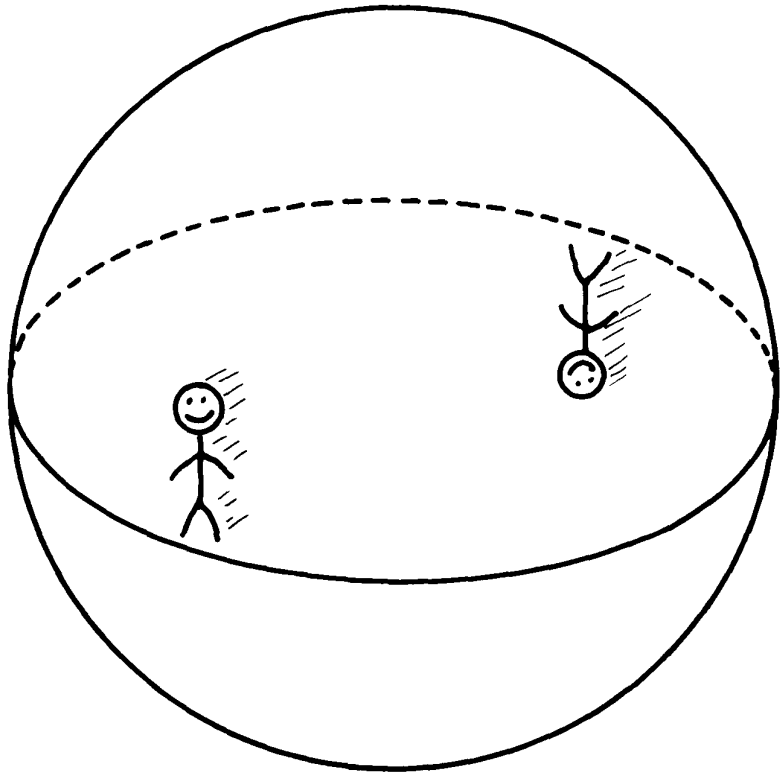


Figure 14.8: When you cross the "seam" of P^3 you come back with your head where your feet were and your left side where your right side was. In effect you rotate a half-turn.

the curvature of a small S^3-universe than a large one. Do you think it's likely that the universe is this small?

I'm sure Gauss was well aware of the above considerations. He may not have been trying to measure the curvature of the universe at all, and if he was he was probably hoping to detect a hyperbolic universe (in hyperbolic space the angles of a triangle add up to less than π). By the way, modern cosmologists attempt to measure the curvature of the universe by very different means (Chapter 19). □

Projective Three-Space

Back in Chapter 4 you constructed the projective plane (P^2) by gluing together opposite points on the circular rim of a hemisphere of S^2 (recall Figure 4.11). The hemisphere's local geometry matched up nicely across the "seam" (Figure 4.12), so you got a surface with the same local geometry as S^2, but a different global topology.

You can make **projective three-space** (P^3) in the same way. Start with a hemisphere of S^3 and glue together opposite points on its (spherical) boundary. It should be clear (at least from analogy) that the hemisphere's local geometry matches up nicely across the spherical "seam". Thus P^3 is a three-manifold with the same local geometry as S^3, but a different global topology.

In Chapter 4 we noted that for topological purposes we can think of P^2 as a disk with opposite boundary points glued, and we can think of P^3 as a solid ball with opposite boundary points glued. The projective plane is nonorientable: when a Flatlander crosses the "seam" he comes back left-right reversed. Projective three-space, on the other hand, *is* orientable. When you cross the "seam" you come back both left-right reversed *and* top-bottom reversed. In effect, you get mirror-reversed two ways, so you come back as your old self! The only difference is that you've been rotated 180°. (See Figure 14.8.)

Exercise 14.5. See Figure 14.9. Does the cylinder form a torus or

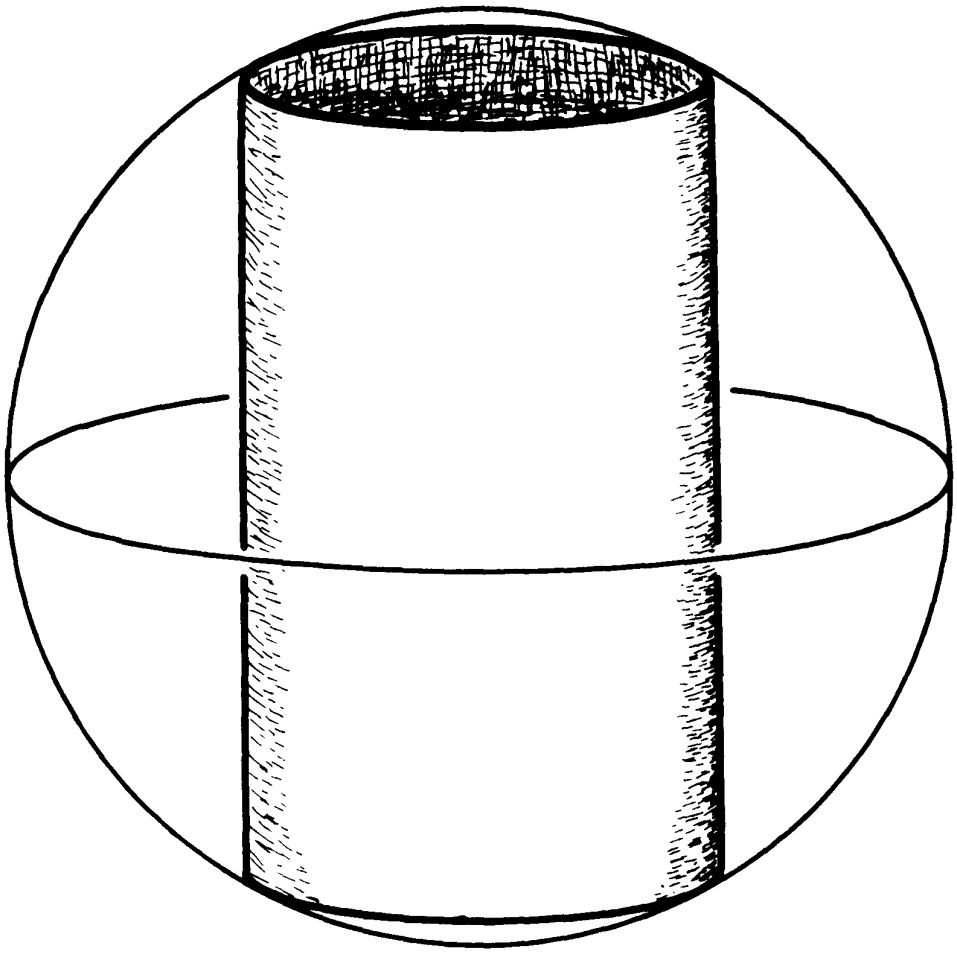


Figure 14.9: Does the cylinder become a torus or a Klein bottle in projective three-space?

a Klein bottle in projective three-space? Is it orientable? Is it two-sided? □

Exercise 14.6. Find a copy of P^2 embedded in P^3. Is it orientable? Is it two-sided? □

15
Hyperbolic Space

Hyperbolic space is just like the hyperbolic plane, only one dimension bigger. In fact, every two-dimensional slice of hyperbolic space is a hyperbolic plane (Figure 15.1) in the same way that every two-dimensional slice of Euclidean space is a Euclidean plane and every two-dimensional slice of a hypersphere is a two-sphere. Hyperbolic space is homogeneous. It is often abbreviated as H^3.

A polyhedron in H^3 has smaller angles than a polyhedron in Euclidean space (Figure 15.2). This fact will be crucial in the next chapter when we find homogeneous geometries for certain three-manifolds.

Figure 15.3 shows a series of successively larger hyperbolic triangles. Note that larger triangles have smaller angles. The figure sheds some light on why no triangle in the (standard)

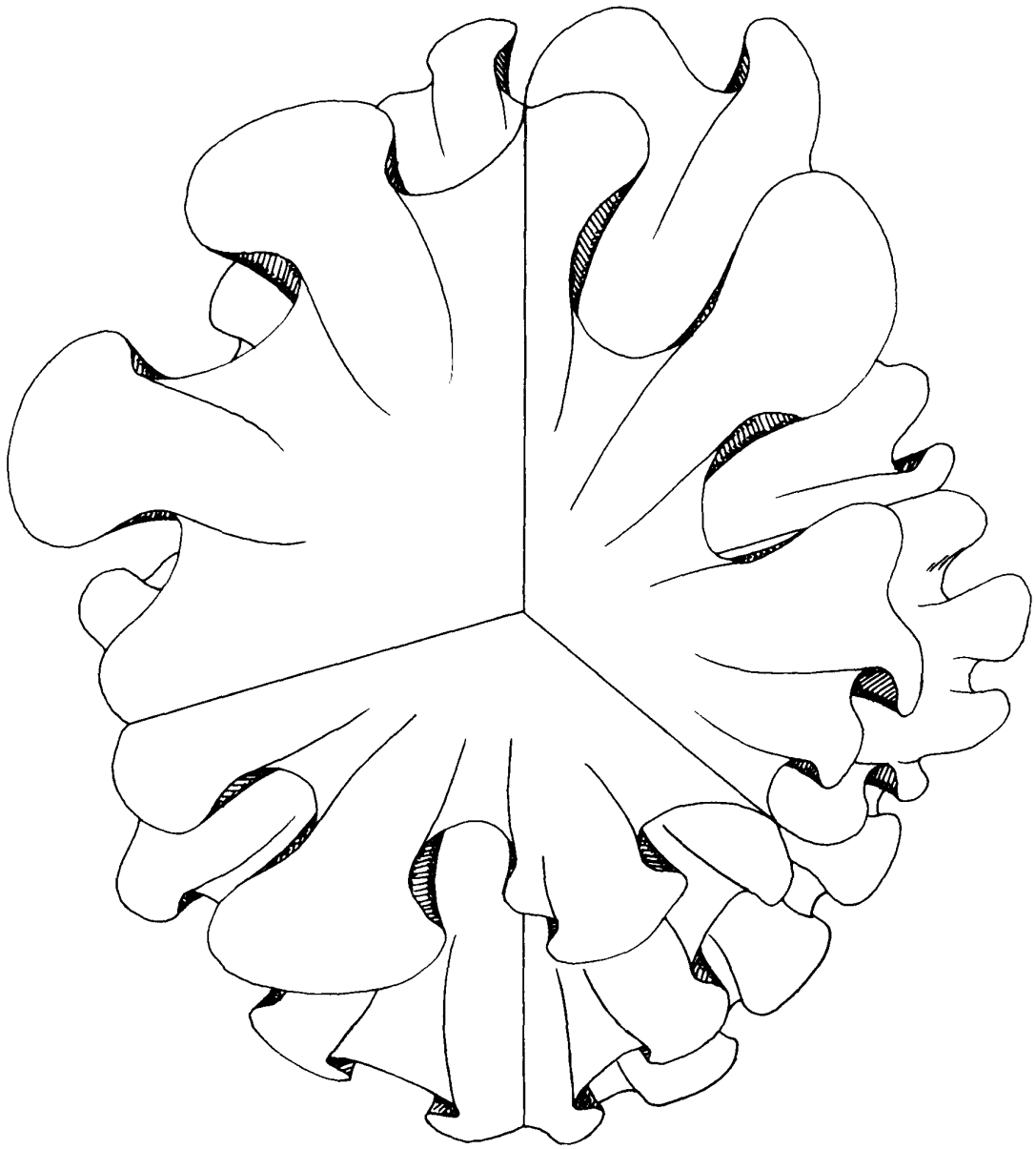


Figure 15.1: Every slice of hyperbolic space is a hyperbolic plane.

hyperbolic plane can have area greater than π. When you try to draw a triangle with more area you find that the triangle's sides don't meet. In other words, you end up with three nice straight geodesics, no two of which intersect! The hyperbolic plane has some strange properties.

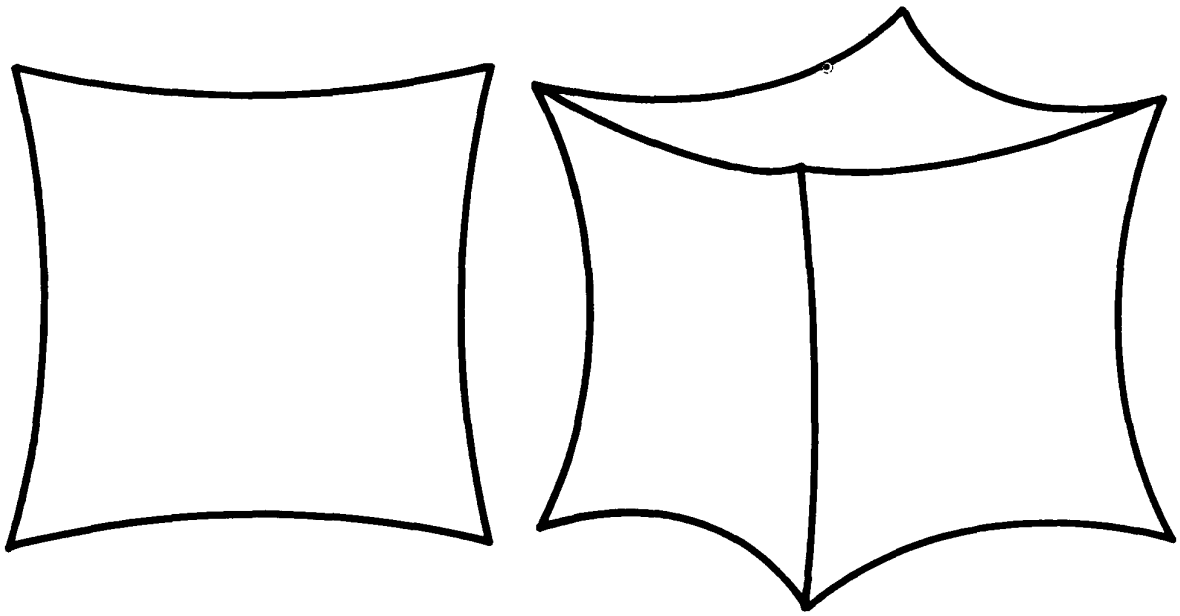


Figure 15.2: Polyhedra in hyperbolic space have smaller angles than do polyhedra in Euclidean space, just as polygons in the hyperbolic plane have smaller angles than do polygons in the Euclidean plane. Note: When representing a "hyperbolic polygon" in the Euclidean plane it is customary to make the sides bend inward so that the angles come out right. In the hyperbolic plane itself, of course, the polygon's sides do not bend inward--they are perfect geodesics. Similarly, when representing a polyhedron from hyperbolic space it is customary to make the sides bend inward so that the edge and corner angles come out right, even though the sides do not bend inward in hyperbolic space itself.

Imagine that you have encountered a party of extracosmic aliens. The aliens come from a highly curved hyperbolic universe, and are in our universe only for a visit. They have obviously mastered the secrets of interuniversal travel. You have heard that hyperbolic universes are somehow more spacious than Euclidean ones, so you ask the aliens to take you home with them. They oblige. When you get to their universe you look out into the sky at a distant galaxy. The light reaching your eyes naturally travels along nice

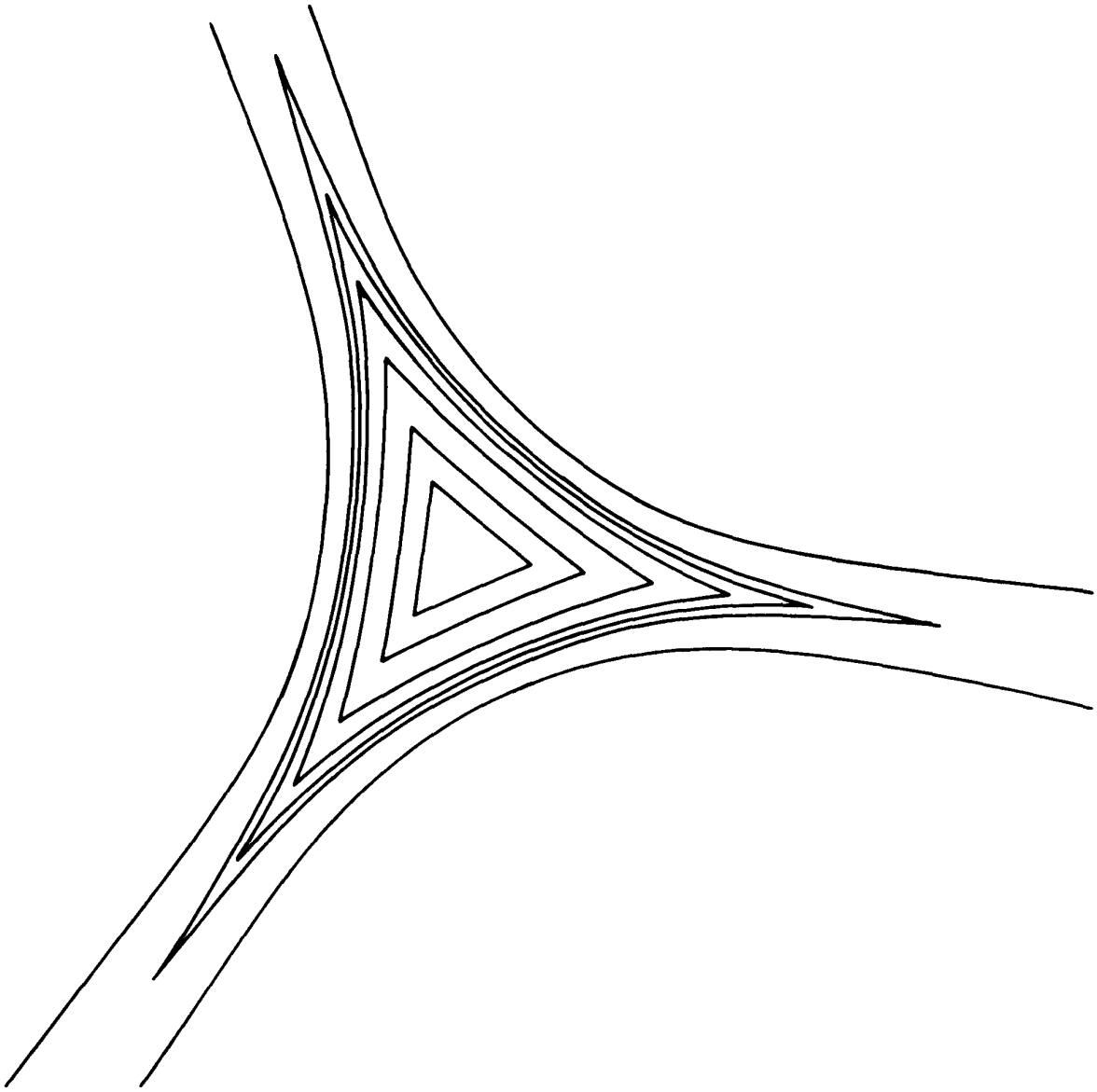


Figure 15.3: A series of successively larger hyperbolic triangles represented in the Euclidean plane. As explained in the caption to Figure 15.2, the triangles' sides don't really bend inward in the hyperbolic plane itself, but the angles shown here are correct.

straight geodesics, but in hyperbolic space geodesics do weird things. The light reaches your left eye at a slightly different angle than it reaches your right eye: you have to look somewhat cross-eyed to focus on the galaxy! (See Figure 15.4.) Your

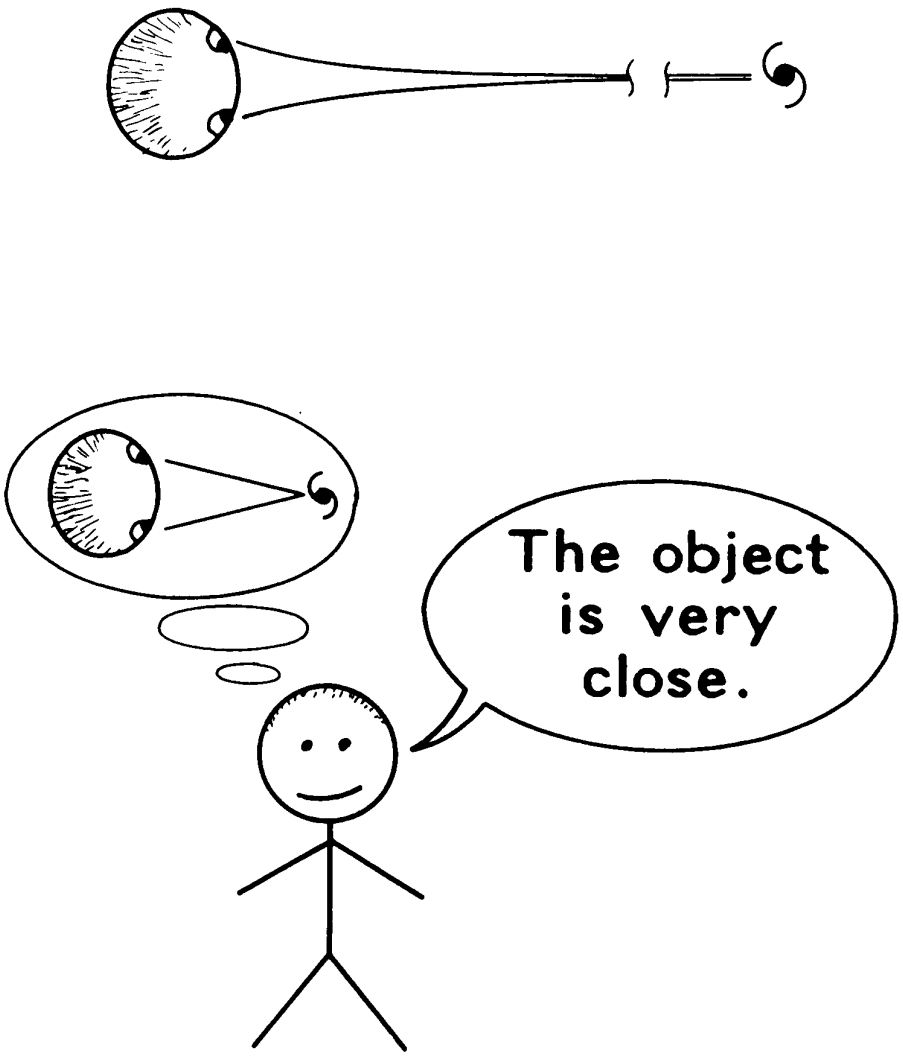


Figure 15.4: According to your binocular vision, very distant objects look close.

brain, used to interpreting visual data in an approximately Euclidean universe, decides that since you have to look cross-eyed the galaxy must be very close. In fact everything in the hyperbolic universe seems to be within a few meters of you. Even though it's really very spacious, a hyperbolic universe can *appear* very cramped.

Exercise 15.1. Since the effects described in the preceding paragraphs have not been observed in the real universe, does this mean that it cannot have the geometry of H^3? □

Exercise 15.2. How would an H^3-universe appear to its own inhabitants? (Hint: It needn't appear the same to them as it would to an outsider!) □

16

Geometries on Three-Manifolds I

The Seifert-Weber space consists of a dodecahedron whose opposite faces are glued with three-tenths turns (Figure 16.1). This three-manifold fails to have a Euclidean geometry for essentially the same reason that the third surface in Figure 11.1 failed to have one: the dodecahedron's twenty corners all come together at a single point, and they are much too fat to fit together properly. The solution is the same as in Chapter 11. Put the dodecahedron in hyperbolic space and let it expand until its corners are skinny enough that they do fit together (Figure 16.2). A Seifert-Weber space made from the appropriate dodecahedron has a homogeneous hyperbolic geometry.

The Poincaré dodecahedral space consists of a dodecahedron whose opposite faces are glued with one-tenth turns (Figure 16.3). This three-manifold fails to have a Euclidean geometry for essentially the same reason that the *first* surface in Figure 11.1

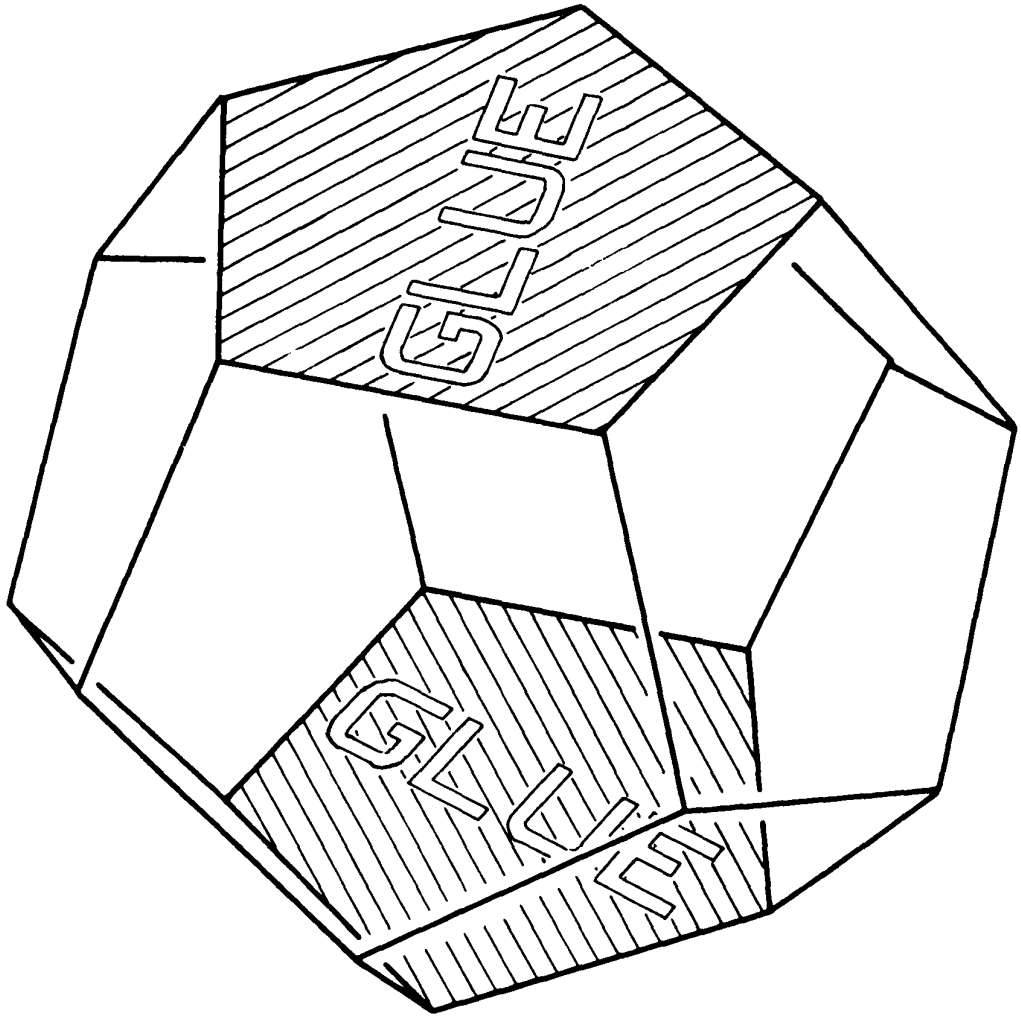


Figure 16.1: In the Seifert-Weber space every face of the dodecahedron is glued to the opposite face with a three-tenths clockwise turn. [A technical point: You might think that gluing the top to the bottom with a clockwise turn would be the same as gluing the bottom to the top with a counterclockwise turn, but this is not the case. Study the figure and you will see that gluing the top to the bottom with a clockwise turn (as viewed from above) works out the same as gluing the bottom to the top with a clockwise turn (as viewed from below). Thus the description of the Seifert-Weber space is self-consistent.]

failed to have one: the dodecahedron's twenty corners come together in five groups of four corners each, and they are a little too skinny to fit together properly. The solution is the same as in

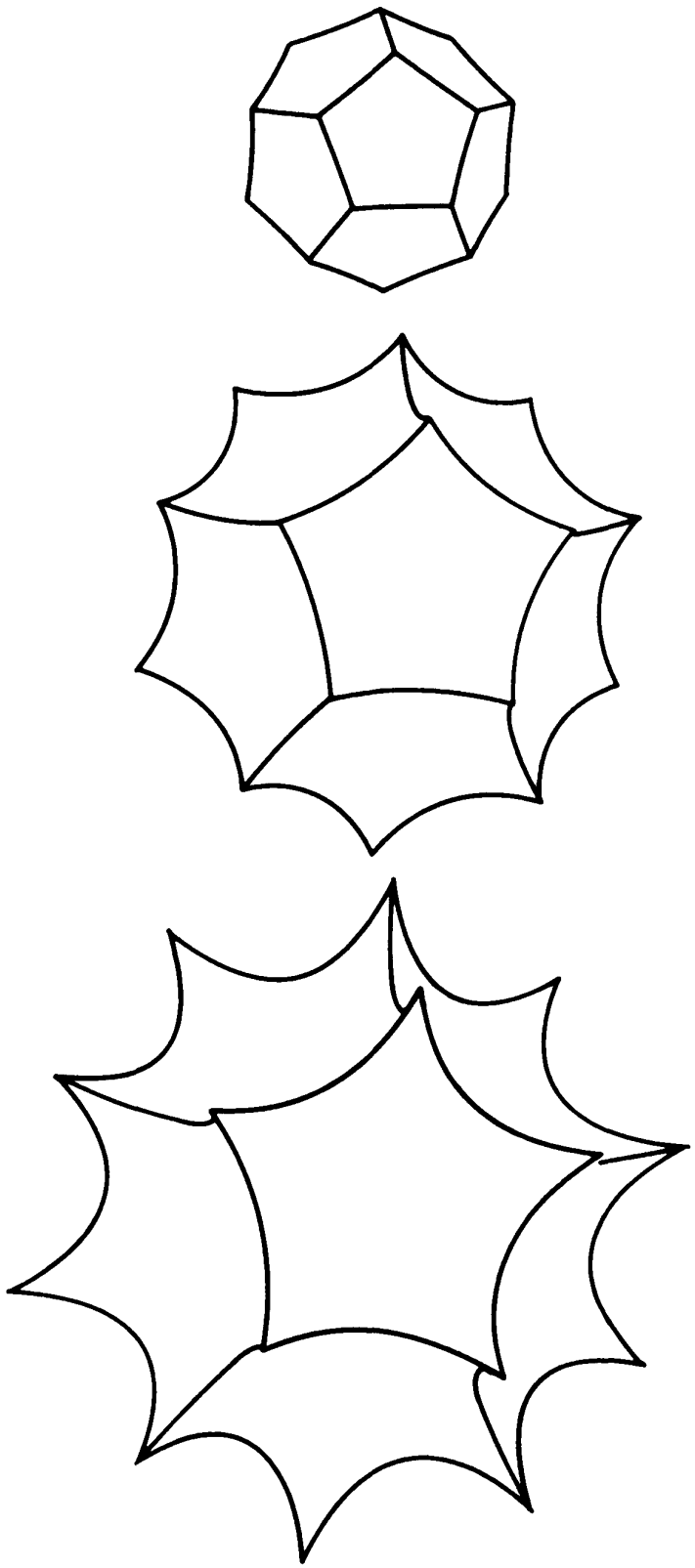


Figure 16.2: Let a dodecahedron expand in hyperbolic space until its corners are the right size to all fit together at a single point. The angles shown here are accurate, but in hyperbolic space itself the dodecahedron's faces do not bend inward.

