Quaternion Reweighted Low-rank Factorization with Deep Denoising Prior for Color Image Inpainting (Supplementary Material)

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Appendix A. The proof of Theorem 5

We first introduce the following lemma, which serves as the cornerstone of the subsequent proof.

Lemma 1. *Let $\dot{\mathbf{U}} \in \mathbb{H}^{M\times r}$, $\dot{\mathbf{V}} \in \mathbb{H}^{N\times r}$ be two quaternion matrices, and let $g(\cdot) : \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi(x) = g(e^x)$ is convex and increasing on the interval $[\min\{\sigma_l(\dot{\mathbf{U}}\dot{\mathbf{V}}^H), \sigma_l(\dot{\mathbf{U}})\sigma_l(\dot{\mathbf{V}})\}, \sigma_1(\dot{\mathbf{U}})\sigma_1(\dot{\mathbf{V}})]$, where $l = \min\{M, N, r\}$. Then, for all $j = 1, 2, \cdots, l$, the following inequality holds:*

$$\sum_{i=1}^{j} g\left(\sigma_i(\dot{\mathbf{U}}\dot{\mathbf{V}}^H)\right) \leqslant \sum_{i=1}^{j} g\left(\sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}})\right).$$

Proof. According to the transformed complex adjoint matrix of quaternion matrix, we have:

$$\begin{aligned}\sum_{i=1}^{j} g\left(\sigma_i(\dot{\mathbf{U}}\dot{\mathbf{V}}^H)\right) &= \frac{1}{2}\sum_{i=1}^{j} g\left(\sigma_i(\mathbf{U}\mathbf{V}^H)_c\right) = \frac{1}{2}\sum_{i=1}^{j} g\left(\sigma_i(\mathbf{U}_c(\mathbf{V}^H)_c)\right)\\ &\leqslant \frac{1}{2}\sum_{i=1}^{j} g\left(\sigma_i(\mathbf{U}_c)\sigma_i((\mathbf{V}^H)_c)\right) = \sum_{i=1}^{j} g\left(\sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}})\right),\end{aligned}$$

where the first and second equalities can be directly verified based on (3), (4) and (5) (or refer to [1], and the inequality follows from [2] (Theorem 3.3.14). □

According to Lemma 1, by defining $g(x) = c(x+\epsilon)^{p-1}x$ with $0 < p \leqslant 1$, it is evident that $\varphi(x) = c(e^x+\epsilon)^{p-1}e^x$ is convex and increasing on the corresponding

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interval. Applying a special case of $j = \min\{M, N, r\}$ to Lemma 1 leads to the following corollary.

Corollary 1. *For two quaternion matrices* $\dot{\mathbf{U}} \in \mathbb{H}^{M\times r}, \dot{\mathbf{V}} \in \mathbb{H}^{N\times r}$ *and* $0 < p \leqslant 1$*, the following inequality holds:*

$$\sum_{i=1}^{\min\{M,N,r\}} \alpha_i \sigma_i(\dot{\mathbf{U}}\dot{\mathbf{V}}^H) \leqslant \sum_{i=1}^{\min\{M,N,r\}} \hat{\alpha}_i \left(\sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}})\right),$$

where the scalars

$$\alpha_i = c\left(\sigma_i(\dot{\mathbf{U}}\dot{\mathbf{V}}^H) + \epsilon\right)^{p-1}, \quad \hat{\alpha}_i = c\left(\sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}}) + \epsilon\right)^{p-1}, \quad i = 1, \cdots, r.$$

Next, we incorporate Corollary 1 to derive the upper bound of the proposed quaternion reweighted nuclear norm. Without loss of generality, we consider two distinct cases: $r \leqslant \min\{M, N\}$ and $r > \min\{M, N\}$.

1) In the first case, where $r \leqslant \min\{M, N\}$, the following inequalities hold:

$$\begin{aligned} ||\dot{\mathbf{X}}||_{\mathbf{w},*} &= \sum_{i=1}^{r} \alpha_i \sigma_i(\dot{\mathbf{X}}) = \sum_{i=1}^{r} \alpha_i \sigma_i(\dot{\mathbf{U}}\dot{\mathbf{V}}^H) \leqslant \sum_{i=1}^{r} \hat{\alpha}_i \left(\sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}})\right) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{r} \hat{\alpha}_i \left(\sigma_i^2(\dot{\mathbf{U}}) + \sigma_i^2(\dot{\mathbf{V}})\right) = \frac{1}{2}\left(||\dot{\mathbf{U}}||_{\mathbf{w},F}^2 + ||\dot{\mathbf{V}}||_{\mathbf{w},F}^2\right). \end{aligned}$$

Suppose the QSVD of $\dot{\mathbf{X}}$ is given by $\dot{\mathbf{X}} = \dot{\mathbf{L}}\mathbf{\Sigma}\dot{\mathbf{R}}^H$. We define $\dot{\mathbf{U}}^\star = \dot{\mathbf{L}}\mathbf{\Sigma}^{\frac{1}{2}}\dot{\mathbf{E}}^H$ and $\dot{\mathbf{V}}^\star = \dot{\mathbf{E}}\mathbf{\Sigma}^{\frac{1}{2}}\dot{\mathbf{R}}^H$, where $\dot{\mathbf{E}}$ is a unitary quaternion matrix. Then, it holds that $\dot{\mathbf{X}} = \dot{\mathbf{U}}^\star(\dot{\mathbf{V}}^\star)^H$, and we obtain the equality:

$$||\dot{\mathbf{X}}||_{\mathbf{w},*} = \frac{1}{2}\left(||\dot{\mathbf{U}}^\star||_{\mathbf{w},F}^2 + ||\dot{\mathbf{V}}^\star||_{\mathbf{w},F}^2\right).$$

2) In the second case, where $r > \min\{M, N\}$, we have:

$$\begin{aligned} ||\dot{\mathbf{X}}||_{\mathbf{w},*} &= \sum_{i=1}^{\min\{M,N\}} \alpha_i \sigma_i(\dot{\mathbf{X}}) \leqslant \sum_{i=1}^{\min\{M,N\}} \hat{\alpha}_i \sigma_i(\dot{\mathbf{U}})\sigma_i(\dot{\mathbf{V}}) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{\min\{M,N\}} \hat{\alpha}_i \left(\sigma_i^2(\dot{\mathbf{U}}) + \sigma_i^2(\dot{\mathbf{V}})\right) = \frac{1}{2}\left(||\dot{\mathbf{U}}||_{\mathbf{w},F}^2 + ||\dot{\mathbf{V}}||_{\mathbf{w},F}^2\right). \end{aligned}$$

