

Computation with Absolutely No Space Overhead

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Technical University of Berlin

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Outline

- 1 The Model of Overhead-Free Computation
 - The Standard Model of Linear Space
 - Our Model of Absolutely No Space Overhead
- 2 The Power of Overhead-Free Computation
 - Palindromes
 - Linear Languages
 - Context-Free Languages with a Forbidden Subword
 - Languages Complete for Polynomial Space
- 3 Limitations of Overhead-Free Computation
 - Linear Space is Strictly More Powerful



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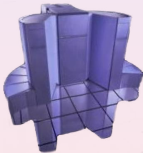
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The Standard Model of Linear Space

tape

0	0	1	0	0	1	0	0
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Turing machine

Characteristics

- Input fills **fixed-size** tape
- Input may be **modified**
- Tape alphabet **is larger than** input alphabet

The Standard Model of Linear Space

tape

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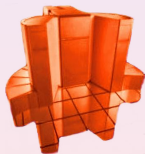
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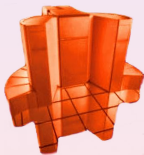
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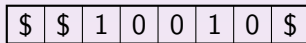
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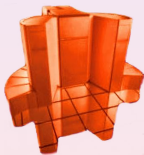
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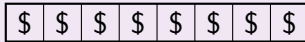
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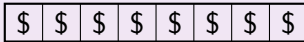
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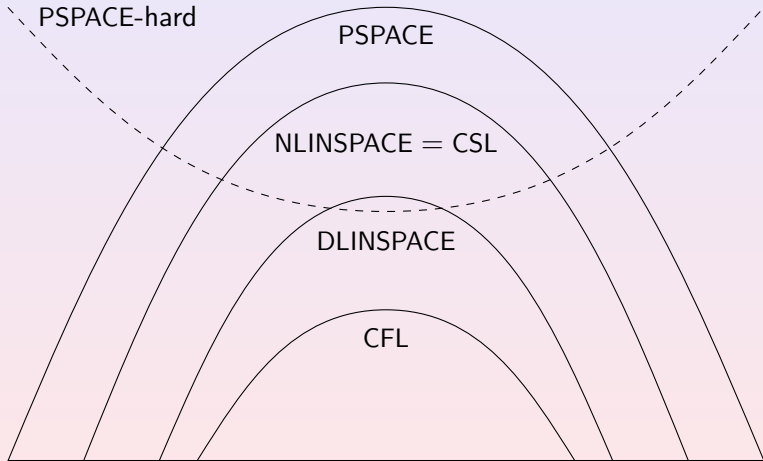


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Linear Space is a Powerful Model



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Our Model of “Absolutely No Space Overhead”

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Our Model of “Absolutely No Space Overhead”



Turing machine

Intuition

- Tape is used like a RAM module.

Definition of Overhead-Free Computations

Definition

A Turing machine is **overhead-free** if

- it has only a single tape,
- writes only on input cells,
- writes only symbols drawn from the input alphabet.

Overhead-Free Computation Complexity Classes

Definition

A language $L \subseteq \Sigma^*$ is in

DOF if L is accepted by a deterministic overhead-free machine with input alphabet Σ ,

DOF_{poly} if L is accepted by a deterministic overhead-free machine with input alphabet Σ in polynomial time.

NOF is the nondeterministic version of DOF ,

NOF_{poly} is the nondeterministic version of DOF_{poly} .



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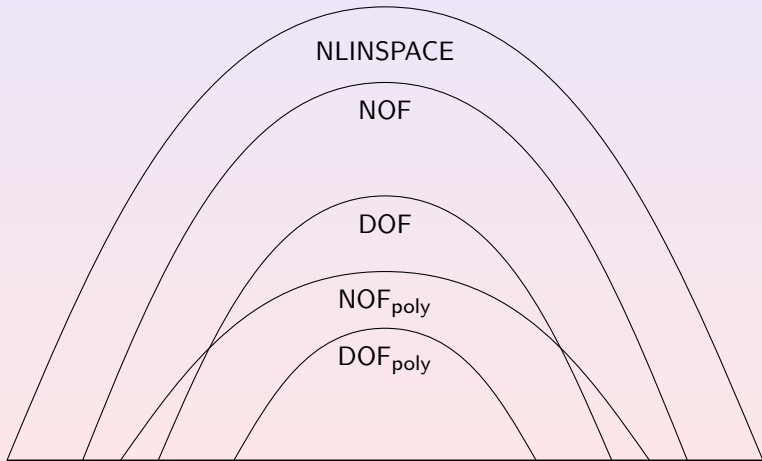
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Simple Relationships among Overhead-Free Computation Classes



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Palindromes Can be Accepted in an Overhead-Free Way

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overhead-free machine

Algorithm

Phase 1:

Compare first and last bit

Place left end marker

Place right end marker

Phase 2:

Compare bits next to end markers

Find left end marker

Advance left end marker

Find right end marker

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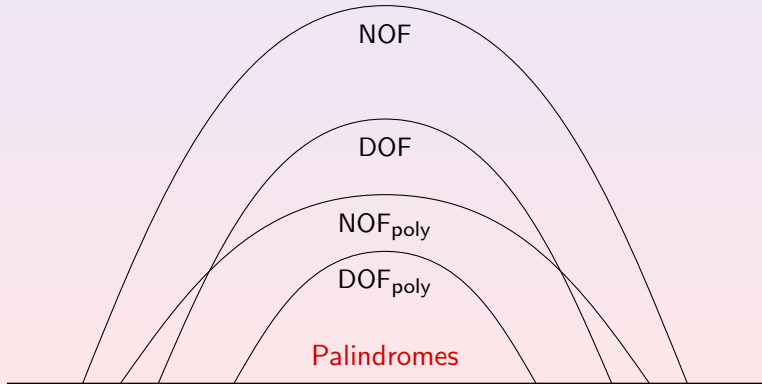
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Relationships among Overhead-Free Computation Classes



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A Review of Linear Grammars

Definition

A grammar is **linear** if it is context-free and there is only one nonterminal per right-hand side.

Example

$G_1: S \rightarrow 00S0 \mid 1$ and $G_2: S \rightarrow 0S10 \mid 0$.

Definition

A grammar is **deterministic** if
“there is always only one rule that can be applied.”

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$G_1: S \rightarrow 00S0 \mid 1$ is deterministic.
 $G_2: S \rightarrow 0S10 \mid 0$ is **not** deterministic.



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Deterministic Linear Languages Can Be Accepted in an Overhead-Free Way

Theorem

Every deterministic linear language is in DOF_{poly} .

Metalinear Languages Can Be Accepted in an Overhead-Free Way

Definition

A language is **metalinear** if it is the concatenation of linear languages.

Example

$\text{TRIPLE-PALINDROME} = \{uvw \mid u, v, \text{ and } w \text{ are palindromes}\}.$

Theorem

Every metalinear language is in NOF_{poly} .



Metalinear Languages Can Be Accepted in an Overhead-Free Way

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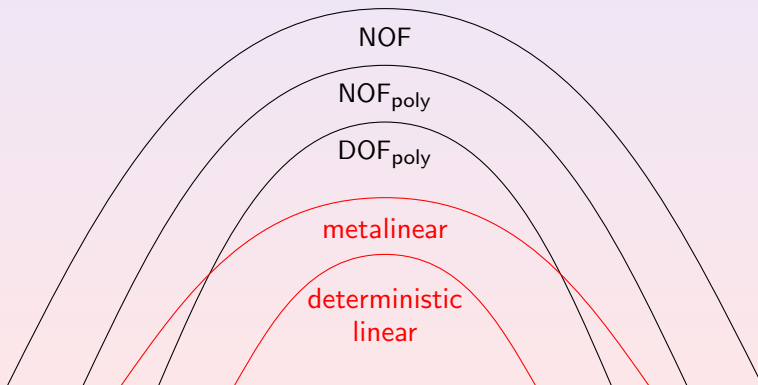
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Definition of Almost-Overhead-Free Computations

Definition

A Turing machine is **almost-overhead-free** if

- it has only a single tape,
- writes only on input cells,
- writes only symbols drawn from the input alphabet
plus one special symbol.

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Context-Free Languages with a Forbidden Subword Can Be Accepted in an Overhead-Free Way

Theorem

Let L be a context-free language with a forbidden word.
Then $L \in \text{NOF}_{\text{poly}}$.

► Skip proof

Context-Free Languages with a Forbidden Subword Can Be Accepted in an Overhead-Free Way

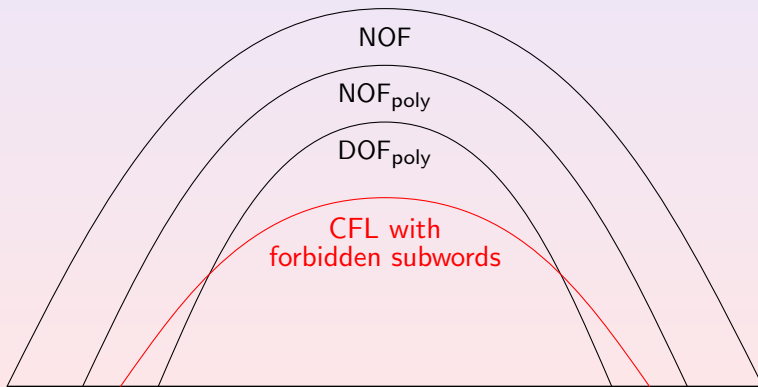
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Proof.

Every context-free language can be accepted by a nondeterministic almost-overhead-free machine in polynomial time.