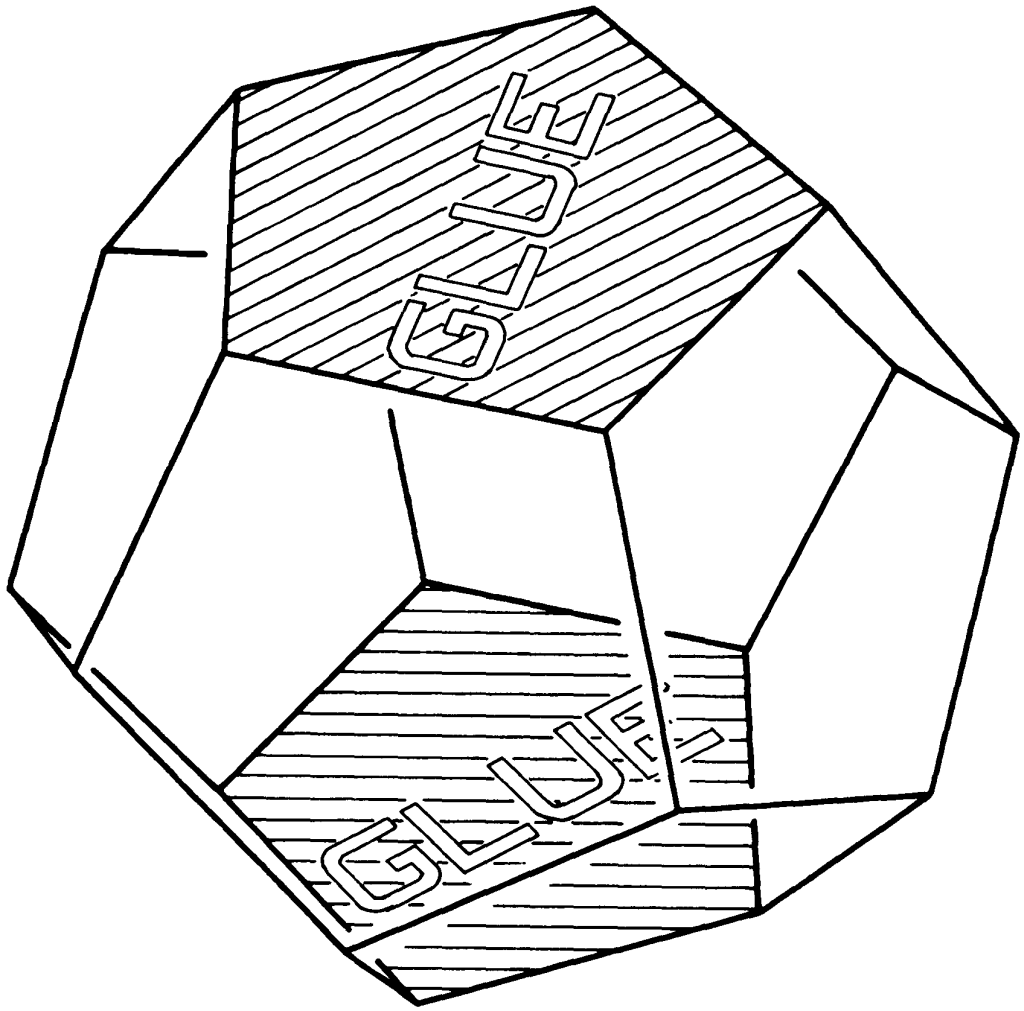


Figure 16.3: In the Poincaré dodecahedral space every face of a dodecahedron is glued to the opposite face with a one-tenth clockwise turn.

Chapter 11. Put the dodecahedron in a hypersphere and let it expand until its corners are fat enough that they do fit together (Figure 16.4). A Poincaré dodecahedral space made from the appropriate dodecahedron has the homogeneous geometry of the hypersphere (i.e. three-dimensional elliptic geometry).

Both the Seifert-Weber space and the Poincaré dodecahedral space appeared in 1933 in C. Weber and H. Seifert's article "The

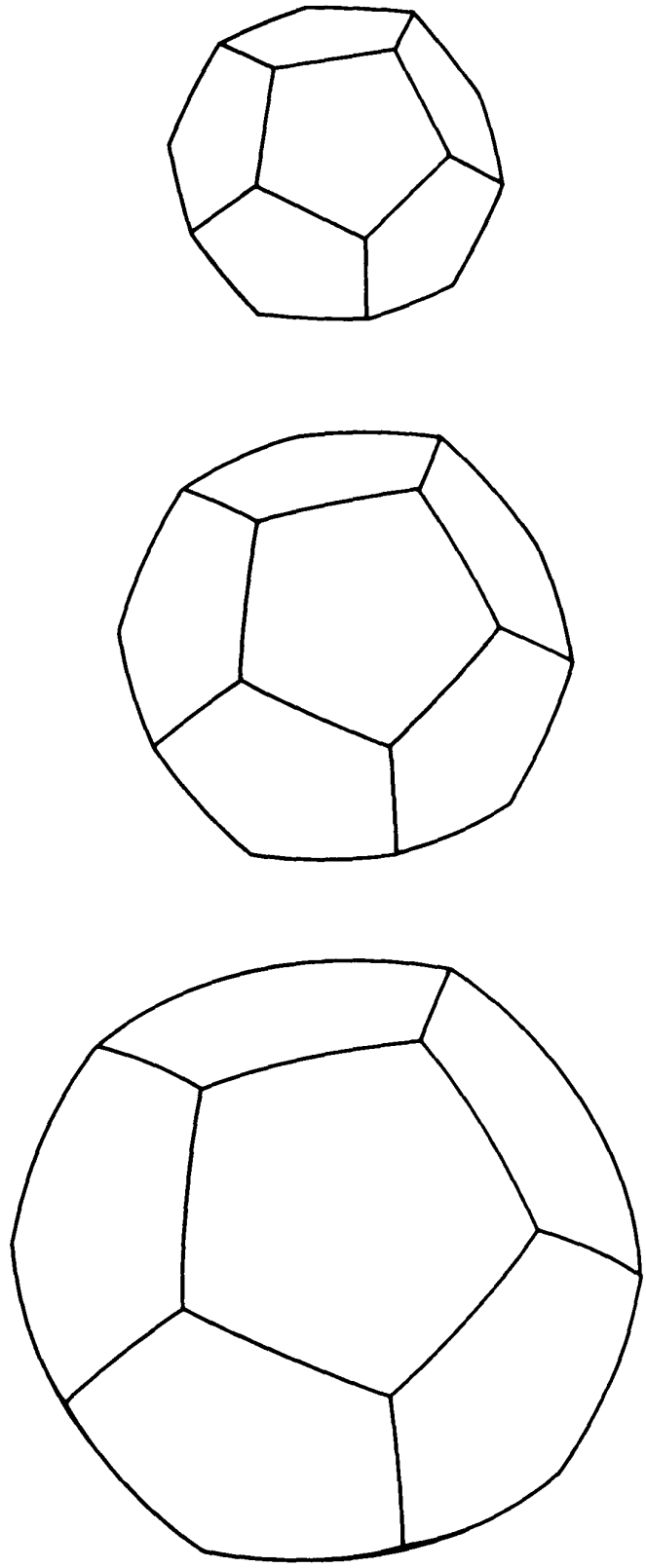


Figure 16.4: Let a dodecahedron expand in a three-sphere until its corners are the right size to fit together in groups of four. The angles shown here are accurate, but in the three-sphere itself the dodecahedron's faces do not bulge outward.

two dodecahedral spaces" (Die beiden Dodekaederräume, Mathematische Zeitschrift, Vol. 37, no. 2, p. 237). The Poincaré dodecahedral space is named in honor of Henri Poincaré (pronounced "pwan-ka-RAY") because it is topologically the same as a three-manifold Poincaré discovered around the turn of the century. Poincaré, though, didn't know that his manifold could be made from a dodecahedron! He was interested in it because it had certain properties in common with the hypersphere, namely the same "homology". (He had previously thought that the only three-manifold with the homology of the three-sphere was the three-sphere itself.)

At this point it's appropriate to note that in the three-torus a cube's eight corners all come together at a single point, and they fit perfectly (Figure 16.5). This is why the three-torus has Euclidean geometry. By the way, all the other three-manifolds in Chapter 7 have Euclidean geometry for the same reason.

We have seen that the Seifert-Weber space admits three-dimensional hyperbolic geometry, the Poincaré dodecahedral space admits three-dimensional elliptic geometry, and the three-torus admits three-dimensional Euclidean geometry. It would be nice if every three-manifold admitted one of these three geometries, but the actual situation is not that simple. For example, $S^2 \times S^1$ has a homogeneous geometry different from the three just mentioned. It's only in the last ten years that people have come to understand the situation more fully. Chapter 18 explains what is known and/or conjectured. Chapter 17 provides the examples necessary for

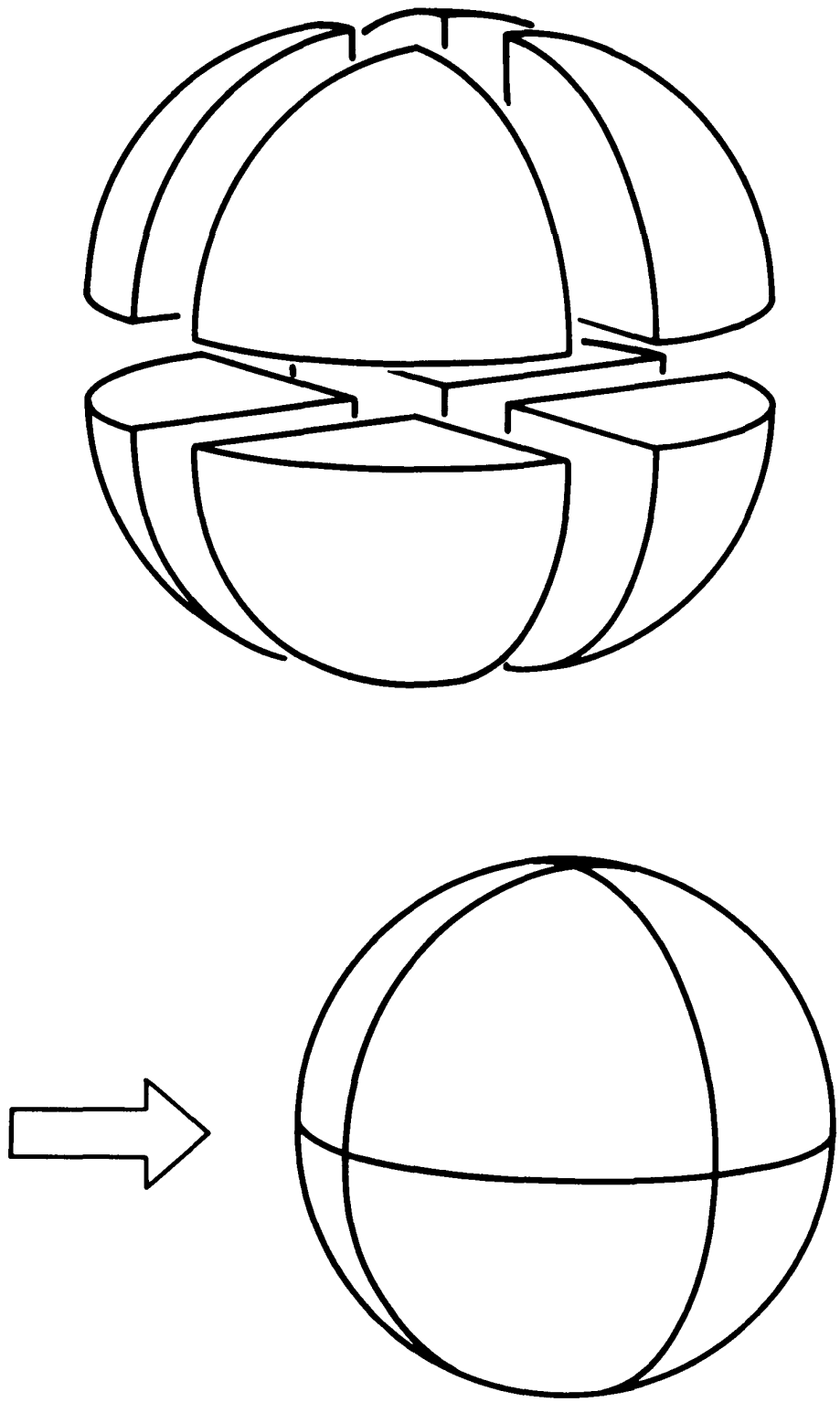


Figure 16.5: Eight corners of a cube fit together perfectly just as they are.

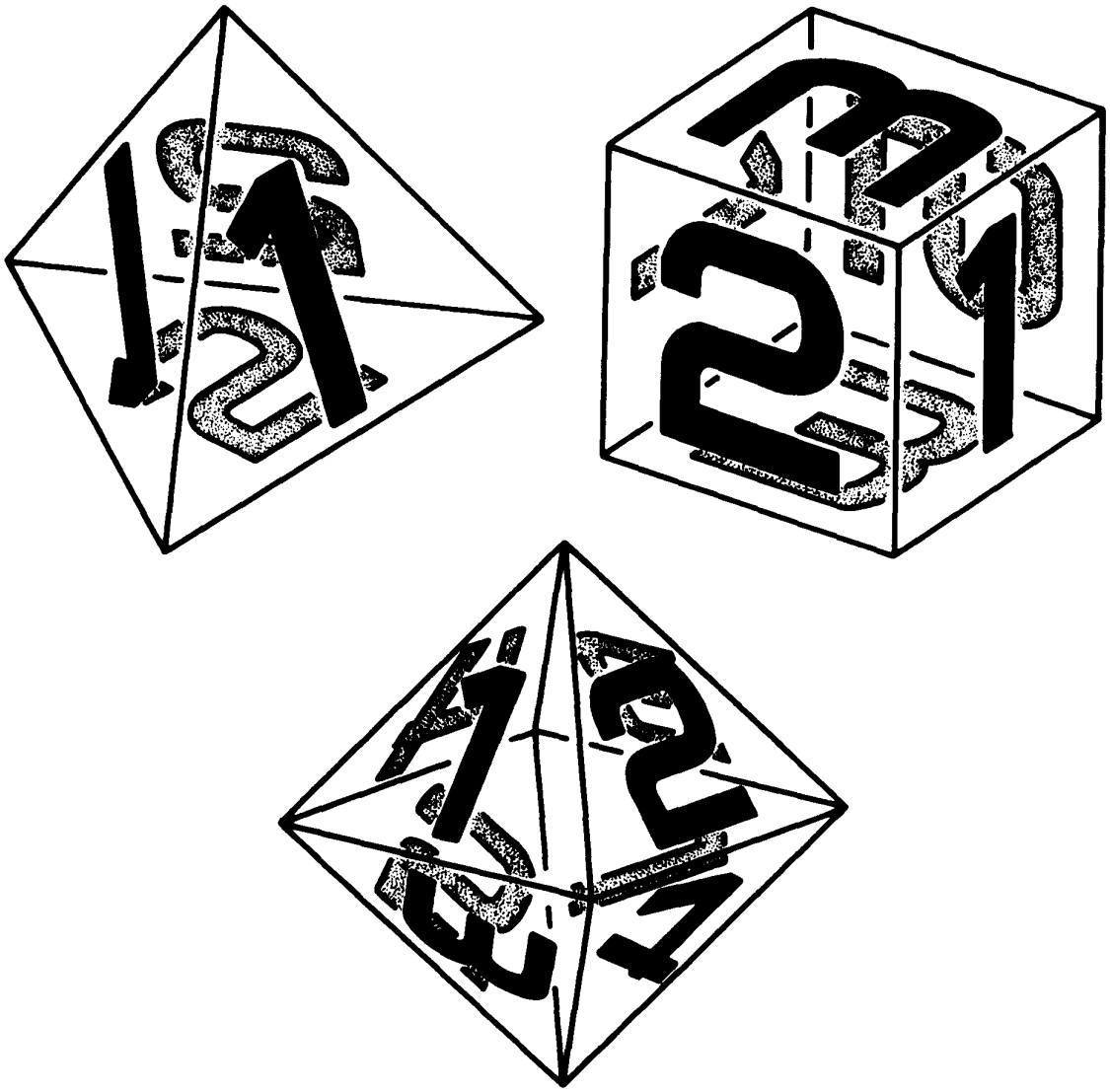


Figure 16.6: The manifolds for Exercises 16.1, 16.2 and 16.3.

Chapter 18. Chapter 19 discusses the nature of the universe, drawing on what you now know about three-manifolds. Chapter 19 does not depend on Chapters 17 or 18, so you can read it immediately if you want.

Exercise 16.1. Consider the three-manifold consisting of a tetrahedron with faces glued as indicated in Figure 16.6(a). How

do the tetrahedron's corners fit together, i.e. how many groups of how many corners each? (Hint: Start in one corner and see which of the other three corners you can reach by passing through a face.) Do the corners have to expand or shrink to fit properly? What homogeneous geometry does this manifold admit? □

Exercise 16.2. The **quaternionic manifold** is a cube with each face glued to the opposite face with a one-quarter clockwise turn (Figure 16.6(b)). How do the cube's corners fit together? What homogeneous geometry does this manifold admit? By the way, the manifold's funny name arises from the fact that its symmetries can be modelled in the quaternions, a number system like the complex numbers but with three imaginary quantities instead of just one. □

Exercise 16.3. The **octahedral space** is an octahedron with each face glued to the opposite face with a one-sixth clockwise turn (Figure 16.6(c)). Find a homogeneous geometry for the octahedral space. (This exercise is a little harder than the preceding two. Even after you figure out how the corners fit, it's still not obvious whether they are too fat, too skinny, or just right. You can work it out by elementary means, but you have to get your hands dirty.) □

17
Bundles

A cylinder is the product of an interval and a circle because it is both an interval of circles and a circle of intervals (Figure 17.1). (For a review of products, see Chapter 6.)

A Möbius band is also a circle of intervals (Figure 17.2), but it fails to be an interval of circles. It is almost a product, but not quite. It therefore qualifies as an interval bundle over a circle. In general a **bundle** over a circle is a bunch of things smoothly arranged in a circle, whether or not they form a product. For example, the quarter turn manifold from Exercise 7.3 is a torus bundle over a circle. Figure 17.3 reviews the construction of the quarter turn manifold: the front and back, and left and right, faces of a cube are glued in the straightforward way, but the top is glued to the bottom with a quarter turn. Figure 17.4 will help you understand the manifold's global topology. Every flat three-manifold in Chapter 7 was either a torus bundle over a circle

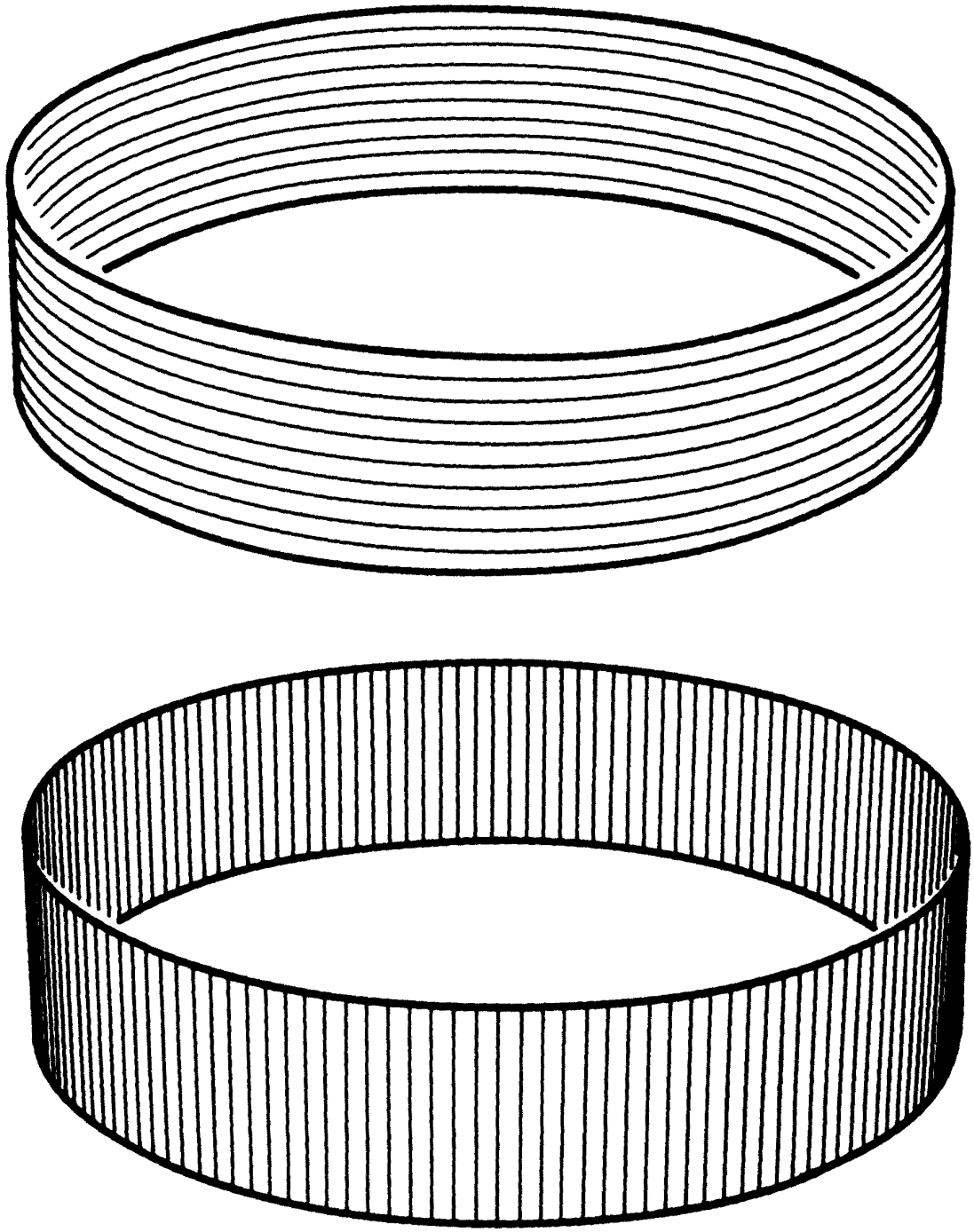


Figure 17.1: A cylinder is both an interval of circles and a circle of intervals.

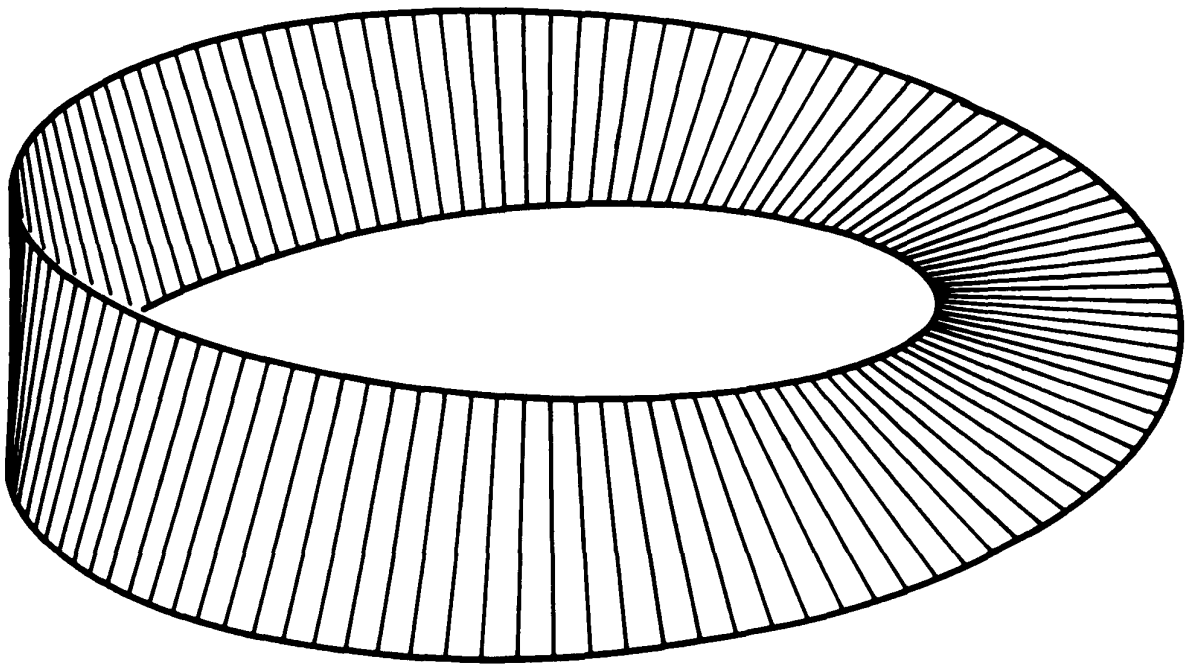


Figure 17.2: A Möbius band is almost the product of an interval and a circle.

(T^2-bundle over S^1) or a Klein bottle bundle over a circle (K^2-bundle over S^1).

Exercise 17.1. The 1/3 turn manifold, introduced in Exercise 7.13, is a (hexagonal) torus bundle over a circle. Draw a picture of it analogous to the picture of the 1/4 turn manifold in Figure 17.4. Do the same for the 1/6 turn manifold. □

The 1/4 turn manifold, the 1/3 turn manifold and the 1/6 turn manifold can each be represented as a circle of tori in three-dimensional space, as in Figure 17.4. On the other hand, when the top of a cube or prism is glued to the bottom with a side-to-side

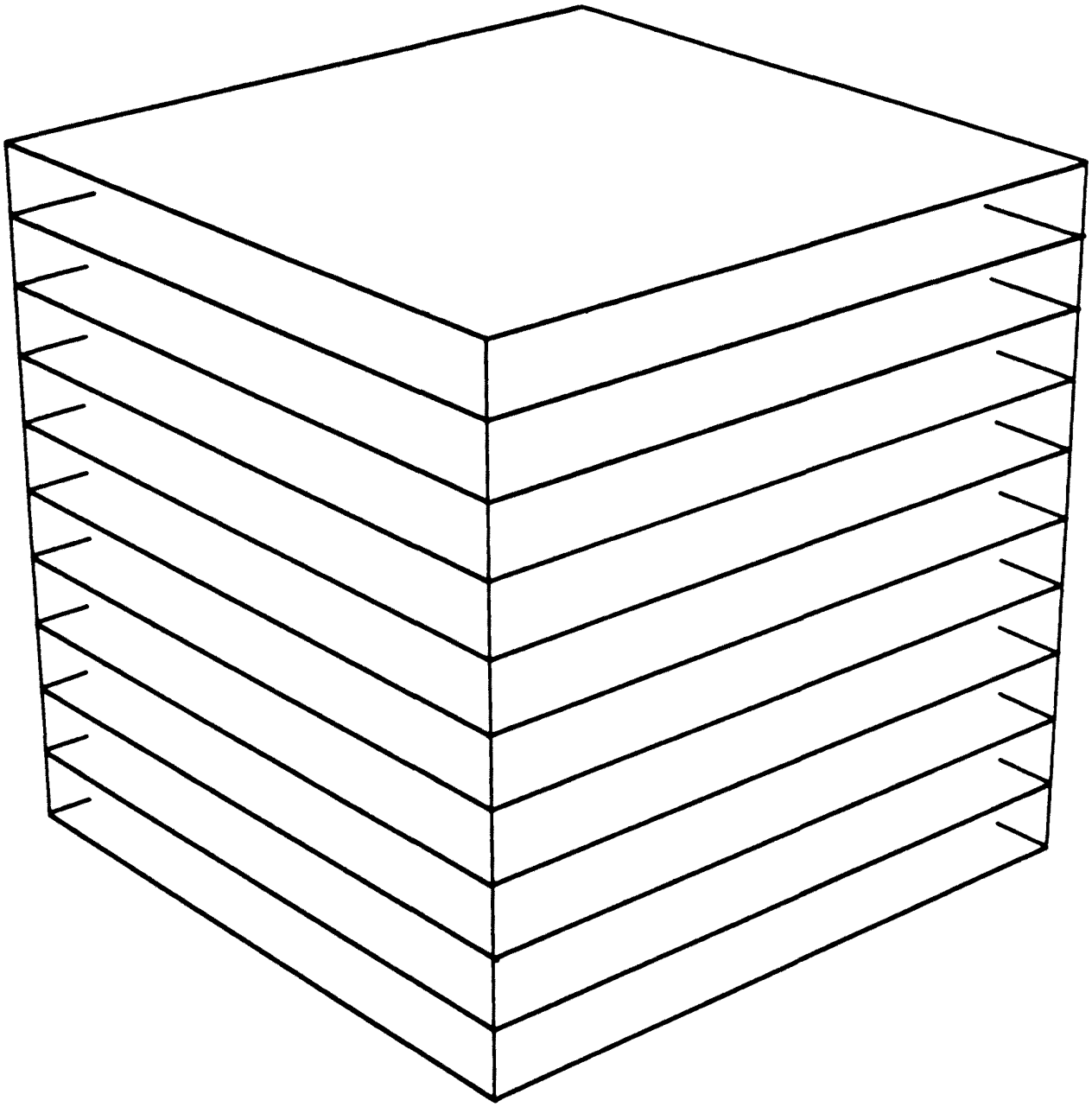


Figure 17.3: Opposite sides of the cube are glued in the straightforward way, but the top is glued to the bottom with a quarter turn. Each horizontal layer forms a torus.