Again, we use the QSVD of $\dot{\mathbf{X}} = \dot{\mathbf{L}}\mathbf{\Sigma}\dot{\mathbf{R}}^H$ to define $\dot{\mathbf{U}}^\star = \dot{\mathbf{L}}_{\mathrm{a}}\mathbf{\Sigma}_{\mathrm{a}}^{\frac{1}{2}}\dot{\mathbf{E}}^H$ and $\dot{\mathbf{V}}^\star = \dot{\mathbf{E}}\mathbf{\Sigma}_{\mathrm{a}}^{\frac{1}{2}}\dot{\mathbf{R}}_{\mathrm{a}}^H$, where $\dot{\mathbf{L}}_{\mathrm{a}}$ and $\dot{\mathbf{R}}_{\mathrm{a}}$ are constructed from $\dot{\mathbf{L}}$ and $\dot{\mathbf{R}}$ by appending $(r - \min\{M, N\})$ zero columns to $\dot{\mathbf{L}}$ and $\dot{\mathbf{R}}$, respectively, and $\mathbf{\Sigma}_{\mathrm{a}}$ is also constructed similarly from $\mathbf{\Sigma}$ through the zero padding. In this way, $\dot{\mathbf{X}} = \dot{\mathbf{U}}^\star(\dot{\mathbf{V}}^\star)^H$ holds, thereby completing the proof.

Appendix B. The proof of Corollary 1

According to Theorem 5, we have:

$$\begin{aligned}&\min_{\dot{\mathbf{U}},\dot{\mathbf{V}}} F_1(\dot{\mathbf{U}},\dot{\mathbf{V}}) \triangleq \frac{\lambda}{2}(||\dot{\mathbf{U}}||_{\mathbf{w},F}^2 + ||\dot{\mathbf{V}}||_{\mathbf{w},F}^2) + \frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{U}}\dot{\mathbf{V}}^H - \dot{\mathbf{Y}})||_F^2 \\ =&\min_{\dot{\mathbf{U}},\dot{\mathbf{V}}} F_2(\dot{\mathbf{U}},\dot{\mathbf{V}}) \triangleq \lambda||\dot{\mathbf{U}}\dot{\mathbf{V}}^H||_{\mathbf{w},*} + \frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{U}}\dot{\mathbf{V}}^H - \dot{\mathbf{Y}})||_F^2,\end{aligned} \tag{B.1}$$

which can be proved by contradiction as follows.

Assume that $\min_{\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1} F_1(\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1) < \min_{\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2} F_2(\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2)$. Then, we can construct a compared solution $F_2(\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1)$, which leads to $\min_{\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1} F_1(\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1) \geqslant F_2(\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1)$ based on Theorem 5, thus contradicting with the assumption.

Conversely, assume that $\min_{\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1} F_1(\dot{\mathbf{U}}_1,\dot{\mathbf{V}}_1) > \min_{\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2} F_2(\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2)$, we can also construct a compared solution $F_1(\dot{\mathbf{U}}^\star,\dot{\mathbf{V}}^\star)$, where $\dot{\mathbf{U}}^\star = \dot{\mathbf{L}}\mathbf{\Sigma}^{\frac{1}{2}}\dot{\mathbf{E}}^H$, $\dot{\mathbf{V}}^\star = \dot{\mathbf{E}}\mathbf{\Sigma}^{\frac{1}{2}}\dot{\mathbf{R}}^H$, $\dot{\mathbf{E}}$ is a unitary quaternion matrix, and $\dot{\mathbf{U}}_2\dot{\mathbf{V}}_2^H = \dot{\mathbf{L}}\mathbf{\Sigma}\dot{\mathbf{R}}^H$ denotes the QSVD of $(\dot{\mathbf{U}}_2\dot{\mathbf{V}}_2^H)$. By Theorem 5, we have $F_1(\dot{\mathbf{U}}^\star,\dot{\mathbf{V}}^\star) = \min_{\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2} F_2(\dot{\mathbf{U}}_2,\dot{\mathbf{V}}_2)$, which contradicts with the assumption again and thus (B.1) is proved.

Furthermore, for any quaternion matrix $\dot{\mathbf{X}} \in \mathbb{H}^{M\times N}$ with $\text{rank}(\dot{\mathbf{X}}) = r^\star \leqslant r$, it can be decomposed into $\dot{\mathbf{X}} = \dot{\mathbf{U}}\dot{\mathbf{V}}^H$ with $\dot{\mathbf{U}} \in \mathbb{H}^{M\times r}$ and $\dot{\mathbf{V}} \in \mathbb{H}^{N\times r}$. Therefore, we have:

$$\begin{aligned}\min_{\dot{\mathbf{X}}} \lambda||\dot{\mathbf{X}}||_{\mathbf{w},*} + \frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{X}} - \dot{\mathbf{Y}})||_F^2 &= \min_{\dot{\mathbf{X}}:\text{rank}(\dot{\mathbf{X}})=r^\star} \lambda||\dot{\mathbf{X}}||_{\mathbf{w},*} + \frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{X}} - \dot{\mathbf{Y}})||_F^2 \\ &= \min_{\dot{\mathbf{U}},\dot{\mathbf{V}}} \lambda||\dot{\mathbf{U}}\dot{\mathbf{V}}^H||_{\mathbf{w},*} + \frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{U}}\dot{\mathbf{V}}^H - \dot{\mathbf{Y}})||_F^2,\end{aligned} \tag{B.2}$$

where the first equality holds since the rank of the solution to (14) is $r^\star$. Combining (B.1) and (B.2) completes the proof of Corollary 1.

Appendix C. The proof of Theorem 6

It is evident that Theorem 6 is equivalent to proving the following formula:

$$\arg\min_{\dot{\mathbf{A}}} F(\dot{\mathbf{A}}) \triangleq \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + ||\dot{\mathbf{A}} - \dot{\mathbf{B}}||_F^2, \tag{C.1}$$

where $\dot{\mathbf{B}} = \dot{\mathbf{L}}\mathbf{\Sigma}\dot{\mathbf{R}}^H$ is the QSVD of $\dot{\mathbf{B}}$, and $\mathbf{W} = diag\{w_1, w_2, \cdots, w_s\}$.