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Overhead-Free Languages can be PSPACE-Complete

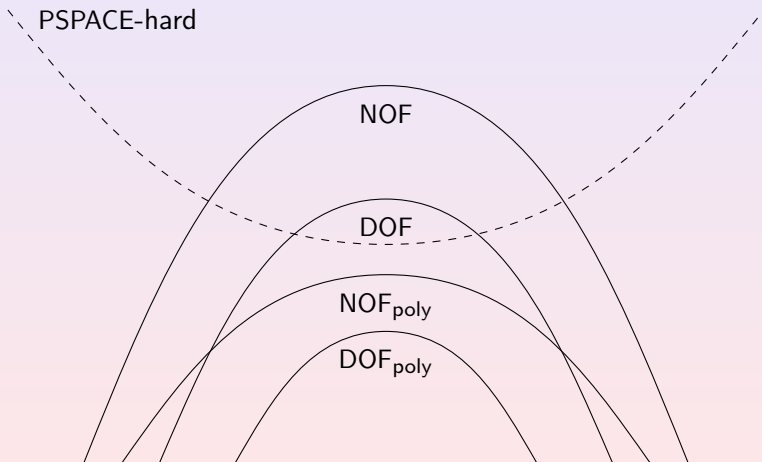
Theorem

DOF contains languages that are complete for PSPACE.

► Proof details



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Some Context-Sensitive Languages Cannot be Accepted in an Overhead-Free Way

Theorem

$\text{DOF} \subsetneq \text{DLINSPACE}$.

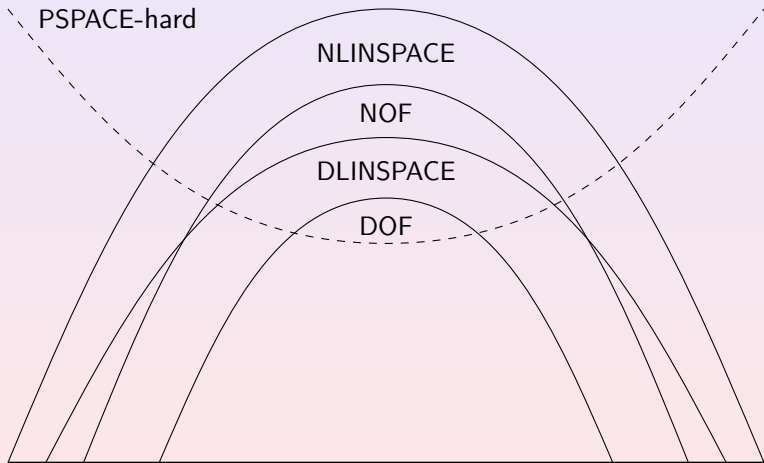
Theorem

$\text{NOF} \subsetneq \text{NLINSPACE}$.

The proofs are based on old diagonalisations due to Feldman, Owings, and Seiferas.



Relationships among Overhead-Free Computation Classes



Candidates for Languages that Cannot be Accepted in an Overhead-Free Way

Conjecture

DOUBLE-PALINDROMES \notin DOF.

Conjecture

$\{ww \mid w \in \{0,1\}^*\} \notin$ NOF.

Proving the first conjecture would show $\text{DOF} \subsetneq \text{NOF}$.



Candidates for Languages that Cannot be Accepted in an Overhead-Free Way

Theorem

$\text{DOUBLE-PALINDROMES} \in \text{DOF}.$

Conjecture

$\{ww \mid w \in \{0,1\}^*\} \notin \text{NOF}.$

Proving the first conjecture would show $\text{DOF} \subsetneq \text{NOF}.$



Summary

- Overhead-free computation is a more faithful **model of fixed-size memory**.
- Overhead-free computation is **less powerful** than linear space.
- **Many** context-free languages can be accepted by overhead-free machines.
- We conjecture that **all** context-free languages are in NOF_{poly} .
- Our results can be seen as new results on the power of **linear bounded automata with fixed alphabet** size.



For Further Reading



A. Salomaa.

Formal Languages.

Academic Press, 1973.



E. Dijkstra.

Smoothsort, an alternative for sorting in situ.

Science of Computer Programming, 1(3):223–233, 1982.



E. Feldman and J. Owings, Jr.

A class of universal linear bounded automata.

Information Sciences, 6:187–190, 1973.



P. Jančar, F. Mráz, M. Plátek, and J. Vogel.

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Appendix Outline

4 Appendix

- Complete Languages
- Improvements for Context-Free Languages
- Abbreviations



Overhead-Free Languages can be PSPACE-Complete

Theorem

DOF contains languages that are complete for PSPACE.

Proof.

- Let $A \in \text{DLINSPACE}$ be PSPACE-complete.
Such languages are known to exist.
- Let M be a linear space machine that accepts $A \subseteq \{0, 1\}^*$ with tape alphabet Γ .
- Let $h: \Gamma \rightarrow \{0, 1\}^*$ be an isometric, injective homomorphism.
- Then $h(L)$ is in DOF and it is PSPACE-complete.

◀ Return



Improvements

Theorem

1. $\text{DCFL} \subseteq \text{DOF}_{\text{poly}}$.
2. $\text{CFL} \subseteq \text{NOF}_{\text{poly}}$.

Explanation of Different Abbreviations

DOF	Deterministic Overhead-Free.
NOF	Nondeterministic Overhead-Free.
DOF _{poly}	Deterministic Overhead-Free, polynomial time.
DOF _{poly}	Nondeterministic Overhead-Free, polynomial time.

Table: Explanation of what different abbreviations mean.