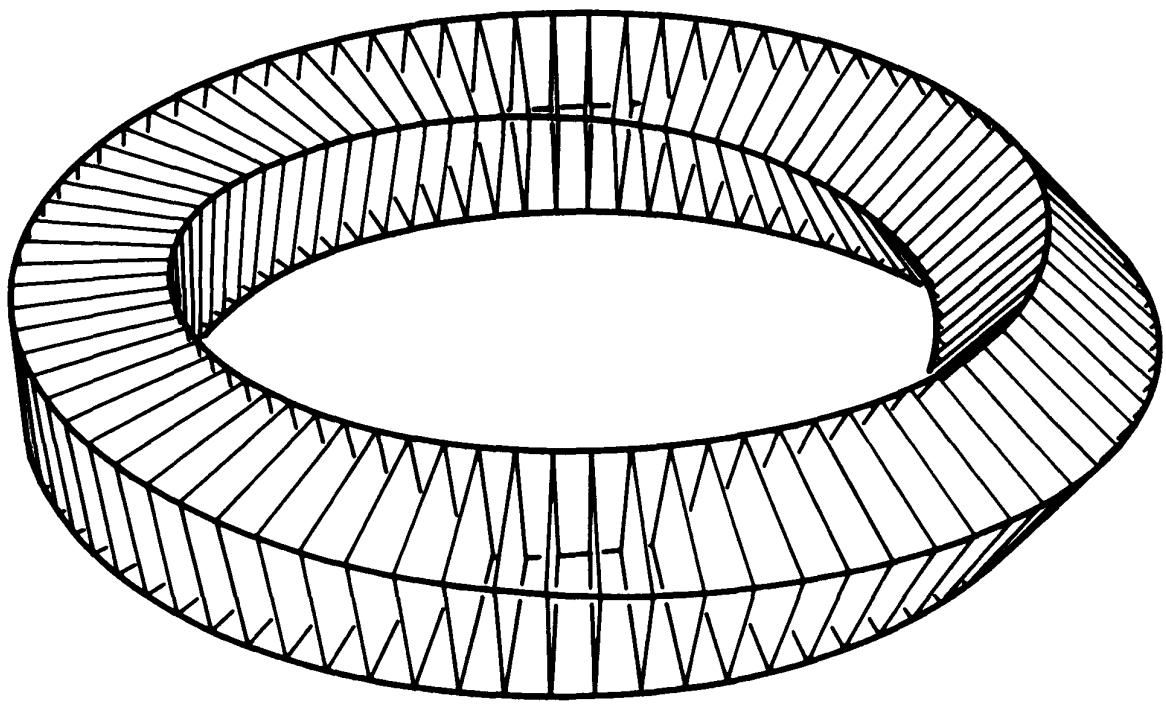


Figure 17.4: If you physically glue a cube's top to its bottom with a quarter turn you'll get a solid like this. Technically the solid has only one side which wraps around four times. But locally it has four sides. If you glue each side to its opposite you convert each square cross-section into a torus. Topologically you get the quarter-turn manifold, a circle of tori with a quarter turn in it. Note that the quarter turn is a global property of the manifold, and has nothing to do with any particular cross-section. Compare this example to the Möbius strip. Technically the Mobius strip has one edge which wraps around twice, but locally it has two edges. What surface do you get when you glue together opposite edges of the Möbius strip?

flip, then you cannot physically carry out the gluing in three-dimensional space to get a picture like Figure 17.4 and you must fall back to a picture like Figure 17.3 to understand the bundle more abstractly.

Exercise 17.2. $K^2 \times S^1$ is both a T^2-bundle over S^1 *and* a K^2-bundle over S^1. Draw one picture representing it as a circle of

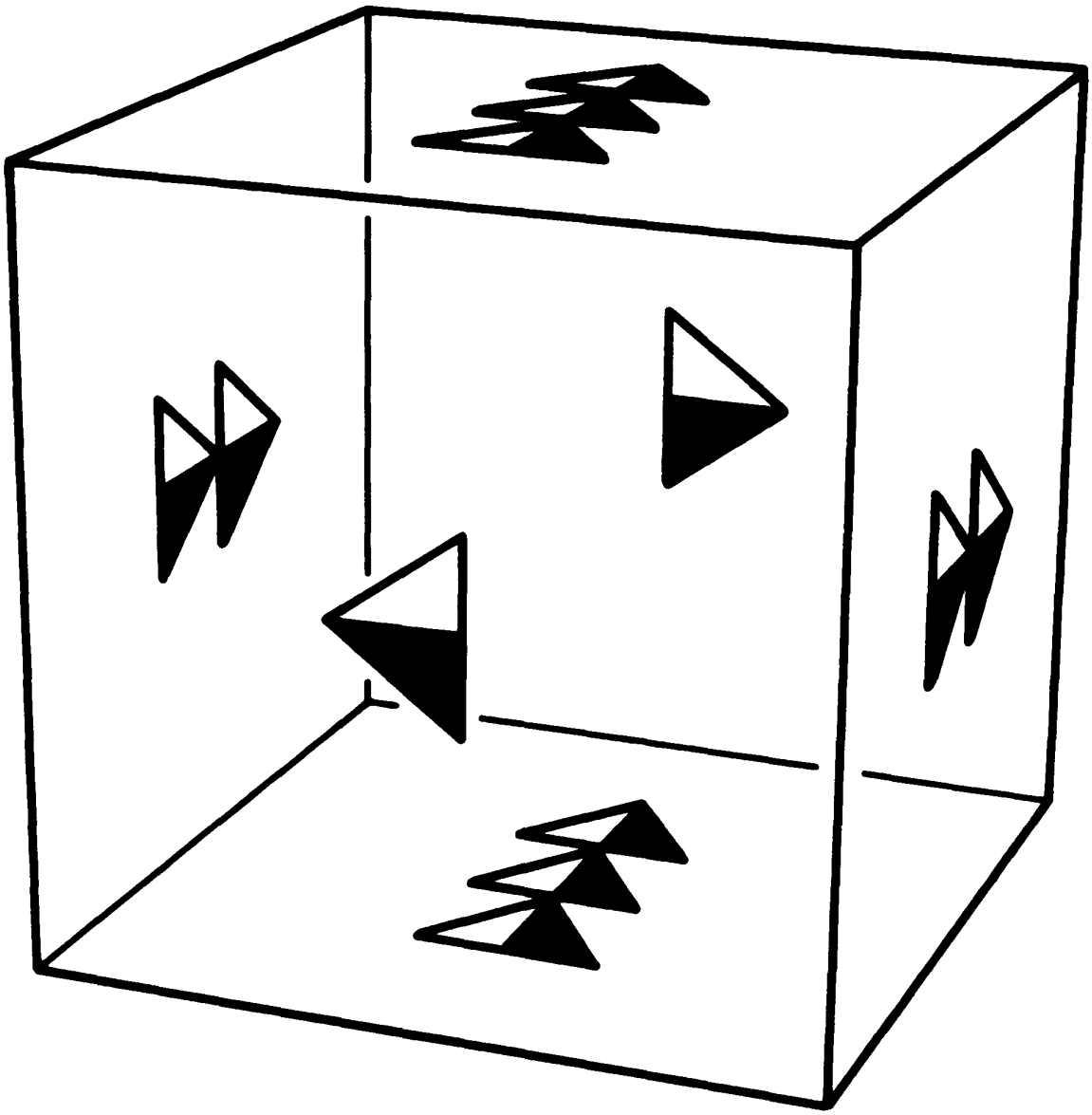


Figure 17.5: To construct $K^2 \times S^1$ you glue the cube's top to its bottom, and its left side to its right side, in the straightforward way, but you glue its front to its back with a side-to-side flip.

tori, and another representing it as a circle of Klein bottles. Your pictures should be analogous to Figure 17.3 rather than 17.4. (You can draw a picture analogous to Figure 17.4 for the K^2-bundle but not for the T^2-bundle.) Figure 17.5 reviews the construction of $K^2 \times S^1$. □

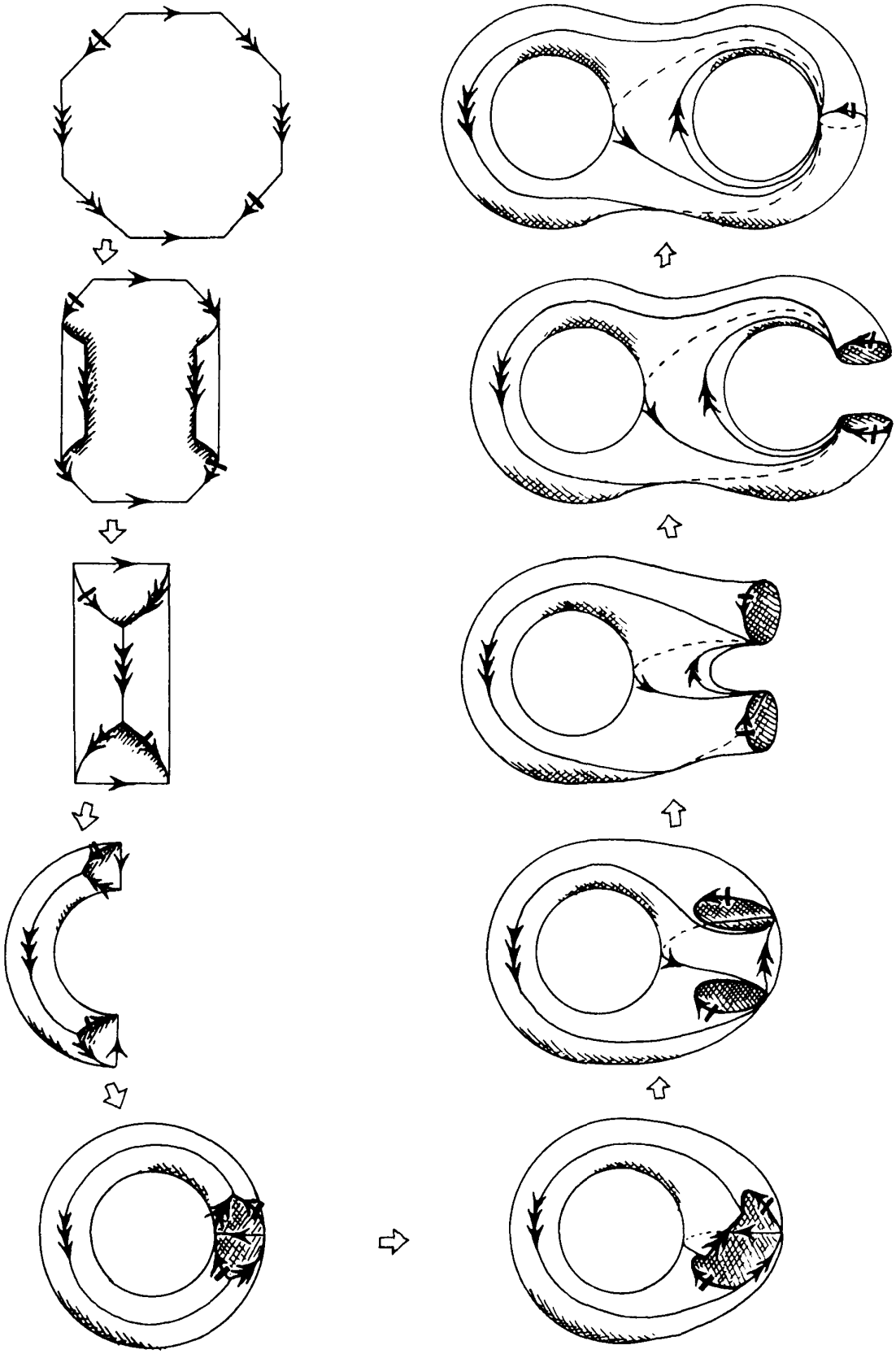


Figure 17.6: If you physically glue together opposite edges of an octagon, you'll get a two-holed doughnut surface.

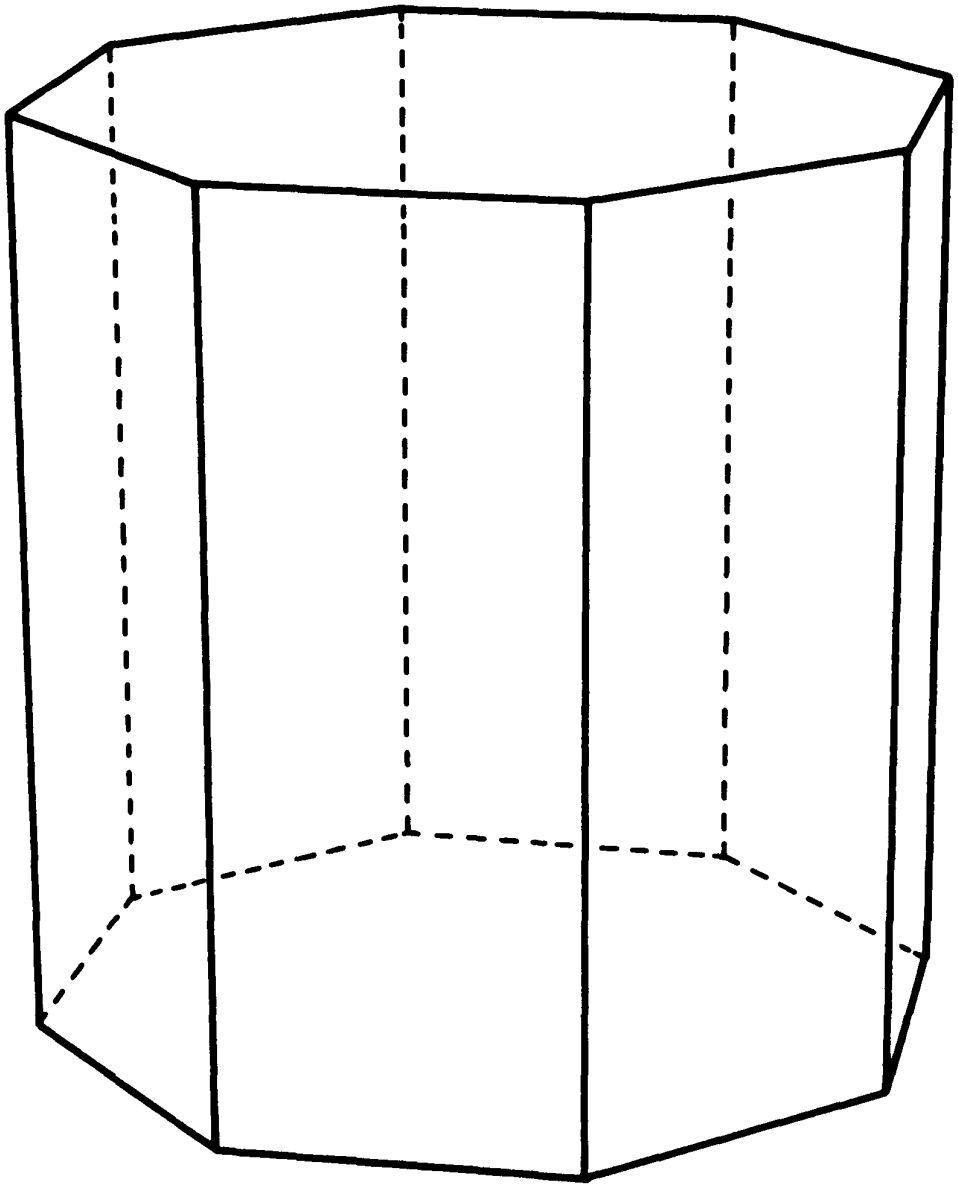


Figure 17.7: An octagonal prism with opposite sides glued is a two-holed doughnut surface cross an interval.

Exercise 17.3. An octagon with opposite edges glued is topologically a two-holed doughnut surface (Figure 17.6). Therefore gluing opposite side faces of an octagonal prism (Figure 17.7) produces a two-holed doughnut surface cross an interval, i.e.

$(T^2 \# T^2) \times I$. Describe several ways in which you can glue the prism's top to its bottom to make a two-holed torus bundle over a circle. □

Each of the following exercises involves a bundle which is difficult to draw in three-dimensional space, but can be described fairly easily by gluings. If you get stuck, look up the answers.

Exercise 17.4. Name two surfaces which are circle bundles over circles. (One is a product and one isn't.) □

Exercise 17.5. A solid doughnut is topologically a disk cross a circle ($D^2 \times S^1$); its surface is topologically a torus. Describe how you would construct a "**solid Klein bottle**", a disk bundle over a circle whose boundary is a Klein bottle.

Exercise 17.6. Describe two ways in which $S^2 \times I$ can be glued up to make an S^2-bundle over S^1. One of these bundles is orientable (it's $S^2 \times S^1$). The other one is nonorientable--in fact, it's the three-dimensional analog of a Klein bottle, so we'll denote it K^3.

What would a "solid" $S^2 \times S^1$ and a "solid" K^3 be like? I put "solid" in quotes here because $S^2 \times S^1$ and K^3 are already three-dimensional (= solid); the manifolds referred to are four-dimensional. Perhaps "hypersolid" would be more accurate? □

Exercise 17.7. Which of the flat manifolds of Chapter 7 are K^2-bundles over S^1? (There are four of them lurking in that chapter.) □

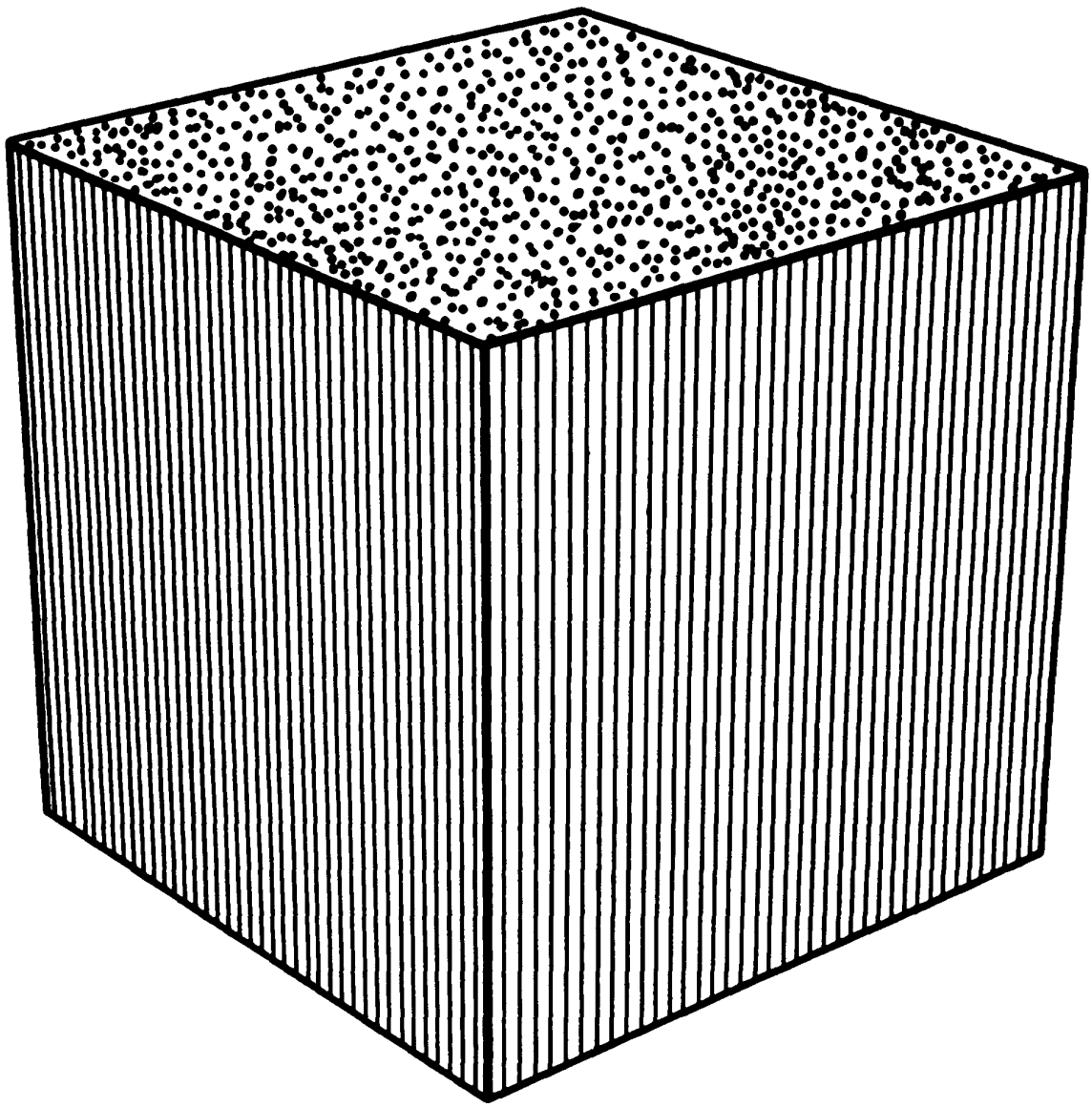


Figure 17.8: A cube filled with spaghetti standing on end.

So far we've concentrated on surface bundles over circles. Now we'll switch things around and look at circle bundles over surfaces. Here's how to construct a simple example of one.

Start by packing together lots of spaghetti into the shape of a cube--the spaghetti should all be standing on end as in Figure

17.8. Thus, mathematically speaking we have a square of intervals. Glue the top of the cube to the bottom. Intrinsically this changes each piece of spaghetti from an interval into a circle, so we now have a square of circles. Glue the cube's sides together in the standard way, so that the circles, rather than being arranged in a square, are now arranged in a torus configuration. Voilà--a circle bundle over a torus! (The underlying three-manifold is, of course, just our old friend T^3.)

Exercise 17.8. Construct a circle bundle over $T^2 \# T^2$. Start with spaghetti packed in the shape of an octagonal prism (Figure 17.7). The resulting three-manifold will be $(T^2 \# T^2) \times S^1$. □

When we draw a picture of a circle bundle over a surface we usually won't draw in the vertical pieces of spaghetti--they tend to clutter the picture and obscure whatever else is going on. But they're there. And even if nothing is said explicitly, the top of the cube or prism will *always* be glued to the bottom, so as to convert each piece of spaghetti into a circle.

Here's a different circle bundle over a torus. Start with the cube of Figure 17.8 and glue the top to the bottom, and the left side to the right side, in the straightforward way, but glue the front to the back with a top-to-bottom flip. You'll get a torus of circles, but the circles will connect up in a strange way. (If you pass through the right face of the cube and return from the left

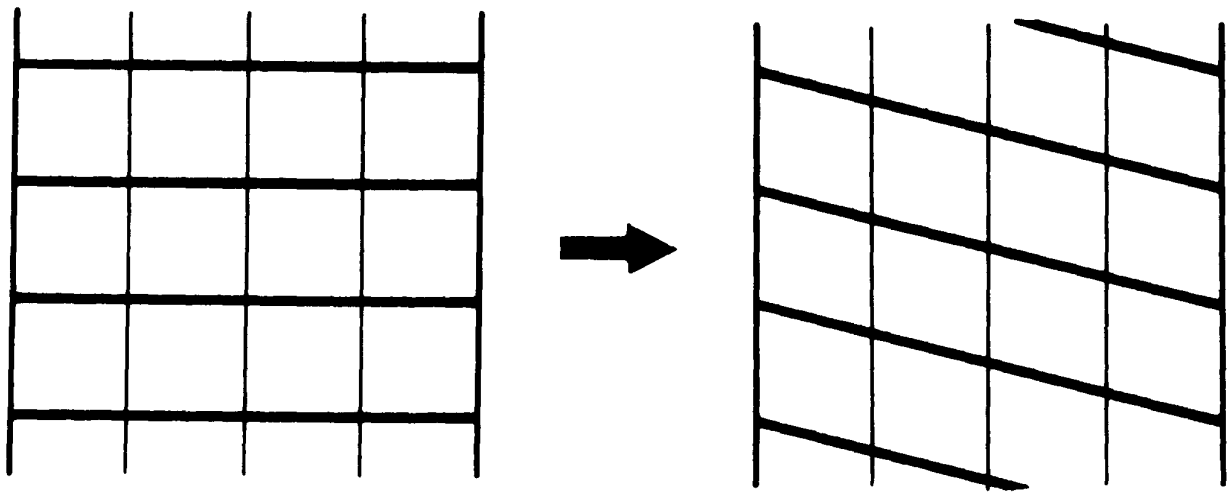


Figure 17.9: A shear.

you'll find that the circles connect up normally, but if you pass through the back face and return from the front face you'll find they connect up with a flip.) The underlying three-manifold in this case is nonorientable--it's $K^2 \times S^1$.

Exercise 17.9. Can you find examples of orientable and/or nonorientable circle bundles over the Klein bottle? (Hint: The manifolds of Figure 8.2 will come in handy.) □

Now for a really weird example! The following circle bundle over a torus, which we'll call a **twisted (three-dimensional) torus**, will play a crucial role in the theory of geometries on three-manifolds (Chapter 18).

Start with a cube as before, and, of course, glue the top to the bottom to convert the vertical intervals into circles. The sides will

be glued not in any of the usual ways, but with a "shear" (Figure 17.9). By this we mean that each vertical circle gets glued to the corresponding vertical circle on the opposite side of the cube, only they get slid up or down in such a way that a horizontal line segment gets tilted. Note that for this shearing to work, it's imperative that the top of the cube be glued to the bottom. To make the twisted torus, glue opposite sides of the cube with a shear as prescribed in Figure 17.10.

This is a funny sort of gluing, and it's not at all clear that what we get is even a manifold. The following exercise deals with this (potential) problem.

Exercise 17.10. In an attempt to construct a circle bundle over a torus, the opposite faces of the cube in Figure 17.11 are glued so that the tilted lines match up. Investigate how the four vertical edges fit together. You'll discover a problem. Now check that this problem doesn't arise when the four edges come together to form the twisted torus. Conclude that the twisted torus is a bona fide three-manifold and is, in fact, a circle bundle over T^2. □

So the twisted torus is a legitimate three-manifold. But is it really something new, or is it merely a distorted representation of good old T^3? If you try looking for any sort of horizontal cross-section in the twisted torus, you'll quickly become convinced that this manifold definitely *isn't* T^3.

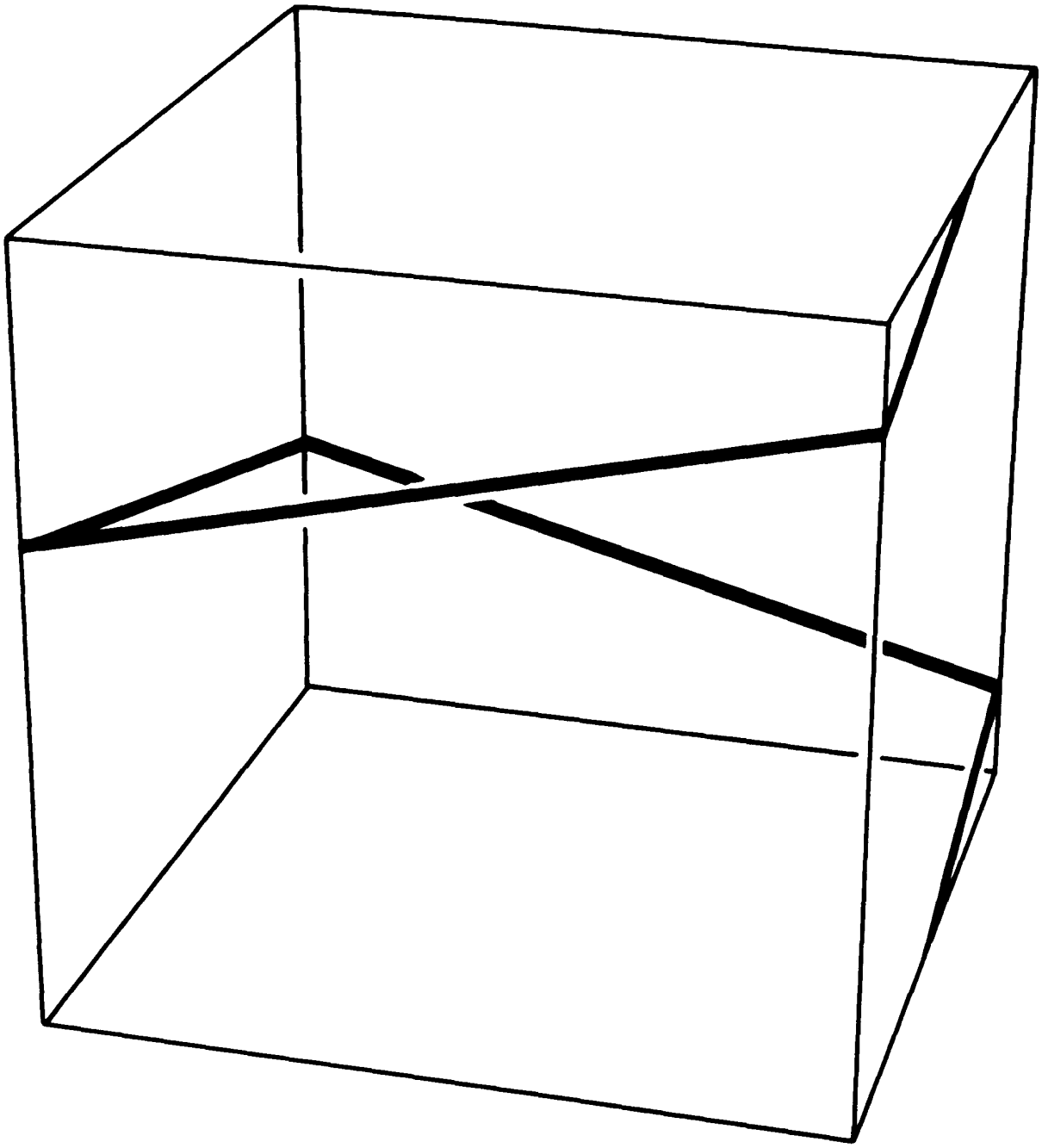


Figure 17.10: To make a twisted torus, glue the cube's opposite sides with a shear so that the tilted line segments match up. (Note: The apparently broken line segment on the right side stops being broken when the cube's top is glued to its bottom.)

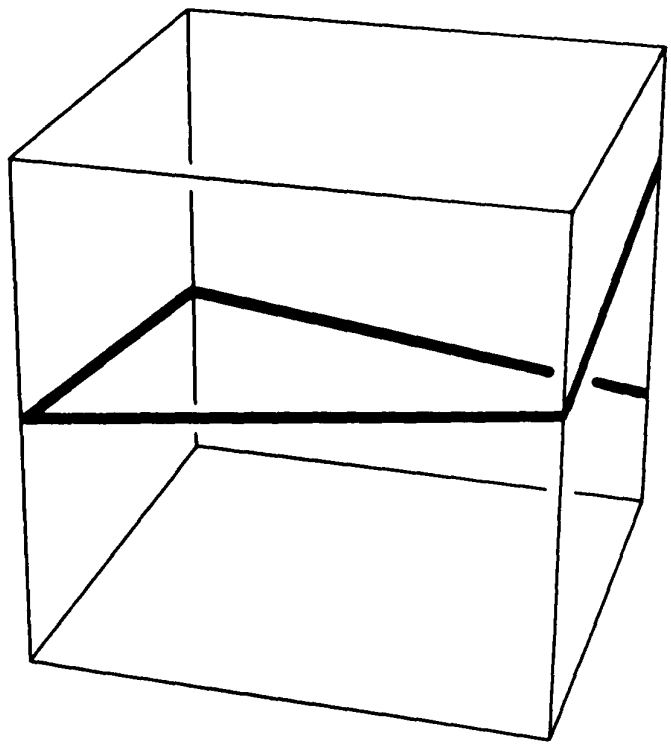


Figure 17.11: Here opposite faces are glued with only half as much shear as in the twisted torus.