Specifically, the objective function can be rewritten as:

$$
\begin{aligned}
F(\dot{\mathbf{A}}) &= \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + ||\dot{\mathbf{A}} - \dot{\mathbf{B}}||_F^2 \\
&= \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + (||\dot{\mathbf{A}}||_F^2 + ||\dot{\mathbf{B}}||_F^2 - 2\langle \dot{\mathbf{A}}, \dot{\mathbf{B}} \rangle) \\
&= \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + \left(||\dot{\mathbf{A}}||_F^2 + ||\dot{\mathbf{B}}||_F^2 - 2tr(\dot{\mathbf{A}}^H \dot{\mathbf{B}})\right).
\end{aligned}
\tag{C.2}
$$

According to the Von Neumann's Trace Inequality in Theorem 4, we have:

$$
\begin{aligned}
F(\dot{\mathbf{A}}) &\geqslant \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + \left(||\sigma(\dot{\mathbf{A}})||_F^2 + ||\sigma(\dot{\mathbf{B}})||_F^2 - 2tr(\sigma(\dot{\mathbf{A}})\sigma(\dot{\mathbf{B}}))\right) \\
&= \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + ||\sigma(\dot{\mathbf{A}}) - \sigma(\dot{\mathbf{B}})||_F^2,
\end{aligned}
\tag{C.3}
$$

where $\sigma(\dot{\mathbf{A}})$ and $\sigma(\dot{\mathbf{B}})$ are the ordered singular values of $\dot{\mathbf{A}}$ and $\dot{\mathbf{B}}$ with the same order, respectively.

Let $\dot{\mathbf{B}} = \dot{\mathbf{L}} diag\{\sigma(\dot{\mathbf{B}})\}\dot{\mathbf{R}}^H$ be the QSVD of $\dot{\mathbf{B}}$. The equation in (C.3) holds if and only if $\dot{\mathbf{L}}$ and $\dot{\mathbf{R}}$ are the unitary quaternion matrices from the QSVD $\dot{\mathbf{A}} = \dot{\mathbf{L}} diag\{\sigma(\dot{\mathbf{A}})\}\dot{\mathbf{R}}^H$. Subsequently, we investigate the optimal solution of the lower bound in (C.3), that is,

$$
\begin{aligned}
\hat{\sigma}(\dot{\mathbf{A}}) &= \arg\min_{\sigma(\dot{\mathbf{A}})} F(\sigma(\dot{\mathbf{A}})) \\
&= \arg\min_{\sigma(\dot{\mathbf{A}})} \gamma \sum_{i=1}^{s} w_i \sigma_i^2(\dot{\mathbf{A}}) + ||\sigma(\dot{\mathbf{A}}) - \sigma(\dot{\mathbf{B}})||_F^2.
\end{aligned}
\tag{C.4}
$$

Let $\frac{\partial F(\delta(\dot{\mathbf{A}}))}{\partial(\sigma(\dot{\mathbf{A}}))} = \dot{\mathbf{0}}$, we have:

$$
\hat{\sigma}(\dot{\mathbf{A}}) = (\gamma \mathbf{W} + \mathbf{I}_s)^{-1}\sigma(\dot{\mathbf{B}}),
$$

where $\mathbf{W} = diag\{w_1, w_2, \cdots, w_s\}$ and thus the optimum is given by:

$$
\begin{aligned}
\dot{\mathbf{A}}^\star = \arg\min_{\dot{\mathbf{A}}} F(\dot{\mathbf{A}}) &= \dot{\mathbf{L}} diag\{\hat{\sigma}(\dot{\mathbf{A}})\}\dot{\mathbf{R}}^H \\
&= \dot{\mathbf{L}} \left((\gamma \mathbf{W} + \mathbf{I}_s)^{-1}\mathbf{\Sigma}\right)\dot{\mathbf{R}}^H.
\end{aligned}
\tag{C.5}
$$

Appendix D. The proof of Theorem 7

1. We begin by proving that sequences $\{\dot{\mathbf{Q}}_U^k\}$, $\{\dot{\mathbf{Q}}_V^k\}$, $\{\dot{\mathbf{U}}^k\}$, $\{\dot{\mathbf{V}}^k\}$, and $\{\dot{\mathbf{X}}^k\}$ are Cauchy sequences. The Lagrange multipliers of the QRLMF method are updated according to equation (31), from which it follows that:

$$\left\{\begin{array}{l}\lim\limits_{k\to+\infty}||\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}||_F=0,\\ \lim\limits_{k\to+\infty}||\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}||_F=0,\\ \lim\limits_{k\to+\infty}||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H||_F=0.\end{array}\right. \tag{D.1}$$

Therefore, the sequence $\{(\dot{\mathbf{U}}^k,\dot{\mathbf{V}}^k,\dot{\mathbf{Q}}_U^k,\dot{\mathbf{Q}}_V^k,\dot{\mathbf{X}}^k)\}$ converges to a feasible solution.

To prove that sequences $\{\dot{\mathbf{Q}}_U^k\}$, $\{\dot{\mathbf{Q}}_V^k\}$, $\{\dot{\mathbf{U}}^k\}$, $\{\dot{\mathbf{V}}^k\}$ and $\{\dot{\mathbf{X}}^k\}$ are Cauchy sequences, we first demonstrate that these sequences are bounded. An effective approach is to show that the Lagrangian function associated with each iteration is bounded. Specifically, based on the update rules for $\dot{\mathbf{L}}_1^k$, $\dot{\mathbf{L}}_2^k$ and $\dot{\mathbf{L}}_3^k$, we have:

$$\begin{aligned}
&\mathcal{L}_{\mu^{k+1}}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^{k+1},\dot{\mathbf{L}}_2^{k+1},\dot{\mathbf{L}}_3^{k+1}\right)\\
&=\frac{\lambda}{2}\left(||\dot{\mathbf{Q}}_U^{k+1}||_{\mathbf{w},F}^2+||\dot{\mathbf{Q}}_V^{k+1}||_{\mathbf{w},F}^2\right)+\Re\left(\langle\dot{\mathbf{L}}_1^{k+1},\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}\rangle\right)+\Re\left(\langle\dot{\mathbf{L}}_2^{k+1},\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}\rangle\right)\\
&+\Re\left(\langle\dot{\mathbf{L}}_3^{k+1},\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H\rangle\right)+\frac{\mu^{k+1}}{2}||\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}||_F^2+\frac{\mu^{k+1}}{2}||\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}||_F^2\\
&+\frac{\mu^{k+1}}{2}||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H||_F^2+\frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{Y}})||_F^2\\
&=\mathcal{L}_{\mu^k}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^k,\dot{\mathbf{L}}_2^k,\dot{\mathbf{L}}_3^k\right)+\Re\left(\langle\dot{\mathbf{L}}_1^{k+1}-\dot{\mathbf{L}}_1^k,\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}\rangle\right)\\
&+\Re\left(\langle\dot{\mathbf{L}}_2^{k+1}-\dot{\mathbf{L}}_2^k,\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}\rangle\right)+\Re\left(\langle\dot{\mathbf{L}}_3^{k+1}-\dot{\mathbf{L}}_3^k,\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H\rangle\right)\\
&+\frac{\mu^{k+1}-\mu^k}{2}\left(||\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}||_F^2+||\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}||_F^2+||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H||_F^2\right)\\
&=\mathcal{L}_{\mu^k}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^k,\dot{\mathbf{L}}_2^k,\dot{\mathbf{L}}_3^k\right)+\frac{\mu^{k+1}+\mu^k}{2(\mu^k)^2}\left(||\dot{\mathbf{L}}_1^{k+1}-\dot{\mathbf{L}}_1^k||_F^2+||\dot{\mathbf{L}}_2^{k+1}-\dot{\mathbf{L}}_2^k||_F^2+||\dot{\mathbf{L}}_3^{k+1}-\dot{\mathbf{L}}_3^k||_F^2\right).
\end{aligned}$$

Let $M_j^2=\max\left\{||\dot{\mathbf{L}}_j^{k+1}-\dot{\mathbf{L}}_j^k||_F^2,k=1,2,\cdots\right\}$ $(j=1,2,3)$, it follows that:

$$\begin{aligned}
&\mathcal{L}_{\mu^{k+1}}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^{k+1},\dot{\mathbf{L}}_2^{k+1},\dot{\mathbf{L}}_3^{k+1}\right)\\
&\leqslant\mathcal{L}_{\mu^k}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^k,\dot{\mathbf{L}}_2^k,\dot{\mathbf{L}}_3^k\right)+\frac{\mu^{k+1}+\mu^k}{2(\mu^k)^2}\left(M_1^2+M_2^2+M_3^2\right)\\
&\leqslant\mathcal{L}_{\mu^0}\left(\dot{\mathbf{Q}}_U^1,\dot{\mathbf{Q}}_V^1,\dot{\mathbf{U}}^1,\dot{\mathbf{V}}^1,\dot{\mathbf{X}}^1,\dot{\mathbf{L}}_1^0,\dot{\mathbf{L}}_2^0,\dot{\mathbf{L}}_3^0\right)+\left(M_1^2+M_2^2+M_3^2\right)\sum_{i=0}^k\frac{1+\rho}{2\mu^0\rho^i}\\
&\leqslant\mathcal{L}_{\mu^0}\left(\dot{\mathbf{Q}}_U^1,\dot{\mathbf{Q}}_V^1,\dot{\mathbf{U}}^1,\dot{\mathbf{V}}^1,\dot{\mathbf{X}}^1,\dot{\mathbf{L}}_1^0,\dot{\mathbf{L}}_2^0,\dot{\mathbf{L}}_3^0\right)+\frac{\left(M_1^2+M_2^2+M_3^2\right)}{\mu^0}\sum_{i=0}^k\frac{1}{\rho^{i-1}}<+\infty.
\end{aligned}$$

It is noted that the second inequality holds universally, since the globally optimal solutions for $\{\dot{\mathbf{X}}^k\},\{\dot{\mathbf{Q}}_U^k\}$ and $\{\dot{\mathbf{Q}}_V^k\}$ are obtained in their respective subproblems. This ensures that the sequence of the Lagrangian function is bounded. Furthermore, we proceed to show that the sequences $\{\dot{\mathbf{X}}^k\}$, $\{\dot{\mathbf{Q}}_U^k\}$ and $\{\dot{\mathbf{Q}}_V^k\}$

are bounded.

$$
\begin{aligned}
&\frac{1}{2}||\mathcal{P}_\Omega(\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{Y}})||_F^2+\frac{\lambda}{2}\left(||\dot{\mathbf{Q}}_U^{k+1}||_{\mathbf{w},F}^2+||\dot{\mathbf{Q}}_V^{k+1}||_{\mathbf{w},F}^2\right)\\
&=\mathcal{L}_{\mu^k}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^k,\dot{\mathbf{L}}_2^k,\dot{\mathbf{L}}_3^k\right)-\Re\left(\langle\dot{\mathbf{L}}_1^k,\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}\rangle\right)-\Re\left(\langle\dot{\mathbf{L}}_2^k,\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}\rangle\right)\\
&-\Re\left(\langle\dot{\mathbf{L}}_3^k,\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H\rangle\right)-\frac{\mu^k}{2}||\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^{k+1}||_F^2-\frac{\mu^k}{2}||\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^{k+1}||_F^2-\frac{\mu^k}{2}||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H||_F^2\\
&=\mathcal{L}_{\mu^k}\left(\dot{\mathbf{Q}}_U^{k+1},\dot{\mathbf{Q}}_V^{k+1},\dot{\mathbf{U}}^{k+1},\dot{\mathbf{V}}^{k+1},\dot{\mathbf{X}}^{k+1},\dot{\mathbf{L}}_1^k,\dot{\mathbf{L}}_2^k,\dot{\mathbf{L}}_3^k\right)\\
&+\frac{1}{2\mu^k}\left(||\dot{\mathbf{L}}_1^k||_F^2-||\dot{\mathbf{L}}_1^{k+1}||_F^2+||\dot{\mathbf{L}}_2^k||_F^2-||\dot{\mathbf{L}}_2^{k+1}||_F^2+||\dot{\mathbf{L}}_3^k||_F^2-||\dot{\mathbf{L}}_3^{k+1}||_F^2\right).
\end{aligned}
$$

Note that $||\mathcal{P}_\Omega(\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{Y}})||_F^2$ and the reweighted Frobenius norm are nonnegative. Therefore, the sequences $\{\dot{\mathbf{X}}^k\},\{\dot{\mathbf{Q}}_U^k\}$ and $\{\dot{\mathbf{Q}}_V^k\}$ are bounded. Furthermore, based on (D.1), it follows that sequences $\{\dot{\mathbf{U}}^k\}$ and $\{\dot{\mathbf{V}}^k\}$ are also bounded.

From (22), by applying quaternion matrix derivatives and setting them to zero, we substitute

$$
\dot{\mathbf{X}}^k=\dot{\mathbf{U}}^k(\dot{\mathbf{V}}^k)^H+\frac{\dot{\mathbf{L}}_3^k}{\mu^{k-1}}-\frac{\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}
$$

to derive the following computational procedure:

$$
\begin{aligned}
&\left(\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{Q}}_U^k+\frac{\dot{\mathbf{L}}_1^k}{\mu^k}\right)+\left(\dot{\mathbf{U}}^{k+1}\left(\dot{\mathbf{V}}^k\right)^H-\dot{\mathbf{X}}^k-\frac{\dot{\mathbf{L}}_3^k}{\mu^k}\right)\dot{\mathbf{V}}^k\\
&=\left(\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{U}}^k+\dot{\mathbf{U}}^k-\dot{\mathbf{Q}}_U^k+\frac{\dot{\mathbf{L}}_1^k}{\mu^k}\right)+\left(\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^k)^H-\dot{\mathbf{U}}^k(\dot{\mathbf{V}}^k)^H-\frac{\dot{\mathbf{L}}_3^k}{\mu^{k-1}}+\frac{\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}-\frac{\dot{\mathbf{L}}_3^k}{\mu^k}\right)\dot{\mathbf{V}}^k\\
&=(\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{U}}^k)\left(\dot{\mathbf{I}}+(\dot{\mathbf{V}}^k)^H\dot{\mathbf{V}}^k\right)+\frac{\dot{\mathbf{L}}_1^k-\dot{\mathbf{L}}_1^{k-1}}{\mu^{k-1}}+\frac{\dot{\mathbf{L}}_1^k}{\mu^k}-\left(\frac{\dot{\mathbf{L}}_3^k-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}+\frac{\dot{\mathbf{L}}_3^k}{\mu^k}\right)\dot{\mathbf{V}}^k=\dot{\mathbf{0}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{Q}}_V^k+\frac{\dot{\mathbf{L}}_2^k}{\mu^k}\right)+\left(\dot{\mathbf{V}}^{k+1}(\dot{\mathbf{U}}^{k+1})^H-(\dot{\mathbf{X}}^k)^H-\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^k}\right)\dot{\mathbf{U}}^{k+1}\\
&=\left(\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^k+\dot{\mathbf{V}}^k-\dot{\mathbf{Q}}_V^k+\frac{\dot{\mathbf{L}}_2^k}{\mu^k}\right)+\left(\dot{\mathbf{V}}^{k+1}(\dot{\mathbf{U}}^{k+1})^H-\dot{\mathbf{V}}^k(\dot{\mathbf{U}}^k)^H-\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^{k-1}}+\frac{(\dot{\mathbf{L}}_3^{k-1})^H}{\mu^{k-1}}-\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^k}\right)\dot{\mathbf{U}}^{k+1}\\
&=\left(\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^k+\dot{\mathbf{V}}^k-\dot{\mathbf{Q}}_V^k+\frac{\dot{\mathbf{L}}_2^k}{\mu^k}\right)+(\dot{\mathbf{V}}^{k+1}(\dot{\mathbf{U}}^{k+1})^H\\
&-\dot{\mathbf{V}}^k(\dot{\mathbf{U}}^{k+1})^H+\dot{\mathbf{V}}^k(\dot{\mathbf{U}}^{k+1})^H-\dot{\mathbf{V}}^k(\dot{\mathbf{U}}^k)^H-\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^{k-1}}+\frac{(\dot{\mathbf{L}}_3^{k-1})^H}{\mu^{k-1}}-\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^k})\dot{\mathbf{U}}^{k+1}\\
&=(\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^k)\left(\dot{\mathbf{I}}+(\dot{\mathbf{U}}^{k+1})^H\dot{\mathbf{U}}^{k+1}\right)+\frac{\dot{\mathbf{L}}_2^k-\dot{\mathbf{L}}_2^{k-1}}{\mu^{k-1}}+\frac{\dot{\mathbf{L}}_2^k}{\mu^k}+\dot{\mathbf{V}}^k(\dot{\mathbf{U}}^{k+1}\\
&-\dot{\mathbf{U}}^k)^H\dot{\mathbf{U}}^{k+1}-\left(\frac{(\dot{\mathbf{L}}_3^k-\dot{\mathbf{L}}_3^{k-1})^H}{\mu^{k-1}}+\frac{(\dot{\mathbf{L}}_3^k)^H}{\mu^k}\right)\dot{\mathbf{U}}^{k+1}=\dot{\mathbf{0}}.
\end{aligned}
$$

To proceed, we isolate the differences $\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{U}}^{k}$ and $\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^{k}$ on one side of the equation, which yields:

$$
\begin{aligned}
&\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{U}}^{k}\\
&=\left(\frac{\dot{\mathbf{L}}_1^{k-1}-\dot{\mathbf{L}}_1^{k}}{\mu^{k-1}}-\frac{\dot{\mathbf{L}}_1^{k}}{\mu^{k}}+\left(\frac{\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}+\frac{\dot{\mathbf{L}}_3^{k}}{\mu^{k}}\right)\dot{\mathbf{V}}^{k}\right)\left(\dot{\mathbf{I}}+\left(\dot{\mathbf{V}}^{k}\right)^{H}\dot{\mathbf{V}}^{k}\right)^{-1}\\
&=\frac{1}{\mu^{k}}\left(\rho(\dot{\mathbf{L}}_1^{k-1}-\dot{\mathbf{L}}_1^{k})-\dot{\mathbf{L}}_1^{k}+\left(\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})+\dot{\mathbf{L}}_3^{k}\right)\dot{\mathbf{V}}^{k}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{V}}^{k})^{H}\dot{\mathbf{V}}^{k}\right)^{-1}=\frac{1}{\mu^{k}}\dot{\mathbf{C}}_1^{k},
\end{aligned}
$$

and

$$
\begin{aligned}
&\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^{k}\\
&=\left(\frac{\dot{\mathbf{L}}_2^{k-1}-\dot{\mathbf{L}}_2^{k}}{\mu^{k-1}}-\frac{\dot{\mathbf{L}}_2^{k}}{\mu^{k}}+\dot{\mathbf{V}}^{k}(\dot{\mathbf{U}}^{k}-\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}+\left(\frac{(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})^{H}}{\mu^{k-1}}+\frac{(\dot{\mathbf{L}}_3^{k})^{H}}{\mu^{k}}\right)\dot{\mathbf{U}}^{k+1}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}\right)^{-1}\\
&=\frac{1}{\mu^{k}}\left(\rho(\dot{\mathbf{L}}_2^{k-1}-\dot{\mathbf{L}}_2^{k})-\dot{\mathbf{L}}_2^{k}+\mu^{k}\dot{\mathbf{V}}^{k}(\dot{\mathbf{U}}^{k}-\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}+\left(\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})^{H}+(\dot{\mathbf{L}}_3^{k})^{H}\right)\dot{\mathbf{U}}^{k+1}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}\right)^{-1}\\
&=\frac{1}{\mu^{k}}\left(\rho(\dot{\mathbf{L}}_2^{k-1}-\dot{\mathbf{L}}_2^{k})-\dot{\mathbf{L}}_2^{k}-\dot{\mathbf{V}}^{k}(\dot{\mathbf{C}}_1^{k})^{H}\dot{\mathbf{U}}^{k+1}+\left(\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})^{H}+(\dot{\mathbf{L}}_3^{k})^{H}\right)\dot{\mathbf{U}}^{k+1}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}\right)^{-1}\\
&=\frac{1}{\mu^{k}}\dot{\mathbf{C}}_2^{k},
\end{aligned}
$$