Exercise 17.11. Construct a circle bundle over a torus that's twice as twisted as the twisted torus described above. □

Exercise 17.12. Construct a twisted circle bundle over $T^2 \# T^2$. □

Exercise 17.13. We constructed the twisted torus as a circle bundle over T^2. Is it also a T^2-bundle over a circle? □

18

Geometries on Three-Manifolds II

In Chapter 11 we found that every surface admits one of the three homogeneous two-dimensional geometries. The sphere and the projective plane admit elliptic geometry, the torus and the Klein bottle admit Euclidean geometry, and all other surfaces admit hyperbolic geometry. In Chapters 14, 15 and 16 we discussed the analogous three-dimensional geometries, and found some examples of three-manifolds which admit them. It turns out that five more homogeneous geometries arise in the study of closed three-manifolds. One of them is the local geometry of $S^2 \times E$. ($S^2 \times E$, $S^2 \times I$ and $S^2 \times S^1$ all have the same local geometry. It's traditional to name the geometry after $S^2 \times E$ because $S^2 \times E$ is the "biggest" manifold having it.) This geometry is homogeneous but not isotropic. It's homogeneous because it's everywhere the same. But it's not isotropic because at any given point we can distinguish some directions from others. Recall from Figure 6.10 that some

cross-sections of $S^2 \times S^1$ are spheres while others are flat tori. Locally one observes that some two-dimensional slices have positive curvature while others have zero curvature (Figure 18.1). The term "**sectional curvature**" refers to the curvature of a two-dimensional slice of a manifold. The word "section" comes from the latin "sectio" which means "slice" (more or less). Thus $S^2 \times S^1$ has positive sectional curvature in the horizontal direction, but zero sectional curvature in any vertical slice. An isotropic geometry has the same sectional curvature in all directions; the sectional curvatures of three-dimensional elliptic geometry are all positive, those of three-dimensional Euclidean geometry are all zero, and those of three-dimensional hyperbolic geometry are all negative.

Exercise 18.1. Back in Exercise 6.7 you found that $P^2 \times S^1$ has $S^2 \times E$ geometry. Name another nonorientable manifold with this geometry. (Hint: It first appeared in Chapter 17.) □

The geometry of $H^2 \times E$ is also homogeneous but not isotropic. Like $S^2 \times E$, it has different sectional curvatures in different directions. Vertical slices have zero curvature, while horizontal slices have negative curvature. Figure 18.2 provides a rough illustration of $H^2 \times E$ geometry.

Exercise 18.2. Name several manifolds with $H^2 \times E$ geometry. □

I should mention in passing that the *only* homogeneous two-dimensional geometries are elliptic, Euclidean and hyperbolic

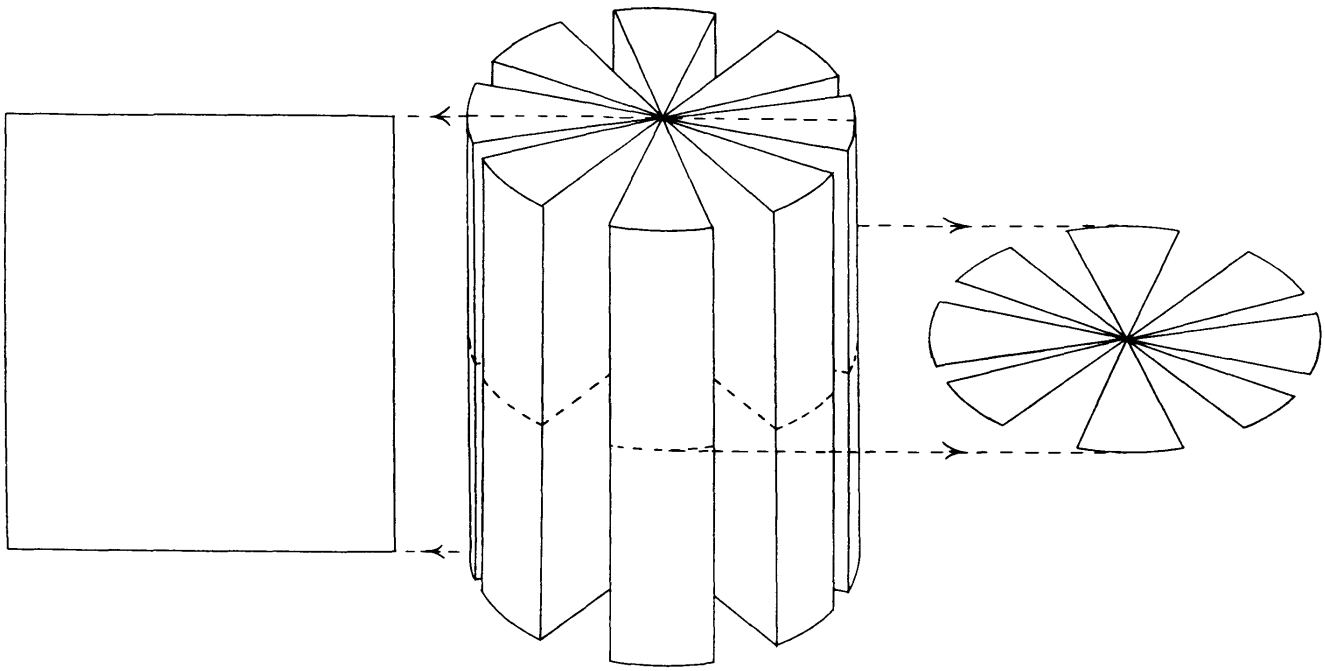


Figure 18.1: A vertical cross-section of $S^2 \times S^1$ has Euclidean geometry but a horizontal cross-section has elliptic geometry. A piece of $S^2 \times S^1$ splits vertically in Euclidean space if not allowed to stretch, but it doesn't split horizontally.

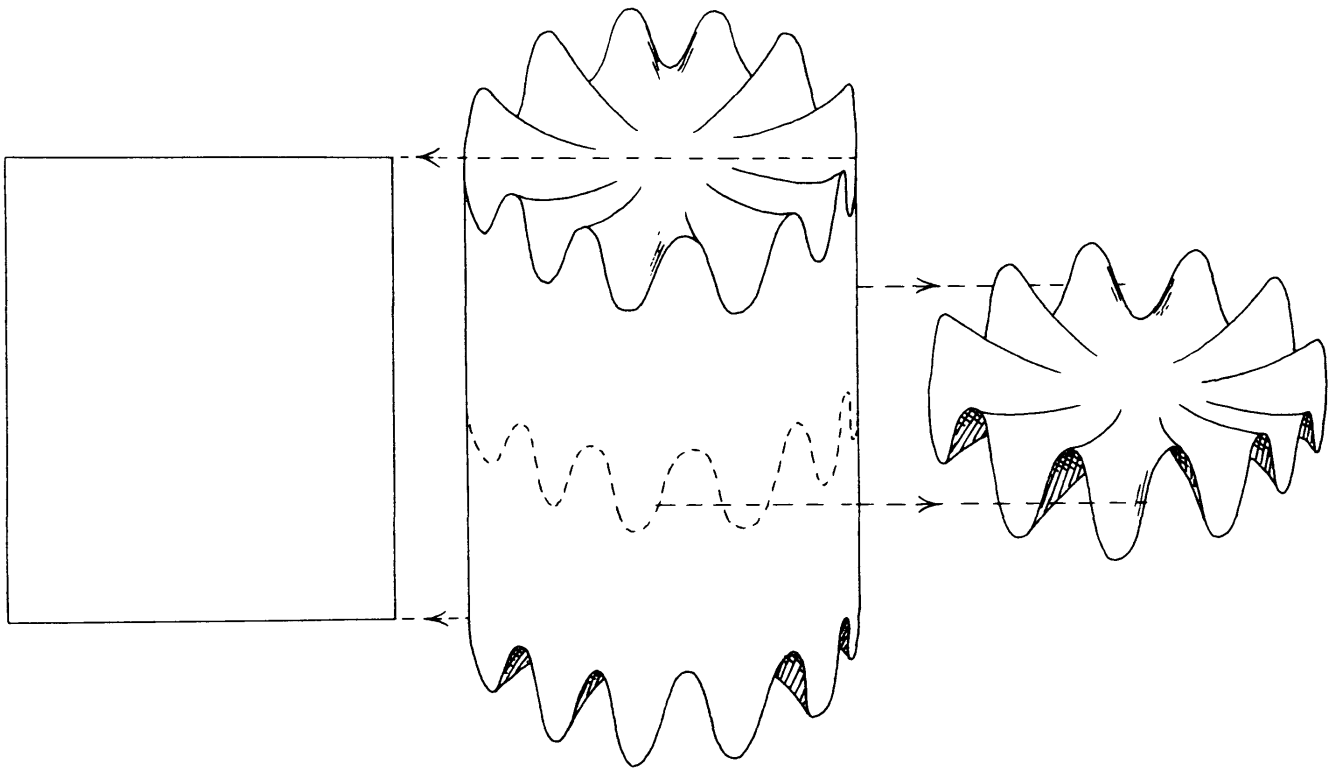


Figure 18.2: A vertical cross-section of $H^2 \times S^1$ has Euclidean geometry but a horizontal cross-section has hyperbolic geometry. By the way, the horizontal slices should really be stacked up in a different dimension than the one they wrinkle into, but this picture is the best we can do with only three dimensions available.

geometry, each of which happens to be isotropic. Geometries which are homogeneous but not isotropic occur only in manifolds of three or more dimensions.

The contemporary theory of three-manifolds deals with homogeneous geometries, without regard to isotropy. Isotropy becomes important only when one applies the mathematical theory to the study of the real universe, which appears isotropic according to current observational data. The present chapter explores homogeneous geometries in general. The cosmological applications of isotropic geometries will be treated in Chapter 19.

A couple technical points are in order before we move on to a list of the eight homogeneous geometries.

Technical Point #1: When I say "geometry" in this chapter I really mean "class of geometries". For example, the geometry of a two-sphere of radius three is, strictly speaking, different than the geometry of a two-sphere of radius seventeen; yet they are similar enough that they're included in the same class of two-dimensional geometries. Three-dimensional geometries admit more variation within each class. We'll see an example of this later in the chapter.

Technical Point #2: Really there are more than eight classes of homogeneous geometries. In fact, there are infinitely many. The catch is that only eight of them occur as homogeneous geometries of

closed three-manifolds. (As you may have noticed, this book is heavily biased towards closed manifolds!)

Here's a list of the eight homogeneous geometries, along with some sample manifolds having each one.

(1) **Elliptic Geometry.** The geometry of S^3.

Sample Elliptic Manifolds: The three-sphere, projective three-space, the Poincaré dodecahedral space.

(2) **Euclidean Geometry.** The geometry of childhood.

Complete List of Euclidean Manifolds: Altogether there are only ten topologically different Euclidean three-manifolds! Six are orientable and four are nonorientable. The nonorientable ones are $K^2 \times S^1$ and the three manifolds of Exercise 7.4; all are K^2-bundles over S^1. The orientable Euclidean manifolds are the three-torus, the quarter turn manifold, the half turn manifold, the one-sixth turn manifold, the one-third turn manifold, and another manifold we haven't seen. The hexagonal three-torus is not included because it's topologically the same as the ordinary three-torus.

(3) **Hyperbolic Geometry.** See Chapter 15 for a description.

Sample Hyperbolic Manifold: Hyperbolic geometry is somewhat enigmatic. So far we've seen only one three-manifold which has it,

namely the Seifert-Weber dodecahedral space. Yet recent research by Bill Thurston suggests that three-dimensional hyperbolic geometry is by far the most common geometry for three-manifolds, just as two-dimensional hyperbolic geometry is the most common geometry for surfaces. If hyperbolic geometry is so common, why haven't we seen more three-manifolds which have it? The reason is that the easiest manifolds to study are not the typical ones, but rather the ones with special symmetry. The first surfaces for which we found geometries were the two simplest ones, the sphere and the torus. Their homogeneous geometries (elliptic and Euclidean, respectively) are atypical precisely because these surfaces are so simple. Not until Chapter 11 did we discover that most surfaces admit hyperbolic geometry. This same phenomenon crops up in the study of three-manifolds. All the simple manifolds admit atypical geometries by virtue of their simplicity. It's the typical--but less simple--manifolds which admit hyperbolic geometry.

(4) S^2 x E Geometry. This geometry was described in Chapter 6 and at the beginning of the present chapter.

Complete List of S^2 x E Manifolds: There are only four manifolds with this geometry. They are $S^2 \times S^1$, K^3, $P^2 \times S^1$, and one other manifold. This last manifold is made from $S^2 \times I$, but in this case each end is glued only to itself! Specifically, every point on an end gets glued to its antipodal point *on the same end;* the gluing resembles the gluing used to turn a ball into P^3. Note that $S^2 \times I$'s intrinsic geometry matches up nicely at the resulting "seams"; if

you are confused, think about how the geometry matches up when you glue antipodal points on each end of a cylinder.

Exercise 18.3. Which of the four $S^2 \times E$ manifolds are orientable? □

(5) H^2 x E Geometry. This geometry was discussed earlier in the chapter.

Sample H^2 x E Manifolds: Here we have a little more variety than in the case of $S^2 \times E$. To begin with, any surface cross a circle will admit $H^2 \times E$ geometry, just so long as the surface *isn't* S^2, P^2, T^2 or K^2. Many other surface bundles work too. For example, to make a $(T^2 \# T^2)$-bundle with $H^2 \times E$ geometry, start with a two-holed doughnut cross an interval as described in Figure 17.7, and glue the top to the bottom with either a 1/8 turn, a 1/4 turn, a 3/8 turn, a half turn, or one of two possible reflections.

(6) Twisted Euclidean Geometry. Figure 17.8 suggests visualizing Euclidean geometry as a bundle of vertical lines; i.e. one thinks of Euclidean space as $E^2 \times E$. Twisted Euclidean geometry may also be thought of as a bundle of vertical lines, only now the lines are bundled together in a strange new way. If you take a trip in a twisted Euclidean manifold, always travelling "horizontally" ("horizontal" means perpendicular to the vertical lines), you'll find that when you return to the line you started on you're some distance above or below your starting point! See Figure 18.3. You'll be above your starting point if you traversed a

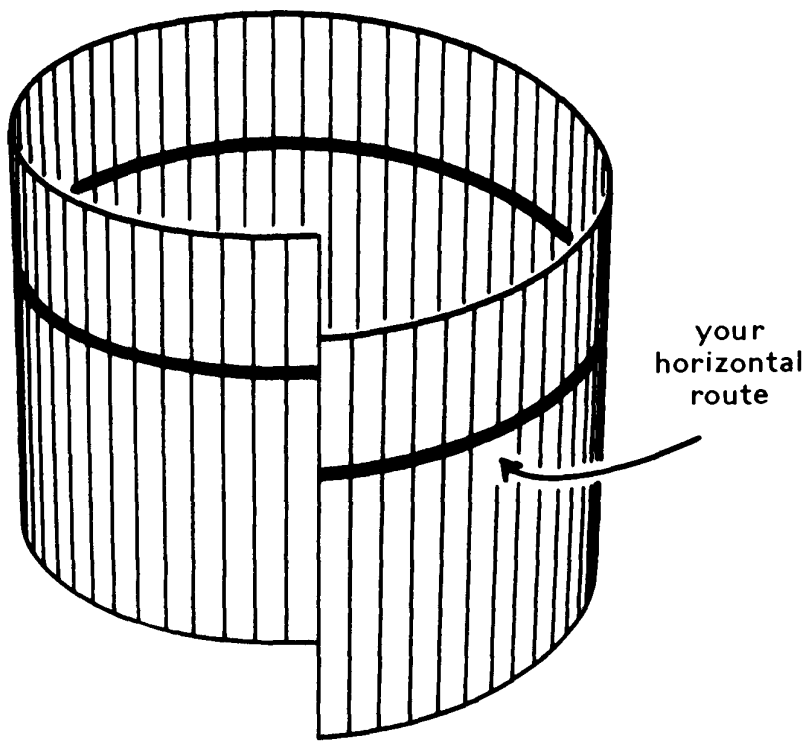


Figure 18.3: If you travel horizontally in a twisted Euclidean manifold you'll come back to a point above or below where you started.

counterclockwise loop, but you'll be below it if you went clockwise. Your exact distance above or below your starting point will be proportional to the area your route encloses.

Figure 18.4 shows a jungle gym in a twisted Euclidean manifold. All the bars have the same length, and they all meet at right angles. Some bars *appear* tilted because the artist had to distort the twisted Euclidean jungle gym to draw it in ordinary Euclidean space.

Sample Twisted Euclidean Manifolds: You can give the twisted torus of Chapter 17 a twisted Euclidean geometry. First put a

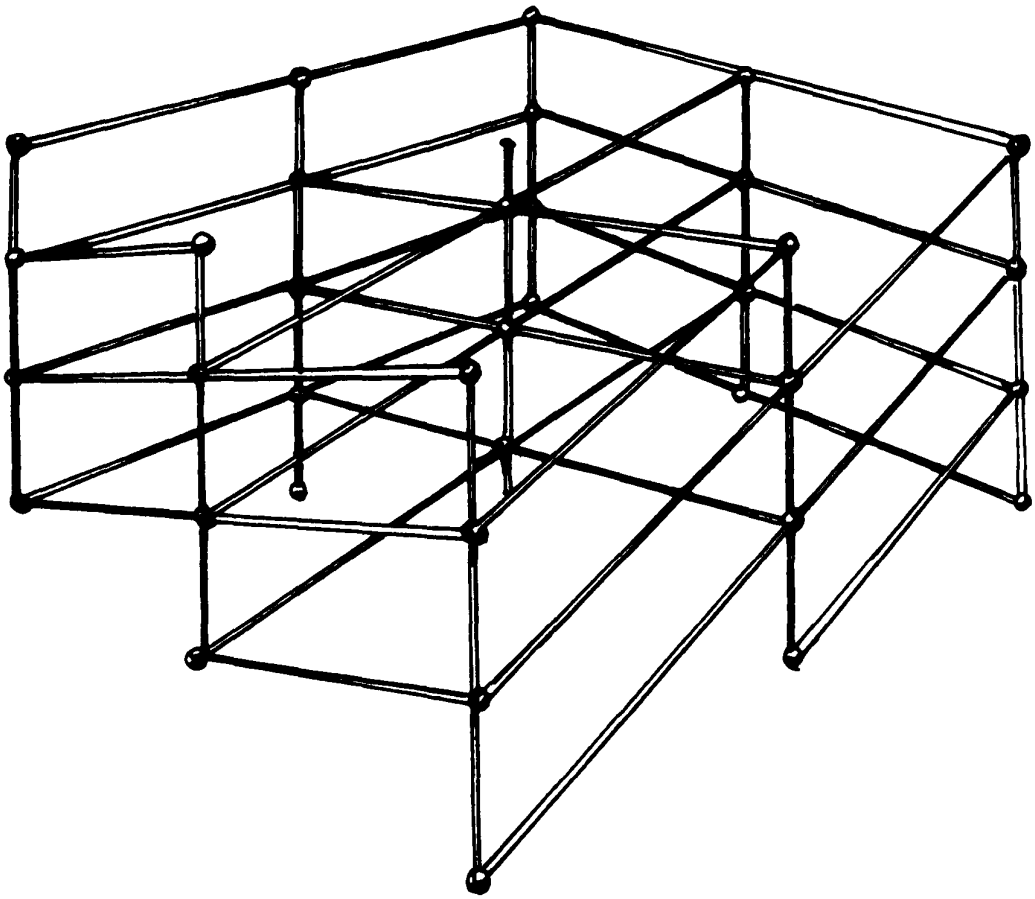


Figure 18.4: A jungle gym in a twisted Euclidean manifold. (Drawing by Bill Thurston.)

twisted Euclidean geometry on the cube of Figure 17.10. With this new geometry the previously tilted lines on the sides of the cube become intrinsically horizontal. When you glue opposite sides of the cube with a shear, you glue the vertical lines on one side to the vertical lines on the other, and the horizontal lines on one side to the horizontal lines on the other. Thus, in terms of the twisted Euclidean geometry you're gluing one side rigidly to the other, with no "shearing" or other abnormalities. The corners and edges still

fit fine, so you've given the twisted torus a twisted Euclidean geometry.

The doubly twisted torus of Exercise 17.11 also admits twisted Euclidean geometry. In fact *every* circle bundle over a torus or Klein bottle admits either Euclidean geometry or twisted Euclidean geometry.

Exercise 18.4. Can a nonorientable three-manifold have a twisted Euclidean geometry? □

(7) Twisted H^2 x E Geometry. This geometry bears the same relation to standard $H^2 \times E$ geometry as twisted Euclidean geometry does to standard Euclidean geometry. Specifically, one thinks of both the twisted and standard $H^2 \times E$ geometries as vertical line bundles over H^2. In the twisted case the lines connect up as in Figure 18.3.

Sample Twisted H^2 x E Manifolds: Any circle bundle over any surface except S^2, P^2, T^2 or K^2 admits either standard or twisted $H^2 \times E$ geometry.

Exercise 18.5. Give explicit instructions for constructing a sample twisted $H^2 \times E$ manifold. □

(Twisted S^2 x E Geometry) Amazingly enough, if you put the right amount of twist into $S^2 \times E$ you'll get elliptic geometry! The amount of twist is right when a traveller travelling horizontally around a region of area x returns to a point x units below where she started

(here I assume that $S^2 \times E$ is a bundle over a *unit* two-sphere). Even when the amount of twist is wrong this geometry is still classified as elliptic geometry, as per Technical Note #1, because its group of symmetries is contained in the group of symmetries of the three-sphere. Note, though, that you can always adjust the twist to the right value by stretching or compressing the vertical lines.

(8) Solve Geometry. This is the real weirdo. Unlike the previous geometries, solve geometry isn't even rotationally symmetric. I don't know any good intrinsic way to understand it. (The name "solve" geometry has to do with "solvable Lie groups".)

Sample Solve Manifolds: Most torus bundles over S^1 admit solve geometry. (None of the ones we've seen in this book do, though, because none of them distort the geometry of the two-dimensional cross-section.)

It would be nice if every three-manifold admitted one of the above homogeneous geometries. Alas, this is not the case. For example, a connected sum of two three-manifolds *never* admits a homogeneous geometry (unless either (a) one of the original manifolds is S^3, or (b) both original manifolds are P^3). Fortunately the situation isn't quite as bad as it sounds. Thurston's work suggests that most three-manifolds admit hyperbolic geometry, and those that don't either admit one of the other seven homogeneous geometries, or can be cut into pieces

which admit homogeneous geometries. (In this context one "cuts a manifold into pieces" by cutting it along spheres--in effect undoing a connected sum--and/or by cutting it along tori.) This view of three-manifolds has not yet been proven correct, but there is a good deal of evidence in its favor. For a summary of the evidence, see Thurston's article *Three dimensional manifolds, Kleinian groups and hyperbolic geometry* (Bulletin of the AMS 6(1982), pp. 357-381). An interesting corollary of Thurston's ideas is that a "randomly chosen" three-manifold is unlikely to be a connected sum.

Part IV
The Universe

19

The Universe

The universe has existed for only about 10 billion years. It began in a very hot, dense, contracted state, and has been expanding ever since. This chapter discusses the beginning of the universe (the so-called big bang), the ensuing expansion, the shape of the universe, and the question of whether the universe will someday recollapse, or whether it will keep expanding forever. Experimental evidence suggests that the universe is isotropic. If it has elliptic geometry it will recollapse someday, if it has hyperbolic geometry it will keep expanding forever, and if its geometry is Euclidean it will expand forever, but just barely (the rate of expansion approaches zero). Current evidence hints that the universe has hyperbolic geometry and will expand forever, but no one really knows for sure.

This chapter is organized around the following five questions:

1. What do we know about the universe?

2. In what sense is the universe expanding?
3. What does the geometry of the universe have to do with whether it recollapses or not?
4. Is the universe closed or open? In other words, is space finite or infinite?
5. What came before the big bang?

Question #1: What do we know about the universe?

We live on the Earth, which is one of (at least) nine planets orbiting our sun. The sun is grouped together with about 100 billion other stars in our galaxy the Milky Way. The Milky Way is shaped like a giant disk 50 or 100 thousand light-years across,[3] with us about 25 thousand light-years from the center. If you go out on a clear, moonless night the stars of our galaxy will appear as a splotchy white band running across the sky (hence the name Milky Way). The other stars you normally see are all in our galaxy too.

There are lots of other galaxies in the universe--billions of them, at least. Some are off by themselves, but most of them are in clusters of anywhere from a few to a few thousand galaxies each.

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[3] A light-year is the distance light can travel in one year. By way of comparison, the sun's light takes about 8 minutes to reach us, so the sun is 8 light-minutes away. Similarly, the moon is about 1 light-second away, and San Diego is about .02 light-seconds from Boston.

On the average, galaxies are distributed in space roughly like dimes spaced a meter apart.

A striking feature of these other galaxies is that they are moving away from us! Furthermore, the rate at which they are receding is directly proportional to their distance from us:

$$(\text{recession velocity}) = (H)(\text{distance})$$

This fact is called the Hubble law, and the proportionality constant H is called the Hubble constant in honor of Edwin Hubble, who discovered the expansion of the universe in the 1920's. (In 1917 Einstein's general relativity predicted the expansion of the universe, but this idea was so counter-intuitive that even Einstein himself couldn't accept it.) The Hubble constant, which has undergone many revisions over the years, is currently thought to be in the neighborhood of 15 to 30 (kilometers/second) per million light-years. So, for example, a galaxy located 20 million light-years away should be receding from us at (20) × (15 to 30) = 300 to 600 km/sec. The reason that the Hubble constant is known only approximately is that it's difficult to judge the distance to a far off galaxy. The galaxy's recession velocity, on the other hand, can be determined accurately by seeing how the spectral lines in the light from the galaxy are "redshifted". For details, see Berry's *Principles of Cosmology and Gravitation*, or almost any other astronomy book.

The fact that the other galaxies are moving away from us doesn't, as one might at first suspect, mean that we occupy some

privileged position in the universe. These other galaxies are moving away from each other, too. More on this under Question #2.

The fact that the galaxies are all moving away from each other means that in the past they must have been closer together. If they had always been moving apart at their current speeds, then about 10 or 20 billion years ago they would have all started from the same place! In reality, their mutual gravitational attraction slows them down, so the further you go into the past, the faster the galaxies were receding. The conclusion is that all the matter in the universe came flying apart even more recently than 10 or 20 billion years ago! This was the big bang, the beginning of the universe, which occurred sometime between 8 and 18 billion years ago.

For the first million years or so after the big bang, the matter in the universe was so hot that atoms couldn't form without being smashed apart almost immediately. Radiation, however, was abundant. Theoretical considerations show that this radiation should still fill the universe. And sure enough, in 1965 Penzias and Wilson discovered this "microwave background radiation". Even though the idea of the big bang was already accepted in 1965, the study of this "fossil radiation" gave physicists a more detailed understanding of the early universe than had previously been possible.

Most cosmologists now believe that our universe is both homogeneous and isotropic. Further explanation of these terms is in order:

Homogeneity. This means that any two regions of the universe are basically alike. Of course, we have to look on a sufficiently large scale, so that "local" fluctuations in the number of galaxies get averaged out. The situation is analogous to saying that a roomfull of air is homogeneous, even though one cubic microcentimeter might contain 17 molecules of nitrogen and 4 of oxygen, while a different cubic microcentimeter might contain 8 of nitrogen and 11 of oxygen.

Isotropy. This means that no matter where you are in the universe, things look basically the same in all directions. An isotropic universe is of necessity homogeneous (to see that conditions must be the same at any two locations A and B, note that the universe is isotropic about the point lying halfway between A and B). The real universe has long been thought to be isotropic--and therefore homogeneous--both for aesthetic reasons and because of the observed rough isotropy in the distribution of galaxies and in the value of the Hubble constant. More recently, the microwave background radiation has been measured to be isotropic to within a fraction of a percent, giving even greater support to the hypothesis of isotropy. Of course, we still don't *know* that the universe is isotropic when viewed from distant galaxies, but it seems unlikely that we should have an isotropic vantage point when no one else does. (Those who worry about the

fine points will be relieved to know that, according to general relativity, the isotropy of the Hubble constant together with an assumption of zero (or at least isotropic) pressure in the universe implies that the geometry of space is also isotropic.)

The foregoing evidence explains why astrophysicists and mathematicians put so much emphasis on three-manifolds which are both homogeneous and isotropic. It is fortunate that there are only three different homogeneous, isotropic geometries for us to consider, namely elliptic, Euclidean and hyperbolic geometry. Of course, there are many different three-manifolds having each geometry, and thus many possible global topologies for the universe.

Exercise 19.1. If a galaxy is n million light-years away, then, according to the Hubble law, it is receding from us at a rate of Hn km/sec. If it had always been travelling at this rate, then we could compute the time since it left us as

$$\begin{aligned} \text{time} &= (\text{distance travelled})/\text{rate} \\ &= (n \text{ million light-years})/(Hn \text{ km/sec}) \\ &= 1/H \ \ (\text{million light-years})(\text{sec})/(\text{km}). \end{aligned}$$

This time is known as the Hubble time, and, as pointed out in the text, it's a slight overestimate of the age of the universe because the galaxies are continually slowing down under the braking effect of their own gravity. It's not surprising that this Hubble time doesn't depend on the distance to the galaxy, since a galaxy which

is three times as far away is receding from us three times as fast. Compute the Hubble time in years, given that the speed of light is 300,000 km/sec. First assume the Hubble constant to be 15 (km/sec)/(million light-years), and then assume it to be 30 (km/sec)/(million light-years). □

Question #2: In what sense is the universe expanding?

Figure 19.1 illustrates an *incorrect* answer to this question. The big bang was *not* like a giant firecracker exploding into an already existing space. There was *no* center to the big bang.

Figure 19.2 illustrates the correct answer to the question. Space itself was very small right after the big bang, and didn't even exist before it! Note that each galaxy sees all the other galaxies moving away from it. In fact, you can deduce from the picture that a galaxy twice as far away will be receding twice as fast, thus "proving" the Hubble law.

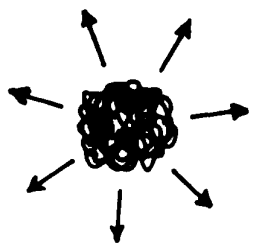
The expansion of a three-dimensional torus-universe is shown in Figure 19.3. Naturally this idea applies to a universe based on any three-manifold.

One question that often comes up is that if everything is expanding--houses, people, atoms, tape measures, *everything*--then how can we tell that things have changed at all? The answer is that not everything is expanding. Houses, people, atoms and tape measures are not expanding. Planets, stars and even galaxies are

An Incorrect Picture of the Big Bang

(1) At the moment of the big bang, all matter starts out at a single point in space.