where $\dot{\mathbf{C}}_1^{k}=\left(\rho(\dot{\mathbf{L}}_1^{k-1}-\dot{\mathbf{L}}_1^{k})-\dot{\mathbf{L}}_1^{k}+\left(\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})+\dot{\mathbf{L}}_3^{k}\right)\dot{\mathbf{V}}^{k}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{V}}^{k})^{H}\dot{\mathbf{V}}^{k}\right)^{-1}$,
$\dot{\mathbf{C}}_2^{k}=\left(\rho(\dot{\mathbf{L}}_2^{k-1}-\dot{\mathbf{L}}_2^{k})-\dot{\mathbf{L}}_2^{k}-\dot{\mathbf{V}}^{k}(\dot{\mathbf{C}}_1^{k})^{H}\dot{\mathbf{U}}^{k+1}+\left(\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})^{H}+(\dot{\mathbf{L}}_3^{k})^{H}\right)\dot{\mathbf{U}}^{k+1}\right)\left(\dot{\mathbf{I}}+(\dot{\mathbf{U}}^{k+1})^{H}\dot{\mathbf{U}}^{k+1}\right)^{-1}$. Therefore, the sequences $\{\dot{\mathbf{U}}^{k}\}$ and $\{\dot{\mathbf{V}}^{k}\}$ are Cauchy sequences.

In the following, we demonstrate that the sequence $\{\dot{\mathbf{X}}^{k}\}$ is also a Cauchy sequence. Based on the update step for the sequence $\{\dot{\mathbf{L}}_3^{k}\}$, we have:

$$
\dot{\mathbf{X}}^{k+1}=\frac{\dot{\mathbf{L}}_3^{k+1}-\dot{\mathbf{L}}_3^{k}}{\mu^{k}}+\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H},
$$

and

$$
\begin{aligned}
&||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{X}}^{k}||_F\\
&=||\mathcal{P}_{\Omega}\left(\frac{\mu^{k}\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H}-\dot{\mathbf{L}}_3^{k}+\dot{\mathbf{Y}}}{1+\mu^{k}}\right)+\mathcal{P}_{\Omega^c}\left(\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H}-\frac{\dot{\mathbf{L}}_3^{k}}{\mu^{k}}\right)-\frac{\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}-\dot{\mathbf{U}}^{k}(\dot{\mathbf{V}}^{k})^{H}||_F\\
&=||\mathcal{P}_{\Omega}\left(\frac{\dot{\mathbf{Y}}-\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H}}{1+\mu^{k}}\right)-\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{L}}_3^{k}}{\mu^{k}}\right)-\frac{\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}+(\dot{\mathbf{U}}^{k+1}-\dot{\mathbf{U}}^{k})(\dot{\mathbf{V}}^{k+1})^{H}+\dot{\mathbf{U}}^{k}(\dot{\mathbf{V}}^{k+1}-\dot{\mathbf{V}}^{k})^{H}||_F\\
&=||\mathcal{P}_{\Omega}\left(\frac{\dot{\mathbf{Y}}-\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H}}{1+\mu^{k}}\right)-\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{L}}_3^{k}}{\mu^{k}}\right)-\frac{\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}+\frac{\dot{\mathbf{C}}_1^{k}}{\mu^{k}}(\dot{\mathbf{V}}^{k+1})^{H}+\frac{\dot{\mathbf{U}}^{k}}{\mu^{k}}(\dot{\mathbf{C}}_2^{k})^{H}||_F\\
&=\frac{1}{\mu^{k}}||\dot{\mathbf{C}}_3^{k}||_F,
\end{aligned}
$$

where $\dot{\mathbf{C}}_3^{k}=\mathcal{P}_{\Omega}\left(\frac{\mu^{k}}{1+\mu^{k}}\left(\dot{\mathbf{Y}}-\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^{H}\right)\right)-\mathcal{P}_{\Omega^c}(\dot{\mathbf{L}}_3^{k})-\rho(\dot{\mathbf{L}}_3^{k}-\dot{\mathbf{L}}_3^{k-1})+\dot{\mathbf{C}}_1^{k}(\dot{\mathbf{V}}^{k+1})^{H}+\dot{\mathbf{U}}^{k}(\dot{\mathbf{C}}_2^{k})^{H}$. Thus, $\{\dot{\mathbf{X}}^{k}\}$ is a Cauchy sequence.

2. By applying the first-order optimization condition to equations (22), (26) and (29), we have:

$$\begin{cases} \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_U^{k+1}||_{\mathbf{w},F} + \mu^k\left(\dot{\mathbf{Q}}_U^{k+1} - \dot{\mathbf{U}}^{k+1} - \frac{\dot{\mathbf{L}}_1^k}{\mu^k}\right), \\ \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_V^{k+1}||_{\mathbf{w},F} + \mu^k\left(\dot{\mathbf{Q}}_V^{k+1} - \dot{\mathbf{V}}^{k+1} - \frac{\dot{\mathbf{L}}_2^k}{\mu^k}\right), \\ \mathcal{P}_\Omega\left(\dot{\mathbf{X}}^{k+1} - \frac{\mu^k\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H - \dot{\mathbf{L}}_3^k + \dot{\mathbf{Y}}}{1+\mu^k}\right) + \mathcal{P}_{\Omega^c}\left(\dot{\mathbf{X}}^{k+1} - \dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H + \frac{\dot{\mathbf{L}}_3^k}{\mu^k}\right) = \dot{\mathbf{0}}. \end{cases}$$

Let $\dot{\mathbf{U}}^\star$, $\dot{\mathbf{V}}^\star$, $\dot{\mathbf{Q}}_U^\star$, $\dot{\mathbf{Q}}_V^\star$ and $\dot{\mathbf{X}}^\star$ denote the accumulation points of the sequences $\{\dot{\mathbf{U}}^k\}$, $\{\dot{\mathbf{V}}^k\}$, $\{\dot{\mathbf{Q}}_U^k\}$, $\{\dot{\mathbf{Q}}_V^k\}$ and $\{\dot{\mathbf{X}}^k\}$, respectively. It follows that $\dot{\mathbf{U}}^\star = \dot{\mathbf{Q}}_U^\star$, $\dot{\mathbf{V}}^\star = \dot{\mathbf{Q}}_V^\star$ and $\dot{\mathbf{X}}^\star = \dot{\mathbf{U}}^\star(\dot{\mathbf{V}}^\star)^H$. Therefore, as $k \to +\infty$, it obtains:

$$\begin{cases} \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_U^\star||_{\mathbf{w},F} - \dot{\mathbf{L}}_1^\star, \\ \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_V^\star||_{\mathbf{w},F} - \dot{\mathbf{L}}_2^\star, \\ \mathcal{P}_\Omega(\dot{\mathbf{X}}^\star) - \dot{\mathbf{Y}} + \mathcal{P}_\Omega(\dot{\mathbf{L}}_3^\star) = \dot{\mathbf{0}}, \\ \mathcal{P}_{\Omega^c}(\dot{\mathbf{L}}_3^\star) = \dot{\mathbf{0}}. \end{cases}$$

Hence, it concludes that any accumulation points $\{(\dot{\mathbf{U}}^\star, \dot{\mathbf{V}}^\star, \dot{\mathbf{Q}}_U^\star, \dot{\mathbf{A}}_V^\star, \dot{\mathbf{X}}^\star)\}$ of the sequence $\{(\dot{\mathbf{U}}^k, \dot{\mathbf{V}}^k, \dot{\mathbf{Q}}_U^k, \dot{\mathbf{Q}}_V^k, \dot{\mathbf{X}}^k)\}$ generated by QRLMF satisfies the KKT conditions.

Appendix E. The proof of Theorem 8

1. As in the proof of Theorem 7, it can also be verified that the sequences $\{\dot{\mathbf{Q}}_U^k\}$, $\{\dot{\mathbf{Q}}_V^k\}$, $\{\dot{\mathbf{U}}^k\}$ and $\{\dot{\mathbf{V}}^k\}$ are Cauchy sequences. In the following, we proceed to show that the sequences $\{\dot{\mathbf{X}}^k\}$ and $\{\mathcal{S}^k\}$ also are Cauchy sequences.

By analyzing equation (39), from which we can derive that:

$$\begin{cases} \dot{\mathbf{X}}^{k+1} = \frac{\dot{\mathbf{L}}_3^{k+1} - \dot{\mathbf{L}}_3^k}{\mu^k} + \dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H, \\ \Gamma^{-1}(\dot{\mathbf{X}}^{k+1}) = \frac{\mathcal{T}^{k+1} - \mathcal{T}^k}{\mu^k} + \mathcal{S}^{k+1}. \end{cases}$$

Convert $\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})$ into quaternion form. Note that $\dot{\mathbf{X}}^k$ and $\Gamma^{-1}(\dot{\mathbf{X}}^k)$ represent the same color image but in different formats. Consequently, we have:

$$\begin{aligned} \dot{\mathbf{X}}^{k+1} &= \frac{\dot{\mathbf{X}}^{k+1} + \Gamma\left(\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})\right)}{2} \\ &= \frac{\frac{\dot{\mathbf{L}}_3^{k+1} - \dot{\mathbf{L}}_3^k}{\mu^k} + \dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H + \Gamma\left(\frac{\mathcal{T}^{k+1} - \mathcal{T}^k}{\mu^k}\right) + \Gamma(\mathcal{S}^{k+1})}{2}. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
&||\dot{\mathbf{X}}^{k+1}-\dot{\mathbf{X}}^{k}||_F\\
&=||\mathcal{P}_{\Omega}\left(\frac{\mu^k\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H+\mu^k\Gamma(\mathcal{S}^k)-\dot{\mathbf{L}}_3^k-\Gamma(\mathcal{T}^k)+\dot{\mathbf{Y}}}{1+2\mu^k}\right)+\mathcal{P}_{\Omega^c}\left(\frac{1}{2}\left(\dot{\mathbf{U}}^{k+1}(\dot{\mathbf{V}}^{k+1})^H+\Gamma(\mathcal{S}^k)-\frac{\dot{\mathbf{L}}_3^k}{\mu^k}-\Gamma\left(\frac{\mathcal{T}^k}{\mu^k}\right)\right)\right)\\
&-\frac{1}{2}\left(\dot{\mathbf{U}}^{k}(\dot{\mathbf{V}}^{k})^H+\Gamma(\mathcal{S}^k)+\frac{\dot{\mathbf{L}}_3^k-\dot{\mathbf{L}}_3^{k-1}}{\mu^{k-1}}+\Gamma\left(\frac{\mathcal{T}^k-\mathcal{T}^{k-1}}{\mu^{k-1}}\right)\right)||_F\\
&=||\mathcal{P}_{\Omega}\left(\frac{2\dot{\mathbf{Y}}-\Gamma(\mathcal{S}^k)-2\dot{\mathbf{L}}_3^k-2\Gamma(\mathcal{T}^k)-\dot{\mathbf{U}}^{k}(\dot{\mathbf{V}}^{k})^H}{2(1+2\mu^k)}-\frac{\dot{\mathbf{C}}_1^k(\dot{\mathbf{V}}^{k+1})^H-\dot{\mathbf{U}}^{k}(\dot{\mathbf{C}}_2^k)^H}{(1+2\mu^k)}\right)+\frac{1}{2}\mathcal{P}_{\Omega^c}\left(\frac{1}{\mu^k}\left(\dot{\mathbf{C}}_1^k(\dot{\mathbf{V}}^{k+1})^H-\dot{\mathbf{U}}^{k}(\dot{\mathbf{C}}_2^k)^H\right)\right)\\
&-\frac{1}{2}\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{L}}_3^k}{\mu^k}+\Gamma\left(\frac{\mathcal{T}^k}{\mu^k}\right)\right)-\frac{\dot{\mathbf{L}}_3^k-\dot{\mathbf{L}}_3^{k-1}}{2\mu^{k-1}}-\Gamma\left(\frac{\mathcal{T}^k-\mathcal{T}^{k-1}}{2\mu^{k-1}}\right)||_F=\frac{1}{\mu^k}||\dot{\mathbf{C}}_4^k||_F,
\end{aligned}
$$