(2) It goes flying off into space in all directions,



(3) and eventually forms galaxies which continue to move further out into space.

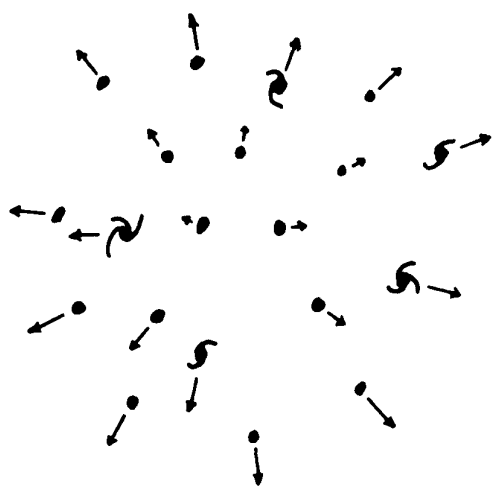


Figure 19.1

A Correct Picture of the Big Bang
(illustrated via a two-dimensional universe)

(1) Space itself starts off being very small. All the matter of the universe is crammed into it.

(2) Space expands very rapidly at first.

(3) Eventually the matter is cool enough to begin forming galaxies.

(4) The galaxies continue to move away from each other. The size of each galaxy stays the same.

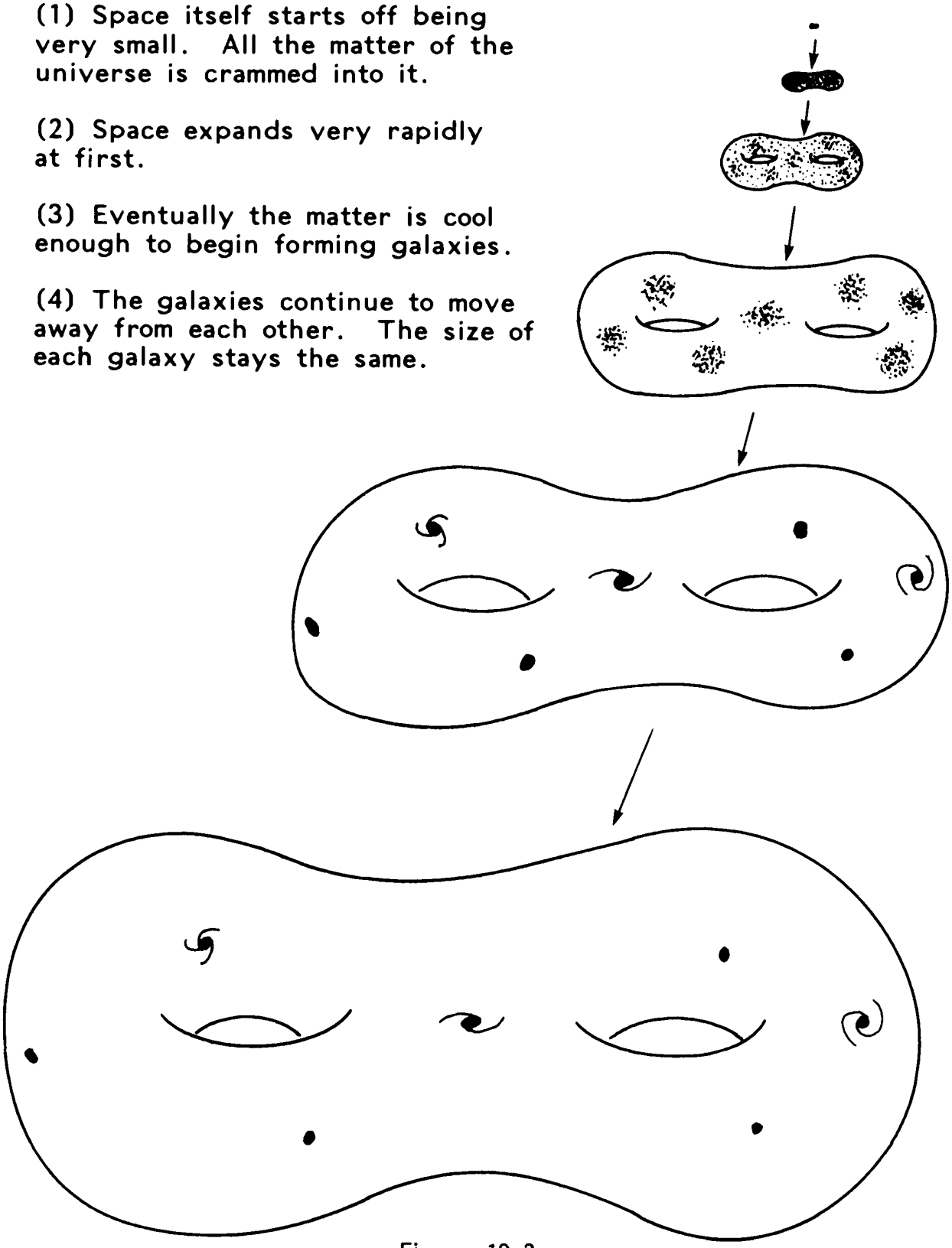


Figure 19.2

not expanding. Space *is* expanding, and so is the distance between galaxies, but that's about it.

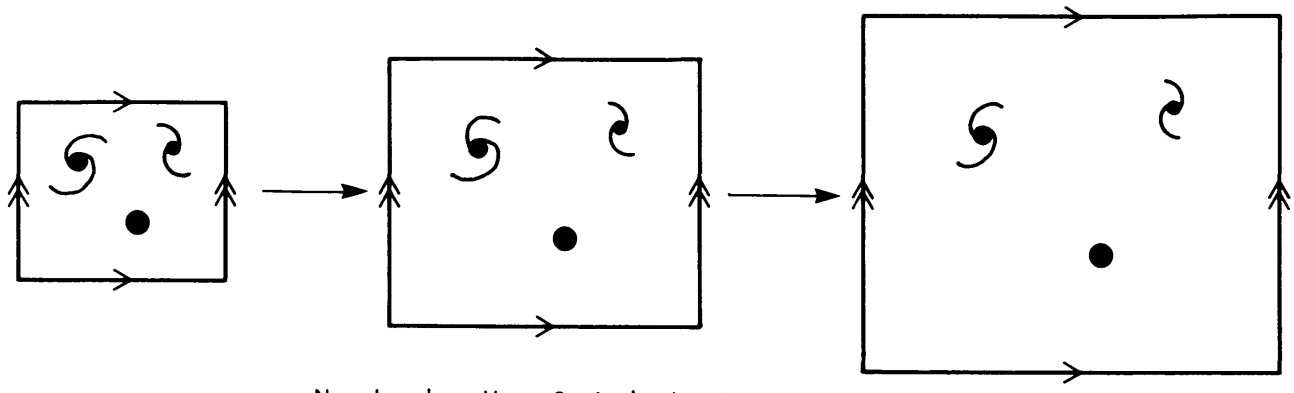
Another question that comes up is that if the universe has infinite volume (i.e. if it's an open three-manifold), then how can it expand and get any bigger? The answer is that its total volume doesn't increase, but the galaxies *do* still move away from each other, just as they would in a closed universe. The expansion of an infinite universe is *locally* identical to the expansion of a finite one (you can, for example, reinterpret Figure 19.3 as an expanding chunk of space in an infinite universe). The difficulties which arise when contemplating the total volume of an infinite expanding universe are difficulties with the concept of infinity, not difficulties with the behavior of the universe.

Question #3: What does the geometry of the universe have to do with whether it recollapses or not?

We'll first look at the technical answer to this question, and then interpret part of that answer.

One applies the field equations of general relativity to each of the three homogeneous, isotropic geometries. (General relativity is concerned only with the universe's local geometry; the global topology is irrelevant.) In the case of elliptic geometry the field equations predict that the universe will eventually recollapse; in the case of hyperbolic geometry they predict that it will expand

For comparison, here's a Flatlander's drawing of an expanding T^2-universe:



Now here's a Homo Sapien's drawing of an expanding T^3-universe:

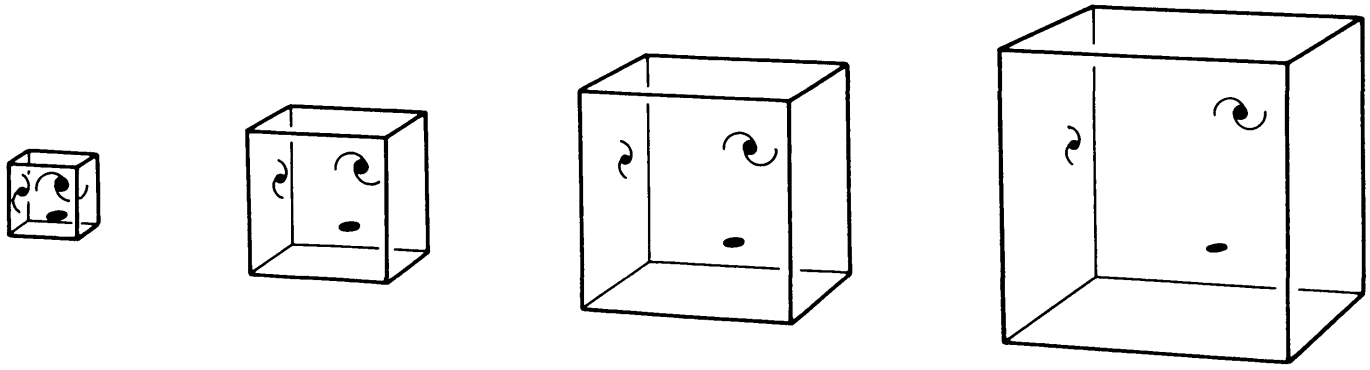


Figure 19.3

forever, gradually approaching a constant rate of expansion; and in the borderline case of Euclidean geometry they predict that the universe will expand forever, but just barely (the rate of expansion approaches zero). In addition, the field equations predict a relationship between the density ρ of matter in the universe and the Hubble constant H. The density ρ will be greater than $[3/(8\pi G)]H^2$ in an elliptic universe, equal to $[3/(8\pi G)]H^2$ in a Euclidean universe, and less than $[3/(8\pi G)]H^2$ in a hyperbolic universe [$G = 6.67 \times 10^{-11}$ $m^3/(kg\ sec^2)$ is a constant of nature, the so-called "gravitational constant"]. The following table summarizes these facts. (The last column will be discussed in Question #4.)

Geometries for the Universe

Geometry of Universe	Fate of Universe	Density of Universe	Volume of Universe
Elliptic	Eventual Recollapse	Dense $\rho > \frac{3}{8\pi G} H^2$	Finite (closed)
Euclidean	Eternal Expansion (just barely)	Borderline $\rho = \frac{3}{8\pi G} H^2$	Finite or Infinite (closed or open)
Hyperbolic	Eternal Expansion	Sparse $\rho < \frac{3}{8\pi G} H^2$	Finite or Infinite (closed or open)

I have no intuitive explanation for the amazingly simple relationship between the universe's geometry and its eventual fate (if you know anything about this, please write to me as soon as

possible). I can, however, say a little more about what the relationship between H and ρ has to do with the fate of the universe. The Hubble constant H represents the rate at which the galaxies are moving away from each other. The density ρ represents how much matter--and therefore how much gravitational attraction--is present to slow the expansion. If ρ is large, or if H is small, then the gravitational attraction will be strong enough to slow the galaxies to a halt and begin pulling them back together. In this case the universe recollapses, as in Figure 19.4. If, on the other hand, ρ is small or H is large, then the motion of the galaxies is too great to be overcome by the comparatively feeble gravitational attraction, and the universe continues expanding forever (gravity at first exerts a significant braking effect on the galaxies, but eventually they get far enough apart that their gravitational attraction is minimal, and the rate of expansion steadies out). Finally, in the case where $\rho = [3/(8\pi G)]H^2$, the galaxies are moving apart fast enough to overcome the gravitational attraction, but just barely--they continue to slow down without quite coming to a halt.

The reason we study these ideas is that we want to understand the real universe. In my opinion the question of whether the universe has elliptic geometry and is destined to recollapse or has hyperbolic geometry and is destined to expand forever is one of the two most fascinating questions raised in this book (the other question being the global topology of the universe). As stated at

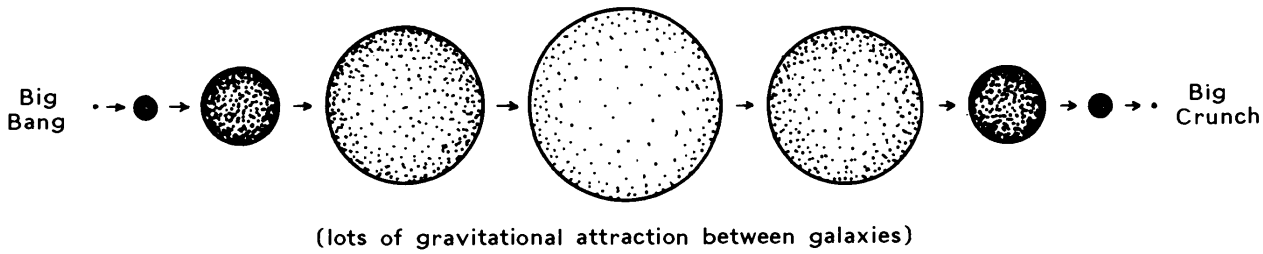


Figure 19.4: A two-dimensional spherical universe which recollapses.

the beginning of the chapter, we don't yet know what the fate of the universe will be, but we can at least rephrase the question in quantitative terms. Astrophysicists define the number Ω (pronounced "omega") to be the actual density of matter in the universe divided by the "critical density":

$$\Omega = \frac{\rho}{(\frac{3}{8\pi G})H^2}$$

Thus, if $\Omega > 1$, then $\rho > [3/(8\pi G)]H^2$ and we live in an elliptic universe which will recollapse. Similarly, if $\Omega < 1$, then $\rho < [3/(8\pi G)]H^2$ and we live in a hyperbolic, eternally expanding universe. And if $\Omega = 1$, then we live in a borderline Euclidean universe. A safe estimate would put $.02 < \Omega < 4$. This obviously isn't very informative, although it is very likely correct. A more daring guess would put $.1 < \Omega < .3$. This guess--which might someday be proved wrong--suggests that the universe has a hyperbolic geometry and will expand forever. And if I may indulge in some wild speculation, I'd like to add that a hyperbolic universe fits in well with the viewpoint presented in the preceding chapter that most closed three-manifolds admit a hyperbolic geometry.

Question #4: Is the universe closed or open? In other words, is space finite or infinite?

Put briefly, we don't know.

We do know, however, that if the universe has an elliptic geometry than it must be closed. If, on the other hand, the

universe has Euclidean or hyperbolic geometry, then it can be either closed or open. The following table gives some sample topologies for the different types of universes.

Possible Global Topologies for the Universe

	Elliptic Geometry	Euclidean Geometry	Hyperbolic Geometry
Closed	S^3, P^3, Poincare dodecahedral space	T^3, 1/4 turn manifold, 1/6 turn manifold	Seifert-Weber space, "most" closed 3-manifolds (see Chapter 18)
Open	NONE	E^3, $T^2 \times E$, $E^2 \times S^1$	H^3 (there are other possibilities as well)

Unfortunately the many examples of closed hyperbolic three-manifolds are little known among cosmologists. You are likely to find books on cosmology stating--incorrectly--that a hyperbolic (= negatively curved) universe *must* be open. The terminology that has grown up around this misconception is particularly unfortunate: in the cosmological literature "closed" is used to mean "elliptic" (= positively curved) and "open" is used to mean "Euclidean or hyperbolic" (= flat or negatively curved). This terminology precludes the very mention of a closed hyperbolic universe.

Question #5: What came before the big bang?

Sagredo: What came before the big bang?

Salviati: Nothing did.

Sagredo: You mean space was entirely empty then?

Salviati: No, space didn't even exist!

Sagredo: Oh, I see: at times before the big bang there simply was no space. What a curious thought.

Salviati: It's worse than that: "before" the big bang there wasn't any time either!

Sagredo: What? No time?! Even if there was no matter and no space, surely there would have been time.

Salviati: Allow me to draw you some pictures. They'll be **spacetime diagrams** somewhat like the one in Figure 13.8, only these will be pictures of a one-dimensional circular universe rather than a two-dimensional planar one. If this circular universe always stays the same size, then its spacetime diagram will be a cylinder, as in Figure 19.5(a). On the other hand, if it's an expanding circular universe, then its spacetime diagram will be a cone, as in Figure 19.5(b).

The import of these pictures is that space and time have been wrapped into a unified **spacetime**. So where there's no space there's no time, and vice-versa. I might add that Einstein's relativity not only permits us, but actually forces us to think of space and time in this way.

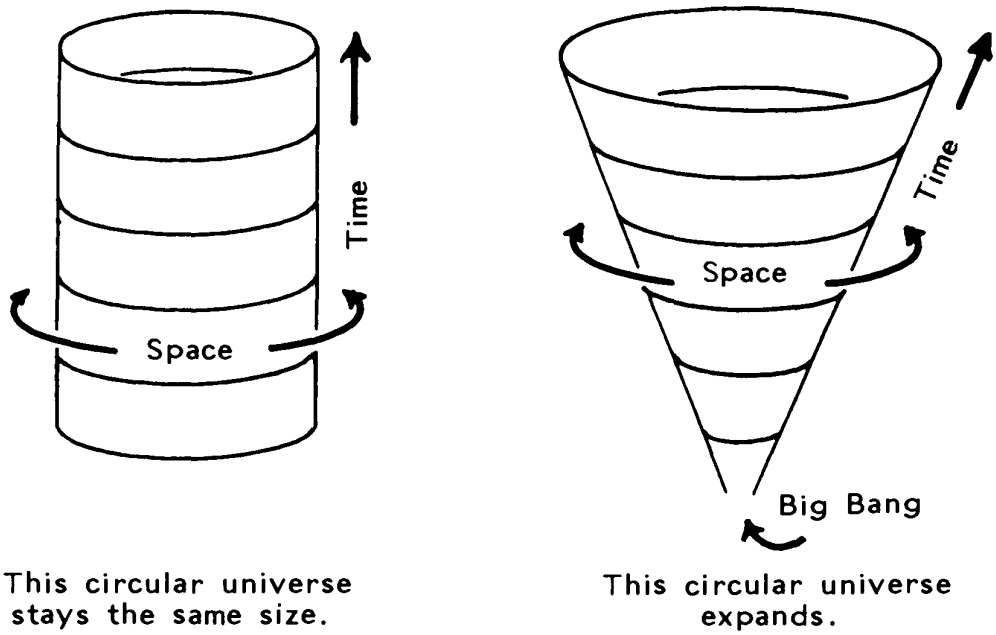


Figure 19.5: Spacetime diagrams for (a) a static and (b) an expanding circular universe.

Sagredo: I see. Spacetime includes *all* matter, *all* space and *all* time--in short, *all* of physical reality.

Salviati: Right.

Sagredo: And all of physical reality comes *after* the big bang.

Salviati: Exactly.

Sagredo: So what *caused* the big bang?

Salviati: Nothing did.

Sagredo: What?! Something must have caused it!

Salviati: Physical events are caused by other physical events,

but the big bang wasn't a physical event, and doesn't demand a cause.

Sagredo: But the big bang appears in the spacetime diagram of Figure 19.5(b), so doesn't that make it a physical event?

Salviati: I'm sorry--I should have explained that earlier. The spacetime diagram you refer to is meant to be a cone with no vertex. Thus, strictly speaking *there is no big bang,* there are only events following it. Every event is caused by earlier events, yet there is no earliest event in the same way that there is no smallest positive number.

Sagredo: So what caused spacetime?

Salviati: Now that's a difficult question! I myself have no idea. In fact, I'm not even sure that the question is a meaningful one. But if you want to postulate a god as the cause of the universe, then according to relativity theory you should imagine Him to have caused spacetime as a whole, rather than just the big bang.

Sagredo: You mean He created the past and the future at the same time?

Salviati: That's the idea, although "at the same time" is a misleading choice of words. Just as God doesn't have a

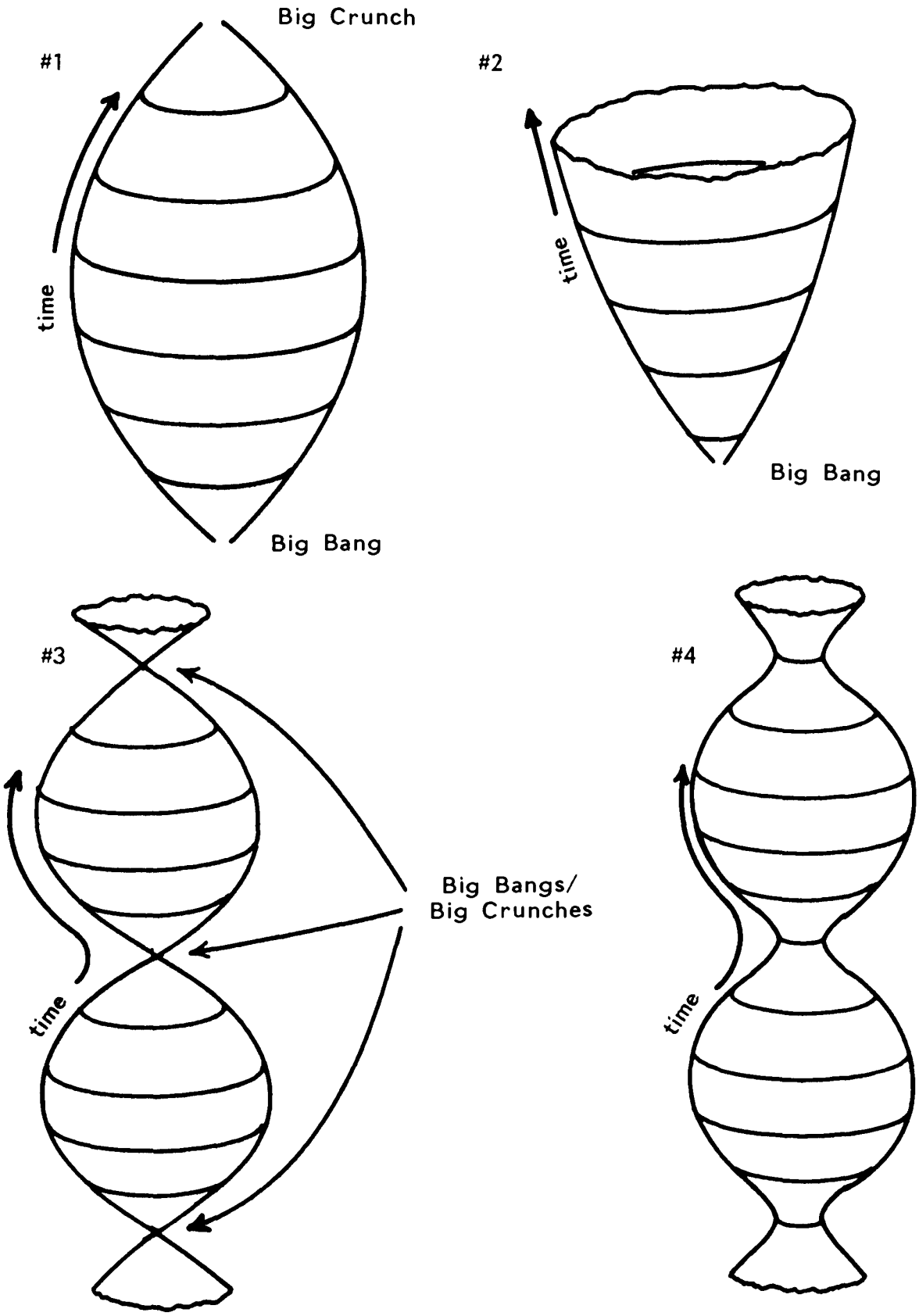
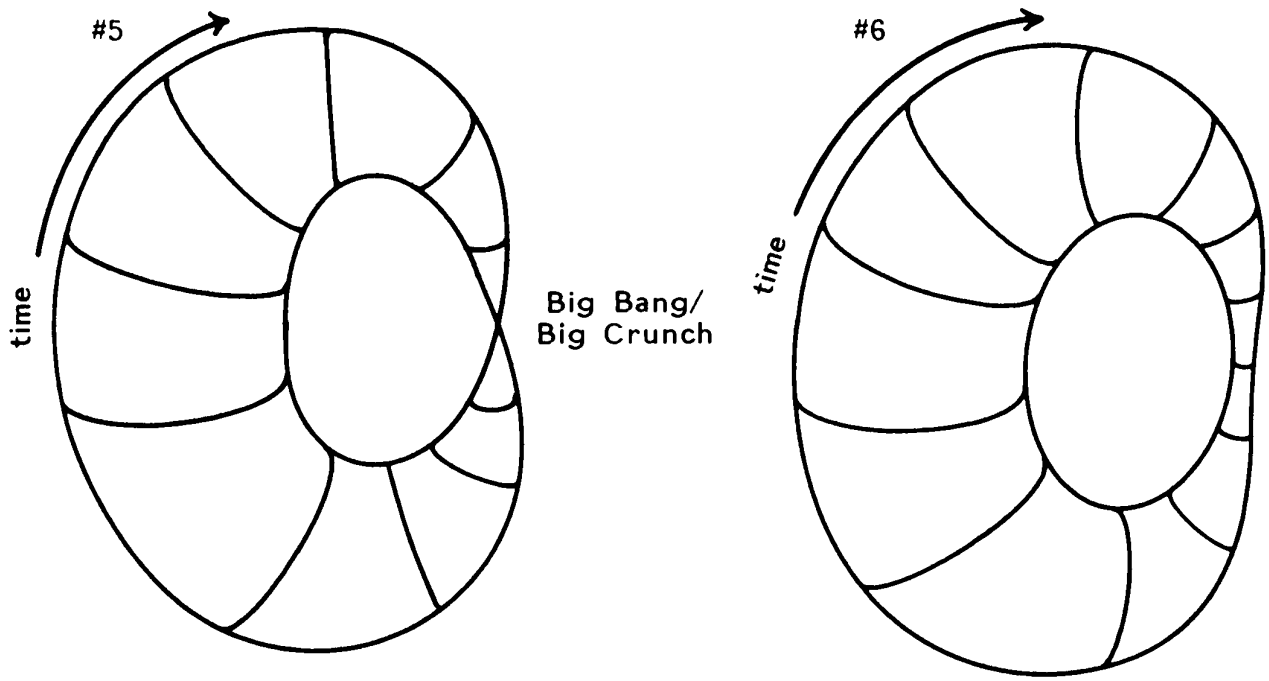


Figure 19.6: Different models of the universe.



location in space, neither does He experience time. A god would have to be outside spacetime, and being outside spacetime means being outside both space *and time.*

Sagredo: That's certainly a switch from the traditional view, in which God is the creator of space, but nevertheless lives in time just as we do.

If you don't mind, Salviati, I'd like to leave these theological issues aside and ask you one more question.

Salviati: Please do.

Sagredo: How do you know that your description of the big bang is correct?

Salviati: I don't! The *nature* of the big bang is very much a matter of speculation, even though its *existence* is supported by overwhelming evidence.

Sagredo: Could you tell me about some competing descriptions of the big bang?

Salviati: Certainly. In Figure 19.6 I've sketched a number of spacetime diagrams, each of which gives a different picture of the universe. The first diagram depicts a "standard" recollapsing universe, and the second depicts a "standard" eternally expanding one. Diagrams #3 and #4 depict oscillating universes--in one case the universe collapses to a point at the end of each cycle, and in the other it "bounces" before it gets that far. Finally, diagrams #5 and #6 depict cyclic universes in which the "big crunch" is the cause of the big bang.

Sagredo: Don't the laws of physics tell us which of these models is most likely to be correct?

Salviati: Unfortunately not. Strange things happened in the first zillionth of a second after the big bang.[4] Temperatures, pressures and densities were enormous,

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[4] In this case a zillionth is approximately 10^{-43}.

perhaps unboundedly so. Under these conditions gravity takes on a quantum nature, but we have as yet no quantum theory of gravity. Thus we can't say with any certainty what the universe was like at times close to the big bang, and so we must content ourselves with idle speculation.

Sagredo: So tell me, why do you advocate the picture that you do? (Namely diagrams #1 and #2 in Figure 19.6.)

Salviati: I feel it's the simplest picture suggested by Einstein's general relativity. The chief appeal of the competing pictures is that one needn't part with the cherished notion that time has neither a beginning nor an end. But these other pictures introduce other complications--namely one must either interpret the point representing the big bang (pictures #3 and #5) or explain why the universe "bounces" (pictures #4 and #6). I myself would rather accept the fact that time may be finite.