where $\dot{\mathbf{C}}_4^k=\mathcal{P}_{\Omega}\left(\frac{\mu^k}{2(1+2\mu^k)}\left(2\dot{\mathbf{Y}}-\Gamma(\mathcal{S}^k)-2\dot{\mathbf{L}}_3^k-2\Gamma(\mathcal{T}^k)-\dot{\mathbf{U}}^k(\dot{\mathbf{V}}^k)^H-2\dot{\mathbf{C}}_1^k(\dot{\mathbf{V}}^{k+1})^H-2\dot{\mathbf{U}}^k(\dot{\mathbf{C}}_2^k)^H\right)\right)$ $+\frac{1}{2}\mathcal{P}_{\Omega^c}\left(\dot{\mathbf{C}}_1^k(\dot{\mathbf{V}}^{k+1})^H+\dot{\mathbf{U}}^k(\dot{\mathbf{C}}_2^k)^H\right)-\frac{1}{2}\mathcal{P}_{\Omega^c}\left(\dot{\mathbf{L}}_3^k+\Gamma(\mathcal{T}^k)\right)-\frac{\rho(\dot{\mathbf{L}}_3^k-\dot{\mathbf{L}}_3^{k-1})}{2}-\rho\Gamma\left(\frac{\mathcal{T}^k-\mathcal{T}^{k-1}}{2}\right)$. Therefore, it follows that $\{\dot{\mathbf{X}}^k\}$ is a Cauchy sequence.

Based on (39), we have:

$$
\mathcal{T}^{k+1}=\mathcal{T}^k+\mu^k\left(\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})-\mathcal{S}^{k+1}\right),
$$

it is easy to obtain:

$$
\mathcal{S}^{k+1}=\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})+\frac{\mathcal{T}^k}{\mu^k}-\frac{\mathcal{T}^{k+1}}{\mu^k}.
$$

Furthermore, we prove that the sequence $\{\mathcal{S}^k\}$ is a Cauchy sequence.

$$
\begin{aligned}
||\mathcal{S}^{k+1}-\mathcal{S}^k||_F&=||\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})+\frac{\mathcal{T}^k}{\mu^k}-\frac{\mathcal{T}^{k+1}}{\mu^k}-\mathcal{S}^k||_F\\
&=||\Gamma^{-1}(\dot{\mathbf{X}}^{k})-\mathcal{S}^k+\Gamma^{-1}(\dot{\mathbf{X}}^{k+1})-\Gamma^{-1}(\dot{\mathbf{X}}^{k})+\frac{\mathcal{T}^k-\mathcal{T}^{k+1}}{\mu^k}||_F\\
&=||\frac{\mathcal{T}^k-\mathcal{T}^{k-1}}{\mu^{k-1}}+\frac{\Gamma^{-1}(\dot{\mathbf{C}}_4^k)}{\mu^k}+\frac{\mathcal{T}^k-\mathcal{T}^{k+1}}{\mu^k}||_F=\frac{1}{\mu^k}||\mathcal{C}_5^k||_F,
\end{aligned}
$$

where $\mathcal{C}_5^k=\rho(\mathcal{T}^k-\mathcal{T}^{k-1})+\Gamma^{-1}(\dot{\mathbf{C}}_4^k)+\mathcal{T}^k-\mathcal{T}^{k+1}$. Therefore, it follows that $\{\mathcal{S}^k\}$ is a Cauchy sequence.

2. By applying the first-order optimization condition to equations (26), (37) and (38), and let $\dot{\mathbf{U}}^\star$, $\dot{\mathbf{V}}^\star$, $\dot{\mathbf{Q}}_U^\star$, $\dot{\mathbf{Q}}_V^\star$, $\dot{\mathbf{X}}^\star$ and $\mathcal{S}^\star$ denote the accumulation points of the sequences $\{\dot{\mathbf{U}}^k\}$, $\{\dot{\mathbf{V}}^k\}$, $\{\dot{\mathbf{Q}}_U^k\}$, $\{\dot{\mathbf{Q}}_V^k\}$, $\{\dot{\mathbf{X}}^k\}$ and $\{\mathcal{S}^k\}$, respectively. We observe that since these sequences are Cauchy sequences, it follows that $\dot{\mathbf{U}}^\star=\dot{\mathbf{Q}}_U^\star$, $\dot{\mathbf{V}}^\star=\dot{\mathbf{Q}}_V^\star$, $\dot{\mathbf{X}}^\star=\dot{\mathbf{U}}^\star(\dot{\mathbf{V}}^\star)^H$ and $\Gamma^{-1}(\dot{\mathbf{X}}^\star)=\mathcal{S}^\star$. Thus, as $k\to+\infty$, we have:

$$\begin{cases} \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_U^\star||_{\mathbf{w},F} - \dot{\mathbf{L}}_1^\star, \\ \dot{\mathbf{0}} \in \lambda\partial||\dot{\mathbf{Q}}_V^\star||_{\mathbf{w},F} - \dot{\mathbf{L}}_2^\star, \\ \mathbf{0} \in \alpha\partial\Phi_{\mathsf{pnp}}(\mathcal{S}^\star) - \mathcal{T}^\star, \\ \mathcal{P}_\Omega(\dot{\mathbf{X}}^\star) - \dot{\mathbf{Y}} + \mathcal{P}_\Omega(\dot{\mathbf{L}}_3^\star) + \mathcal{P}_\Omega(\Gamma(\mathcal{T}^\star)) = \dot{\mathbf{0}}, \\ \mathcal{P}_{\Omega^c}(\dot{\mathbf{L}}_3^\star) + \mathcal{P}_{\Omega^c}(\Gamma(\mathcal{T}^\star)) = \dot{\mathbf{0}}. \end{cases}$$

Therefore, any accumulation points $\{(\dot{\mathbf{U}}^\star,\ \dot{\mathbf{V}}^\star,\ \dot{\mathbf{Q}}_U^\star,\ \dot{\mathbf{A}}_V^\star,\ \dot{\mathbf{X}}^\star,\ \mathcal{S}^\star)\}$ of the sequence $\{(\dot{\mathbf{U}}^k,\ \dot{\mathbf{V}}^k,\ \dot{\mathbf{Q}}_U^k,\ \dot{\mathbf{Q}}_V^k,\ \dot{\mathbf{X}}^k,\ \mathcal{S}^k)\}$ generated by DeepQRLMF satisfies the KKT conditions.

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