Appendix A

Answers

Chapter 2

2.2. Player X can win by taking the middle square of the bottom row. Player O wins by taking the bottom left square.
2.3. Positions *a*, *b*, *c*, *e* and *g* are all equivalent, as are positions *d*, *f* and *h*. Position *c* is obtained from *a* by moving everything down one notch. Position *g* is obtained from *a* by moving everything to the left one notch. Position *b* is obtained from *a* by moving everything down a notch *and* to the left a notch. Position *e* comes from rotating *a* a quarter turn. Position *f* is obtained from *d* by rotating it a half turn (or by moving it down and and to the right), and *h* is obtained from *f* by moving everything one notch to the right.
2.4. The first player has, in effect, only one possible move: any move she makes can be shifted upward and/or to the right until it appears in the center of the board. Her opponent has two possible responses: either directly next to her or on a diagonal. With optimal play the first player can always win.
2.5. The white knight threatens a bishop, a king, and two knights, while a queen, a rook, a bishop, and the knights threaten it.
2.6. Every black piece is threatened by both the white knight and the white queen.
2.8. The lower left-hand corner. (Imagine him going first up one square and then to the right one square.)
2.9. They cannot, for the same reason as in conventional chess: a bishop threatens only pieces on the same color square, while a knight threatens only pieces on an oppositely-colored square.
2.10. When you look through a wall you see what appears to be another copy of the room. You see yourself from behind. When

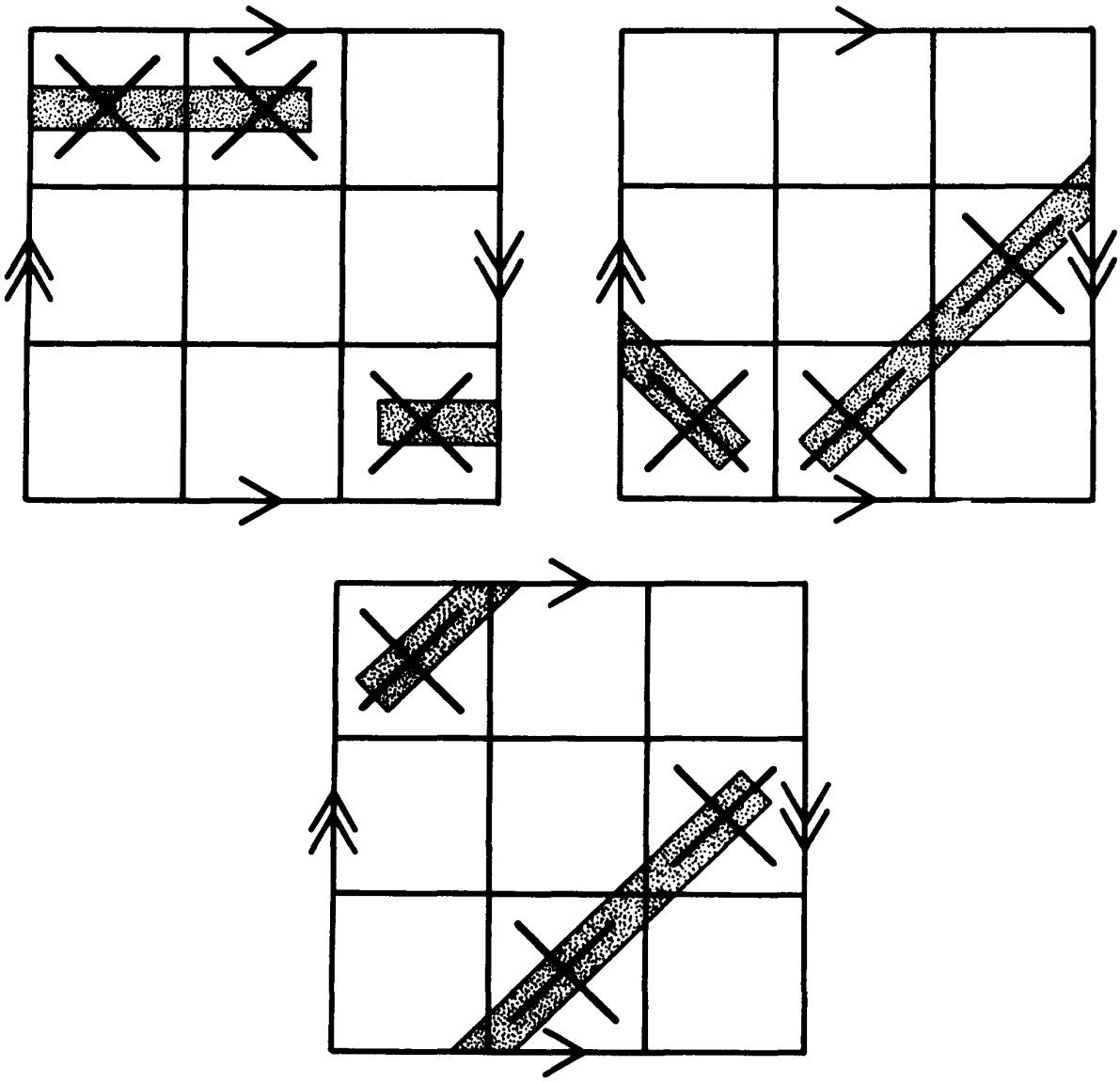
you through the floor you see the top of your head, and when you look through the ceiling you see the bottoms of your feet.
2.12. Throw the ball and turn around quick. If you're fast enough you can even get in a little batting practice.

Chapter 3

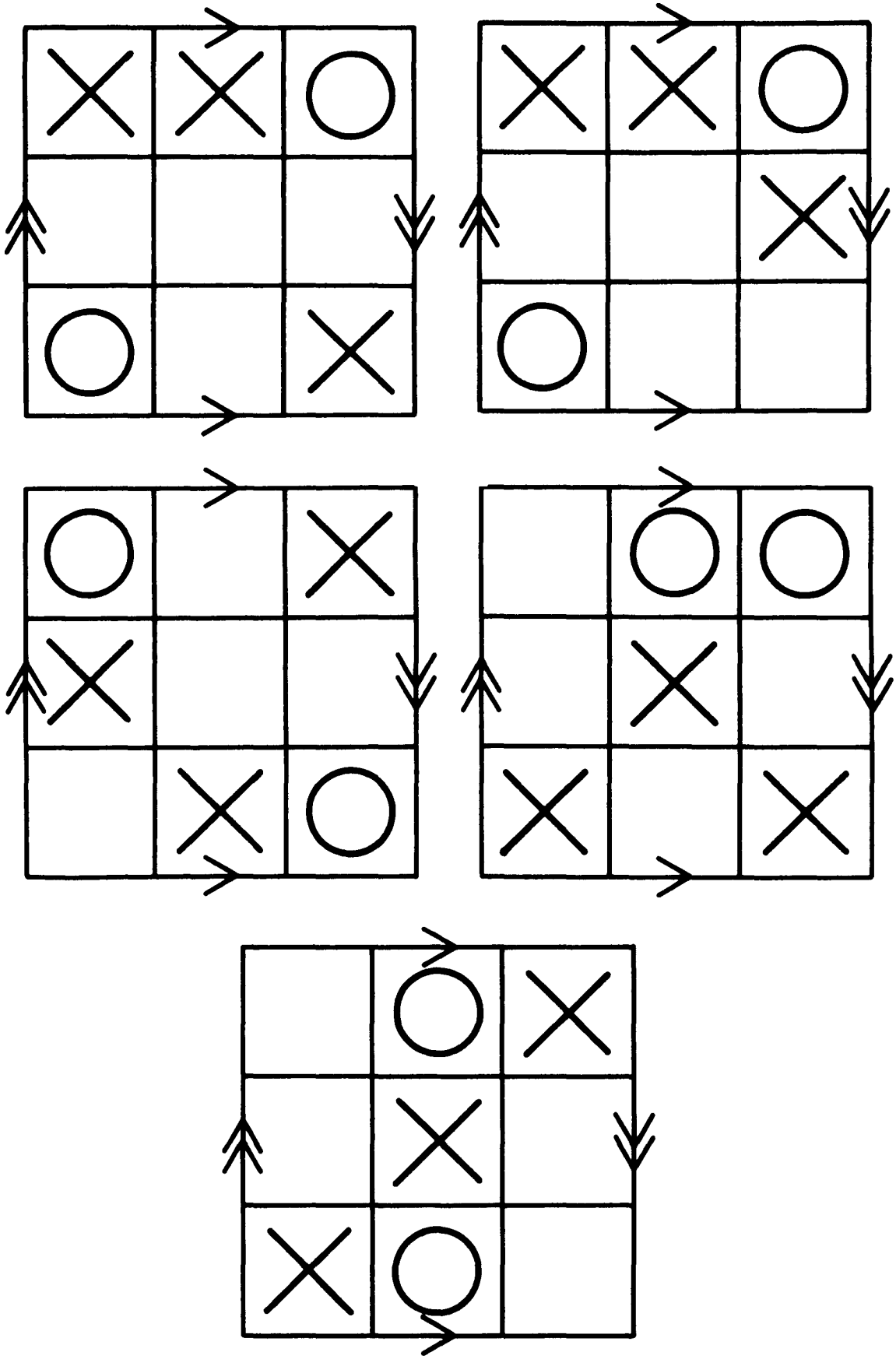
3.1. Surfaces *a* and *c* have the same topology. So do *b* and *d*, and *e*, *f*, *g* and *h*. Surface *h* may look like it has four holes, but if you flatten it out you'll see that it has only three. For *g* you need to deform the surface to uncross the cross pieces.
3.2. He discovered a topological property. The property remains even if Flatland is distorted.
3.3. Surfaces *a*, *c*, *d* and *f* have the same extrinsic topology, as do surfaces *b* and *e*. The hardest part is seeing that *c* and *f* are the same as *a*: shrink the connecting piece in *f* to make it look like *c*, and then pull the inner loop out from the outer loop to make *c* look like *a*. Good luck!
3.4. Reglue the band with a half-twist instead of a full twist. (You get a "Mobius strip".)
3.5. You can roll the paper into a cone, but you can't wrap it smoothly onto a basketball. Try it! If you don't have a basketball, use somebody's head. The paper, the cylinder and the cone have the same intrinsic geometry, but the intrinsic geometry of the basketball is different.
3.6. Imagine the universe as a room with opposite walls glued. Home is next to one wall, your friends are in the middle, and you do your exploration over by the opposite wall. After you're done exploring you simply pass through the wall and you are home.
3.7. Discovery #1 is local, while discoveries #2 and #3 are global.
3.8. A one-dimensional manifold is a space with the local topology of a line. A circle is an example of a one-dimensional manifold. (In fact, a circle is the *only* "closed" one-dimensional manifold.)
3.9. The cylinder and the plane have the same intrinsic local geometry (but different extrinsic local geometries). They have different global topologies, but they have the same local topology as do all surfaces. Only discovery #2 might distinguish a cylinder from a plane, and even then Flatlanders on a cylinder would have to travel in exactly the right direction.
3.10. They have the same local topology (as do all three-manifolds), the same local geometry (namely that of ordinary Euclidean space), and the same global topology (one can easily be deformed to the exact shape and proportions of the other). However, the fact that their overall dimensions are different means that they have different global geometries.
3.11. Yes. At any point it has the local geometry of ordinary Euclidean space.
3.12. 1, 3, 4, 7 and 9 are closed; 2, 5, 6 and 8 are open.
3.13. He bumps into himself.

Chapter 4

4.2. They all do!



4.3. The white knight threatens a bishop, a knight and a rook. The same bishop, knight and rook threaten it.
4.4. On all boards X can win immediately. See page 290. (Incidentally, on each of the first four boards X will lose if he tries to block O, because O has two separate ways to win.)



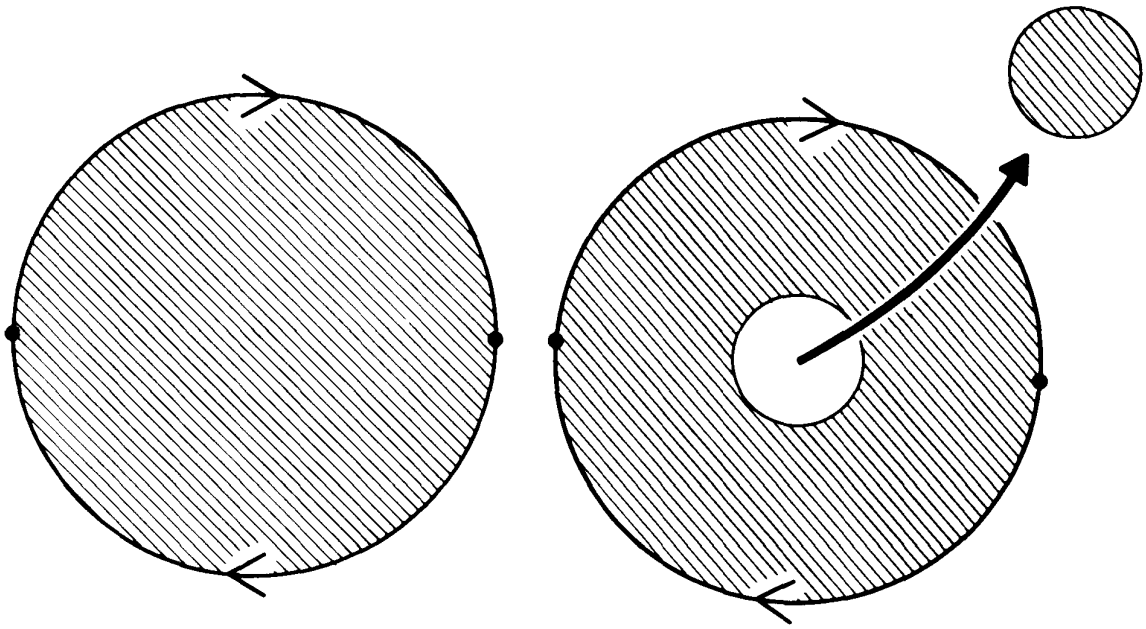
4.7. The bishop returns from the *upper* left hand corner.
4.8. Each square on the right side of the chessboard is adjacent to a square *of the same color* on the left side of the board. So when the bishop passes from one side of the board to the other it returns on a different color square. This phenomenon no longer occurs on the Klein bottle chessboard of Figure 4.8, because there every square is adjancent only to squares of the opposite color. A knight and a rook can still simultaneously threaten each other on this new board: place the rook in the row just above the center, and place the knight one space over and two spaces down.
4.9. When you look through the back wall you see a mirror-reversed copy of the room. When you look through the other walls you see normal copies of the room, just like in the three-torus.
4.11. The projective plane is nonorientable: A Flatlander crossing the rim comes back with his left and right sides interchanged.
4.12. He's furthest from home when he's crossing the rim. After that he's getting closer again.
4.13. Two fire stations should be positioned "90° apart". For example, they could be 90° apart on the rim, or one could be on the rim with the other at the south pole. Three fire stations should be positioned so that any two are 90° apart. One way to do this is to have two of them 90° apart on the rim with the third at the south pole.
4.14. To decide whether he is on a projective plane or a sphere, a Flatlander can walk in a straight line until he gets back to his starting point: on a projective plane he comes back as his mirror image, but on a sphere he doesn't. This trick won't work to tell a projective plane from a Klein bottle; instead the second Flatlander can measure the angles of a triangle and see whether they add up to 180° or not.
4.15. Sphere: curved and orientable. Torus: flat and orientable. Klein bottle: flat and nonorientable. Projective plane: curved and nonorientable.
4.16. Projective three-space is orientable. If you cross the "seam" you come back rotated 180°, but not mirror-reversed. For more explanation, see Figure 14.8 and the corresponding paragraph in the text of Chapter 14.
4.17. Orientability is a global property because it says something about a manifold as a whole. It is a topological property because deforming a manifold does not affect it.

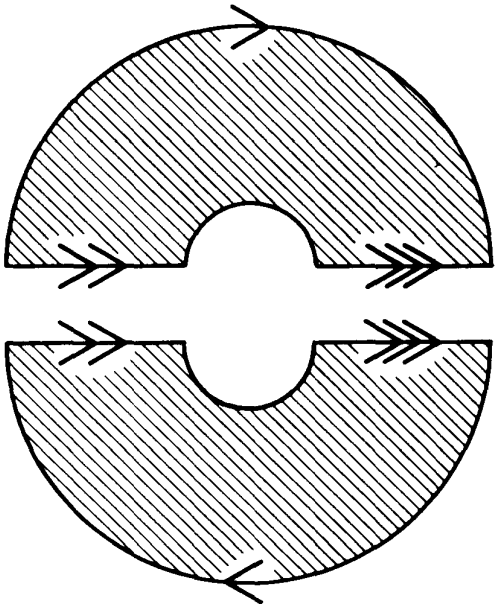
Chapter 5

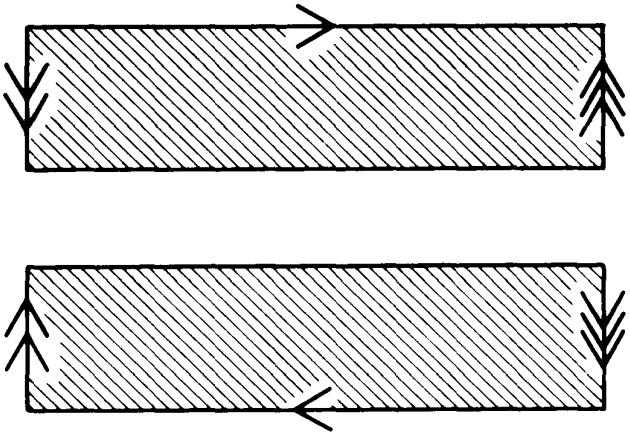
5.1. The connected sum of a two-holed doughnut surface and a one-holed doughnut surface is a three-holed doughnut surface (you can make a drawing just like Figure 5.4). Similarly, a six-holed doughnut surface and an eleven-holed doughnut surface form a seventeen-holed doughnut surface.

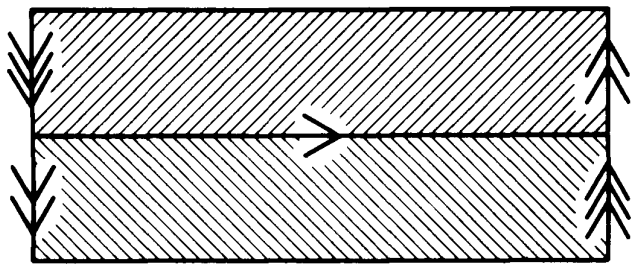
5.2. The connected sum of any surface and a sphere is just the same surface you started with. The reason is that a sphere with a disk cut out is topologically a disk, so when you form the connected sum you are removing one disk from your surface and replacing it with a different one.

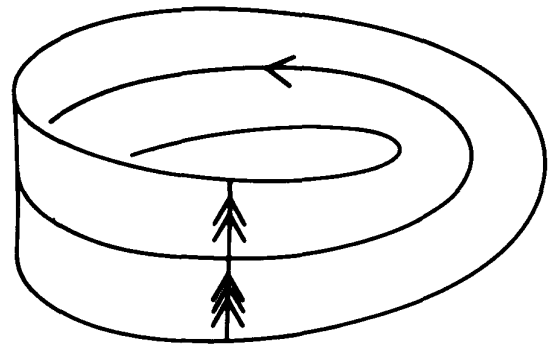
5.3. A projective plane with a disk cut out is a Möbius strip.









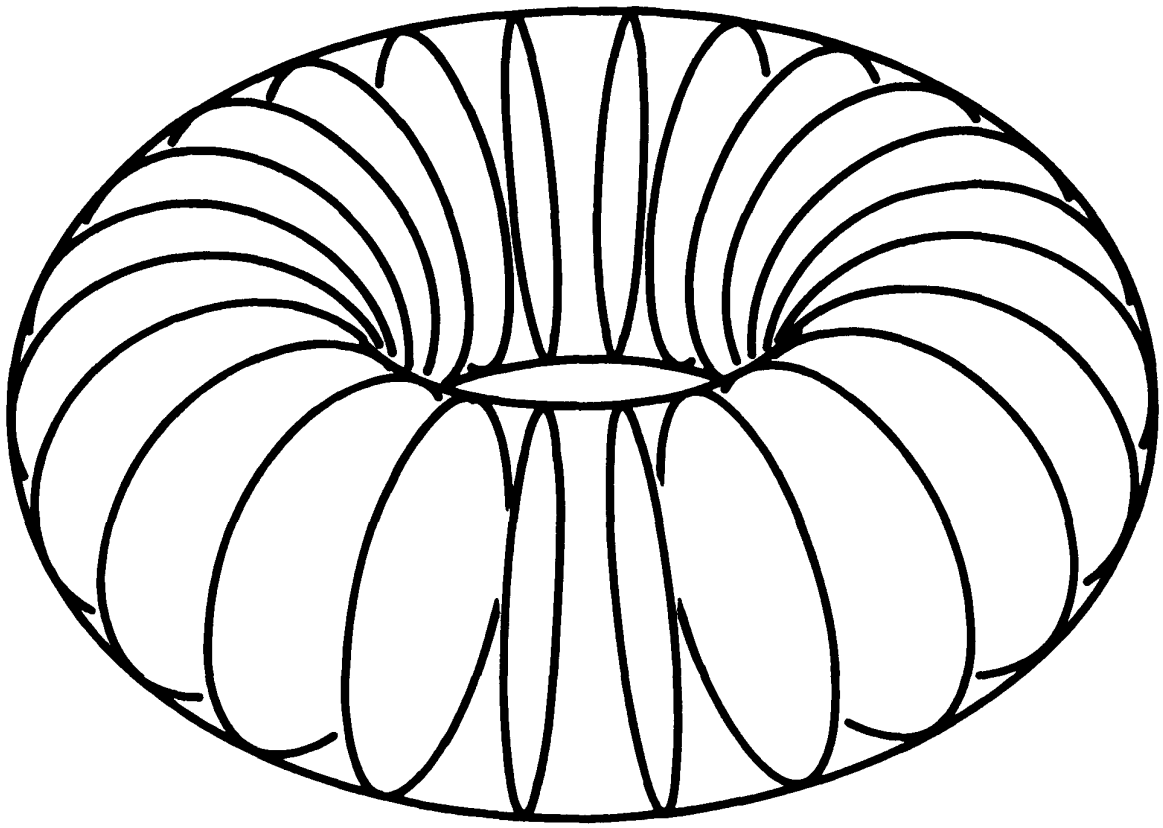


5.4. The connected sum of two projective planes is a Klein bottle. You start with two projective planes, cut a disk out of each to get two Möbius strips (as per Exercise 5.3), and then glue the Möbius strips' edges together to get a Klein bottle (as per Figure 5.6).
5.5. Flatland was the connected sum of a torus and a projective plane. The safe region was a torus with a disk cut out, and the reversing region was a Möbius strip, which is a projective plane with a disk cut out.
5.6. $(T^2 \# T^2) \# S^2 = T^2 \# T^2$, $K^2 \# S^2 = K^2$ and $P^2 \# S^2 = P^2$. (In general for any surface X, $X \# S^2 = X$.)
5.7. (a) $K^2 \# P^2 = P^2 \# P^2 \# P^2$, (b) $K^2 \# T^2 = P^2 \# P^2 \# T^2 = T^2 \# P^2 \# P^2$, and (c) $K^2 \# K^2 = P^2 \# P^2 \# P^2 \# P^2$.
5.9. Exercise 5.8 says that $T^2 \# P^2 = K^2 \# P^2$, and Exercise 5.4 implies that $K^2 \# P^2 = P^2 \# P^2 \# P^2$.
5.10. A surface written as a connected sum of both tori and projective planes can always be rewritten as a connected sum of projective planes only. Without changing the surface's global topology you can convert the tori to Klein bottles (by Exercise 5.8) and then convert the Klein bottles to projective planes (by Exercise 5.4). Note, though, that you cannot convert a surface consisting of tori only into a surface consisting of projective planes only, because Exercise 5.8 does not apply when no projective planes are present. (Besides, you know a connected sum of tori cannot be the same as a connected sum of projective planes because one is orientable while the other is not.) $T^2 \# P^2 = K^2 \# P^2 = P^2 \# P^2 \# P^2$, $T^2 \# K^2 = T^2 \# P^2 \# P^2 = K^2 \# P^2 \# P^2 = P^2 \# P^2 \# P^2 \# P^2$, $P^2 \# S^2 = P^2$ and $S^2 \# S^2 = S^2$. The sphere and the connected sums of tori are all orientable, while the connected sums of projective planes are all nonorientable.
5.11. $T^2 \# S^2 = T^2$, $K^2 = P^2 \# P^2$, $S^2 \# S^2 \# S^2 = S^2 \# S^2$, $P^2 \# T^2 = K^2 \# P^2$ and $K^2 \# T^2 \# P^2 = P^2 \# P^2 \# P^2 \# K^2$.

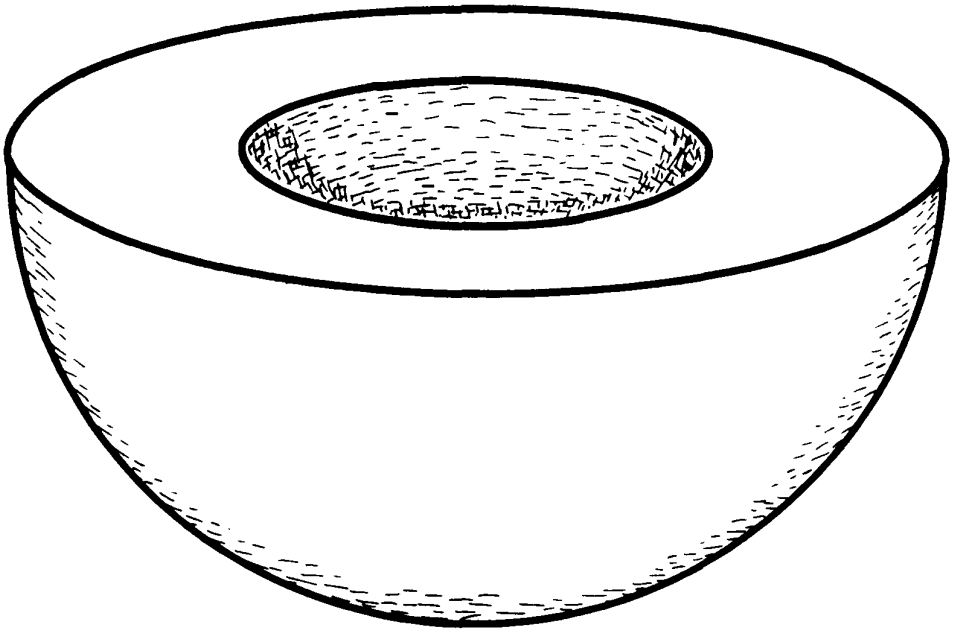
Chapter 6

6.1. 1. Square, 2. Plane, 3. Infinite cylinder, 4. Infinite strip.
6.2. No: it's a circle of intervals but not it's not an interval of circles. (Chapter 17 will address this issue in more detail.)

6.3. $D^2 \times S^1$ is topologically a solid doughnut. Here's a picture showing that it's a circle of disks; it's also a disk of circles, but that's a little harder to draw.

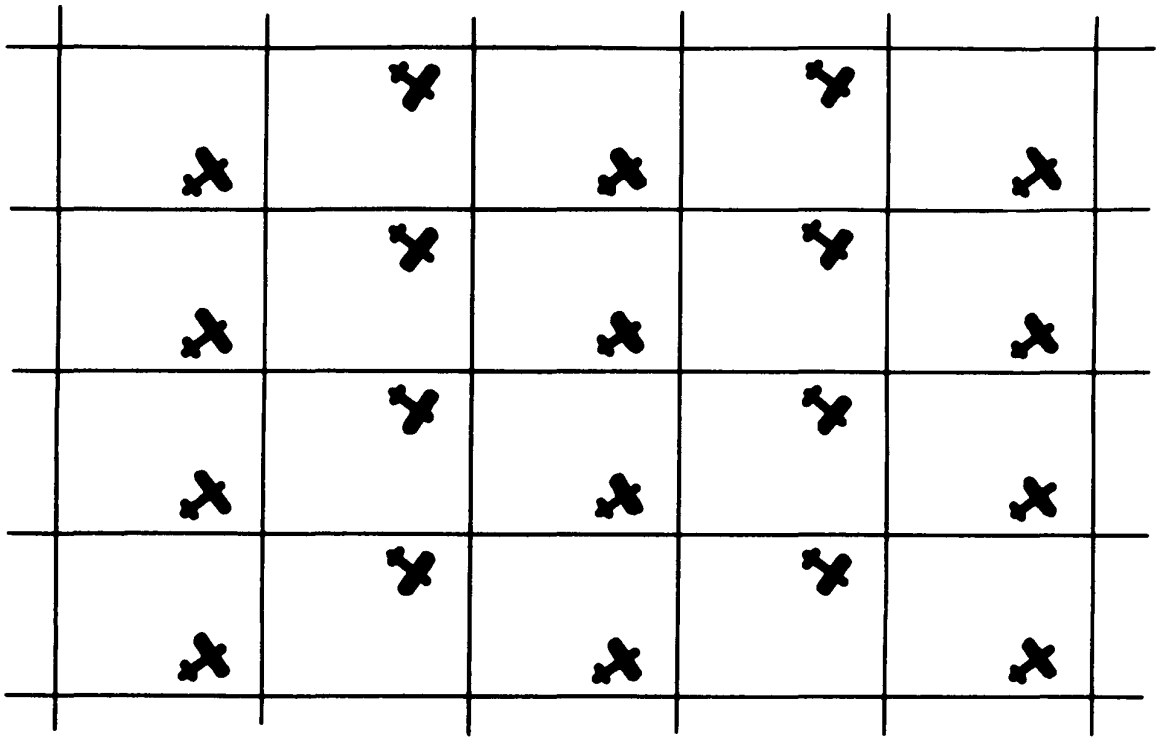


6.4. In the first example the circles are not all the same size, in the last example the intervals are not all the same size, and in the middle example the circles are not perpendicular to the intervals.
6.5. The geometrical product is a perfect square. *Any* deformed square will be only a topological product.
6.6. It's a Klein bottle cross a circle ($K^2 \times S^1$). (Each horizontal layer in Figure 6.5. becomes a Klein bottle when the cubes sides are glued.) It's a geometrical product because the horizontal Klein bottles are all the same size, the vertical circles are all the same size, and the Klein bottles and the circles are perpendicular to each other.
6.7. $P^2 \times S^1$. You can visualize $P^2 \times S^1$ as a thickened hemisphere, as shown in the top illustration on page 295. The inside surface is glued to the outside surface in the obvious way, and points on the "rim surface" are glued so that each hemispherical layer becomes a projective plane. This manifold is locally identical to $S^2 \times S^1$, but its global topology is different. Can you find an orientation reversing path?

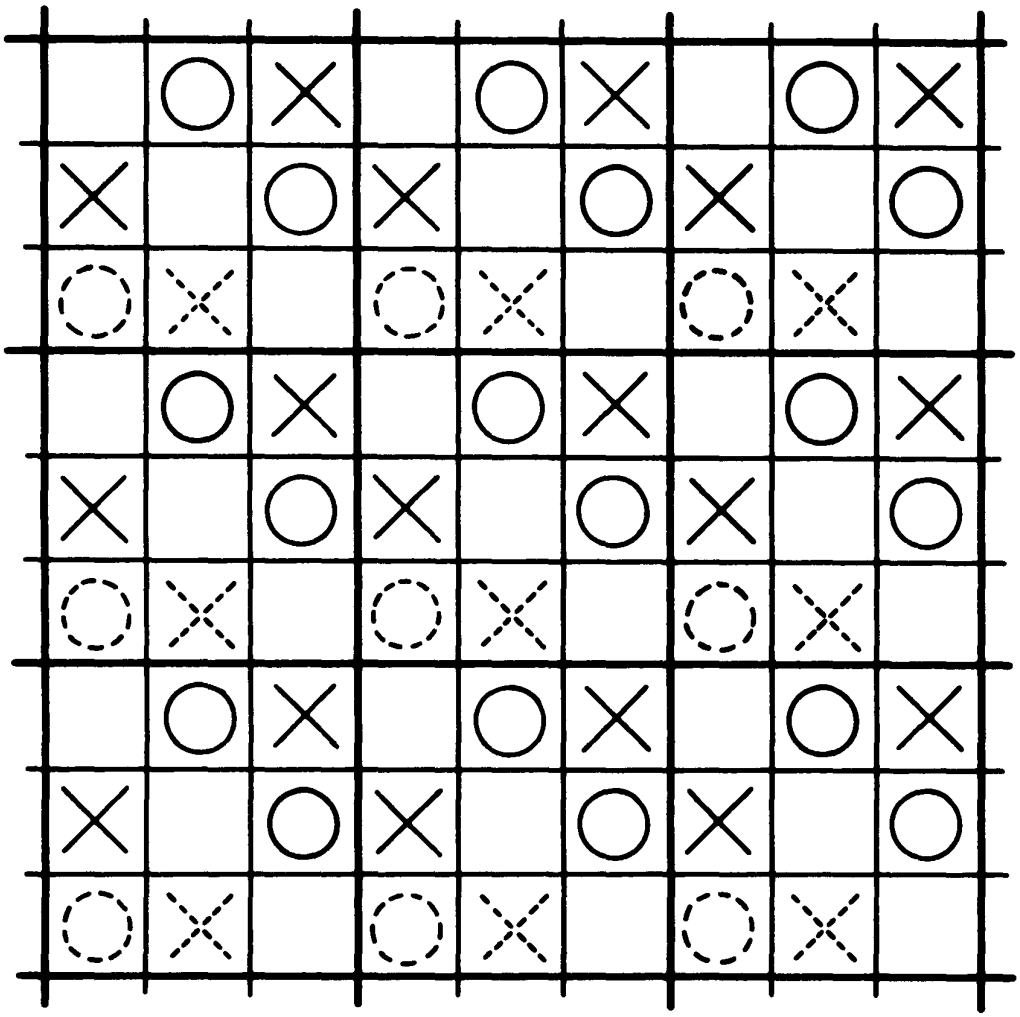


Chapter 7

7.1.



7.2.



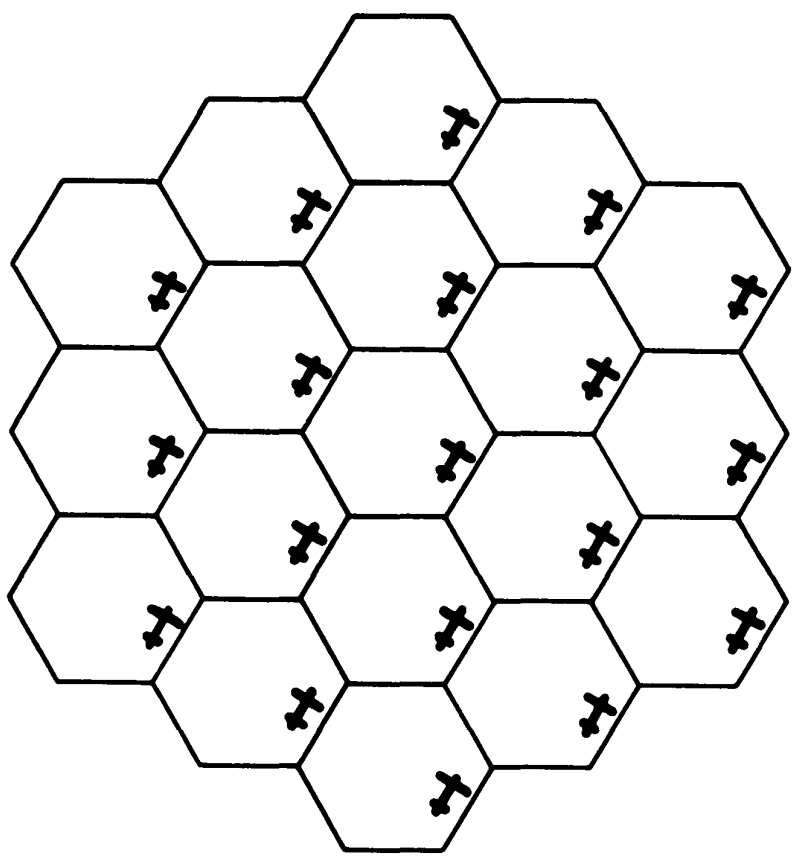
7.3. In the quarter turn manifold you see what appears to be an infinite lattice of cubes: all the cubes on a given level are colored identically, all the cubes on the next higher level appear to be rotated a quarter turn, the cubes on the level after that are rotated an additional quarter turn, etc. The views in the half turn and three-quarters turn manifolds are analogous. The one-quarter turn and three-quarters turn manifolds are mirror images of each other. Surprisingly, you can't tell one from the other intrinsically! The reason is that you have no standard by which to judge clockwise and counterclockwise; looking at the hands on your watch will do no good because you don't know whether you showed up in the manifold as your "normal self" or your "mirror image". Philosophically inclined readers may wish to ponder this matter some more: the heart of the problem is that in everyday speech we use

terms like clockwise and counterclockwise, and right and left, in an absolute way, but really they are relative terms and can be used only to compare one object to another object at the same location.

7.4. Figure 7.10: The front is glued to the back with a side-to-side flip, the top is glued to the bottom with a side-to-side flip, but the left side is glued to the right side normally. Figure 7.11: The front is glued to the back with a side-to-side flip, the top is glued to the bottom with a front-to-back flip, and the left side is glued to the right side normally. Figure 7.12: The front is glued to the back with a side-to-side flip, the top is glued to the bottom with a half turn, and the left side is glued to the right side normally. (You are correct if you said that the top is glued to the bottom in 7.12 with both a side-to-side and a front-to-back flip, because this is the same as a half turn.)

7.5. The manifolds in Exercise 7.3 are all orientable; if you go on a trip in one of them you might come back rotated but you won't come back mirror reversed. The manifolds in Exercise 7.4 are all nonorientable; in each one at least one pair of faces (e.g. the front and back) are glued with a flip.

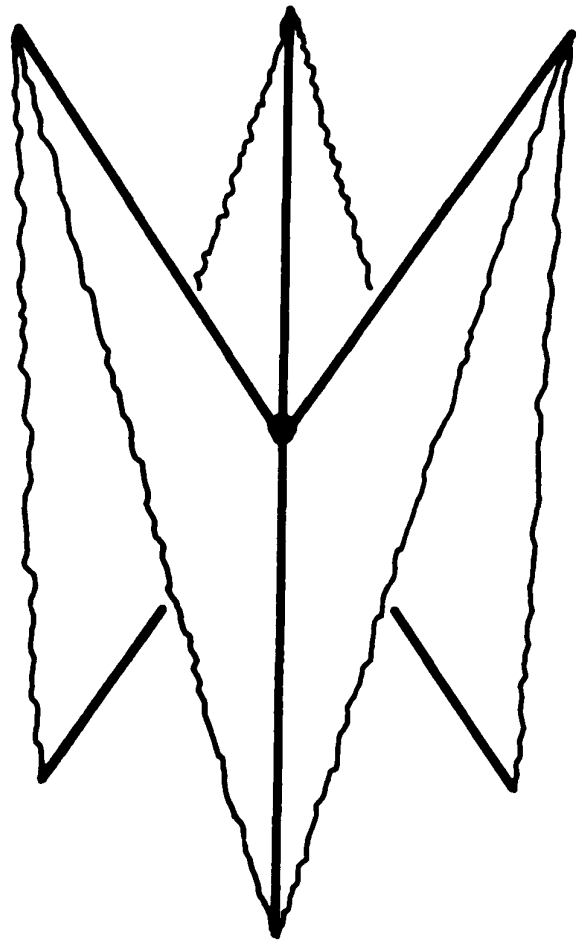
7.6.



If a second biplane is flying around in the hexagonal torus, then of course it too will appear once in each hexagonal cell.

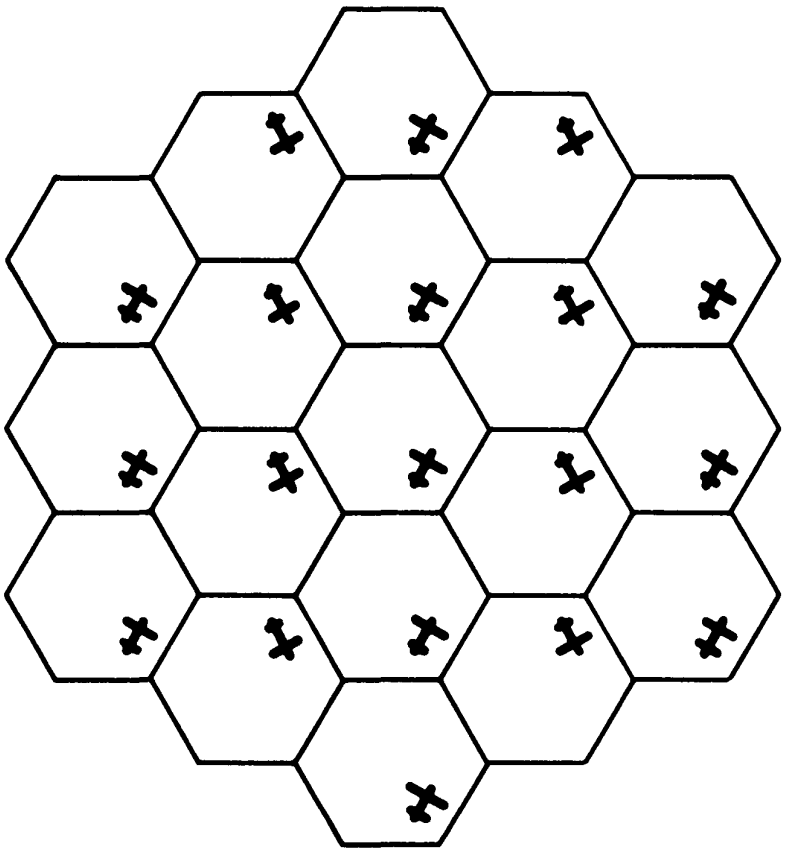
7.7. In the first surface the hexagon's corners meet in three groups of only two corners each, so the surface has three cone points. In the second surface the hexagon's corners fit perfectly in two groups of three corners each; there are no cone points. In the third surface the hexagon's six corners meet at a single point! You get the opposite of a cone point: there is too much angle surrounding the vertex instead of too little.

Opposite of a cone point:



7.8. None of them.

7.9. The surface is a hexagonal Klein bottle.



7.10. The hexagonal torus and the ordinary flat torus have the same local topology (as do all surfaces), the same local geometry (both are flat) and the same global topology (both can be deformed to a doughnut shape). On the other hand they have different global geometrical properties, as evidenced by the pattern of the images in Exercise 7.6.

7.11. Only the second surface counts as flat. The first is disqualified by its cone points, and the third by its opposite-of-a-cone-point.

7.12. The hexagonal three-torus and the usual three-torus have the same local geometry; both are flat, that is, they have the same local geometry as ordinary Euclidean space. They both have the same global topology because each is a torus cross a circle (in one case it's a hexagonal torus cross a circle and in the other it's an ordinary flat torus cross a circle, but those are geometrical differences, not topological ones). Their global geometries are different for the reason just mentioned. When you look around in a hexagonal three-torus you see copies of yourself arranged in layers; each layer forms a hexagonal lattice.

7.13. You'd see copies of yourself arranged in layers, and each layer would still be a hexagonal lattice just like in the hexagonal three-torus. Only now the copies of you in each layer are rotated a one-third (or a one-sixth) turn relative to the copies in the layer below.

Chapter 8

8.1. (1) Torus, orientable and two-sided. (2) Klein bottle, nonorientable and two-sided. (3) Torus, orientable and one-sided. (4) Klein bottle, nonorientable and one-sided.
8.2. Sidedness is an extrinsic property of a surface because it has to do with how the surface is embedded in a three-manifold (for a specific example, note that the tori in the first and third drawings of Figure 8.2 are intrinsically identical, even though one is two-sided and the other is one-sided). Orientability is an intrinsic property because Flatlanders living in a surface can tell whether it's orientable or not. All Klein bottles are nonorientable, but some are one-sided (e.g. the one in the last drawing of Figure 8.2) while others are two-sided (e.g. the one in the second drawing of Figure 8.2).
8.3. Use the center strip (running front to back) of the two-sided Klein bottle in the second drawing of Figure 8.2.

Chapter 9

9.1. First triangle: area = $\pi/2$, angle-sum = $3\pi/2$. Second triangle: area = $\pi/6$, angle-sum = $7\pi/6$. Third triangle: area = $2\pi/3$, angle-sum = $5\pi/3$. Fourth triangle: area = π, angle-sum = 2π. Fifth triangle: area = $\pi/2$, angle-sum = $3\pi/2$. Sixth triangle: area = 2π, angle-sum = 3π. The formula relating the area to the angle-sum appears later in the text.
9.2. The sum of the angles of the first triangle is $\pi/2 + \pi/3 + \pi/4 = 13\pi/12$, so its area must be $13\pi/12 - \pi = \pi/12$. For the second triangle you must convert each degree measurement to radians by multiplying by $\pi/180$. The area works out to be $A = (1.065 + 1.082 + 1.100) - \pi = 3.247 - 3.142 = 0.105$. (Computational note: when the angles are given in degrees, it is often more efficient to subtract 180° from the angle sum before converting to radians. Thus in this problem you would first compute $(61° + 62° + 63°) - 180° = 6°$, and then convert 6° to radians to get the answer, $6° \times (\pi/180) = 0.105$.)
9.3. *Direct method:* Modify the book's derivation of the formula $A = (\alpha+\beta+\gamma) - \pi$ to apply on a sphere of radius two, radius three or radius r. Start by finding the correct formula for the area of a double lune, and work from there. Your final result, for a triangle on a sphere of radius r, should be $A = r^2[(\alpha+\beta+\gamma) - \pi]$.
Fancy method: Consider a triangle on a sphere or radius one. Now

enlarge the sphere by a factor of two, and let the triangle get enlarged along with it. The angles of the triangle do not change, but its area increases by a factor of four (because both its height and its width have doubled). In general, when you increase the radius of the sphere by a factor of r, the area of the triangle, just like the area of the whole sphere, increases by a factor of r^2. In other words, a triangle on a sphere of radius r has r^2 times the area of a similar triangle on a sphere of radius one. The formula for its area is therefore $A = r^2[(\alpha+\beta+\gamma) - \pi]$.

9.4. The field's area is $A = r^2[(\alpha+\beta+\gamma) - \pi] = (1000\text{ m})^2[(0.76138 + 1.48567 + 0.89483) - 3.14159] = (1000000\text{ m}^2)[0.00029] = 290\text{ m}^2$. You start off with five significant figures, but after the subtraction you are left with only two! Thus you know the correct answer only to within a few percent. What if the original data had had only three digits of accuracy?

9.5. Plug into $A = r^2[(\alpha+\beta+\gamma) - \pi]$ to find the sphere's radius, which works out to be r = 10,000 meters. The sphere's area is therefore $A = 4\pi r^2 = 1.2 \times 10^9$ square meters.

9.6. (1) $\pi + (2 \times 10^{-5})$ radians, (2) $\pi + (.01)$ radians, (3) $\pi + \frac{1}{2}$ radians.

9.7. A projective plane has the same local geometry as a sphere, and the formulas for spherical triangles apply on it too. Flatlanders on a projective plane couldn't tell locally that they weren't on a sphere, but they could tell globally because a projective plane is nonorientable, and long-distance travellers can come back mirror-reversed.

Chapter 10

10.2. You get an icosahedron. (An icosahedron is a regular polyhedron with twenty triangular faces.) The icosahedron approximates the geometry of a sphere in exactly the same way hyperbolic paper approximates the geometry of the hyperbolic plane.

10.3. $A = \pi - (\pi/3 + \pi/4 + \pi/6) = \pi - 3\pi/4 = \pi/4$.

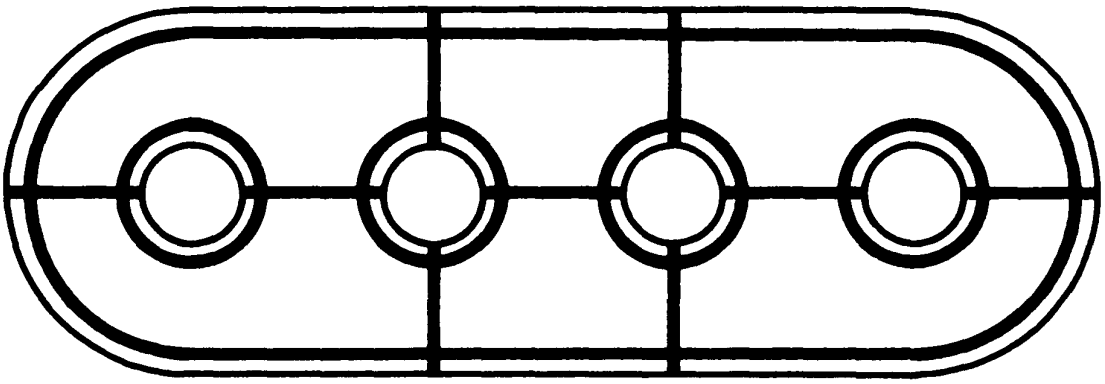
Chapter 11

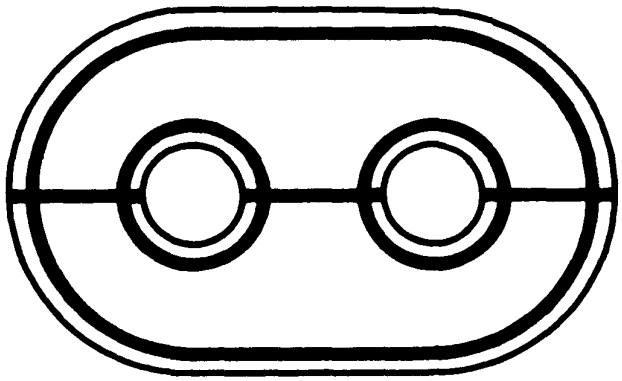
11.1. To eliminate the cone points, put the square on a sphere and let it expand until its corners fit properly (this occurs when the square fills an entire hemisphere and each corner angle is 180°). The surface now has a homogeneous elliptic geometry. You should recognize it as a projective plane (it had the global topology of a projective plane all along, but it just now got the right geometry). The first surface in Figure 11.1 is a projective plane too.

11.2. *First surface:* all six corners come together, hyperbolic geometry eliminates the opposite-of-a-cone-point. *Second surface:*

the corners meet in two groups of three, there are no cone points so the surface has a Euclidean (= flat) geometry. *Third surface:* all eight corners come together, hyperbolic geometry eliminates the opposite-of-a-cone-point. *Fourth surface:* all eight corners come together, hyperbolic geometry eliminates the opposite-of-a-cone-point.

11.3.





When you cut an n-holed doughnut surface into hexagons whose corners meet in groups of four you invariably get 4n - 4 hexagons. (Even though there are different ways to do the cutting, you always get the same number of hexagons!)

11.4. P^2 has an elliptic geometry to begin with. $P^2\#P^2$ can be given a Euclidean geometry, because the 90° corners of flat squares fit nicely in groups of four. ($P^2\#P^2$ is topologically the same as K^2, so we knew all along that this surface could be given a Euclidean geometry.) In all the other cases the polygons' corners must be shrunk to fit together in groups of four, so all the other connected sums of projective planes can be given hyperbolic geometry.

11.5.

	orientable	nonorientable
elliptic	S^2	P^2
Euclidean	T^2	$P^2\#P^2$ (= K^2)
hyperbolic	$T^2\#T^2$ $T^2\#T^2\#T^2$ etc.	$P^2\#P^2\#P^2$ $P^2\#P^2\#P^2\#P^2$ etc.

Chapter 12

12.1. The cell-division has nine vertices, fifteen edges and seven faces.

12.2. An n-gon can be broken down into n - 2 triangles. The sum of the angles of the n-gon is the sum of the angles of all the triangles put together, and the angles of each triangle add up to π.

12.3. The area of the n-gon is the sum of the areas of the n - 2 triangles. The area of each triangle is the sum of its angles minus π. Therefore the area of the n-gon is the sum of all the angles minus n - 2 times π.

12.4. The method is the same as for a polygon on a sphere, only now the area of each triangle is $\pi - (\alpha+\beta+\gamma)$ instead of $(\alpha+\beta+\gamma) - \pi$. The final formula is $A = (n - 2)\pi$ - (sum of all angles).

12.7. The only change necessary is to use the formula $A = (n - 2)\pi$ - (sum of all angles) instead of A = (sum of all angles) - $(n - 2)\pi$ in Step 2. This is just the negative of what you had before, so the final formula comes out as $A = -2\pi\chi$.

12.8. Using the cell-division of Figure 12.1(b), $\chi(T^2\#T^2) = 8 - 16 + 6 = -2$. The cell-division of Exercise 11.3 gets to the same answer via a different route: $\chi(T^2\#T^2) = 6 - 12 + 4 = -2$.

12.9. $A = -2\pi\chi = -2\pi(-2) = 4\pi$.

12.10. If you use the cell-division from Figure 11.3 you get $\chi = 12 - 24 + 8 = -4$. (Other cell-divisions will have different numbers of vertices, edges and faces, but they all give the same answer for the Euler number.) Area $= -2\pi\chi = -2\pi(-4) = 8\pi$.

12.11. $\chi(T^2) = v - e + f = 1 - 2 + 1 = 0$. The Klein bottle's Euler number also works out to be zero no matter what cell-division you use.

12.12. The Euler number decreases by two each time. Presumably the Euler number of $T^2\#T^2\#T^2\#T^2$ is -6.

12.13. The reasoning is the same as that used to compute the Euler number of a connected sum of tori. In the cell-division for a connected sum of projective planes $v = 2n \times 2 \div 4 = n$, $e = 2 \times 2n \div 2 = 2n$ and of course $f = 2$. So $\chi = v - e + f = n - 2n + 2 = 2 - n$.

orientability

Euler number	orientable	nonorientable
2	S^2	
1		P^2
0	T^2	$P^2\#P^2$
-1		$P^2\#P^2\#P^2$
-2	$T^2\#T^2$	$P^2\#P^2\#P^2\#P^2$
-3		$P^2\#P^2\#P^2\#P^2\#P^2$
-4	$T^2\#T^2\#T^2$	etc.
-5		
-6	$T^2\#T^2\#T^2\#T^2$	
-7		
-8	$T^2\#T^2\#T^2\#T^2\#T^2$	
.		
.	etc.	
.		

12.14. *Figure 11.1:* The first surface has $\chi = 3 - 3 + 1 = 1$, is nonorientable, and is therefore P^2. The second surface has $\chi = 2 - 3 + 1 = 0$, is nonorientable, and is therefore $P^2\#P^2$ (i.e. it's topologically a Klein bottle). The third surface has $\chi = 1 - 3 + 1 = -1$, is nonorientable, and is therefore $P^2\#P^2\#P^2$. *Figure 11.2:* The first surface has $\chi = 1 - 3 + 1 = -1$, is nonorientable, and is therefore $P^2\#P^2\#P^2$. The second surface has $\chi = 2 - 3 + 1 = 0$, is nonorientable, and is therefore $P^2\#P^2$. The third surface has $\chi = 1 - 4 + 1 = -2$, is orientable, and is therefore $T^2\#T^2$. The fourth surface has $\chi = 1 - 4 + 1 = -2$, is nonorientable, and is therefore $P^2\#P^2\#P^2\#P^2$.

12.15. Curvature is $k = 1/r^2$, so it is measured in $(\text{meters})^{-2}$. The surface's area A is measured in $(\text{meters})^2$, so kA is a dimensionless quantity, as is $2\pi\chi$.

12.16. The curvature of a projective plane is the same as the curvature of a sphere of the same radius; in this case $k = 1/r^2 = 1/(2 \text{ meters})^2 = \frac{1}{4} \text{ m}^{-2}$. The area of a projective plane is half the area of a sphere of the same radius; in this case $A = 2\pi r^2 = 2\pi(2 \text{ m.})^2 = 8\pi \text{ m}^2$. A projective plane's Euler number is 1 no matter what the radius is, so the Gauss-Bonnet formula reads $(\frac{1}{4} \text{ m}^{-2})(8\pi \text{ m}^2) = 2\pi(1)$.

12.17. Use the formula $kA = 2\pi\chi$ as usual. Plug in $k = -.00001 \text{ m}^{-2}$ and $\chi = -4$, and solve to get $2500000 \text{ m}^2 = 2.5 \text{ km}^2$.

12.18. Plug $k = -3.1658 \times 10^{-6} \text{ m}^{-2}$ and $A = 1{,}984{,}707 \text{ m}^2$ into $kA = 2\pi\chi$ and solve for $\chi = -1$. Consult the table in Figure 12.3 to see that the only surface with $\chi = -1$ is $P^2\#P^2\#P^2$. (It's amazing how far you can get on so little initial information.)

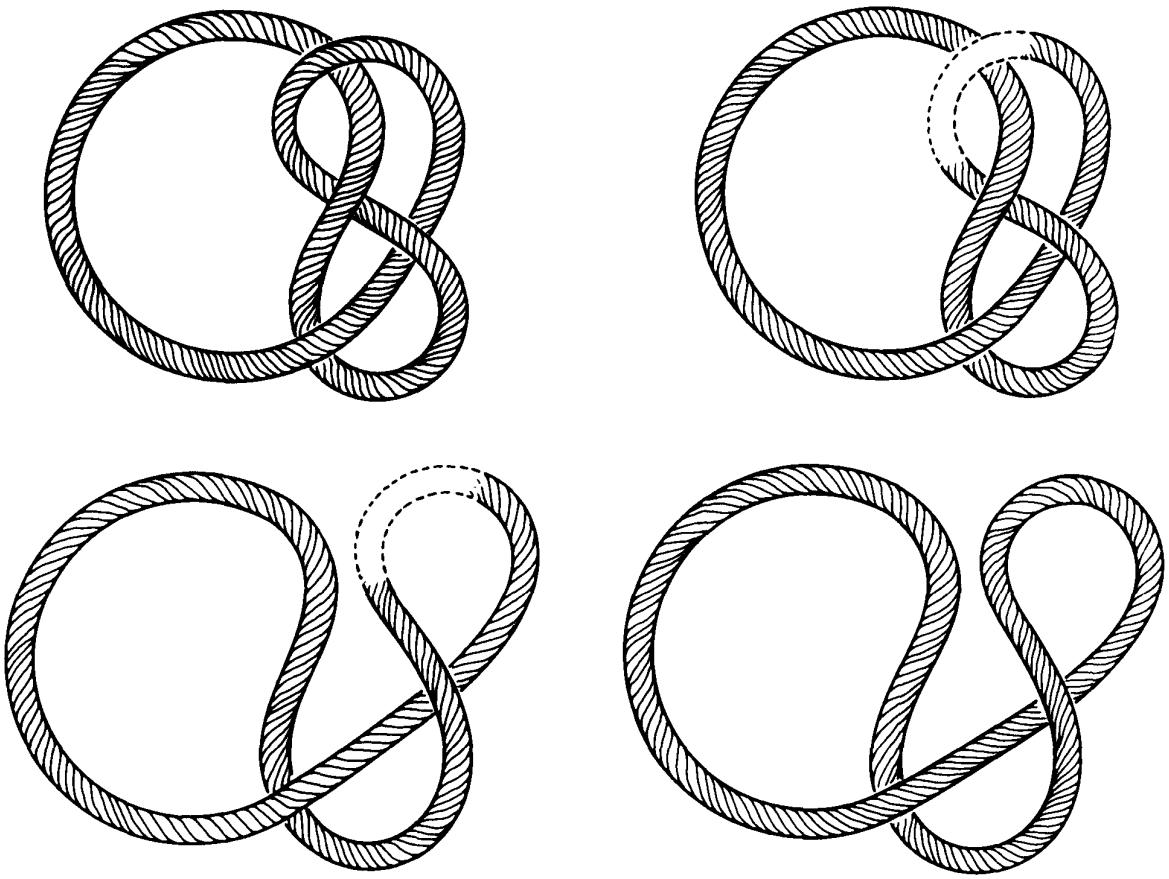
12.19. The $T^2\#T^2$ is more curved. You can compute the curvature of each surface directly, or if you want to get fancy you can reason that the $T^2\#T^2$ has twice the Euler number but less than twice the area so it must have a greater curvature.

12.20. Plug into $kA_\Delta = (\alpha + \beta + \gamma) - \pi$ to solve for $k = -1.02 \times 10^{-4} \text{ km}^{-2}$. Then use $kA = 2\pi\chi$ to compute $\chi = -4.05$. Presumably χ is -4, and the Flatlanders' (orientable!) universe has the global topology of $T^2\#T^2\#T^2$ (cf. the table in Figure 12.3). The error in the computed value of χ is most likely due to the uncertainty in the total area.

12.21. The center of the blip has positive curvature (it's convex), but the periphery has negative curvature (it's locally saddle shaped).

Chapter 13

13.1. A spirit could pass one section of rope "over" another section in the fourth dimension. See page 306. The effect is the same as if the one section passed through the other. The knotted loop becomes unknotted.



13.2. Push the "neck" of the Klein bottle into the fourth dimension.

Chapter 14

14.3. You will find yourself trapped on what used to be the outside of the balloon, but is now effectively its inside! What do you have to do to escape? (Assume you can't fit through the mouthpiece and don't want to damage the balloon.)

14.4. First of all, 10 minutes of arc = 1/6 of a degree = .003 radians. The angles of Gauss' triangle must exceed π by at least .003 radians if Gauss is to detect the difference with his equipment. The area of the triangle is about 4000 km^2, so Gauss can use the formula $kA_\Delta = (\alpha+\beta+\gamma) - \pi$ to deduce that the curvature k must be at least $[(\alpha+\beta+\gamma) - \pi]/A_\Delta = [.003]/(4000\ \text{km}^2) = .000001\ \text{km}^{-2}$ if he is to detect it. This corresponds to a universe of radius at most 1000 km! (Use the formula $k = 1/r^2$.) The real universe is obviously not this small!

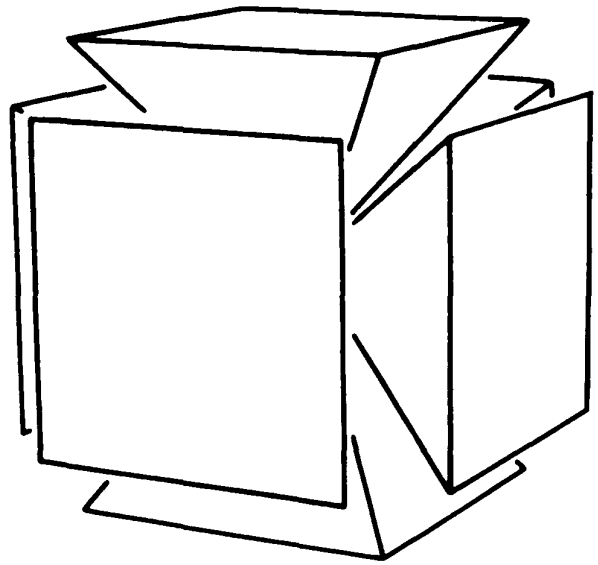
14.5. The ends of the cylinder are glued to each other with a 180° rotation (not a flip!) so the glued cylinder is topologically a torus. It is orientable because a Flatlander never returns to his starting point mirror-reversed, and it is two-sided because a three-dimensional ant can't get from one side of the surface to the other.
14.6. A disk passing through the center of the ball forms a projective plane in P^3. Any such projective plane is nonorientable and one-sided. (In an orientable three-manifold--like P^3--every orientable surface is two-sided and every nonorientable surface is one-sided.)

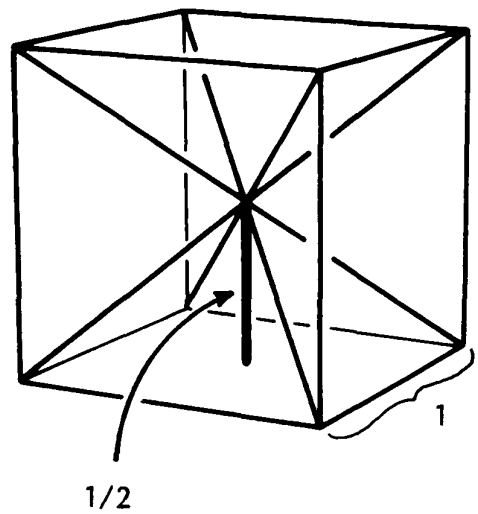
Chapter 15

15.1. Our universe might have the local geometry of H^3, but its curvature would be so slight that we haven't yet detected it. (By the way, what is the smallest negative curvature which Gauss could have detected? Cf. Exercise 14.4.)
15.2. It would appear infinite to them. From experience their brains would accurately judge distances based on how cross-eyed their eyes are. How would our universe appear to them?

Chapter 16

16.1. All four corners come together in one group. (The way you can tell is that when the gluings are taken into account each corner is "adjacent" to the other three.) The corners are too pointy to fit together snugly, so the tetrahedron must be expanded in a hypersphere. This gives the manifold an elliptic geometry.
16.2. The cube's corners come together in two groups of four corners each. (Each corner is adjacent to the other three in its group, but to no others.) The corners are too small to fit snugly in groups of four, and the manifold ends up with an elliptic geometry.
16.3. The octahedron's six corners meet in a single group. The exact pattern is shown in the first two drawings on page 308. Each corner is adjacent to four of the remaining five corners, and each of those four is adjacent to the fifth. The question now is whether the corners are too skinny, too fat, or just right. The corners would be just right if each were exactly as pointy as one of the six pyramids which comprise the cube in the middle drawing below. One way to tell (certainly not the only way) is to use the Pythagorean theorem to discover that the top half of an octahedron is a pyramid whose altitude is about .7 times the width of the base, while each of the six pyramids in the middle drawing has an altitude which is exactly .5 times the width of its base. This tells you that the octahedron's corners are a little too pointy, so the octahedral space ends up with an elliptic geometry.

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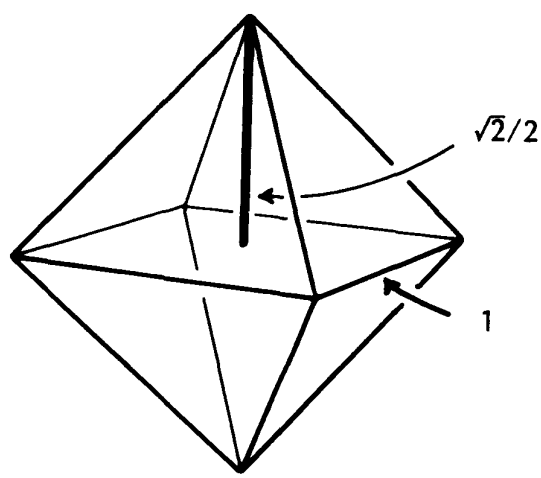
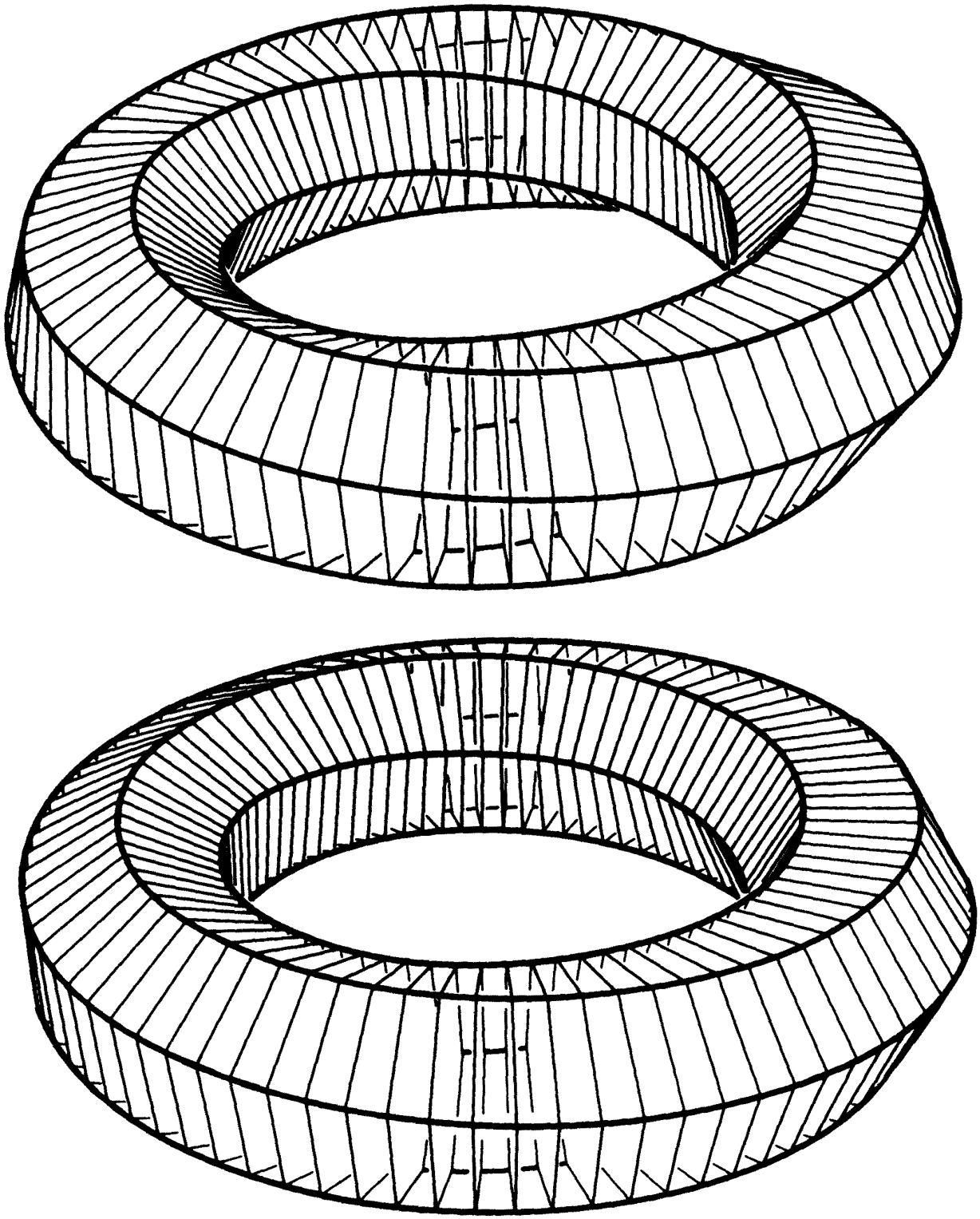
1/2

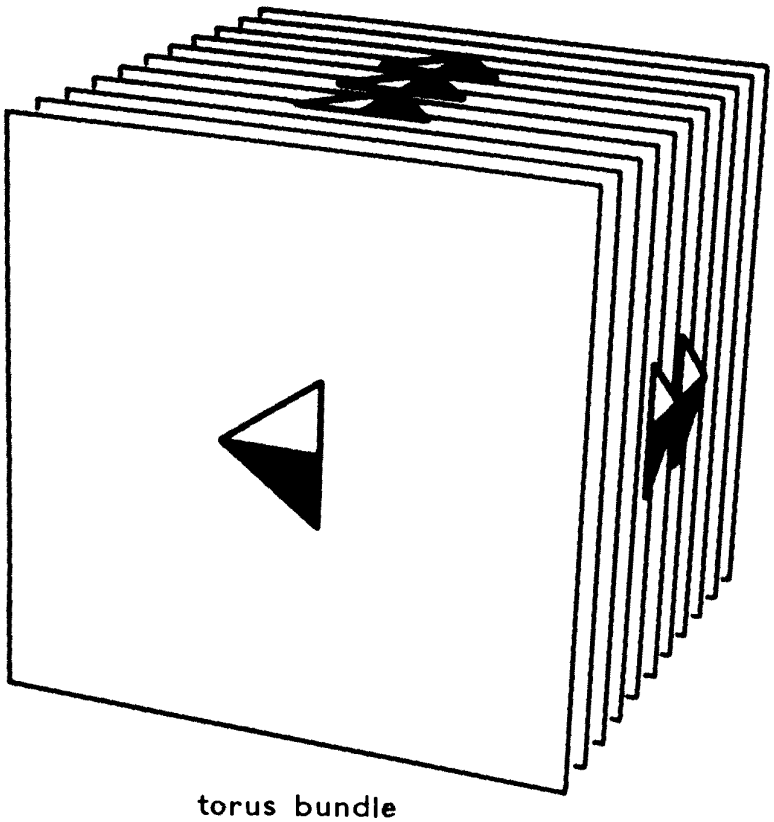

√2/2
1

Chapter 17

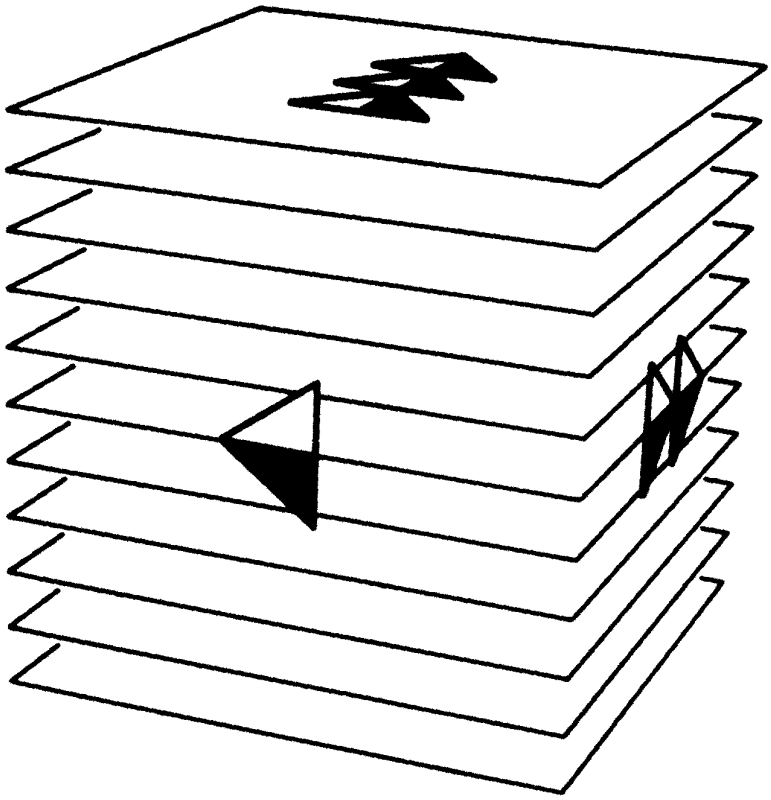
17.1.



17.2.

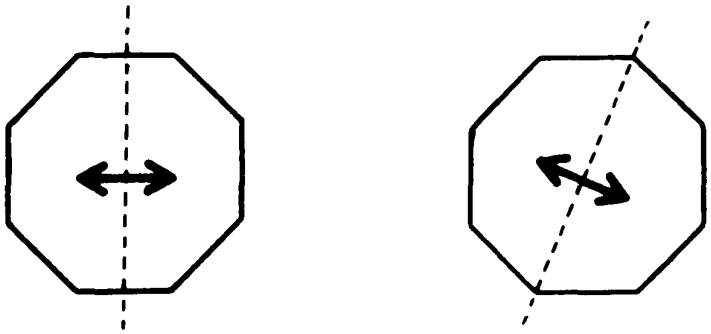


torus bundle
over a circle



Klein bottle bundle
over a circle

17.3. You can glue the top to the bottom with a 1/8, 1/4, 3/8 or 1/2 turn. You can also glue them with one of the two types of flips shown below, although for the flips you can't make a nice picture in three-dimensional space like Figure 17.4. (Interestingly enough, the two flips give topologically different manifolds.)



flip flip

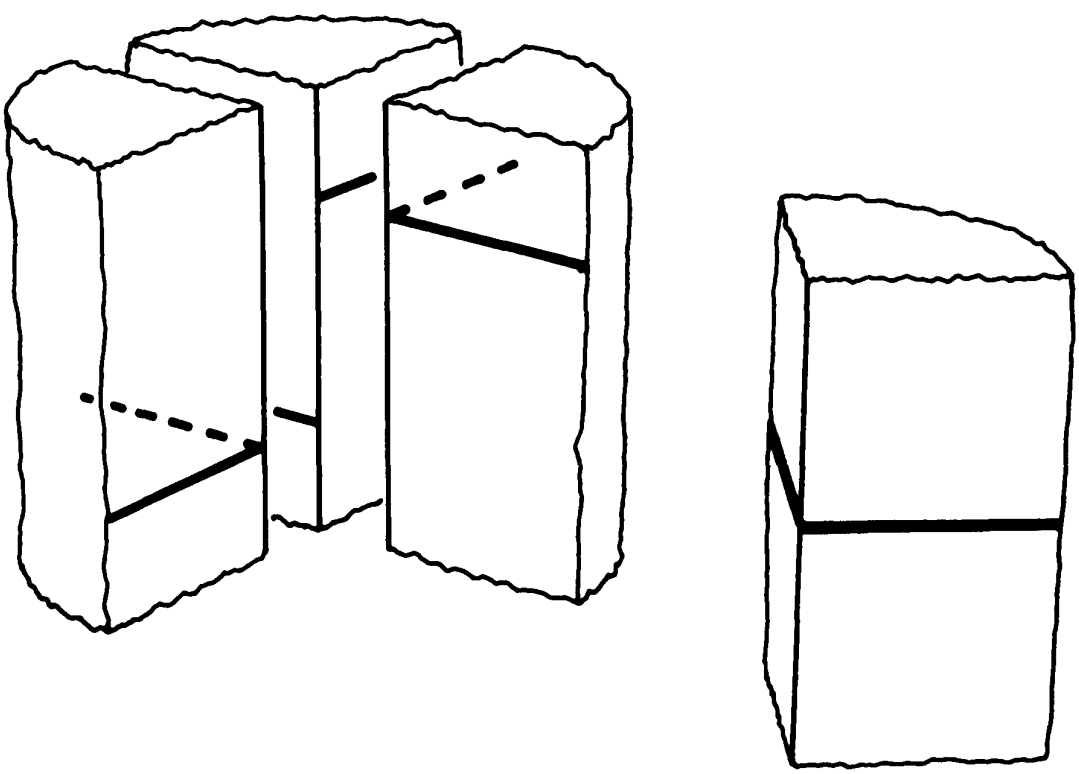
17.4. A torus and a Klein bottle. By the way, a Möbius strip with opposite edges glued is a Klein bottle:

Möbius strip Klein bottle

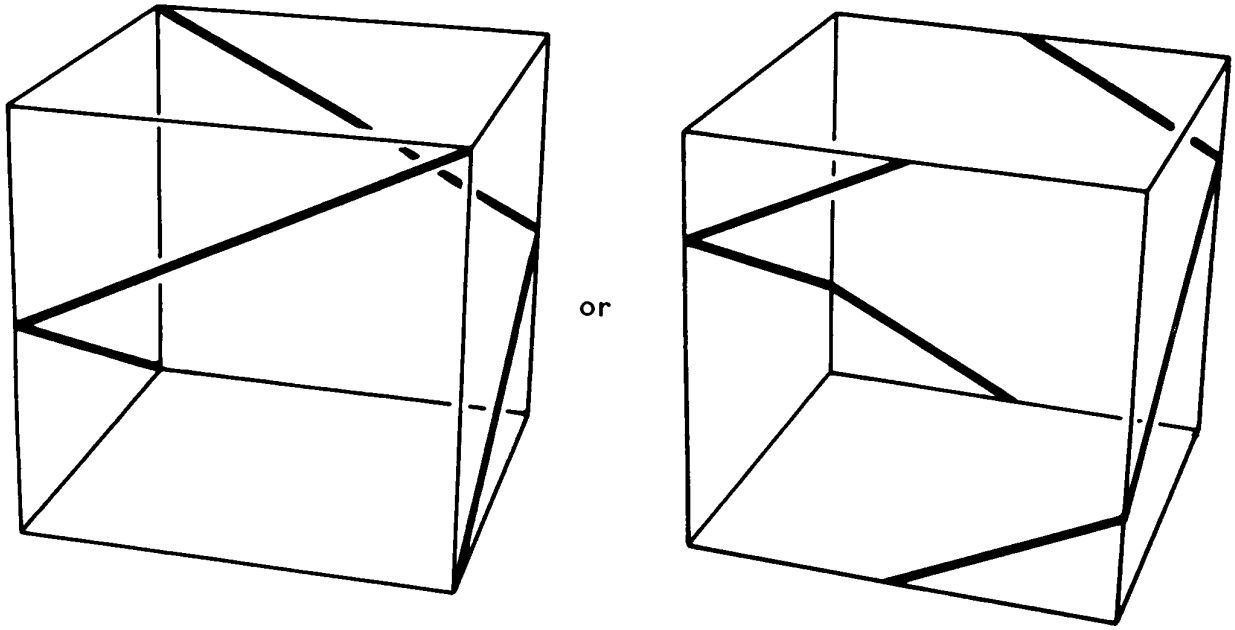
17.5. Start with a solid cylinder and imagine one end to be glued to the other with a side-to-side flip. What you get is a circle of disks, but it's not a product. (You won't be able to physically carry out the gluing in three-dimensional space, just as a Flatlander can't physically assemble a Möbius strip in the plane.)

17.6. Imagine $S^2 \times I$ as a thickened spherical shell. You can glue the inner surface to the outer surface in the most straightforward way to get $S^2 \times S^1$, or you can glue the inner surface to the outer surface with a side-to-side flip to get a nonorientable sphere bundle over a circle (K^3). Any other way of gluing the inner surface to the outer surface--such as gluing each point on the inner sphere to the diametrically opposite point on the outer sphere--gives a manifold which is topologically the same as one of the two already mentioned (it's like gluing the ends of a cylinder with a fraction of

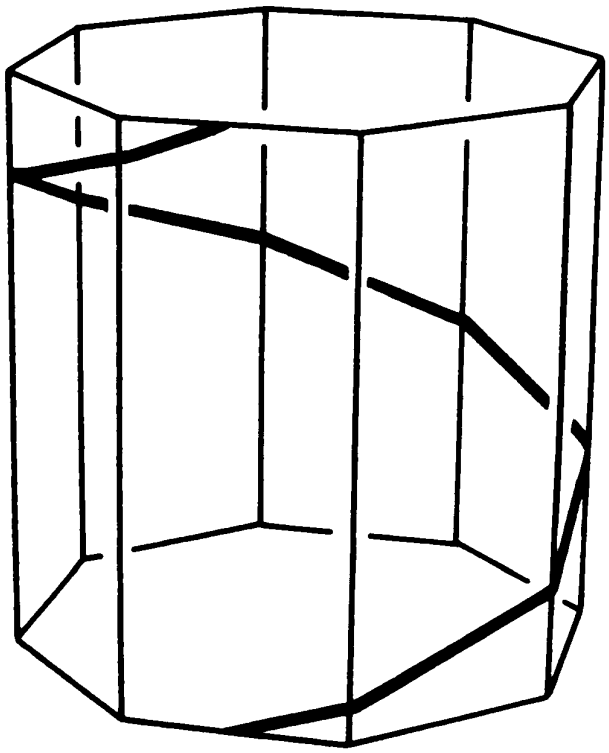
a turn to get a surface which is geometrically different from, but topologically the same as, the torus obtained by gluing the cylinder's ends without the turn; to get a topologically different surface you have to do something more drastic, like glue the cylinder's ends with a side-to-side flip).
You can make a "hypersolid" $S^2 \times S^1$ by gluing together the ends of a solid ball cross an interval in the simplest way. To make a "hypersolid" K^3 you'd have to glue them together with a side-to-side flip. To understand these four-dimensional manifolds, think about how A Square might try to understand an ordinary solid torus or solid Klein bottle.
17.7. The three manifolds of Exercise 7.4 plus $K^2 \times S^1$.
17.8. Glue the top of the prism to the bottom to get an octagon of circles, and then glue opposite sides to get a two-holed doughnut of circles. (An octagon with opposite sides glued has the topology of $T^2 \# T^2$.)
17.9. Imagine each cube in Figure 8.2 to be filled with vertical spaghetti. When a cube's top is glued to its bottom the spaghetti become circles, so you will get some sort of circle bundle in each case. The first is an orientable circle bundle over a torus, the second is a nonorientable circle bundle over a Klein bottle, the third is a nonorientable circle bundle over a torus, and the fourth is an orientable circle bundle over a Klein bottle.
17.10. Any three vertical edges can be glued together so that the horizontal lines match up, but you can never glue in the fourth edge because it has to be at two different "heights" at once. In the regular twisted torus this problem does not arise.



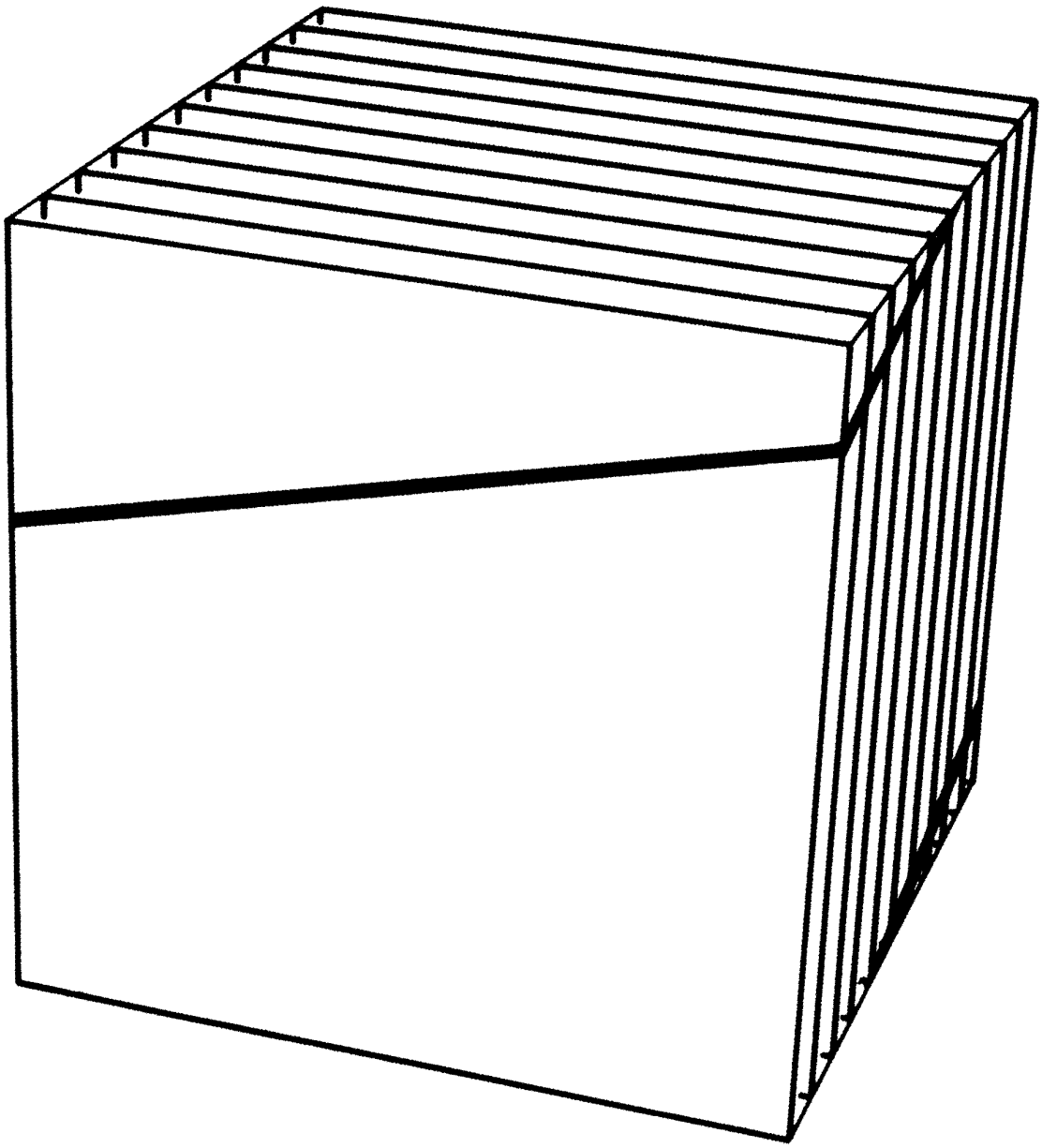
17.11



17.12.



17.13. Surprisingly, the twisted torus *is* a torus bundle over a circle: just slice it up as shown below and note that each slice is topologically a torus in spite of the shear. The twisted torus still isn't a product, though, because the way it's a circle of tori is unrelated to the way it's a torus of circles.



Chapter 18

18.1. K^3 (see Exercise 17.6).
18.2. Any surface cross a circle, just so the surface isn't S^2, P^2,

T^2 or K^2. Many surface bundles over circles also have $H^2 \times E$ geometry; see Exercise 17.3 for some specific examples.

18.3. K^3 and $P^2 \times S^1$ are nonorientable, while $S^2 \times S^1$ and the new manifold are orientable. The new manifold is in some ways similar to projective three-space: when you leave either the inner or the outer surface of $S^2 \times I$ (thought of as a thickened spherical shell) you come back rotated, but not mirror reversed, at the diametrically opposite point on the same surface.

18.4. No. In a three-manifold with twisted Euclidean geometry you can easily observe whether you have to travel clockwise or counterclockwise to come back below where you started. Say you are in such a manifold, and say you observe that travelling horizontally in a small clockwise circle brings you back below where you started. A friend of yours sets off on a long journey through distant regions of the manifold. All along the way he continues to observe that motion in a small clockwise circle brings one back below where one starts. When he returns from the long journey he still observes that clockwise motion brings one back below where one starts. This means that he isn't mirror-reversed! For if he were mirror-reversed it would seem to him that motion in a *counterclockwise* circle would be required to bring one back below where one started. This proves that a traveller never comes home mirror-reversed, so the manifold must be orientable.

18.5. The manifold from Exercise 17.12 will do nicely.

Chapter 19

19.1. According to the formula derived in the statement of the exercise, the Hubble time is 1/15 (million light-years)(sec)/(km) when the Hubble constant is 15 (km/sec)/(million light-years). The challenge is to put the answer into some sort of humanly comprehensible units, like years. To do this we'll need to know that 1 year = 365.25 days = 8766 hours = 525960 minutes = 31557600 seconds. This lets us deduce that if light can travel 300000 kilometers in one second, then it can travel $300000 \times 31557600 = 9.47 \times 10^{12}$ kilometers in a year. Thus one million light-years is 9.47×10^{18} kilometers. The Hubble time can now be converted to seconds: 1/15 (million light-years)(sec)/(km) = 1/15 (9.47×10^{18} km)(sec)/(km) = $(1/15)(9.47 \times 10^{18})$ (km)(sec)/(km) = 6.31×10^{17} sec. Finally, divide through by 31557600 to get the Hubble time in years: $(6.31 \times 10^{17}$ sec$) \div (31557600$ sec/year$) = 2.0 \times 10^{10}$ years. In words, that's twenty billion years. If you double the Hubble constant to 30 (km/sec)/(million light-years), then you halve the Hubble time to ten billion years.

Appendix B
Bibliography

1. Abbott, E.A., *Flatland: A Romance of Many Dimensions*, Dover, New York, 1952.

2. Burger, D., *Sphereland*, Barnes and Noble, New York, 1983.

3. Gamow, G., *Mr. Tompkins in Paperback*, Cambridge, 1971.

Flatland and *Sphereland* are required reading for all readers of this book! *Mr. Tompkins* is an amusing and informative collection of stories in which an innocent bank clerk dreams of worlds where relativistic and quantum effects are everyday phenomena; Story #5 "The Pulasating Universe" gives a vivid account of an expanding--and recollapsing--hypersphere.

4. Griffiths, *Surfaces*, 2nd ed., Cambridge, 1981.

This book is easy to read, and discusses some interesting and important stuff. Specifically, it contains a good simple explanation of Morse theory.

5. Rucker, R., *Geometry, Relativity and the Fourth Dimension*, Dover, New York, 1977.

This is a fun book. It bounces around between geometry, physics and philosophy.

Rucker has recently overhauled his book and republished it as

6. Rucker, R., *The Fourth Dimension*, Houghton-Mifflin, 1984.

This is the place to go if you want to understand the philosophical implications of spacetime.

7. Hilbert, D. and S. Cohn-Vossen, *Geometry and the Imagination* (translation of *Anschauliche Geometrie*, 1932), Chelsea, New York, 1952.

8. Coxeter, H.S.M., *Introduction to Geometry*, Wiley, New York, 1969.

These two books are good general references on geometry. You can read them cover-to-cover, but I prefer to browse around, reading whatever sections catch my interest.

9. Guilleman, V. and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

This is an excellent introduction to modern differential topology. It covers many beautiful topics--my favorites are vector fields and degrees of mappings--yet it requires only a minimum background, namely calculus and linear algebra (believe me, compared to other books on the topic this is a minimum).

10. Armstrong, M.A., *Basic Topology*, McGraw-Hill, England, 1979.

This is a good introduction to both point-set and algebraic topology. It requires prior knowledge of linear algebra and elementary group theory.

11. Rolfsen, D., *Knots and Links*, Publish or Perish, Berkeley, 1976.

I highly recommend *Knots and Links* as a follow-up to the present book. Prerequisites: abstract algebra and some algebraic topology.

12. Seifert, H. and W. Threlfall, *A Textbook of Topology* (translation of *Lehrbuch der Topologie*, 1934), Academic Press, New York, 1980.

Don't be put off by the age of this book. The ideas it contains (homology theory, fundamental group, covering spaces, 3-manifolds, Poincare duality, linking numbers) are as interesting and relevant today as they were in 1934.

13. Thurston, W., *Three-dimensional Geometry and Topology*, Princeton Univ. Press, to appear.

If you think of our Chapter 18 as a preview, then *Three-dimensional Geometry and Topology* is the feature film. You'll need a strong background in mathematics to understand all the details, but even without such a background you can still enjoy many of the examples.